Article

Chaotic dynamics of the fractional-love model with an external force

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Abstract: Based on the fractional order of nonlinear system for love model with a periodic function as an external force, analyzed the characteristics of the chaotic dynamic in this study. The relationship between the chaotic dynamic of the fractional-love model with the external force and the fractional-order system was analyzed when the parameters are fixed. Further, we also studied the relationship between the chaotic systemic dynamic and the parameters when the fractional-order system is fixed. The results show that when the parameters are fixed, the fractional-order system exhibited segmented chaotic states for the different fractional orders of the system. When fixed the fractional-order system, the system exhibited the periodic and chaotic states as parameter changes.

Keywords: Chaotic dynamic; nonlinear system; fractional order; Love model; parameter;

1. Introduction

Fractional calculus is a generalization of the integer-order calculus, which shares the same history length as the study of integer-calculus theory. Before 1960, though, the study of fractional-order systems rarely attracted the attention of researchers, until several decades later, especially after the discovery of some physical systems to show the fractional-order dynamic characteristics, the fractional-system research received an increasing level of attention; now, the fractional system has become a hot global research topic.

In recent years, with the deep research and exploration of chaos systems, many researchers have proposed a number of fractional chaotic systems based on the integer-chaos system such as the fractional Rössler system [1-2], fractional Chen system [3-4], fractional Liu system [5-6], and fractional Lorenz system [7-8], among others. In this paper, the focus is the fractional love model with an external force.

Over the last three decades, many researchers studied chaotic dynamics in numerous fields such as mathematics, physics, chemistry, engineering, and social science [9-14]. In particular, the chaotic behaviors of the habits and minds of humans like addiction [15-16], happiness [17-18], and the "love model" [19-21] in terms of the social sciences have been studied.

Strogatz [18] and Sprott [19], who also explained the behavior of linear and nonlinear systems with respect to the love model, proposed the love model based on Shakespeare's "Romeo and Juliet". Actually in mathematics, the love model is not only based on Romeo and Juliet, but it can also be defined as the Laura and Petrarch model [22-23] and the Adam and Eve model [24]; however, the "Romeo and Juliet" model is commonly used in the study of nonlinear dynamics by researchers. There are many researchers do study the "Romeo and Juliet" love model to deal with the existence of periodic and chaotic motions. For example, Wauer et.al [25] proposed and analyzed the dynamical models of love with time-varying fluctuations. Son and Park [26] proposed the time-delay effect on the love-dynamic model with the Hopf bifurcation and a periodic-doubling bifurcation diagram. Bae [27-33] proposed that the existence of the periodic and chaotic behaviors that are based on the love

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model of "Romeo and Juliet" are represented through the time series and phase portraits with the same and different time delays and an external force.

Although, there are many research do study for love by using several of models, actually, in real life, we know the love is so complex and uncertainty, so it is difficult to exactly describe the real love status. Until now, most of model of love are based on the Strogatz and Sprott, who explained the behavior of linear and nonlinear systems of integer-order love model, but no one proposed the fractional-love model. Recently many researchers have proposed a number of fractional chaotic systems based on the integer-chaos system, because fractional order can better reflect the system changes. Particularly, compared to the integer order, the fractional order can reflect the "memory dependency" of certain dynamic processes to a certain extent, which means that the current state is dependent on the past state. In love model, whether two people have a dependency on the memory will have a great impact on the results, so the fractional-love model is more convincing and closer to real life than integer-love model. Therefore, in this paper, we proposed the fractional-love model.

In order to produce chaotic behavior in the dynamic system, the dynamic system needs to be three-dimensional with at least one nonlinear term, so in love model of two people, we need to add the external force to make the system as three-dimensional. There are many functions can be used as external force, however, in order to make the system closer to real life, we choose the sine-wave function be-cause it represents positive and negative characteristics and it is similar to human characteristics. Therefore, in this paper, the focus is the fractional love model with sine wave as external force. To analyze the chaotic dynamic of the present system more effectively, the computer simulation obtained the time series, phase portrait, power spectrum, Poincare map, Maximal Lyapunov exponent (LE), and bifurcation diagram to analyze the fixed parameters. The relationship between the chaotic dynamic of the fractional-love model with the external force and the fractional-order system was analyzed when the parameters are fixed. Further, the relationship between the chaotic systemic dynamic and the parameters was also studied when the fractional-order system is fixed.

2. Love model

Strogatz [18] proposed the love model for "Romeo and Juliet" with the linear-differential of Eq. (1):

$$\mathrm{dR}/dt = aR + bJ,$$
$$\mathrm{dJ}/dt = cR + dJ \tag{1}$$

Where，the parameters a and b describe the Romeo's feelings, and c and d describe the Juliet's feelings. There are four situations for the Romeo romantic style, which are based on the parameters and were suggested by Strogatz and his students [18], including 'eager beaver' (a >0, b>0), 'narcissistic nerd' (a>0, b<0), 'cautious lover' (a<0, b>0), and 'hermit' (a<0, b<0).

