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Analytical and Numerical Treatments of Conservative Diffusions and the Burgers Equation

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Abstract: The present work is concerned with the study of the generalized Langevin equation and its link to the physical theories of statistical mechanics and scale relativity. It is demonstrated that the form of the coefficients of Langevin equation depend critically on the assumption of continuity of the reconstructed trajectory. This in turn demands for the fluctuations of the diffusion term to be discontinuous in time. This paper further investigates the connection between the scale-relativistic and stochastic mechanics approaches, respectively, with the study of the Burgers equation, which in this case appears as a stochastic geodesic equation. By further demanding time reversibility of the drift the Langevin equation can also describe equivalent quantum-mechanical systems in a path-wise manner. The resulting statistical description obeys the Fokker-Plank equation of the probability density of the differential system, which can be readily estimated from Monte Carlo simulations of the random paths. Based on the Fokker-Plank formalism a new derivation of the transient probability densities is presented. Finally, stochastic simulations are compared to the theoretical results.

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1. Introduction

The Langevin equation was introduced in order to describe the motion of a test particle subjected to a fluctuating force and a viscous drag [1]. Its formulation was later generalized to encompass also other types of systems. The Langevin equation is also fundamental for the stochastic interpretation of Quantum Mechanics (QM) [2] and it also appears, in the form of a geodesic equation, in the scale relativity theory (SR) developed by Nottale [3]. The equation represents a substantial theoretical innovation because it was in fact the first stochastic differential equation. The formal theory of stochastic differential equations was developed much later by the works of Itô and Stratonovich (see for example [4] for introduction).

In contrast to the picture of diffusion as an uncorrelated random walk, the theory of dynamical systems makes it possible to treat diffusion as a deterministic dynamical process. There the Langevin dynamics can be also driven by chaotic but deterministic processes [5–7]. Emergence of diffusive behavior and Markovian evolution was also addressed by Gillepsie [8]. The recent study of Tyran-Kaminska demonstrates that simple diffusion processes can emerge as weak limits of piecewise continuous processes constructed within a totally deterministic framework [7] This is a finding which lends credence to the widely used techniques of Monte-Carlo simulations using pseudo-random number generators.

A different way of looking at the Langevin equation is to specify a fractal driving process instead of the Wiener process. Examples can be given by the studies of deterministic diffusion, where generalized Takagi functions appear [9,10]. Using this approach both fractal and linear behaviours

of the diffusion coefficients can be demonstrated. In combination the studies mentioned so far demonstrate a fundamental interplay between emergent stochasticity, chaotic dynamics and fractality, which governs transport phenomena.

Interpretations of quantum mechanics are drawing a reemerging attention in the light of the centennial of David Bohm's anniversary. Part of this work was presented at the Emergent Quantum Mechanics 2017 conference. On the other hand, the present paper does not make strong foundational claims; instead it is concerned with some questions of the mathematical foundations of the scale relativity theory and its link to stochastic mechanics. This work establishes a connection between the scale-relativistic and stochastic mechanics approaches, respectively, and the study of the Burgers equation. Such a link was not recognized before to this author's knowledge. Moreover, it is demonstrated how complex Cole-Hopf transformations map the complex Burgers equation, derived in a variational setting, to the free Schroedinger equation.

The paper starts by presenting briefly stochastic mechanics and the scale relativity theories. Section 2 provides a general result about stochastic representations of the generalized Langevin equation. Section 5 establishes the connection with Burgers equation. Based on the Fokker-Plank formalism a new derivation of the transient probability densities is presented. Finally, in section 9 numerical simulations are compared with the theoretical results.

1.1. Stochastic Mechanics

In the 1930s certain similarities were discovered between the equations of classical statistical mechanics and the Schrödinger equation. From such discoveries arose the stochastic interpretation of quantum mechanics. The formal equations for the diffusions of stochastic mechanics were first formulated by Fényes in 1952 [11] and [12] and later taken up by Nelson [2]. Following this interpretation the trajectories of the configuration, described as a Markov stochastic process, are regarded as physically real. Nelson's derivation demands time reversibility of the process and vanishing of the stochastic acceleration. The Langevin equation can describe equivalent quantum-mechanical systems in a path-wise manner. These are the so-called *conservative diffusions* of Carlen [13]. The existence of so-conceived QM particle paths was proven under certain reasonable conditions [13]. Nelson's original formulation employed a stochastic Newton's law, where the form of the stochastic acceleration had to be postulated. A Lagrangian formulation of stochastic mechanics was achieved by Pavon in complex form [14]. However, the achieved presentation is far from intuitive. In her treatment the stochastic Lagrangian is the classical Lagrangian evaluated on a complex-valued velocity field in place of the real-valued classical velocity, while the dynamics is given by a complex-valued stochastic differential equation, corresponding to the treatment of Nottale. The Lagrangian problem was formulated as a constrained optimization problem, where the dynamics acted as the constraint.

1.2. Scale relativity theory

The scale relativity theory extends the principle of relativity also to resolution scales [3,15,16]. The main tenet of the theory is that there is no preferred scale of description of the physical reality. Therefore, a physical phenomenon must be described simultaneously at all admissible scales. While this is consistent with calculus for differentiable signals, the situation changes if non-differentiable models, such as Brownian motion or turbulence are addressed. For these cases, the scale of observation (or resolution) is present irreducibly in the local description of a phenomenon. This lead Nottale to postulating fractality of the underlying mathematical variety (i.e. a pseudo-manifold) describing the observables. It should be noted that in Nottale's approach only finite differences are admissible. The scale relativistic approach results in corrections of Hamiltonian mechanics, which arise due to non-differentiability of trajectories. Nottale introduces a complex operator, that he calls the scale derivative, which acts as a pseudo-derivative (see Appendix 1.2 for details). Using this tool, Nottale gives an informal derivation of the Schrödinger equation from the classical Newtonian equation

of dynamics, via a quantization procedure which follows from an extension of Einstein's relativity principle called the scale relativity principle.

2. Stochastic Representation of Trajectories

If one considers the Brownian particle as a subsystem and the surrounding particles as an infinite dimensional thermal reservoir the Langevin equation precisely models the situation where a subsystem suitably interacts with a thermal reservoir. The type of the effective random force can be identified with a Wiener process, which has continuous but almost everywhere non-differentiable paths. Mathematical descriptions of strongly non-linear phenomena necessitate relaxation of the global assumption of differentiability. In contrast, classical physics assumes global smoothness of the signals and continuity of their first two derivatives. Therefore, non-smooth phenomena, such as fractals slip through its conceptual net. This argument can be further elaborated as follows. Consider measurement of a trajectory in time $x(t)$. Non-differentiability can occur in three scenarios:

1. divergence of the velocity, that is divergence of the difference quotient,
2. oscillatory singularity or
3. difference between forward and backward velocities.

While for (1) and (2) the velocities (i.e. derivatives) can not be defined mathematically, scenario (3) requires dropping only the assumption of continuity of the resulting velocity. That is, $x'_+(t) \neq x'_-(t)$ at the point of non-differentiability t^1. While scenario (2) is excluded by the scale relativity theory, scenario (1) leads to scale dependence of the difference quotient. Examples of fractal functions, such as the mathematical Brownian motion paths, are typically of divergent length. This at best can be viewed as a mathematical idealization since in this case the work for moving a particle along its trajectory must be infinite. On the other hand, non-differentiability does not need to occur "everywhere" (i.e. with full Lebesgue measure) on a trajectory. In this case the trajectory can be almost everywhere differentiable except on a certain dense set of points. Examples of these are the singular functions, such as the Salem-de Rham's functions [17] or the well known Cantor's function. Singular functions have finite lengths, therefore the exerted displacement work is also finite. This makes them promising candidates for conceptualization of non-smooth phenomena in physics.

