

HYPO- q -NORMS ON A CARTESIAN PRODUCT OF ALGEBRAS OF OPERATORS ON BANACH SPACES

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ABSTRACT. In this paper we introduce the hypo- q -operator norm and hypo- q -numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q -norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given.

1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . On \mathbb{K}^n endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}(\|\cdot\|_n) := \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \mid \|\boldsymbol{\lambda}\|_n \leq 1\}.$$

As an example of such norms we should mention the usual p -norms

$$(1.1) \quad \|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ (\sum_{k=1}^n |\lambda_k|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \dots \times E$ endowed with the canonical linear structure we can define the following p -norms:

$$(1.2) \quad \|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ (\sum_{k=1}^n \|x_k\|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [6], for a given norm $\|\cdot\|_n$ on \mathbb{K}^n , we define the functional $\|\cdot\|_{h,n} : E^n \rightarrow [0, \infty)$ given by

$$(1.3) \quad \|\mathbf{x}\|_{h,n} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}(\|\cdot\|_n)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|,$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

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It is easy to see, by the properties of the norm $\|\cdot\|$, that:

- (i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
- (ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}, \mathbf{y} \in E^n$;
- (iii) $\|\alpha\mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n . This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n .

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0), (0, \lambda_2^0, \dots, 0), \dots, (0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n .

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p -norms $\|\cdot\|_{n,p}$ on \mathbb{K}^n , then we can obtain the following *hypo- q -norms* on E^n :

$$(1.4) \quad \|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|,$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in \mathbb{K}^n , which we denote by \mathbb{B}_n , $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \mid \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right\}$ that generates the *hypo-Euclidean norm* on E^n , i.e.,

$$(1.5) \quad \|\mathbf{x}\|_{h,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|.$$

Moreover, if $E = H$, H is a inner product space over \mathbb{K} , then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$(1.6) \quad \|\mathbf{x}\|_e := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|.$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over \mathbb{K} and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \dots \times H$, for the n -tuples of vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$(1.7) \quad \langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \quad \mathbf{x}, \mathbf{y} \in H^n,$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n , i.e.,

$$(1.8) \quad \|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \quad \mathbf{x} \in H^n.$$

The following result established in [6] connects the usual Euclidean norm $\|\cdot\|_2$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [6]). *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$(1.9) \quad \frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2,$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n .

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [6]). *For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, we have*

$$(1.10) \quad \|\mathbf{x}\|_e = \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}.$$

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|u\|=1} |f(u)| < \infty, \text{ where } f \in E^*.$$

We have the following representation result for the *hypo- q -norms* on E^n plays a key role in obtaining different bounds for these norms, see [8]:

Theorem 3 (Dragomir, 2017, [8]). *Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . For any $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, we have*

$$(1.11) \quad \|\mathbf{x}\|_{h,n,q} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^n |f(x_j)|^q \right)^{1/q} \right\}$$

where $q \geq 1$, and

$$(1.12) \quad \|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\}.$$

We have the following inequalities of interest:

Corollary 1. *With the assumptions of Theorem 3 we have for $q \geq 1$ that*

$$(1.13) \quad \frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{n,q}$$

for any $\mathbf{x} \in E^n$.

We have for $r \geq q \geq 1$ that

$$(1.14) \quad \|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r}$$

for any $\mathbf{x} \in E^n$.

In this paper we introduce the hypo- q -operator norms and hypo- q -numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q -norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given.

2. SEMI-INNER PRODUCTS AND PRELIMINARY RESULTS

In what follows, we assume that E is a linear space over the real or complex number field \mathbb{K} .

The following concept was introduced in 1961 by G. Lumer [11] but the main properties of it were discovered by J. R. Giles [12], P. L. Papini [19], P. M. Miličić [14]–[16], I. Roşca [20], B. Nath [18] and others, see also [2].

In this section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 1. The mapping $[\cdot, \cdot] : E \times E \rightarrow \mathbb{K}$ will be called the semi-inner product in the sense of Lumer-Giles or L-G-s.i.p., for short, if the following properties are satisfied:

- (i) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in E$;
- (ii) $[\lambda x, y] = \lambda [x, y]$ for all $x, y \in E$ and λ a scalar in \mathbb{K} ;
- (iii) $[x, x] \geq 0$ for all $x \in E$ and $[x, x] = 0$ implies that $x = 0$;
- (iv) $|[x, y]|^2 \leq [x, x][y, y]$ (Schwarz's inequality) for all $x, y \in E$;
- (v) $[x, \lambda y] = \lambda [x, y]$ for all $x, y \in E$ and λ a scalar in \mathbb{K} .