The simple system is the linear system for which the allowable dynamics are limited, so Eq. (1) can rewrite as Eq. (2) through the addition of the nonlinear term.

$$\mathrm{dR}/dt = aR + bJ(1-J),$$
$$\mathrm{dJ}/dt = cR(1-R) + dJ \tag{2}$$

Eq. (2), however, cannot use to produce chaotic behavior because the order of Eq. (2) is only two-dimensional. To produce chaotic behavior in the dynamic system, the dynamic system needs to be three-dimensional with at least one nonlinear term. Eq. (3) can rewrite into a third-order system through the addition of the 5sin (πt) as an external force as follows:

$$\mathrm{dR}/dt = aR + bJ(1-J) + 5\sin(\pi t),$$
$$\mathrm{dJ}/dt = cR(1-R) + dJ \tag{3}$$

Bae [30-31] has proposed a love model of "Romeo and Juliet" wherein the sine wave is an existent external force of the periodic and chaotic behaviors. In such a case, the sum of the system

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order is 3. In the next section, the focus is the fraction-love model of "Romeo and Juliet" for which the sine wave is an external force and the sum of the system order is gradually reduced.

3. Chaotic dynamics of the fractional-love model with the external force

Eq. (3) can be modified to Eq. (4) through the addition of the fractional order:

$$d^{\alpha}R/dt^{\alpha} = aR + \mathrm{b}J(1-J) + 5\sin(\pi t),$$
$$d^{\beta}J/dt^{\beta} = \mathrm{c}R(1-R) + \mathrm{d}J \quad (4)$$

Where α, β are the fractional orders of the system.

3.1. Analysis of the systemic dynamics of the fixed parameters

In this section, the parameters were fixed as a = - 1.5, b = - 2, and c = d = 1, followed by the setting of 0<α=β=q≤1, so that Eq. (4) can be rewritten as Eq. (5):

$$d^{q}R/dt^{q} = (-1.5)R + (-2)J(1-J) + 5\sin(\pi t),$$
$$d^{q}J/dt^{q} = R(1-R) + J \quad (5)$$

The fraction q can be changed from 1 to 0.5 by 0.05 steps to study the chaotic dynamic of the system. The time series, phase portrait, power spectrum, Poincaré map, and maximal LE can use to prove the chaotic behavior.

3.1.1. Time series

The chaotic time series is a definite motion that determines the presence of a system. The chaotic time series is a time series with chaotic-model characteristics. The chaotic time series contains the rich dynamic information of the system. The study of chaos from the time series began with Packard et.al. (1980), who proposed the reconstruction-phase space theory [35]. In terms of the chaotic time series, it is known that the periodic motion corresponds to the rules sequence, and the chaotic motion corresponds to the irregular sequence; therefore, it is possible to intuitively determine whether the system is chaotic by observing the systemic time series. The fraction q can be changed from 1 to 0.5 by 0.05 steps to obtain the time series through the implementation of a computer simulation, as shown in Figure 1.

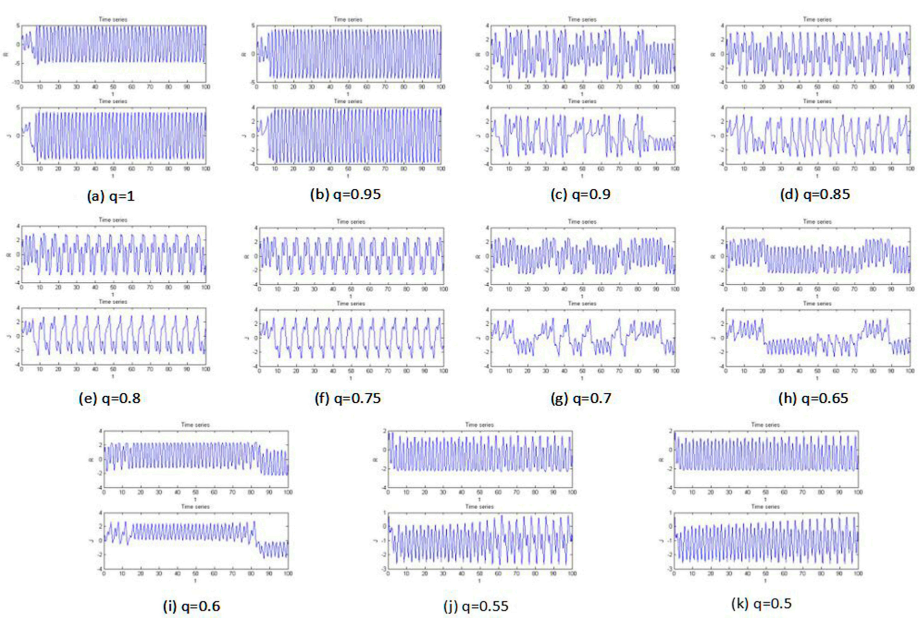

Figure 1. Time series of the system with different fractional-order q values.