The relationship between Nelson's and Nottale's approaches can be established in a formal way. For clarity of the argument we focus on the one dimensional case. First let's establish the concept of stochastic embedding of a signal. In the following we assume that the deterministic signal (i.e. trajectory) will be represented by an **equivalence class of stochastic paths** having the same expectation as the given deterministic signal. A possibly non-differentiable continuous trajectory is represented by a continuous Markov stochastic process valued in a virtual space of paths as follows:

Definition 1 (Markov Stochastic Embedding)**.** *Consider a bounded deterministic signal $x(t)$ on the compact interval $T \subseteq \mathbb{R}$ representing time. Define the stochastic embedding $\mathcal{S}_\rho$ in the probability space $(T \otimes \Omega, \mathcal{F}, \rho)$, where ρ is the probability density, as the isomorphism*

$$\mathcal{S}_\rho : \quad T \otimes \mathbb{R} \mapsto (T \otimes \Omega, \mathcal{F}, \rho)$$
$$\mathcal{S}_\rho : \quad (t, x(t)) \mapsto X(t, \omega), \quad X \in T \otimes \Omega$$

under the constraint

$$\mathbb{E}^\omega X(t, \omega) = x(t),$$

where the random variables sampled at different times t are i.i.d and $\mathcal{F}$ is a σ-algebra.

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1   A simple example of such behavior is $|x|$ around the origin $x = 0$

Note: the ω-index will be skipped from the notation wherever convenient for clarity. Also X_t and $X(t)$ will be used interchangeably. Deterministic signals are denoted by lower case, while the stochastic by upper case letters.

The above definition implicitly assumes that $X_t \in \mathbb{L}^1(T \otimes \Omega, \mathcal{F}, \rho)$ and $\mathbb{E}^\omega X(t, \omega) < \infty$.

The name of the embedding is justified by the following Lemma:

Lemma 1. *The stochastic process under the above definition has the Markov property.*

Proof. By construction for fixed $t, \delta \in \mathcal{F}$

$$\mathbb{E}^\omega X_t = x(t), \quad \mathbb{E}^\omega X_{t+\delta} = x(t+\delta)$$

The conditional expectation is

$$\mathbb{E}^\omega (X_{t+\delta}|X_t) = \int_\Omega x \frac{\rho(X_{t+\delta}, X_t)}{\rho(X_t)} dx$$

where $x = X_{t+\delta}$. However, by independence of the variables $\rho(X_{t+\delta}, X_t) = \rho(X_{t+\delta})\rho(X_t)$. Therefore,

$$\mathbb{E}^\omega X_{t+\delta} = \mathbb{E}^\omega (X_{t+\delta}|X_t)$$

Since δ can be both positive and negative the claim follows. $\square$

Consider the non-linear problem, where the phase-space trajectory of a system is represented by a Hölder or F-analytic function $x(t)$ and t is a real-valued parameter, for example time or curve length. Let's suppose that the continuous temporal evolution of a differential system can be represented by a generalized Langevin equation. This can be written in differential form as

$$dx(t) = a(x,t)dt + B(x,t)dt^\beta, \; \beta < 1 \tag{1}$$

where $a(x,t)$ and B are bounded and measurable functions of the co-ordinates and furthermore $a(x,t)$ is continuous in both x and t. That is for all $0 \le \epsilon \le dt$

$$\Delta_\epsilon^+ [x] (t) = x(t+\epsilon) - x(t) = a(x,t)\epsilon + B(x,t)\epsilon^\beta + \mathcal{O}(\epsilon)$$

This can be recognized as the Hölder growth condition of order β, since $a(x,t)\epsilon$ is an $\mathcal{O}(\epsilon)$ term. The fractional exponent β is left as a free parameter with value to be determined later.

The type of admissible functions coupled to the fractional exponent depends critically on the assumption of continuity of the reconstructed trajectory. This in turn demands for the fluctuations of the fractional term to be discontinuous. The proof technique is introduced in [18], while the argument is similar to the one presented by Gillespie [8].

Without loss of generality set $a = 0$. Let $x_{t+\epsilon} = x_t + B(x_t, t) + \mathcal{O}(\epsilon^\beta)$ and $|\Delta_\epsilon x| \le K\epsilon^\beta$. Fix the interval $[t, t+\epsilon]$ and choose a partition of points $\mathcal{P} = \{t_k = t + \epsilon k / N\}$

$$x_{t_k} = x_{t_{k-1}} + B(x_{t_{k-1}}, t_{k-1}) (\epsilon/N)^\beta + \mathcal{O}\left((\epsilon/N)^\beta\right)$$

Therefore, by induction

$$\Delta_\epsilon x = x_{t+\epsilon} - x_t = \frac{1}{N^\beta} \sum_{k=0}^{N-1} B(x_{t_k}, t_k)\epsilon^\beta + \mathcal{O}\left(N^{1-\beta}\epsilon^\beta\right)$$

If we suppose that B is continuous in x, implying also continuity in t after taking supremum limit on both sides

$$\limsup_{\epsilon \to 0} \frac{\Delta_\epsilon x}{\epsilon^\beta} = N^{1-\beta} B(x_t, t) = B(x_t, t)$$

Therefore, either $\beta = 1$ (which is forbidden by hypothesis) or else $B = 0$. So that $B(x, t)$ must oscillate from point to point if $\beta < 1$. Then let's denote the set $\chi_\beta := \{B(x_t, t) \neq 0\}$.

The argument demonstrates that so-defined set is totally disconnected in the topology of the real line [18]. This allows for the choice of the algebra $\mathcal{F}$, since we can demand that $\Omega \subseteq \chi_\beta$ has for elements the semi-open intervals $[\tau_i, \tau_j)$, $\tau_{i,j} \in \chi_\beta$. Furthermore, the initial system eq . 1 is equivalent to finite existence of the fractional velocity $v_+^\beta x(\tau_i) \neq 0$, as the differential system can be recognized as fractional Taylor series [18]. In oder words, the events in the probability space are the observations of non-vanishing values for the fractional velocity.

From now on let $\mathcal{P}_\tau \equiv \mathcal{P} \subseteq \mathcal{F}$. Without loss of generality, suppose that $\mathcal{O}\left(N^{1-\beta} \epsilon^\beta\right) \leq 1$. By the stochastic representation $x_t \mapsto (X_t(\omega), \rho)$ so that

$$\mathbb{E} \frac{\Delta_\epsilon X}{\epsilon^\beta} - \mathcal{O}_\epsilon = \frac{1}{N^\beta} \sum_{k=0}^{N-1} \mathbb{E} \, B(X_{t_k}, t_k) \quad \forall N$$

Therefore, we demand that $B(X_t, t)$ is $\mathcal{F}$-measurable and $\mathbb{L}^2(\Omega, T)$ as a technical condition.

By the Hölder condition

$$|x_{t_k} - x_{t_{k-1}}| \leq K_k \epsilon^\beta$$

for some set of constants K_k. Then by transfer

$$|\mathbb{E} \, B(X_{t_k}, t_k) - \mathbb{E} \, B(X_{t_k}, t_k)| \leq K_k \epsilon^\beta$$

Since x_{t_k} is arbitrary we can set it to 0. Therefore, $\mathbb{E} \, B(X_{t_k}, t_k)$ exists. By the same argument

$$\mathbb{E} \, (\Delta_\epsilon X)^2 \leq K_k^k \epsilon^{2\beta}$$

Then we proceed by induction. Let $K_s = \sup_i K_i^2$ from the above partition.

$$(\Delta_\epsilon x)^2 = \sum_{i,j=0}^{N-1} \Delta_i x \Delta_j x = \sum_{i=0}^{N-1} \Delta_i x^2 + 2 \sum_{i<j}^{N-1} \Delta_i x \Delta_j x \leq 3K_s \, \epsilon^{2\beta} N^{1-2\beta}$$

Therefore, for the embedded variable

$$\mathbb{E} \, (\Delta_\epsilon X)^2 = \mathbb{E} \sum_{i,j=0}^{N-1} \Delta_i X \Delta_j X = \mathbb{E} \sum_{i=0}^{N-1} \Delta_i X^2 + 2\mathbb{E} \sum_{i<j}^{N-1} \Delta_i X \Delta_j X \leq 3K_s \, \epsilon^{2\beta} N^{1-2\beta}$$

Since $\Delta_i X \Delta_j X$ are independent by Lemma 1 $\mathbb{E}\Delta_i X \Delta_j X = \mathbb{E}\Delta_i X \mathbb{E}\Delta_j X \leq K_s$. Therefore, $Var[\Delta_\epsilon X] \leq 3K_s \, \epsilon^{2\beta} N^{1-2\beta}$.