The following results collect some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 1. Let E be a linear space and $[\cdot, \cdot]$ a L-G-s.i.p. on E . Then the following statements are true:

- (i) The mapping $E \ni x \xrightarrow{\|\cdot\|} [x, x]^{\frac{1}{2}} \in \mathbb{R}_+$ is a norm on E ;
- (ii) For every $y \in E$ the functional $E \ni x \xrightarrow{f_y} [x, y] \in \mathbb{K}$ is a continuous linear functional on E endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality $\|f_y\| = \|y\|$.

Definition 2. The mapping $J : E \rightarrow 2^{E^*}$, where E^* is the dual space of E , given by:

$$J(x) := \{x^* \in E^* \mid \langle x^*, x \rangle = \|x^*\| \|x\|, \|x^*\| = \|x\|\}, \quad x \in E$$

will be called the normalised duality mapping of normed linear space $(E, \|\cdot\|)$.

Definition 3. A mapping $\tilde{J} : E \rightarrow E^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in E .

The following theorem due to I. Roşca [20] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 4. Let $(E, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for all } x, y \text{ in } E,$$

where \tilde{J} is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2. Let $(E, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:

- (i) E is smooth;
- (ii) There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.

We need the following lemma holding for n -tuples of complex numbers:

Lemma 1. Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$.

(i) If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(2.1) \quad \sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,q}.$$

(ii) We have

$$(2.2) \quad \sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,1} \quad \text{and} \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,\infty}.$$

Proof. (i). Using Hölder's discrete inequality for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q},$$

which implies that

$$(2.3) \quad \sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,q}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are n -tuples of complex numbers.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j} |\beta_j|^{q-2}}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}}$$

for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\begin{aligned} \left| \sum_{j=1}^n \alpha_j \beta_j \right| &= \left| \sum_{j=1}^n \frac{\overline{\beta_j} |\beta_j|^{q-2}}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}} \beta_j \right| = \frac{\sum_{j=1}^n |\beta_j|^q}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}} \\ &= \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q} = \|\beta\|_{n,q} \end{aligned}$$

and

$$\begin{aligned} \|\alpha\|_{n,p}^p &= \sum_{j=1}^n |\alpha_j|^p = \sum_{j=1}^n \frac{|\overline{\beta_j} |\beta_j|^{q-2}|^p}{(\sum_{k=1}^n |\beta_k|^q)^p} = \sum_{j=1}^n \frac{(|\beta_j|^{q-1})^p}{(\sum_{k=1}^n |\beta_k|^q)^p} \\ &= \sum_{j=1}^n \frac{|\beta_j|^{qp-p}}{(\sum_{k=1}^n |\beta_k|^q)^p} = \sum_{j=1}^n \frac{|\beta_j|^q}{(\sum_{k=1}^n |\beta_k|^q)^p} = 1. \end{aligned}$$

Therefore, by (2.3) we have the representation (2.1).

(ii). Using the properties of the modulus, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \max_{j \in \{1, \dots, n\}} |\alpha_j| \sum_{j=1}^n |\beta_j|,$$

which implies that

$$(2.4) \quad \sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,1},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \sum_{j=1}^n \frac{\overline{\beta_j}}{|\beta_j|} \beta_j \right| = \sum_{j=1}^n |\beta_j| = \|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} |\alpha_j| = \max_{j \in \{1, \dots, n\}} \left| \frac{\overline{\beta_j}}{|\beta_j|} \right| = 1$$

and by (2.4) we get the first representation in (2.2).

Moreover, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \sum_{j=1}^n |\alpha_j| \max_{j \in \{1, \dots, n\}} |\beta_j|,$$

which implies that

$$(2.5) \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,\infty},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, let $j_0 \in \{1, \dots, n\}$ such that $\|\beta\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} |\beta_j| = |\beta_{j_0}|$. Consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j = 0$ for $j \neq j_0$. For this choice we get

$$\sum_{j=1}^n |\alpha_j| = \frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|} = 1 \quad \text{and} \quad \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|} \beta_{j_0} \right| = |\beta_{j_0}| = \|\beta\|_{n,\infty},$$

therefore by (2.5) we obtain the second representation in (2.2). \square

Theorem 5. Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} and $[\cdot, \cdot]$ a L-G-s.i.p. on E that generates the norm $\|\cdot\|$, i.e. $[x, x]^{1/2} = \|x\|$. For any $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, we have

$$(2.6) \quad \|\mathbf{x}\|_{h,n,q} = \sup_{\|u\|=1} \left\{ \left(\sum_{j=1}^n |[x_j, u]^q \right)^{1/q} \right\},$$

where $q \geq 1$.