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From Figure 1, when the fraction q is equal to the four situations 1, 0.95, 0.8, and 0.75, the time series shows the regular sequence, while the fraction q is equal to 0.9, 0.85, 0.8, 0.65, and 0.6, the time series shows the irregular sequence. However, the q situations that are equal to 0.55 and 0.5 are not easy to distinguish. Therefore, the dynamic characteristics of the fractional-love model with the external force can initially understand, but to further know the dynamic characteristics of the system, the phase portrait must be observed.

3.1.2. Phase portrait

According to the direct-observation method, the periodic motion in the phase space corresponds to the closed curve, and the chaotic motion corresponds to the trajectory of the random separation in a certain region. Therefore, by observing the phase portrait of the fractional-love model with the external-force system, it is possible to further determine whether the system is chaotic or not. The results of the systemic phase portrait shown in Figure 2.

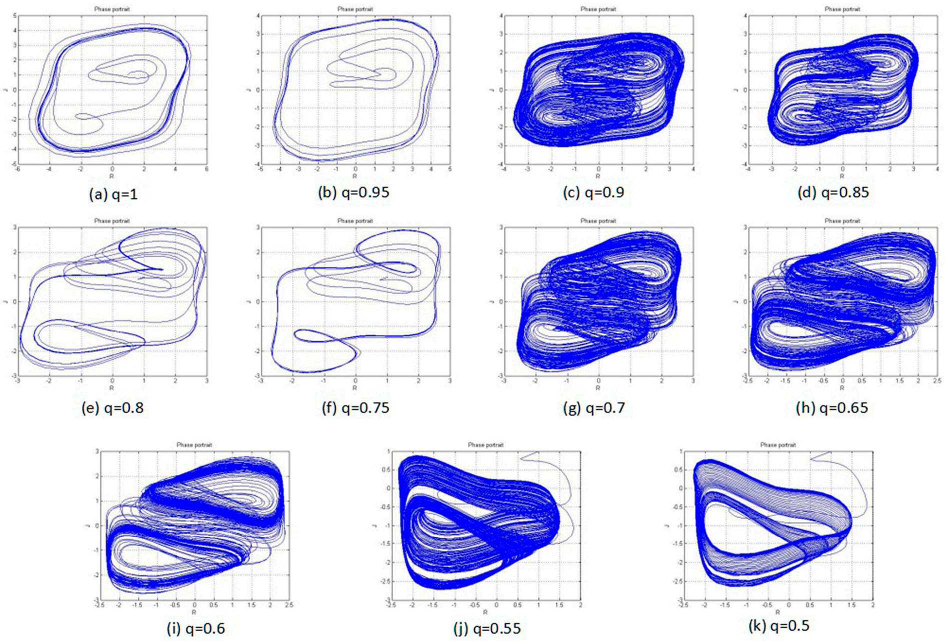

Figure 2. Phase portrait of the system with different fractional-order q values.

From Figure 2, it is possible to see that when the fraction q is equal to the four situations 1, 0.95, 0.8, and 0.75, the phase-portrait curve is a limit cycle or converges to a single point, which indicates that the system is in a periodic state at these moments. In the remaining cases, the phase-portrait variables exist in a random separation state—that is, chaotic attractors—indicating that, for these cases, the system is in a chaotic state. From the time-series and phase-portrait results, the fractional-order system exhibited segmented chaotic states. The observation method here, however, is only a qualitative analysis, so this conclusion needs to be further verified.

3.1.3. Power spectrum

A power-spectrum analysis can provide the frequency-domain information of the signal. From an analysis of the power spectrum, it is possible to observe whether the systemic characteristics are chaotic or not. For the periodic motion, the power spectrum is a discrete spectrum, while for the chaotic motion the power spectrum is a continuous spectrum. It is possible to determine whether the system is chaotic by plotting the power spectrum of the system-generated signal. The power spectrum of the system shown in Figure 3.