The argument can be specialized to $\beta = 1/2$ where $Var[\Delta_\epsilon X] \leq K_s \, \epsilon^{2\beta}$. Therefore, the variance exists $\forall N$ and the Central Limit Theorem holds. Since by Lemma 1 the process is Markov, it must follow that in limit the random process is Wiener.

Now suppose that $a \neq 0$. Then since $a(x)$ is BVC then a.e.

$$\mathbb{E}a(X, t) = \mathbb{E}a(x + Z, t) = \mathbb{E}(a(x, t) + a'_x Z + \mathcal{O}(Z)) = a(x, t), \quad Z = X_t - x_t$$

and

$$\mathbb{E}a(X, t)^2 = a(x, t)^2 + a_x'^2 \sigma^2, \quad \sigma^2 = \mathbb{E}Z^2$$

however, σ^2 exists by the previous argument. Therefore, $Var[\Delta_\epsilon X] \leq 3K_s \, \epsilon^{2\beta} N^{1-2\beta} - a(x,t)^2\epsilon^2 \leq 3K_s \, \epsilon^{2\beta} N^{1-2\beta}$ by the argument as in the previous case. Therefore, for $\beta = 1/2$ the limit of the random process is Wiener.

Denote the limit Wiener process by W_t. Using the stationarity and self-similarity of the increments $\Delta_\epsilon^+ W_t = \sqrt{\epsilon}\, N(0,1)$ where $N(0,1)$ is a standard Gaussian random variable. Therefore, for $\beta = 1/2$ the velocity can be regularized to a finite value if we take the expectation. That is

$$v_+^\beta \mathbb{E} W_t = 0$$

since $\Delta_\epsilon^+ \mathbb{E} W_t = 0$. However,

$$v_+^\beta \, \mathbb{E}|W_t| = \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-z^2/2} dz = 1 \;.$$

The estimate holds a.s. since $\mathbb{P}(W_t = 0) = 0$.

Finally, there is a function $b(X,t)\xi = B(X,t)$, $\xi \sim N(0,1)$. This follows directly from the axiom of choice, since we can always choose $\xi = 1$. Therefore, the last equation can be treated as a definition of $b(X,t)$.

In summary, the following theorem can be formulated:

Theorem 1 (Gaussian embedding)**.** *Suppose that $x(t)$ is β-differentiable of order $\beta = 1/2$ in the interval $T = [t, t + \epsilon]$ and*

$$dx(t) = a(x,t)dt + B(x,t)dt^\beta$$

for $0 < dt \leq \epsilon$, where $a(x,t)$ is continuous in both x and t and $B(x,t) \in \mathbb{L}^2(\Omega, T)$ is bounded but discontinuous. Further, let χ_β is the set of change (Def. A6) of $f[T]$.

Then $x(t)$ can be embedded in a probability space $(T \otimes \Omega, \mathcal{F}, \rho)$, such that

1. *$\Omega \subseteq \chi_\beta$*
2. *X_t has iid Gaussian increments and*
3. *$\mathbb{E} X_t = x(t)$*
4. *$v_+^\beta \mathbb{E}\left(|X_t| \,|\, X_t = x\right) = v_+^\beta |x(t)| = |b(x,t)|$ hold a.s.*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

holds a.s. In the last equation W_t is a standard Wiener process and

$$b(X,t)\xi := B(X,t), \quad \xi \sim N(0,1).$$

Such embedding can be also called a **consistent stochastic embedding**. This theorem allows for Nelson's characterization of the Langevin diffusion process.

3. Nelson's Characterization

Starting from the generalized Langevin equation the argument can be specialized to a Wiener driving process, which can be handled using the apparatus of Itô calculus. Consider the stochastic differential equation with continuous drift and diffusion coefficients

$$dX_t = a(X,t)dt + b(X,t)dW_t$$

where $a(X,t)$ and $b(X,t)$ are smooth functions of the co-ordinates and dW_t are the increments of a Wiener process $dW_{dt} \sim N(0, dt)$ adapted to the past.

Let $\mathbb{E}X_t = x(t)$. Following Nelson [2] the forward and backward and *drift*, respectively *diffusion* coefficients, can be identified as the averaged velocities

$$a = \lim_{dt \to 0} \mathbb{E}\left(\frac{X_{t+dt} - X_t}{dt} \,\middle|\, X_t = x \right) = \frac{d}{dt}(x - b\sqrt{dt}) \tag{2}$$

$$|b| = \lim_{dt \to 0} \mathbb{E}\left(\frac{|X_{t+dt} - X_t|}{\sqrt{dt}} \,\middle|\, X_t = x \right) = v_+^{1/2}|x| \tag{3}$$

The evolution of the density of the process can be computed from the forward Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(a\rho) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b^2\rho\right) = 0 \tag{4}$$

which can be recognized as a conservation law for the probability current *j*:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}j = 0, \quad j := a\rho - \frac{1}{2}\frac{\partial}{\partial x}b^2\rho$$

Under the finite energy technical condition there is a backwards process with the same transition density

$$dX_t = \hat{a}(X,t)dt + b(X,t)d\hat{W}_t$$

which is adapted to the future. This leads to the anticipative (i.e. anti-Itô) stochastic integrals. This process has Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\hat{a}\rho) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b^2\rho\right) = 0 \tag{5}$$

Then it follows that the *osmotic velocity* can be defined from

$$a - \hat{a} = b^2\frac{\partial}{\partial x}\log b^2\rho + \phi(t)$$

where $\phi(t)$ is an arbitrary $\mathbb{C}^1$ function of time as $u := \frac{1}{2}(a - \hat{a})$ and the *current velocity* as

$$v := \frac{1}{2}(a + \hat{a})$$

so that a continuity equation holds

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}v\rho = 0$$

4. The Complex Velocity Operator in SR and SM theories

Scale relativity treats velocity only as a finite difference. This is a necessity due to the assumed non-differentiability of the trajectories. Non-differentiability leads to introduction of the 2 velocity fields – forward and backward, depending on the direction of differentiation in time. These fields are assumed to be finite for small values of the time step dt but they diverge to infinity in the limit $dt \to 0$ or if dt is infinitesimal. Therefore, such velocity fields can be defined only up to a finite resolution underlying the physical phenomenon under study. The velocity fields are assumed to admit representation of the form of a sum of a "classical part" plus a correction of a resolution-dependent and diverging fractal part. The classical part corresponds to the **absolutely continuous** part of the trajectory, while the fractal part corresponds to the **singular** and possibly **oscillatory** parts. This bivariate vector field is represented as a complex-valued vector field [19] as

$$\mathbf{v} = V - iU \quad \in \mathbb{R}^3,$$

with components given by

$$v_+ := \frac{\Delta^+_{dt} x}{dt} \quad | \quad v_+ := \frac{\Delta^-_{dt} x}{dt}$$

$$U := \frac{1}{2}(v_+ + v_-) \quad | \quad V := \frac{1}{2}(v_+ - v_-)$$

where V is the "classical" velocity and U is a new quasi-velocity quantity (i.e. the *osmotic* velocity in the terminology of Nelson). In this treatment the Under this assumptions Nottale introduces a complexified material derivative, which is a pseudo-differential operator

$$\mathcal{D}F = \partial_t F + \mathbf{v} \cdot \nabla F - i\sigma \nabla^2 F$$

where σ is a constant, related to the resolution scale.