Proof. Now if $[\cdot, \cdot]$ is a L-G-s.i.p. that generates the norm $\|\cdot\|$, then

$$(2.7) \quad \sup_{\|u\|=1} |[x, u]| = \|x\| \quad \text{for any } x \in X.$$

Indeed, if $x = 0$ the equality is obvious. If $x \neq 0$, then by Schwarz's inequality we have

$$|[x, u]| \leq \|x\| \|u\| \text{ for any } u \in X.$$

By taking the supremum in this inequality we have

$$\sup_{\|u\|=1} |[x, u]| \leq \|x\|.$$

On the other hand by taking $u_0 := \frac{x}{\|x\|}$ we have that $\|u_0\| = 1$ and since

$$\sup_{\|u\|=1} |[x, u]| \geq |[x, u_0]| = \left| \left[x, \frac{x}{\|x\|} \right] \right| = \frac{\|x\|^2}{\|x\|} = \|x\|,$$

then we get the desired equality (3.19).

Assume that $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by the definition (1.4) and representation (2.7) we have

$$(2.8) \quad \|\mathbf{x}\|_{h,n,q} := \sup_{|\alpha|_p \leq 1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| = \sup_{|\alpha|_p \leq 1} \left(\sup_{\|u\|=1} \left| \left[\left(\sum_{j=1}^n \alpha_j x_j \right), u \right] \right| \right) \\ = \sup_{\|u\|=1} \left(\sup_{|\alpha|_p \leq 1} \left| \sum_{j=1}^n \alpha_j [x_j, u] \right| \right) = \sup_{\|u\|=1} \left(\sum_{j=1}^n |[x_j, u]|^q \right)^{1/q},$$

where the last equality in (2.8) follows by the representation (2.1) for $\beta_j = [x_j, u]$, $j \in \{1, \dots, n\}$.

For $q = 1, p = \infty$ the representation (2.6) follows in a similar way by utilising the first equality in (2.2). We omit the details. \square

Remark 1. If $(E, \|\cdot\|)$ is an inner product space with $\langle \cdot, \cdot \rangle$ generating the norm, then we recapture the representation result obtained in the recent paper [9].

Remark 2. We observe that the representation (2.6) provides a stronger result than the one from Theorem 3 since it makes use of a smaller class of bounded linear functionals, namely the ones generated by a given L-G-s.i.p on E that generates the norm $\|\cdot\|$.

3. THE CASE OF OPERATORS ON BANACH SPACES

A fundamental result due to Lumer [11], in the theory of operators on complex Banach spaces X , is that if $T \in \mathcal{B}(X)$, then

$$(3.1) \quad w(T) \leq \|T\| \leq 4w(T),$$

where $w(T) := \sup_{\|x\|=1} |[Tx, x]|$ is the numerical radius of the operator T and $[\cdot, \cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$.

As shown by Glickfeld [13], the second inequality in (3.1) holds with $e = \exp(1)$ instead of 4 and e is the best possible constant. Therefore we have the sharp inequalities

$$(3.2) \quad \frac{1}{e} \|T\| \leq w(T) \leq \|T\|$$

for any $T \in \mathcal{B}(X)$.

On the Cartesian product $B^{(n)}(X) := \mathcal{B}(X) \times \dots \times \mathcal{B}(X)$ we can define the *hypo- q -operator norms* of $(T_1, \dots, T_n) \in B^{(n)}(X)$ by

$$(3.3) \quad \|(T_1, \dots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \quad \text{where } p, q \in [1, \infty],$$

with the convention that if $p = 1, q = \infty$; if $p = \infty, q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

If $[\cdot, \cdot]$ is a *s-L-G-s.i.p.* that generates the norm $\|\cdot\|$ of X and $w(T) := \sup_{\|x\|=1} |[Tx, x]|$ is the numerical radius of the operator T we can also define the *hypo- q -numerical radius* of $(T_1, \dots, T_n) \in B^{(n)}(X)$ by

$$(3.4) \quad w_{h,n,q}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,p} \leq 1} w \left(\sum_{j=1}^n \lambda_j T_j \right) \quad \text{with } p, q \in [1, \infty],$$

with the convention that if $p = 1, q = \infty$; if $p = \infty, q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

Using (3.2) we have

$$\frac{1}{e} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \leq w \left(\sum_{j=1}^n \lambda_j T_j \right) \leq \left\| \sum_{j=1}^n \lambda_j T_j \right\|$$

and by taking the supremum over $\|\lambda\|_{n,p} \leq 1$ in this inequality, we get the following fundamental result

$$(3.5) \quad \frac{1}{e} \|(T_1, \dots, T_n)\|_{h,n,q} \leq w_{h,n,q}(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $q \geq 1$. The inequalities (3.5) are sharp, which follow by the unidimensional case.