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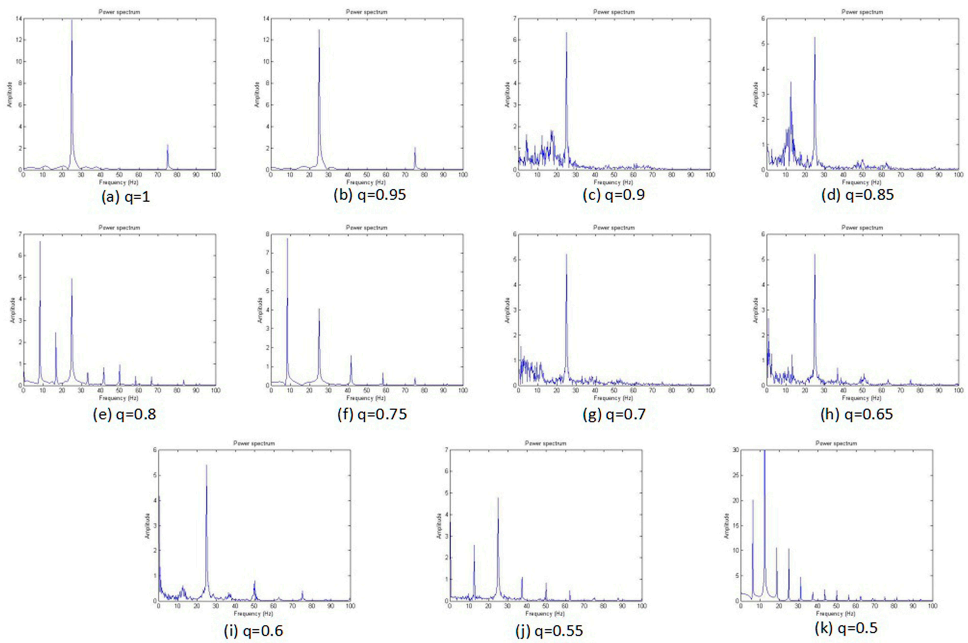

Figure 3. Power spectrum of the system with different fractional-order q values.

Figure 3 shows that when the fraction q is equal to the four cases of 1, 0.95, 0.8, and 0.75, the power spectrum is a discrete spectrum. At these moments, the system is in a periodic state. In the remaining situations, the power spectrum is a continuous spectrum, or a chaotic state, so the system exhibits a segmented chaotic state. This conclusion is consistent with the phase-portrait results.

3.1.4. Poincaré map

For the selection of a cross section in a multi-dimensional phase space, this section can be both a plane and a surface. Then, a point series of the continuous dynamic orbit that intersects with the cross section can consider. From the cut point on the Poincaré map, the motion-characteristics information can obtain. Different forms of motion pass through the cross section, and the intersectional cross section comprises different distribution characteristics, as follows:

(1) The periodic motion leaves a limited number of discrete points on this cross section.

(2) The quasi-periodic motion leaves a closed curve on the cross section.

(3) The chaotic motion is along a line or a curved-arc distribution point that is set on the cross section.

Therefore, the points, which are left on the Poincaré map to judge the system status, can be observed. Figure 4 shows the results of the Poincaré map as the fractional order was changed.

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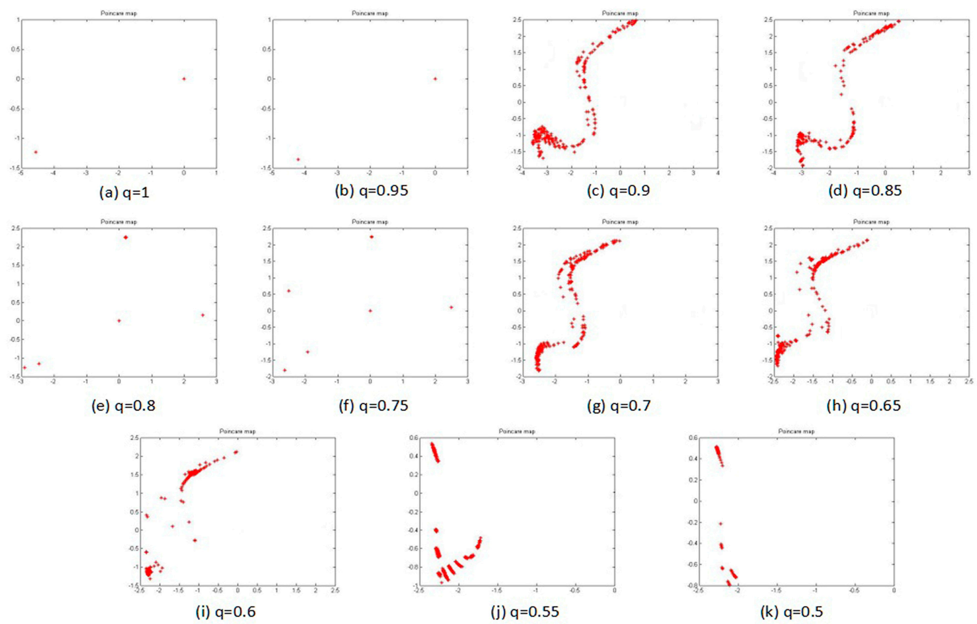

Figure 4. Poincaré map of the system with different fractional-order q values.