The drift, resp. diffusion coefficients can be further embedded in a complex space as proposed by Pavon [14] in stochastic mechanics:

$$a \otimes \hat{a} \mapsto \mathcal{V} := v - iu$$

$$X_{t+dt} \otimes X_{t-dt} \mapsto d\mathcal{X} = \frac{1}{2}(X_{t+dt} + X_{t-dt}) - i\frac{1}{2}(X_{t+dt} - X_{t-dt})$$

so that the diffusion process becomes complex.

$$d\mathcal{X} = \mathcal{V}dt + \frac{1-i}{2}b dW_t + \frac{1+i}{2}\hat{b} d\hat{W}_t$$

In the case of $b = \hat{b}$

$$d\mathcal{X} = \mathcal{V}dt + \frac{1-i}{2}b\left(dW_t + id\hat{W}_t\right) = \mathcal{V}dt + \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}b\left(dW_t + id\hat{W}_t\right)$$

So that finally,

$$d\mathcal{X} = \mathcal{V}dt + \sqrt{-i}b dZ_t, \quad Z_t := \frac{dW_t + id\hat{W}_t}{\sqrt{2}}$$

Therefore, a formal Itô differential can be introduced in exactly the same way

$$dF = \frac{\partial}{\partial t}F + d\mathcal{X}\frac{\partial}{\partial x}F + \frac{1}{2}\left[d\mathcal{X}^2\right]\frac{\partial^2}{\partial x^2}F \tag{6}$$

with quadratic variation $\left[d\mathcal{X}^2\right] = -ib^2 dt$. Since the double adaptation of Z_t it retains its local martingale properties. That is $\mathbb{E}Z_t = 0$, $VarZ_t = 1$.

Therefore, in components

$$dF = \left(\frac{\partial}{\partial t}F + \mathcal{V}\frac{\partial F}{\partial x} - \frac{ib^2}{2}\frac{\partial^2}{\partial x^2}F\right)dt + \sqrt{-i}b\frac{\partial F}{\partial x}dZ_t$$

From this it is apparent that both theories share identical algebraical structure, while SM can be considered as a stochastic representation of scale relativity.

Remark 1. *Conceptually, the forward process can be interpreted as a prediction, while the backward process can be interpreted as a retrodiction.*

Note that in the complex formulation of Pavon the real part of the driving process Z_t corresponds to the forward (i.e. adapted to the past) process, while the imaginary part corresponds to the backward (i.e. adapted to the future) process. This is of course one of infinitely many choices, since the complex factor in the diffusion coefficient is root of unity and hence represent a rotation in the complex plane representation.

The martingale property of the complex Wiener process conceptually means that the knowledge of the past and future of the process do not bias the outcome of at the present time (i.e. at measurement). Note that the mapping is invertible since

$$X_{t+dt} = Re\,(d\mathcal{X}) + Im\,(d\mathcal{X}), \quad | \quad X_{t-dt} = Re\,(d\mathcal{X}) - Im\,(d\mathcal{X})$$

From these formulas it is apparent that the real part (i.e. respectively the imaginary part) of the resulting process do not have separate meanings, as they mix the predictive process with the retrodictive process. To illustrate the point, suppose that $F = F_r + iF_i$ and $a = a_r + ia_i$ and the original process $d\mathcal{X}$ is transformed as $F(\mathcal{X})$. Then a straightforward calculation gives

$$Re(dF) = \left(\frac{\partial F_r}{\partial t} + a_r \frac{\partial}{\partial x} F_r - a_i \frac{\partial}{\partial x} F_i + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} F_i \right) dt$$
$$+ \frac{b}{\sqrt{2}} \left(dW_t \left(\frac{\partial}{\partial x} F_r + \frac{\partial}{\partial x} F_i \right) + d\hat{W}_t \left(\frac{\partial}{\partial x} Fr - \frac{\partial}{\partial x} F_i \right) \right) \quad (7)$$

$$Im(dF) = \left(\frac{\partial F_i}{\partial t} + a_r \frac{\partial}{\partial x} F_r + a_i \frac{\partial}{\partial x} F_i + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} F_r \right) dt$$
$$- \frac{b}{\sqrt{2}} \left(dW_t \left(\frac{\partial}{\partial x} F_r - \frac{\partial}{\partial x} F_i \right) - d\hat{W}_t \left(\frac{\partial}{\partial x} Fr + \frac{\partial}{\partial x} F_i \right) \right) \quad (8)$$

5. The Real Stochastic Geodesic Equations

The appearance of the Wiener process entails application of the fundamental Itô Lemma for the forward (i.e. adapted to the past , $\alpha = 1$) or the backward processes (i.e. adapted to the future, $\alpha = 0$), respectively. In differential notation it reads

$$dF(X) = dX \frac{\partial}{\partial x} F \pm \frac{[dX^2]}{2} \frac{\partial^2}{\partial x^2} F \quad (9)$$

where, $[dX^2] = b^2 dt$ is the quadratic variation of the process. It can be seen that in this case the (forward) differential operator d acts as a material derivative.

The term *geodesic* will be interpreted as a solution of a variational problem [20,21]. A brief treatment is given in Sec. C. By application of Itô's Lemma the forward geodesic equation can be obtained:

$$\frac{\partial}{\partial t} a + a \frac{\partial}{\partial x} a + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} a = 0 \quad (10)$$

This can be recognized as a Burgers equation with negative kinematic viscosity for the drift field [22].

The backward geodesic equation follows from the Itô lemma for the anticipative process

$$\frac{\partial}{\partial t} a' + a' \frac{\partial}{\partial x} a' - \frac{b^2}{2} \frac{\partial^2}{\partial x^2} a' = 0 \quad (11)$$

This can be recognized as a Burgers equation with positive kinematic viscosity for the drift field.

The solution of the Burgers equation is well known and can be given by the convolution integrals (eq. 41) for the case of positive viscosity [22]. The case about the negative viscosity can not be easily solved using Fourier transform. The inapplicability of the standard Fourier transform method comes from the negative viscosity coefficient in the former case. Therefore, a different solution technique will be pursued. Time-varying solutions will be constructed from topological deformation of the stationary solutions.

In Quantum Mechanics applications $b = \hbar/2m$ so normalization to one will be assumed further in most cases.

6. Path-wise separable solutions

In first instance, one can solve the geodesic equation by supposing separability. By making the ansatz $a(x,t) = f(x)g(t)$ we arrive at the equation:

$$\frac{f''(x)}{2f(x)} + g(t)\,f'(x) + \frac{g'(t)}{g(t)} = 0$$

This has the only possible solution

$$a(x,t) = \frac{x}{t+T} \tag{12}$$

The resulting Itô equation can be written as

$$dX = \frac{X}{t+T}dt + dW_t$$

The equation for the drift is therefore

$$da = \frac{1}{t+T}dW_t$$

which can be integrated exactly in Itô sense

$$a(t) = a_0 + \int_0^t \frac{dW_s}{s+T}$$

Therefore,

$$X(t) = \frac{x_0}{T}(t+T) + (t+T)\int_0^t \frac{dW_s}{T+s} \tag{13}$$

where T is the stopping time. Therefore, an exact numerical quadrature can be performed (Fig. 2)

The corresponding density can be obtained from the Fokker-Plank equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(\frac{\rho\,x}{t+T}\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho = 0$$

with solution

$$\rho(x,t) = \frac{1}{\sqrt{\pi\,(T+t)}}\exp\left(\frac{x^2}{t+T}\right)$$

It should be noted that under time reversion we arrive at the same solution, which however leads to a different Fokker-Plank equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(\frac{\rho\,x}{t-T}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho = 0$$

with solution

$$\rho(x,t) = \frac{1}{\sqrt{\pi\,(T-t)}}\exp\left(-\frac{x^2}{t-T}\right)$$

which can be recognized as a Brownian bridge.