Theorem 6. *Let $(X, \|\cdot\|)$ be a Banach space and $[\cdot, \cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X . Let $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $x, y \in X$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$(3.6) \quad \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right| = \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q}$$

and, in particular

$$(3.7) \quad \sup_{\|\alpha\|_{n,2} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right| = \left(\sum_{j=1}^n |[T_j x, y]|^2 \right)^{1/2}.$$

We also have

$$(3.8) \quad \sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right| = \sum_{j=1}^n |[T_j x, y]|$$

and

$$(3.9) \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right| = \max_{j \in \{1, \dots, n\}} \{|[T_j x, y]|\}.$$

Proof. If we take $\beta = ([T_1x, y], \dots, [T_nx, y]) \in \mathbb{C}^n$ in (2.1), then we get

$$\begin{aligned} \left(\sum_{j=1}^n |[T_jx, y]|^q \right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j [T_jx, y] \right| = \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left[\sum_{j=1}^n \alpha_j T_jx, y \right] \right|, \end{aligned}$$

which proves (3.6).

The equalities (3.8) and (3.9) follow by (2.2). \square

Corollary 2. *With the assumptions of Theorem 6, if $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $x \in X$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$(3.10) \quad \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right| = \left(\sum_{j=1}^n |[T_jx, x]|^q \right)^{1/q}$$

and, in particular

$$(3.11) \quad \sup_{\|\alpha\|_{n,2} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right| = \left(\sum_{j=1}^n |[T_jx, x]|^2 \right)^{1/2}.$$

We also have

$$(3.12) \quad \sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right| = \sum_{j=1}^n |[T_jx, x]|$$

and

$$(3.13) \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right| = \max_{j \in \{1, \dots, n\}} \{|[T_jx, x]|\}.$$

Corollary 3. *With the assumptions of Theorem 6, if $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $x \in X$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$(3.14) \quad \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^n |[T_jx, y]|^q \right)^{1/q}$$

and, in particular

$$(3.15) \quad \sup_{\|\alpha\|_{n,2} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^n |[T_jx, y]|^2 \right)^{1/2}.$$

We also have

$$(3.16) \quad \sup_{\|\alpha\|_{n,\infty} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \sum_{j=1}^n |[T_jx, y]|$$

and

$$(3.17) \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \max_{j \in \{1, \dots, n\}} \{\|T_j x\|\}.$$

Proof. By the properties of semi-inner product, we have for any $u \in X$, $u \neq 0$ (see also (3.19)) that

$$(3.18) \quad \|u\| = \sup_{\|y\|=1} |[u, y]|.$$

Let $x \in X$, then by taking the supremum over $\|y\| = 1$ in (3.6) we get for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned} \sup_{\|y\|=1} \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} &= \sup_{\|y\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right\|^q \right)^{1/q} \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|y\|=1} \left\| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right] \right\|^q \right)^{1/q} \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \left(\sum_{j=1}^n \alpha_j T_j \right) x \right\|, \end{aligned}$$

which proves the equality (3.14). We used for the last equality the property (3.18).

The other equalities can be proved in a similar way by using Theorem 6, however the details are omitted. \square

We can state and prove our main representation result.

Theorem 7. Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot, \cdot]$ a s -L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $(T_1, \dots, T_n) \in B^{(n)}(X)$.

(i) For $q \geq 1$ we have the representation for the hypo- q -operator norm

$$(3.19) \quad \|(T_1, \dots, T_n)\|_{h,n,q} = \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q}$$

and

$$(3.20) \quad \|(T_1, \dots, T_n)\|_{h,n,\infty} = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}.$$

(ii) For $q \geq 1$ we have the representation for the hypo- q -numerical radius

$$(3.21) \quad w_{h,n,q}(T_1, \dots, T_n) = \sup_{\|x\|=1} \left(\sum_{j=1}^n |[T_j x, x]|^q \right)^{1/q}$$

and

$$(3.22) \quad w_{h,n,\infty}(T_1, \dots, T_n) = \max_{j \in \{1, \dots, n\}} \{w(T_j)\}.$$

Proof. (i) By using the equality (3.14) we have for $(T_1, \dots, T_n) \in B^{(n)}(X)$ that

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|y\|=1} \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} \right) \\ &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|x\|=1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j \right\| = \|(T_1, \dots, T_n)\|_{h,n,q}, \end{aligned}$$

which proves (3.19). The rest is obvious.

(ii) By using the equality (3.10) we have for $(T_1, \dots, T_n) \in B^{(n)}(X)$ that

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^n |[T_j x, x]|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|x\|=1} \left\| \left[\left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right] \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} w \left(\sum_{j=1}^n \alpha_j T_j \right) = w_{h,n,q}(T_1, \dots, T_n), \end{aligned}$$

which proves (3.21). The rest is obvious. \square

We can consider on $B^{(n)}(X)$ the following usual operator and numerical radius q -norms, for $q \geq 1$

$$\|(T_1, \dots, T_n)\|_{n,q} := \left(\sum_{j=1}^n \|T_j\|^q \right)^{1/q} \quad \text{and} \quad w_{n,q}(T_1, \dots, T_n) := \left(\sum_{j=1}^n w^q(T_j) \right)^{1/q}$$

where $(T_1, \dots, T_n) \in B^{(n)}(X)$. For $q = \infty$ we put

$$\|(T_1, \dots, T_n)\|_{n,\infty} := \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} \quad \text{and} \quad w_{n,\infty}(T_1, \dots, T_n) := \max_{j \in \{1, \dots, n\}} \{w(T_j)\}.$$