From Figure 4, the points when the fraction q is equal to 1, 0.95, 0.8, and 0.75 can be clearly seen, and there is a number of discrete points on the Poincare map that indicate that the system exhibits the periodic motion. The remaining situations are along a line distribution of points on the Poincare map, so the system exhibits the chaotic motion. These results are consistent with the results of the phase portrait and the power spectrum, so until now, the system certainly presents a segmented chaotic state.

So far, all of the methods are used to qualitatively analyze the dynamics of the system. In the next section, the maximal LE of the system is calculated to quantitatively analyze the dynamics of the system.

3.1.5. Maximal Lyapunov exponent (MLE)

The MLE, one of the important dynamic-characteristic measurements of the system, characterizes the average exponential rate of the convergence or the divergence of the system variables in the adjacent phase-space orbits. Especially, the maximal LE determines whether the adjacent trajectories can move closer to each other to form a stable or unstable point. If the maximal LE is less than 0, the system shows the periodic motion, while a maximal LE of more than 0 shows a chaotic systemic motion. Therefore, it is possible to calculate the maximal LE of the system to quantitatively analyze whether the system is in a chaotic state. Table 1 shows the maximal LE of the system as different fractional orders when the parameters are fixed.

Table 1. MLE of the system with different fractional orders when the parameters are fixed.

Fractional order (q)	MLE (λ)	Dynamic state
q = 1	λ= - 0.0458	periodic
q = 0.95	λ= - 0.0299	periodic
q = 0.9	λ= 0.0711	chaotic
q = 0.85	λ= 0.3467	chaotic
q = 0.8	λ= - 0.0331	periodic
q = 0.75	λ= - 0.0344	periodic
q = 0.7	λ= 0.2456	chaotic
q = 0.65	λ= 0.3585	chaotic

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q = 0.6	λ= 0.1835	chaotic
q = 0.55	λ= 0.0754	chaotic
q = 0.5	λ= 0.0357	chaotic

From Table 1, it is possible to clearly know when the fraction q is equal to 1, 0.95, 0.8, or 0.75, and when the maximal LE is less than 0, so a periodic-state system can be identified. In the other situations, the maximal LE is more than 0, so the system is in the chaotic state.

Based on all of the methods that are used, the authors conclude that the state of the fractional-love model with the external-force system is related to the fractional order. When the parameters are fixed, the system exhibited the segmented chaotic state with a different fractional order.

3.2. Analysis of the systemic dynamics of the fixed fractional orders

For this section, the fractional order of the system was fixed as 0.85, and the parameters b, c, and d were also fixed as - 2, 1, and 1, respectively; then, the parameter was changed to a for an analysis of the chaotic dynamics of the system. In addition, the time series, phase portrait, power spectrum, Poincare map, and MLE used to obtain the results. It is worth mentioning that the bifurcation-diagram method was added for this section to verify the results. Figures 5 to 10 show the time series, phase portrait, power spectrum, and Poincare map of the system when parameter a is a=-5, a=-2.42, a=-2, a=-1.76, a=-1.53, and a=-1.45, respectively.

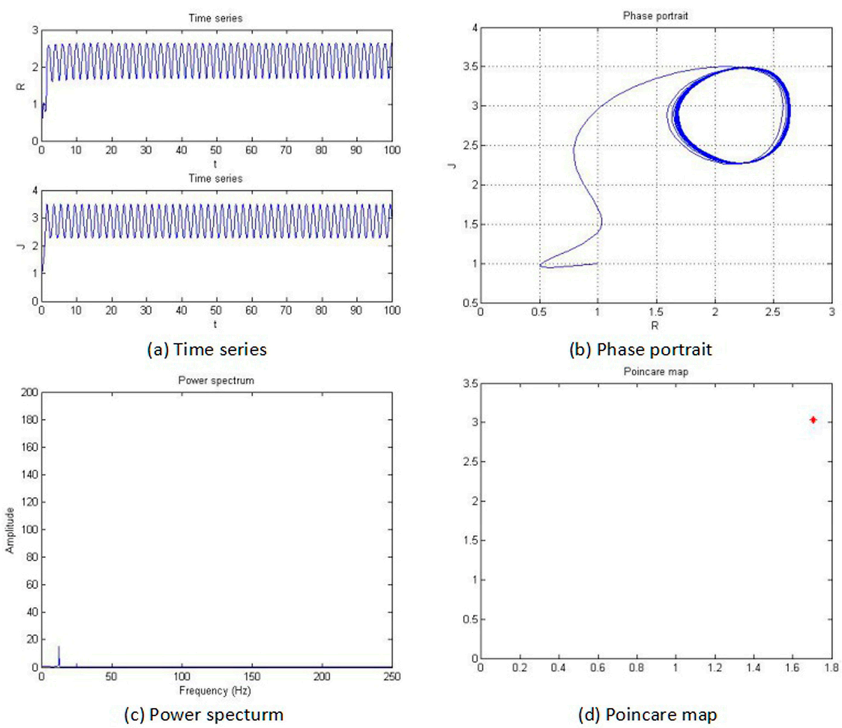

Figure 5. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 5.