6.1. Stationary drift fields

For time-homogeneous diffusion the geodesic equation can be brought into the form

$$\frac{1}{2}\frac{\partial}{\partial x}\left(a^2 + \frac{\partial}{\partial x}a\right) = 0$$

Which can be integrated once to give

$$a^2 + \frac{\partial}{\partial x}a = -E$$

The integration constant E can be identified with the energy. The resulting first order ODE can be solved as

$$a(x) = -\sqrt{E} \tan\left(\sqrt{E}\,x + c\right), \quad E > 0 \tag{14}$$

$$a(x) = \frac{1}{x + c}, \quad E = 0 \tag{15}$$

The solution for $E > 0$ was identified by Herman [23]. By translation invariance of the coordinates $c = 0$ is admissible. This observation will be used further for the transient solution. The link between the two solutions can be established as follows. Note that

$$a(x) = \sqrt{E} \cot \sqrt{E}\,x$$

Then

$$\lim_{E \to 0} \sqrt{E} \cot \sqrt{E}\,x = \frac{1}{x}$$

which is the second solution.

The expectation of the trajectory can be obtained by solving the ODE

$$\frac{dx}{dt} = -\sqrt{E} \tan \sqrt{E}\,x$$

so that

$$\sqrt{E}\,x(t) = \arcsin e^{-Et+c} \tag{16}$$

$$\sqrt{E}\,x(t) = \arccos e^{-Et+c} \tag{17}$$

In accordance with so-developed theory for $c = 0$

$$v_+^{1/2}x(t = 0) = \pm \lim_{h \to 0+} E \frac{2\sqrt{h}\,e^{-Eh}}{\sqrt{1 - e^{-2Eh}}} = \pm\sqrt{2E}$$

Furthermore, for $E = 0$

$$\frac{dx}{dt} = \pm\frac{1}{x}$$

so that in the same way

$$x(t) = c \pm \sqrt{2t}$$

The backward geodesic equation

$$\frac{1}{2}\frac{\partial}{\partial x}\left(a^2 - \frac{\partial}{\partial x}a\right) = 0$$

by the same method leads to

$$a(x) = -\sqrt{E} \tanh\left(\sqrt{E}\,x + c\right), \quad E > 0 \tag{18}$$

$$a(x) = -\frac{1}{x + c}, \quad E = 0 \tag{19}$$

6.2. Stationary Density Solutions

The stationary density $\rho(y)$ is a solution of the forward Kolmogorov (i.e. Fokker-Plank) equations:

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\rho\left(y^2+1\right)^2 = 0, \quad E > 0 \tag{20}$$

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\rho\, y^4 = 0, \quad E = 0 \tag{21}$$

The case $E > 0$ leads to

$$\frac{\partial}{\partial x}\tan(x)\rho + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho = 0 \tag{22}$$

with stationary solution

$$\rho = \cos^2\sqrt{E}x$$

which can be valid on a bounded domain. The case $E = 0$ leads to

$$\frac{\partial}{\partial x}\frac{\rho}{x} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho = 0 \tag{23}$$

with a stationary solution

$$\rho = |x|$$

which can be valid on a bounded domain.

6.3. Transient drift fields

The solution of the Burgers equation is well known and can be given by the convolution integrals (eq. 41) for the case of positive viscosity [22]. The case of negative viscosity emerging here is more challenging and it will be solved by a deformation of the stationary solution, so that in limit

$$\lim_{t\to\infty} a(t,x) = a(x), \quad E > 0$$

which, recovers the stationary solution. The solution is sought in the form (neglecting scale factors)

$$a(t,x) = -\frac{\sin x}{\cos x + f(t)}$$

for the unknown function $f(t)$, which results in a linear PDE:

$$\frac{2 f'(t) + f(t)}{2\left(\cos x + f(t)\right)^2}\sin x = 0$$

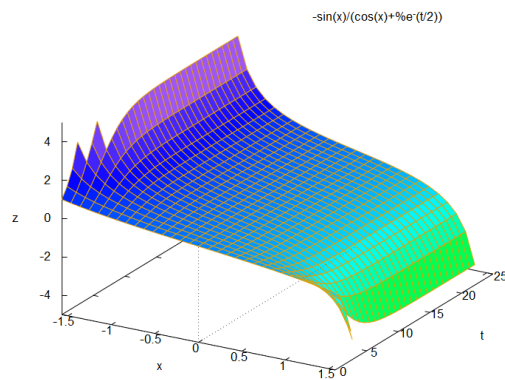By variation of the parameters, the solution for $a(t,x)$ is given as

$$a(t,x) = -\sqrt{E}\frac{\sin\sqrt{E}x}{\cos\sqrt{E}x + ke^{-\frac{Et}{2}}} \tag{24}$$

where the E constant represents an energy scale and k is an arbitrary constant. We can assume normalization, for example $k = \pm\sqrt{E}$, such that $a(t,\frac{\pi}{2E}) = \pm1$.
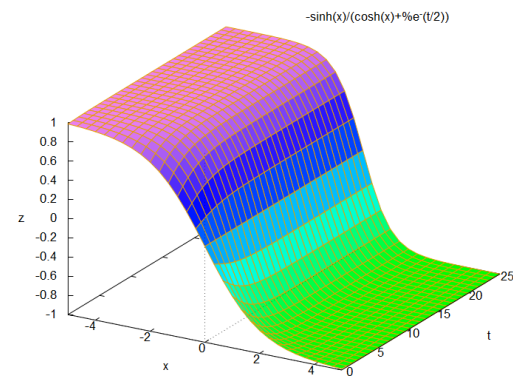
The time-varying solution can be recognized as the Green's function since the denominator vanishes for

$$\sqrt{E}\,x = \pi - \arccos e^{-\frac{Et}{2}}$$

A – Forward drift

B – Backward drift

Figure 1. Time-varying drift fields for $E = 1, k = 1$

The transformed Itô drift equation for k=1 reads

$$da(t,x) = -E \frac{e^{-Et/2}\cos\sqrt{E}x + 1}{\left(e^{-Et/2} + \cos\sqrt{E}x\right)^2}dW_t = -\sqrt{E}\frac{e^{-Et/2}\cos\sqrt{E}x + 1}{\left(e^{-Et/2} + \cos\sqrt{E}x\right)^2}dW_{Et}$$

It can be further noticed that rescaling in a pair of new variables $x' = \sqrt{E},\quad t' = Et$ leaves the ratio

$$z = \frac{x^2}{t} = \frac{x'^2}{t'}$$

invariant so that z becomes a similarity variable.

Further, a formal forward Kolmogorov equation can be written in the $y = a(t,x)$ variable with $E = 1$ as

$$\frac{\partial}{\partial t}\rho - \frac{1}{2}\frac{\partial^2}{\partial y^2}\frac{\rho e^t\left(\cos(x) + e^{\frac{t}{2}}\right)^2}{\left(e^{\frac{t}{2}}\cos(x) + 1\right)^4} = 0, \quad x = \pm\arcsin\left(\frac{e^{-\frac{t}{2}}y\left(\sqrt{(e^t-1)\,y^2 + e^t} - 1\right)}{y^2 + 1}\right)$$

however its solution is challenging due to its mixed non-linearity and will not be attempted here. Nevertheless, the analysis presented so far assures that asymptotically ρ can be obtained as a solution of the stationary equation.