Corollary 4. *With the assumptions of Theorem 7 we have for $q \geq 1$ that*

$$(3.23) \quad \frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{n,q}$$

and

$$(3.24) \quad \frac{1}{n^{1/q}} w_{n,q}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq w_{n,q}(T_1, \dots, T_n)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

Proof. Let $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $x, y \in X$ with $\|x\| = \|y\| = 1$. Then by Schwarz's inequality we have

$$\left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} \leq \left(\sum_{j=1}^n \|T_j x\|^q \|y\|^q \right)^{1/q} = \left(\sum_{j=1}^n \|T_j x\|^q \right)^{1/q}.$$

By the operator norm inequality we also have

$$\left(\sum_{j=1}^n \|T_j x\|^q \right)^{1/q} \leq \left(\sum_{j=1}^n \|T_j\|^q \|x\|^q \right)^{1/q} = \|(T_1, \dots, T_n)\|_{n,q}.$$

Therefore

$$\left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} \leq \|(T_1, \dots, T_n)\|_{n,q}$$

and by taking the supremum over $\|x\| = \|y\| = 1$ we get the second inequality in (3.23).

By the properties of complex numbers, we have

$$\max_{j \in \{1, \dots, n\}} \{|[T_j x, y]|\} \leq \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q}$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$.

Observe also that, by (3.18) we have for any operator $T \in B(X)$ that

$$(3.25) \quad \|Tx\| = \sup_{\|y\|=1} |[Tx, y]| \text{ for any } x \in X$$

and

$$(3.26) \quad \|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \left(\sup_{\|y\|=1} |[Tx, y]| \right) = \sup_{\|x\|=\|y\|=1} |[Tx, y]|.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get

$$(3.27) \quad \sup_{\|x\|=\|y\|=1} \left(\max_{j \in \{1, \dots, n\}} \{|[T_j x, y]|\} \right) \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

and since

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\max_{j \in \{1, \dots, n\}} \{|[T_j x, y]|\} \right) &= \max_{j \in \{1, \dots, n\}} \left\{ \sup_{\|x\|=\|y\|=1} |[T_j x, y]| \right\} \\ &= \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned}$$

then by (3.27) we get

$$(3.28) \quad \|(T_1, \dots, T_n)\|_{n,\infty} \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

Since

$$(3.29) \quad \|(T_1, \dots, T_n)\|_{n,q} := \left(\sum_{j=1}^n \|T_j\|^q \right)^{1/q} \leq \left(n \|(T_1, \dots, T_n)\|_{n,\infty}^q \right)^{1/q} \\ = n^{1/q} \|(T_1, \dots, T_n)\|_{n,\infty},$$

then by (3.28) and (3.29) we get

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

The inequality (3.24) follows in a similar way and we omit the details. \square

Corollary 5. *With the assumptions of Theorem 7 we have for $r \geq q \geq 1$ that*

$$(3.30) \quad \|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|(T_1, \dots, T_n)\|_{h,n,r}$$

and

$$(3.31) \quad w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{r-q}{rq}} w_{h,n,r}(T_1, \dots, T_n)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j , $j = 1, \dots, n$ and $r \geq q > 0$ (see for instance [22] and [17])

$$(3.32) \quad \left(\sum_{j=1}^n a_j^r \right)^{1/r} \leq \left(\sum_{j=1}^n a_j^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n a_j^r \right)^{1/r}.$$

Let $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $x, y \in X$ with $\|x\| = \|y\| = 1$. Then by (3.32) we get

$$\left(\sum_{j=1}^n |[T_j x, y]|^r \right)^{1/r} \leq \left(\sum_{j=1}^n |[T_j x, y]|^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n |[T_j x, y]|^r \right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (3.30).

The inequality (3.31) follows in a similar way and we omit the details. \square

For $q = 2$, we put

$$\|(T_1, \dots, T_n)\|_{h,n,e} := \|(T_1, \dots, T_n)\|_{h,n,2} \quad \text{and} \quad w_{h,n,e}(T_1, \dots, T_n) := w_{h,n,2}(T_1, \dots, T_n).$$

Remark 3. *For $q \geq 2$ we have by (3.30) and (3.31) that*

$$(3.33) \quad \|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq n^{\frac{q-2}{2q}} \|(T_1, \dots, T_n)\|_{h,n,q}$$

and

$$(3.34) \quad w_{h,n,q}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq n^{\frac{q-2}{2q}} w_{h,n,q}(T_1, \dots, T_n)$$

and for $1 \leq q \leq 2$ we have

$$(3.35) \quad \|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|(T_1, \dots, T_n)\|_{h,n,e}$$

and

$$(3.36) \quad w_{h,n,e}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{2-q}{2q}} w_{h,n,e}(T_1, \dots, T_n)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