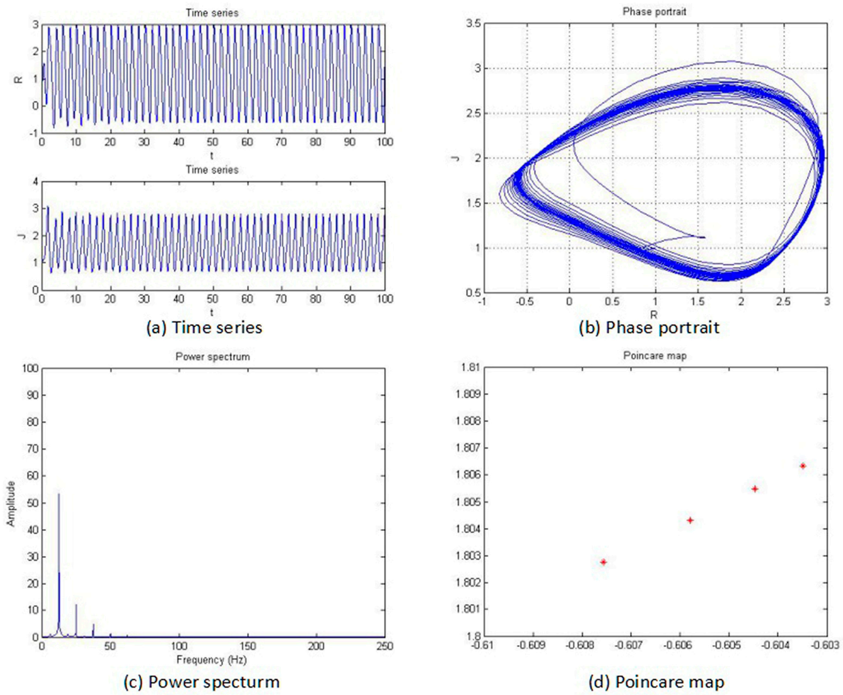

Figure 6. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 2.42.

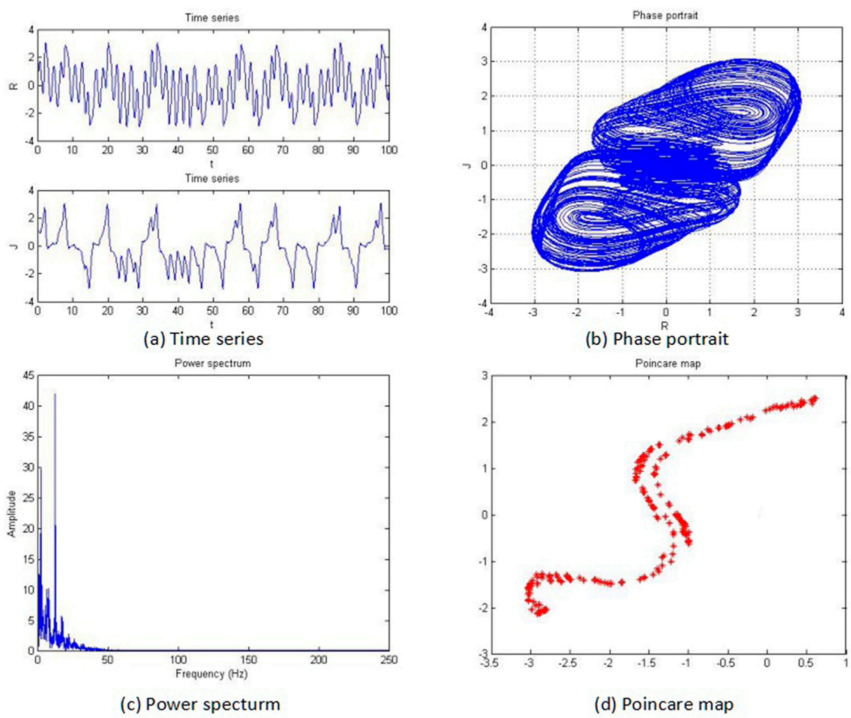

Figure 7. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 2.