The backward equation leads to the following solution

$$a(t,x)' = -\sqrt{E}\frac{\sinh\sqrt{E}x}{\cosh\sqrt{E}x + ke^{-\frac{Et}{2}}} \tag{25}$$

6.4. Asymptotic Density Solutions

The forward drift itself $a \equiv y$ (notation changed!) obeys the transformed stochastic differential equations

$$E > 0: \quad dy = \sqrt{E}\left(y^2 + 1\right)dW_t = \left(y^2 + 1\right)dW_{Et} \tag{26}$$

$$E = 0: \quad dy = y^2\,dW_t \tag{27}$$

The density ρ is a solution of the forward Kolmogorov (i.e. Fokker-Plank) equations:

$$\frac{\partial}{\partial E\, t}\rho = \frac{1}{2}\frac{\partial^2}{\partial y^2}\rho \left(y^2+1\right)^2, \quad E \neq 0 \tag{28}$$

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\frac{\partial^2}{\partial y^2}\rho\, y^4, \quad E = 0 \tag{29}$$

The solutions can be obtained using the Laplace transform. In this way the PDE can be transformed to an ODE for the Laplace variable:

$$-\frac{1}{2}\frac{\partial^2}{\partial y^2}\hat{\rho} y^4 + \hat{\rho}\, s = \rho(0,y) \tag{30}$$

$$-\frac{1}{2}\frac{\partial^2}{\partial y^2}\hat{\rho}\left(y^2+1\right)^2 + \hat{\rho}\, s = \rho(0,y) \tag{31}$$

To obtain the Green's function we take homogeneous initial conditions a.e. The solutions in time domain can be obtained by the inverse Laplace transformation:

$$\hat{\rho}(s,y) = \frac{A e^{-\frac{\sqrt{2s}}{y}}}{\sqrt{s}\, y^3} \xrightarrow{\mathcal{L}_s^{-1}} \rho(t,y) = \frac{A}{\sqrt{t}\, y^3}e^{-\frac{1}{2t y^2}} \tag{32}$$

$$\hat{\rho}(s,y) = \frac{e^{-\sqrt{2s/E-1}\,\arctan(y)}}{\left(y^2+1\right)^{\frac{3}{2}}} \xrightarrow{\mathcal{L}_s^{-1}} \rho(t,y) = \frac{e^{-\frac{\arctan^2 y-(Et)^2}{2Et}}}{\sqrt{\pi\, Et}\,\sqrt{\left(y^2+1\right)^3}} \tag{33}$$

In the position space the solution can be obtained using Grisanov's theorem:

$$\rho(t,x) = \frac{|x|}{\sqrt{t}}\exp\left(-\frac{x^2}{2t}\right) \tag{34}$$

$$\rho(t,x) = \frac{\left|\cos\left(\sqrt{E}\,x\right)\right|}{\sqrt{\pi E\, t}}\exp\left(\frac{Et}{2}-\frac{x^2}{2t}\right) \tag{35}$$

This is not acceptable from physical point of view since $\lim\limits_{t\to\infty}\rho(t,x)$ diverges.

In the same way for the backward drift

$$E > 0: \quad dy = \sqrt{E}\left(y^2-1\right) dW_t = \left(y^2-1\right) dW_{Et} \tag{36}$$

$$E = 0: \quad dy = y^2\, dW_t \tag{37}$$

$$\frac{\partial}{\partial E\, t}\rho = \frac{1}{2}\frac{\partial^2}{\partial y^2}\rho \left(y^2-1\right)^2, \quad E \neq 0 \tag{38}$$

with solutions

$$\rho(t,y) = \frac{1}{\sqrt{\pi\, Et}\,\sqrt{\left(y^2-1\right)^3}}\exp\left(-\frac{\arctan h^2 y+(Et)^2}{2E\, t}\right)$$

in the drift space and in position space

$$\rho(t,x) = \frac{\cosh\left(\sqrt{E}\,x\right)}{\sqrt{\pi E\, t}}\exp\left(-\frac{x^2}{2t}-\frac{Et}{2}\right)$$

respectively. This is acceptable from physical point of view since $\lim_{t \to \infty} \rho(t, x) = 0$, which is a correct asymptotic behavior.

7. The Complex Stochastic Geodesic Equations

The complexification removes the restriction of positive definiteness of the E parameter so that the substitution $t \mapsto \pm Et$ becomes admissible by appropriate cut along the complex plane.

In a similar way, for the complex case we have

$$dX = -i\sqrt{E}\tanh\sqrt{E}Xdt + \sqrt{-i}\,dZ_t$$

which under substitution $y = \tanh x$ leads to

$$dy = -\sqrt{-i}\sqrt{E}\left(1 - y^2\right)dZ_t$$

By the same methods as used above, the asymptotic density for the drift variable can be obtained as

$$\rho(t, y) = Re \frac{1}{\sqrt{\pi t}\,(y^2 - 1)^{\frac{3}{2}}}\exp\left(\frac{iEt}{2} - \frac{i\operatorname{arctanh}^2 y}{2Et}\right)$$

For the resulting density in the position space it can be calculated that

$$\rho(t, x) = Re \frac{i\cosh\left(\sqrt{E}x\right)}{\sqrt{\pi Et}}\exp\left(\frac{iEt}{2} - \frac{ix^2}{2t}\right) \tag{39}$$

In a similar way for the other solution

$$dX = i\sqrt{E}\tan\sqrt{E}Xdt + \sqrt{-i}\,dZ_t$$

which under substitution $y = \tan\sqrt{E}x$ leads to the drift equation

$$dy = \sqrt{-i}\sqrt{E}(1 + y^2)\,dZ_t$$

The density can be readily obtained as

$$\rho(t, y) = \frac{1}{\sqrt{\pi t}\,(y^2 + 1)^{\frac{3}{2}}}\exp\left(-\frac{iEt}{2} - \frac{i\arctan^2 y}{2Et}\right)$$

In the position space the density is of the form

$$\rho(t, x) = Re \frac{|\cos\left(\sqrt{E}x\right)|}{\sqrt{\pi Et}}\exp\left(-\frac{ix^2}{2t} - \frac{iEt}{2}\right) \tag{40}$$

In either case the densities asymptotically approach zero.

8. Real-valued and Complex Cole–Hopf transformations

The Burgers equation can be linearized by the Cole–Hopf transformation [24,25]. This mapping transforms the nonlinear Burgers equation into the linear heat conduction equation in the following way. Let

$$u = \frac{\partial}{\partial x}\log a$$

Substitution into Eq. 10 leads to

$$\frac{1}{2\,u^2}\left(u\,\frac{\partial^3 u}{\partial x^3} + 2u\,\frac{\partial^2 u}{\partial t\partial x} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) = 0$$

This can be recognized as

$$\frac{\partial}{\partial x}\frac{1}{u}\left(\frac{\partial}{\partial t}u + \frac{1}{2}\frac{\partial^2}{\partial x^2}u\right) = 0$$

which is equivalent to a solution of the equation

$$\frac{\partial}{\partial t}u + \frac{1}{2}\frac{\partial^2}{\partial x^2}u = 0$$

It should be noted that if instead of the forward development (i.e prediction) one takes the backward development (i.e. retrodiction) the usual form of the Burgers equation is recovered. This corresponds to the anticipative Wiener process, which is subject to anti-Itô calculus [26].

$$\frac{\partial}{\partial t}\hat{a} + \hat{a}\frac{\partial}{\partial x}\hat{a} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\hat{a} = 0$$

In this case the usual general solution can be revealed

$$\phi_0(x) = \exp\left(\frac{1}{2\nu}\int_0^x \hat{a}_0(u)du\right) \tag{41}$$

$$\hat{a}(x,t) = \frac{\partial}{\partial x}\log\frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty}\phi_0(u)e^{-\frac{(x-u)^2}{4\nu t}}du = \frac{\int_{-\infty}^{\infty}\frac{x-u}{t}\phi_0(u)e^{-\frac{(x-u)^2}{4\nu t}}du}{\int_{-\infty}^{\infty}\phi_0(u)e^{-\frac{(x-u)^2}{4\nu t}}du} \tag{42}$$

where $\nu = 1/2$ is the viscosity coefficient.