Also, if we take $q = 1$ and $r \geq 1$ in (3.30) and (3.31), then we get

$$(3.37) \quad \|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|(T_1, \dots, T_n)\|_{h,n,r}$$

and

$$(3.38) \quad w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq n^{\frac{r-1}{r}} w_{h,n,r}(T_1, \dots, T_n)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

In particular, for $r = 2$ we get

$$(3.39) \quad \|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq \sqrt{n} \|(T_1, \dots, T_n)\|_{h,n,e}$$

and

$$(3.40) \quad w_{n,e}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq \sqrt{n} w_{n,e}(T_1, \dots, T_n)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(X)$.

We have:

Proposition 3. For any $(T_1, \dots, T_n) \in B^{(n)}(X)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$(3.41) \quad \|(T_1, \dots, T_n)\|_{h,n,q} \geq \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\|$$

and

$$(3.42) \quad w_{h,n,q}(T_1, \dots, T_n) \geq \frac{1}{n^{1/p}} w \left(\sum_{j=1}^n T_j \right).$$

Proof. Let $\lambda_j = \frac{1}{n^{1/p}}$ for $j \in \{1, \dots, n\}$, then $\sum_{j=1}^n |\lambda_j|^p = 1$. Therefore by (1.8) we get

$$\|(T_1, \dots, T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \geq \left\| \sum_{j=1}^n \frac{1}{n^{1/p}} T_j \right\| = \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\|.$$

The inequality (3.42) follows in a similar way. \square

We can also introduce the following norms for $(T_1, \dots, T_n) \in B^{(n)}(X)$,

$$\|(T_1, \dots, T_n)\|_{s,n,p} := \sup_{\|x\|=1} \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p},$$

where $p \geq 1$ and

$$\|(T_1, \dots, T_n)\|_{s,n,\infty} := \sup_{\|x\|=1} \left(\max_{j \in \{1, \dots, n\}} \|T_j x\| \right) = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}.$$

The triangle inequality $\|\cdot\|_{s,n,q}$ follows by Minkowski inequality, while the other properties of the norm are obvious.

Proposition 4. Let $(T_1, \dots, T_n) \in B^{(n)}(X)$. We have for $p \geq 1$, that

$$(3.43) \quad \|(T_1, \dots, T_n)\|_{h,n,p} \leq \|(T_1, \dots, T_n)\|_{s,n,p} \leq \|(T_1, \dots, T_n)\|_{n,p}.$$

Proof. We have for $p \geq 2$ and $x, y \in X$ with $\|x\| = \|y\| = 1$, that

$$|[T_j x, y]|^p \leq \|T_j x\|^p \|y\|^p = \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p$$

for $j \in \{1, \dots, n\}$.

This implies

$$\sum_{j=1}^n |[T_j x, y]|^p \leq \sum_{j=1}^n \|T_j x\|^p \leq \sum_{j=1}^n \|T_j\|^p,$$

namely

$$(3.44) \quad \left(\sum_{j=1}^n |[T_j x, y]|^p \right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j\|^p \right)^{1/p},$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.44), we get the desired result (3.43). \square

4. REVERSE INEQUALITIES

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [1] (see also [4, Theorem 5.14]):

Lemma 2. *Let $a, A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:*

$$(4.1) \quad ay_j \leq z_j \leq Ay_j \text{ for each } j \in \{1, \dots, n\}.$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$(4.2) \quad 0 \leq \sum_{j=1}^n w_j z_j^2 \sum_{j=1}^n w_j y_j^2 - \left(\sum_{j=1}^n w_j z_j y_j \right)^2 \leq \frac{1}{4} (A - a)^2 \left(\sum_{j=1}^n w_j y_j^2 \right)^2.$$

The constant $\frac{1}{4}$ is sharp in (4.2).

O. Shisha and B. Mond obtained in 1967 (see [22]) the following counterparts of (CBS)-inequality (see also [4, Theorem 5.20 & 5.21]):

Lemma 3. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists a, A, b, B with the property that:*

$$(4.3) \quad 0 \leq a \leq a_j \leq A \text{ and } 0 < b \leq b_j \leq B \text{ for any } j \in \{1, \dots, n\},$$

then we have the inequality

$$(4.4) \quad \sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 - \left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^n a_j b_j \sum_{j=1}^n b_j^2.$$

and

Lemma 4. *Assume that \mathbf{a}, \mathbf{b} are nonnegative sequences and there exists γ, Γ with the property that*

$$(4.5) \quad 0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \text{ for any } j \in \{1, \dots, n\}.$$

Then we have the inequality

$$(4.6) \quad 0 \leq \left(\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^n b_j^2.$$

We have:

Theorem 8. Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot, \cdot]$ a s -L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $(T_1, \dots, T_n) \in B^{(n)}(X)$.