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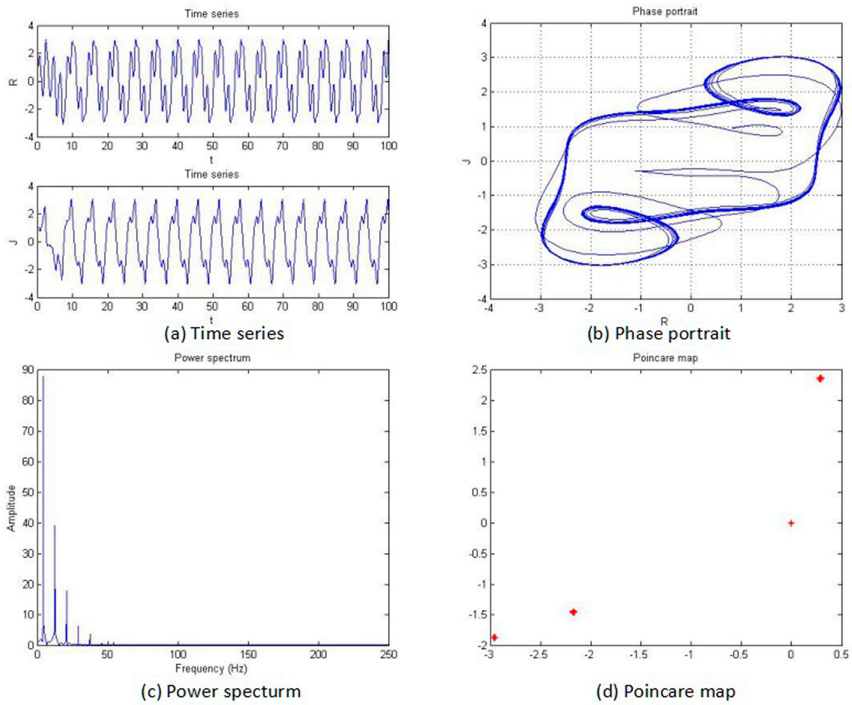

Figure 8. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 1.76.

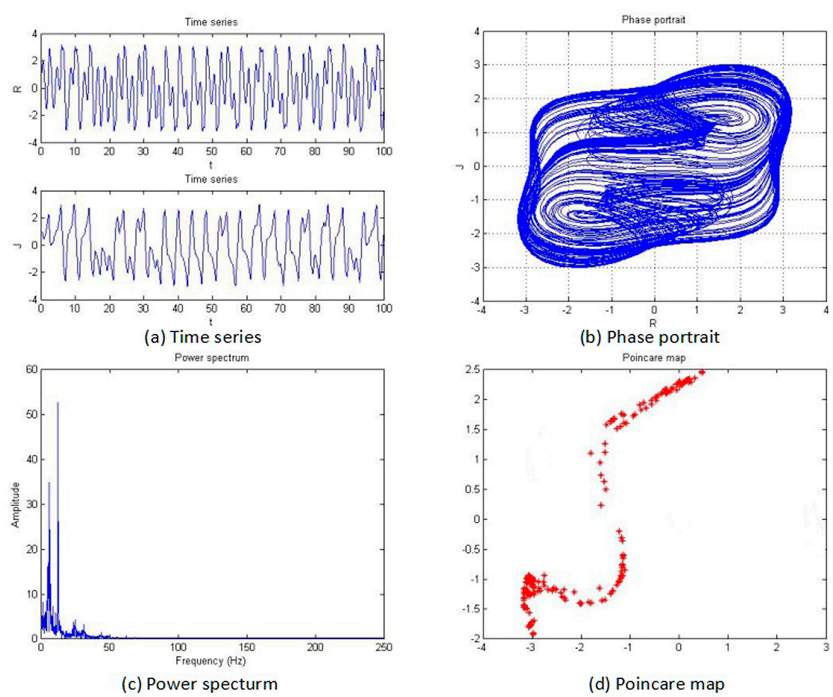

Figure 9. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 1.53.