In the complex case, starting from the generalized Itô differential, the complex velocity field becomes

$$d\mathcal{V} = \left(\frac{\partial}{\partial t}\mathcal{V} + \mathcal{V}\frac{\partial}{\partial x}\mathcal{V} - \frac{ib^2}{2}\frac{\partial^2}{\partial x^2}\mathcal{V}\right)dt + \sqrt{-ib}\frac{\partial}{\partial x}\mathcal{V}\,dZ_t$$

The geodesic equation reads

$$\mathbb{E}\,d\mathcal{V} = 0$$

Therefore, by the martingale property this is equivalent to

$$\frac{\partial}{\partial t}\mathcal{V} + \mathcal{V}\frac{\partial}{\partial x}\mathcal{V} - \frac{ib^2}{2}\frac{\partial^2}{\partial x^2}\mathcal{V} = 0$$
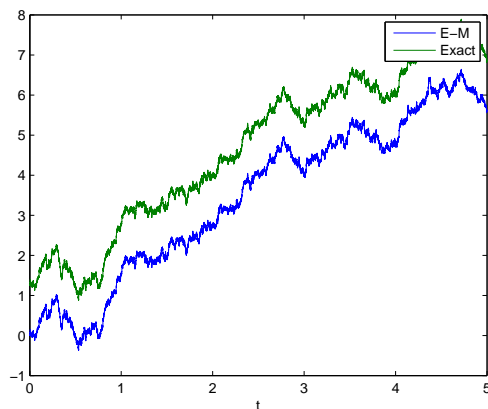
which can be recognized as a generalized Burgers equation with imaginary kinematic viscosity coefficient. Applying the complex Cole–Hopf transformation as [27]

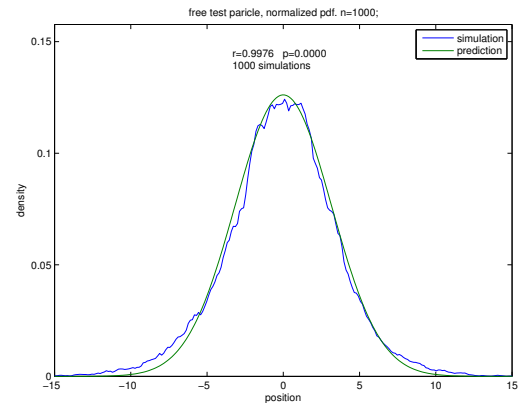$$\mathcal{V} = -i\frac{\partial}{\partial x}\log U, \quad -\pi < \arg U < \pi$$

and specializing to $b = 1$ leads to

$$-\frac{U\left(\frac{\partial^3}{\partial x^3}U\right) - \left(\frac{\partial}{\partial x}U\right)\left(\frac{\partial^2}{\partial x^2}U\right) - 2i\left(\frac{\partial}{\partial t}U\right)\left(\frac{\partial}{\partial x}U\right) + 2iU\left(\frac{\partial^2}{\partial t\partial x}U\right)}{2U^2} = 0$$

A: Virtual trajectories

B: Empirical vs theoretical density

A: Exact simulation of separable process is compared with the Euler - Maruyama algorithm. E - Exact simulation, E-M - Euler - Maruyama simulation; An offset is added to the exact solution for appreciation. Time is given in arbitrary units. B: The empirical *pdf* is estimated from $n = \log^2(N_s N)$ bins. Pearson's correlation is given as an inset – r= 0.9976.

Figure 2. Virtual trajectories of the separable process

Which can be recognized as a gradient

$$-\frac{\partial}{\partial x}\frac{1}{U}\left(i\frac{\partial}{\partial t}U + \frac{1}{2}\frac{\partial^2}{\partial x^2}U\right) = 0$$

The last equation is equivalent to the solution of the free Schrödinger equation. On the other hand, the diffusion part is simply

$$-\sqrt{i}\left(\frac{\partial^2}{\partial x^2}\log U\right)dZ_t$$

since $-i\sqrt{-i} = -\sqrt{i}$.

This corresponds with the arguments given in [28] that the coefficient of stochastic noise should be purely imaginary.

9. Numerical Results

Several different solutions of the stochastic geodesic equation were simulated using the Euler – Maruyama algorithm. Simulations were performed in Matlab. En example of a simulation script is given in Appendix A.

9.1. Exact simulations

Individual simulations were run for the separable solution. A comparison between exact simulations of separable process and Euler - Maruyama algorithm is presented in Fig. 2. Achieved correlation was 1.0 while the mean squared error was on the order $1E - 8$. Density is computed from $N_s = 1000$ simulations.

9.2. Free diffusion

The normalized asymptotic transient density of the free particle distribution can be recognized as the Rayleigh's distribution

$$R(x, t) = \frac{|x|}{2t}e^{-\frac{x^2}{2t}}$$

A: Virtual trajectories: free particles

B: Empirical vs theoretical density
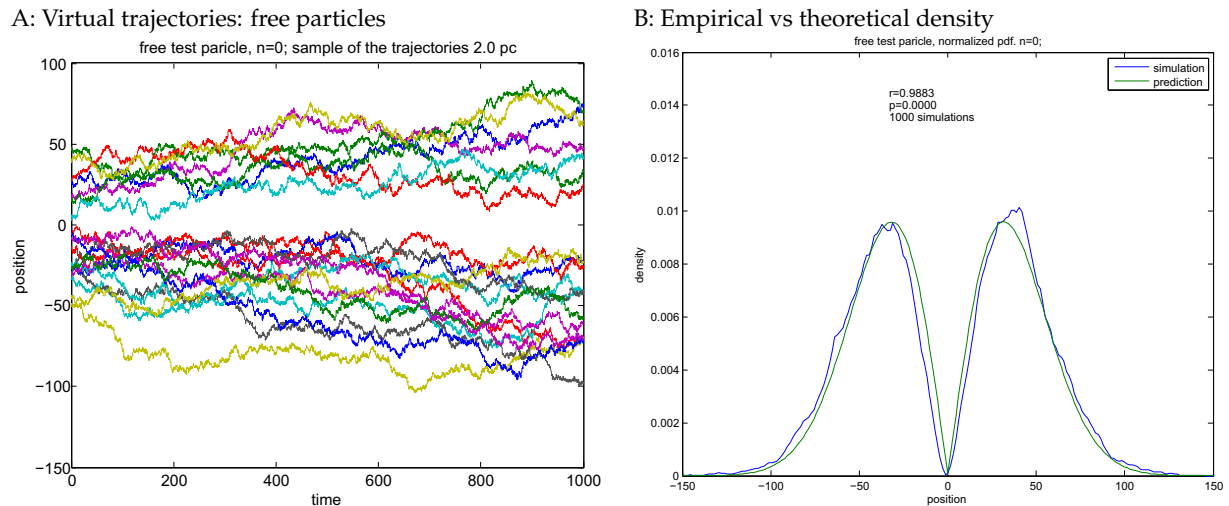
Simulations are based on $N = 10000$ points in $N_s = 1000$ simulations. A, B - width of potential well is $2L = 100$ units. The empirical *pdf* is estimated from $n = \log^2(N_s\, N)$ bins. Pearson's correlations are given as inset – r= 0.9883. Norming of the free particle transient results in Rayleigh density.

Figure 3. Simulations of free particles

9.3. Particle in a box

The third simulated case was a freely diffusing particle in a square potential well of size $2L$. Individual trajectories were simulated according to the fundamental equation according to the scheme of Euler-Maruyama:

$$x_{n+1} = x_n - 2D\Delta t \frac{\pi n}{L} \tan\left(\frac{\pi n}{L} x_n - \pi \frac{n+1}{2}\right) + \sqrt{2D\Delta t}\ \Delta W_n$$

where $\Delta W_n \sim N(0,1)$.

Restarting boundary conditions were used for the simulations.

The initial particle positions were sampled from a uniform distribution between $-L$ and L. The theoretical density for the particle in a box case is given by

$$\rho_s(x) = \frac{2}{L}\ \sin^2\left(n\pi\left(\frac{x}{L} + \frac{1}{2}\right)\right)$$

Results are based on $N = 10000$ points in $N_s = 1000$ simulations.

The empirical *pdf* is estimated from $n = \log(N_s\, N)^2$ bins. Pearson's correlations are given as insets: B – r=0.9939, D – r= 0.9871. For both cases the numerical precision correlates excellently with the analytical solutions (Fig. 4).