(i) We have

$$(4.7) \quad 0 \leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2$$

and

$$(4.8) \quad 0 \leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2.$$

(ii) We have

$$(4.9) \quad \begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} \|(T_1, \dots, T_n)\|_{h,n,1} \end{aligned}$$

and

$$(4.10) \quad \begin{aligned} 0 &\leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} w_{h,n,1}(T_1, \dots, T_n). \end{aligned}$$

(iii) We have

$$(4.11) \quad 0 \leq \|(T_1, \dots, T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty}$$

and

$$(4.12) \quad 0 \leq w_{n,e}(T_1, \dots, T_n) - \frac{1}{\sqrt{n}} w_{h,n,1}(T_1, \dots, T_n) \leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty}.$$

Proof. (i). Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and put

$$R = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}.$$

If $x, y \in H$, with $\|x\| = \|y\| = 1$ then

$$|[T_j x, y]| \leq \|T_j x\| \leq \|T_j\| \leq R$$

for any $j \in \{1, \dots, n\}$.

If we write the inequality (3.2) for $z_j = [T_j x, y]$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^n |[T_j x, y]|^2 - \left(\sum_{j=1}^n |[T_j x, y]| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$(4.13) \quad \sum_{j=1}^n |[T_j x, y]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |[T_j x, y]| \right)^2 + \frac{1}{4} n R^2$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$(4.14) \quad \sum_{j=1}^n |[T_j x, x]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |[T_j x, x]| \right)^2 + \frac{1}{4} n R^2$$

for any $x \in H$, with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.13) and $\|x\| = 1$ in (4.14), then we get (4.7) and (4.8).

(ii). Let $(T_1, \dots, T_n) \in B^{(n)}(H)$. If we write the inequality (3.4) for $a_j = |[T_j x, y]|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^n |[T_j x, y]|^2 - \left(\sum_{j=1}^n |[T_j x, y]| \right)^2 \leq n R \sum_{j=1}^n |[T_j x, y]|,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$(4.15) \quad \sum_{j=1}^n |[T_j x, y]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |[T_j x, y]| \right)^2 + R \sum_{j=1}^n |[T_j x, y]|,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$(4.16) \quad \sum_{j=1}^n |[T_j x, x]|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |[T_j x, x]| \right)^2 + R \sum_{j=1}^n |[T_j x, x]|,$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.15) and $\|x\| = 1$ in (4.16), then we get (4.9) and (4.10).

(iii). If we write the inequality (4.6) for $a_j = |[T_j x, y]|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \leq \left(n \sum_{j=1}^n |[T_j x, y]|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n |[T_j x, y]| \leq \frac{1}{4} n R,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$(4.17) \quad \left(\sum_{j=1}^n |[T_j x, y]|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |[T_j x, y]| + \frac{1}{4} \sqrt{n} R,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$(4.18) \quad \left(\sum_{j=1}^n |[T_j x, x]|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |[T_j x, x]| + \frac{1}{4} \sqrt{n} R,$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.17) and $\|x\| = 1$ in (4.18), then we get (4.11) and (4.12). \square

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If $\gamma, \Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$(4.19) \quad 0 \leq \operatorname{Re}[(\Gamma - \alpha_j)(\overline{\alpha_j} - \bar{\gamma})] \\ = (\operatorname{Re} \Gamma - \operatorname{Re} \alpha_j)(\operatorname{Re} \alpha_j - \operatorname{Re} \gamma) + (\operatorname{Im} \Gamma - \operatorname{Im} \alpha_j)(\operatorname{Im} \alpha_j - \operatorname{Im} \gamma)$$

or, equivalently,

$$(4.20) \quad \left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

for each $j \in \{1, \dots, n\}$, then (see for instance [5, p. 9])

$$(4.21) \quad n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq \frac{1}{4} n^2 |\Gamma - \gamma|^2.$$

In addition, if $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$, then (see for example [5, p. 26]):

$$(4.22) \quad n \sum_{j=1}^n |\alpha_j|^2 \leq \frac{1}{4} \frac{\left\{ \operatorname{Re}[(\bar{\Gamma} + \bar{\gamma}) \sum_{j=1}^n \alpha_j] \right\}^2}{\operatorname{Re}(\Gamma \bar{\gamma})} \\ \leq \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma \bar{\gamma})} \left| \sum_{j=1}^n \alpha_j \right|^2.$$

Also, if $\Gamma \neq -\gamma$, then (see for instance [5, p. 32]):

$$(4.23) \quad \left(n \sum_{j=1}^n |\alpha_j|^2 \right)^{\frac{1}{2}} - \left| \sum_{j=1}^n \alpha_j \right| \leq \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.$$

Finally, from [7] we can also state that

$$(4.24) \quad n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq n \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})} \right] \left| \sum_{j=1}^n \alpha_j \right|,$$

provided $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (4.19) to hold is that

$$(4.25) \quad \operatorname{Re} \Gamma \geq \operatorname{Re} \alpha_j \geq \operatorname{Re} \gamma \quad \text{and} \quad \operatorname{Im} \Gamma \geq \operatorname{Im} \alpha_j \geq \operatorname{Im} \gamma$$

for each $j \in \{1, \dots, n\}$.