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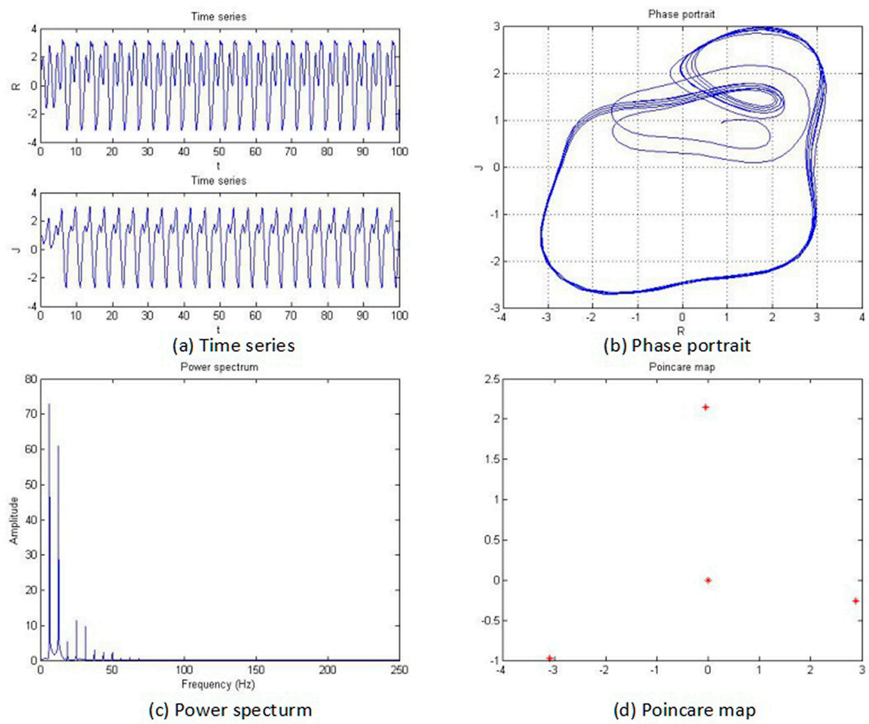

Figure 10. Time series (a), phase portrait (b), power spectrum (c), and Poincare map (d) of the system when a = - 1.45.

From Figures 5 to 10, it is possible to clearly see that when the parameter a is equal to - 2 and - 1.53, the system is in the chaotic state, and in the other situations, the system is in the periodic state. Therefore, it is possible to initially conclude that when the fractional order of the system is fixed, the system shows the periodic and chaotic states as the parameter a is changed. To verify the accuracy of the conclusion, the results of the maximal LE and the bifurcation diagram, as shown in Figure 11.

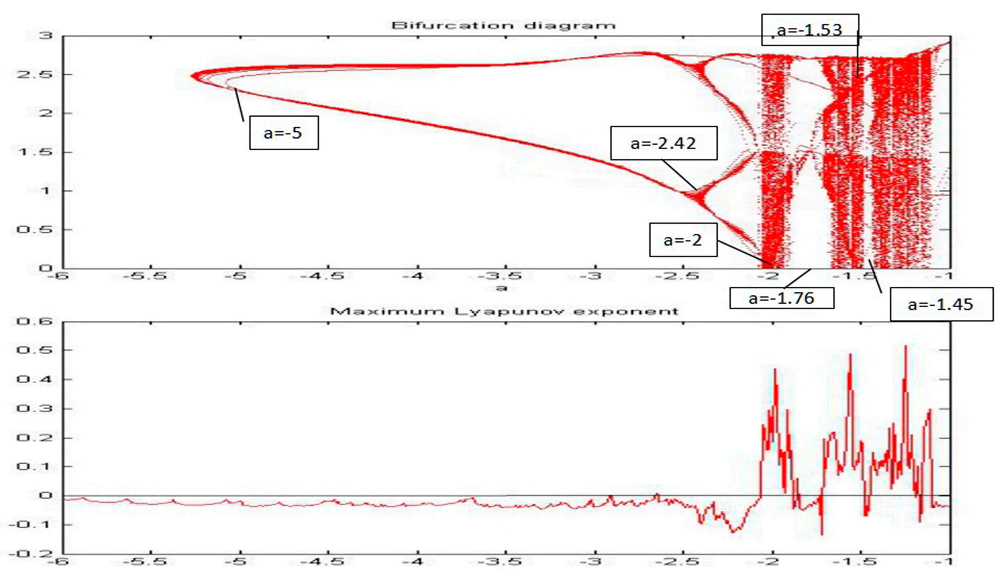

Figure 11. The Maximal Lyapunov exponent (MLE) and the bifurcation diagram of the system when a is changed from - 6 to - 1.

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From the results of the maximal LE and the bifurcation diagram, it is possible to conclude that when the fractional order of the system is fixed, the system shows the periodic and chaotic states as the parameter a is changed.

4. Conclusion

In this paper, the time series, phase portrait, power spectrum, Poincare map, maximal LE, and bifurcation diagram were used to analyze the characteristics of the chaotic dynamic of the fractional-love model with an external-force system. For the analysis, we study the following two aspects of the system: when the parameters were fixed and the fractional order of the system was changed to produce different systemic states and the relationship between the chaotic dynamic of the system and the parameters when the fractional order of the system was fixed. The results show that the chaotic dynamic of the system related to the parameters and the fractional order of the system. When the parameters are fixed, the system exhibited segmented chaotic states with the different fractional orders of the system; When the fractional order of the system was fixed, the system exhibited the periodic state and the chaotic state as the parameter a was changed. Therefore, the characteristics of the fractional-love model that comprises a sine wave as the external-force system are rich dynamic.

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