Theorem 9. Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot, \cdot]$ a s -L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \gamma$. Assume that

$$(4.26) \quad w \left(T_j - \frac{\gamma + \Gamma}{2} I \right) \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}.$$

(i) We have

$$(4.27) \quad w_{h,n,e}^2(T_1, \dots, T_n) \leq \frac{1}{n} w^2 \left(\sum_{j=1}^n T_j \right) + \frac{1}{4} n |\Gamma - \gamma|^2.$$

(ii) If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then

$$(4.28) \quad w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{2\sqrt{n}} \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} w \left(\sum_{j=1}^n T_j \right)$$

and

$$(4.29) \quad w_{h,n,e}^2(T_1, \dots, T_n) \leq \left[\frac{1}{n} w \left(\sum_{j=1}^n T_j \right) + [|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}] \right] \\ \times w \left(\sum_{j=1}^n T_j \right).$$

(iii) If $\Gamma \neq -\gamma$, then

$$(4.30) \quad w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{\sqrt{n}} \left(w \left(\sum_{j=1}^n T_j \right) + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right).$$

Proof. Let $x \in H$ with $\|x\| = 1$ and $(T_1, \dots, T_n) \in B^{(n)}(H)$ with the property (4.26). By taking $\alpha_j = [T_j x, x]$ we have

$$\left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| = \left| [T_j x, x] - \frac{\gamma + \Gamma}{2} [x, x] \right| = \left| \left[\left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right] \right| \\ \leq \sup_{\|x\|=1} \left| \left[\left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right] \right| = w \left(T_j - \frac{\gamma + \Gamma}{2} I \right) \\ \leq \frac{1}{2} |\Gamma - \gamma|$$

for any $j \in \{1, \dots, n\}$.

(i) By using the inequality (4.21), we have

$$(4.31) \quad \sum_{j=1}^n |[T_j x, x]|^2 \leq \frac{1}{n} \left| \sum_{j=1}^n [T_j x, x] \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ = \frac{1}{n} \left| \left[\sum_{j=1}^n T_j x, x \right] \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (4.31) we get

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^n |[T_j x, x]|^2 \right) &\leq \frac{1}{n} \sup_{\|x\|=1} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} w^2 \left(\sum_{j=1}^n T_j \right) + \frac{1}{4} n |\Gamma - \gamma|^2, \end{aligned}$$

which proves (4.27).

(ii) If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then by (4.22) we have for $\alpha_j = [T_j x, x]$, $j \in \{1, \dots, n\}$ that

$$\begin{aligned} (4.32) \quad \sum_{j=1}^n |[T_j x, x]|^2 &\leq \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{j=1}^n [T_j x, x] \right|^2 \\ &= \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|^2 \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

On taking the supremum over $\|x\| = 1$ in (4.32) we get (4.32).

Also, by (4.24) we get

$$\sum_{j=1}^n |[T_j x, x]|^2 \leq \frac{1}{n} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|,$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we have

$$\begin{aligned} &\sup_{\|x\|=1} \sum_{j=1}^n |[T_j x, x]|^2 \\ &\leq \sup_{\|x\|=1} \left[\frac{1}{n} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\| \right] \\ &\leq \frac{1}{n} \sup_{\|x\|=1} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \sup_{\|x\|=1} \left\| \left[\sum_{j=1}^n T_j x, x \right] \right\| \\ &= \frac{1}{n} w^2 \left(\sum_{j=1}^n T_j \right) + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] w \left(\sum_{j=1}^n T_j \right), \end{aligned}$$

which proves (4.29).

(iii) By the inequality (4.23) we have

$$\begin{aligned} \left(\sum_{j=1}^n |[T_j x, x]|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{n}} \left(\left| \sum_{j=1}^n [T_j x, x] \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \\ &= \frac{1}{\sqrt{n}} \left(\left| \left[\sum_{j=1}^n T_j x, x \right] \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we get (4.30). \square

Remark 4. By the use of the elementary inequality $w(T) \leq \|T\|$ that holds for any $T \in B(X)$, a sufficient condition for (4.26) to hold is that

$$(4.33) \quad \left\| T_j - \frac{\gamma + \Gamma}{2} \right\| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for any } j \in \{1, \dots, n\}.$$

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