10. Discussion

The relaxation of the differentiability assumption opens new avenues in describing physical phenomena [2,3] but also challenges existing mathematical methods, which are developed for smooth signals. While this description can be achieved also by fractional differ–integrals, or by multi-scale approaches [29], the present work focuses on a local description. The reason for this choice is that locality provides a direct way of physical interpretation of the obtained results. Hölderian functions in this regard can be used as building blocks of such strongly non-linear models, which give rise to singular [30,31] or non-differentiable models.
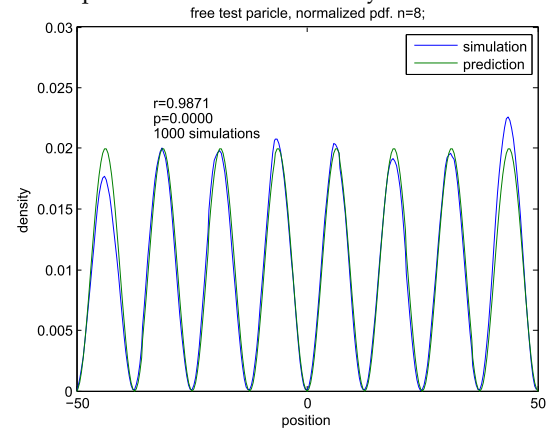
The second motivation of this work was to investigate the potential of stochastic methods for simulations of quantum-mechanical and convection-diffusive systems. Presented numerical

A: Virtual trajectories: particle in a box, n=2
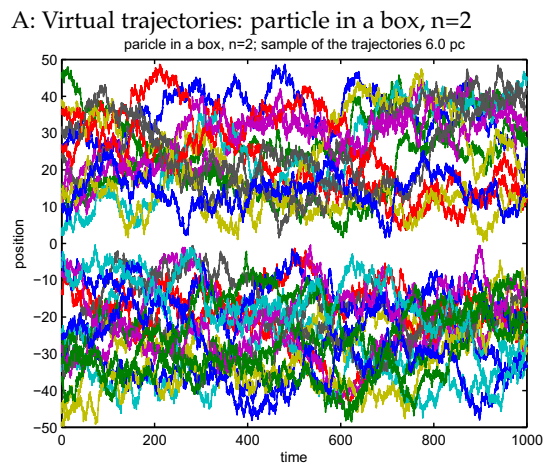
B: Empirical vs theoretical density
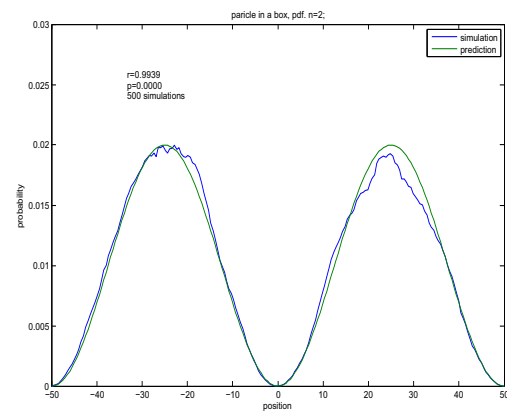
C: Test particles in a box, n=8
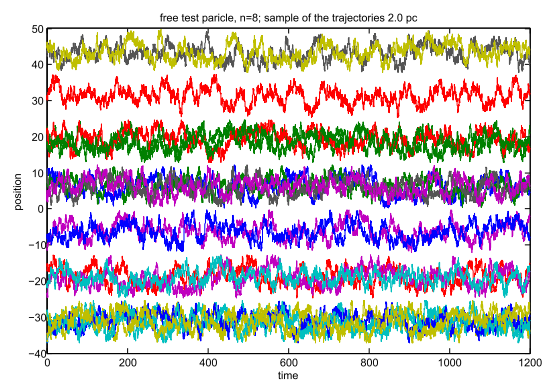
D: Empirical vs theoretical density

A, B - width of potential well is $2L = 100$ units.

Figure 4. Simulations of particles in a box for two quantum numbers

approaches can be used, for example, for simulations of nanoparticles or quantum dots, which are mesoscopic objects and are expected to have properties intermediate between macroscopic and quantum systems. This can be of interest, for example in sedimentation studies, where Langevin dynamics was proposed [32]. In principle, presented results can be extended towards asynchronous simulations using the Gillespie's algorithm [8]. This can be achieved by using time steps distributed exponentially.

10.1. Stochastic representations

The Burgers equation was initially proposed by Bateman while modeling the weakly viscous liquid motion [33]. It was later studied extensively by Burgers as a simple model of turbulence [22]. Presently, the number of applications of the Burgers equation is very diverse. It has been used to model physical systems, such as surface perturbations, acoustical waves, electromagnetic waves, density waves, or traffic (see for example [34]). The stochastic representation of the Burgers equation can be traced back to the seminal works of Busnello et al. [35,36]. Later, Constantin and Iyer derived a probabilistic representation of the deterministic 3-dimensional Navier-Stokes equations [26,37]. The result presented here complements the findings of these authors as incompressibility, and hence the harmonicity of the drift, in the Burgers equation is not required.

While the usual presentation of the stochastic mechanics typically used the Schrödinger equation as a solution device and paths were constructed from solutions of the Schrödinger equation, this is not necessary. McClendon and Rabitz simulated several quantum systems using the differential equations of Nelson's stochastic quantization as a starting point [38]. In the framework of scale relativity, Herman [23] simulated QM particle in a box using the Langevin equations. Later, Al-Rashid et al. [39] simulated the quantum harmonic oscillator extending Herman's approach.

The approach presented here can be used as an alternative to numerical solutions of the Schrödinger equation. In this scenario, the denisty of the solution can be sampled from Monte-Carlo simulations as demonstrated.

10.2. Properties of the coordinate process

In Nottale's treatment the Markov coordinate process is only virtual. A fact that is not fully addressed by both stochastic mechanics and scale relativity is why do the theories work only for (box) fractal dimension 2 of the paths. While Nottale gives an heuristic argument and claims that the prescription of a Wiener process may be generalized he does not proceed to rigorously develop the argument. On the other hand, the stochastic mechanics fixes from the start the Wiener process as a driving noise. While this may look plausible in view of the traditions in the treatment of Brownian motion, it is a choice that should be justified as nowadays anomalous types of diffusion dynamics are also recognized and systematically investigated (overview in [40]). The answer to this question can be given more easily by an approach inspired by Nottale and is partially given by the argument given by Gillepsie [8]. The original argument in [8] contains an explicit assumption of existence of the second moment of the distribution, which amounts to assuming Hölder continuity of order $1/2$ as demonstrated here in Th. 1. Th. 1 corresponds to the result established for fractal interpolation computed via a chaos game where the limit random distribution has been identified with the Gaussian distribution [41,42].

Appendix A. Notation and Properties of Fractional Velocity

Definition A1 (Asymptotic $\mathcal{O}$ notation). *The notation $\mathcal{O}\left(x^{\alpha}\right)$ is interpreted as the convention that*

$$\lim_{x \to 0} \frac{\mathcal{O}\left(x^{\alpha}\right)}{x^{\alpha}} = 0$$

for $\alpha > 0$. The notation $\mathcal{O}_x$ will be interpreted to indicate a Cauchy-null sequence with no particular power dependence of x.

Definition A2. *We say that f is of (point-wise) Hölder class $\mathbb{H}^{\beta}$ if for a given x there exist two positive constants $C, \delta \in \mathbb{R}$ that for an arbitrary $y \in Dom[f]$ and given $|x - y| \leq \delta$ fulfill the inequality $|f(x) - f(y)| \leq C|x - y|^{\beta}$, where $|\cdot|$ denotes the norm of the argument.*

Definition A3. *Define the parametrized difference operators acting on a function $f(x)$ as*

$$\Delta_{\epsilon}^{\pm}[f](x) := sgn(\epsilon)\left(f(x + \epsilon) - f(x)\right).$$

The first one we refer to as forward difference operator, the second one we refer to as backward difference operator.

Definition A4. *Define Fractional Variation operators of order $0 \leq \beta \leq 1$ as*

$$v_{\epsilon\pm}^{\beta}[f](x) := \frac{\Delta_{\epsilon}^{\pm}[f](x)}{|\epsilon|^{\beta}} \tag{A1}$$

Appendix B. Essential properties of fractional velocity

This section follows the presentation given recently in [43].

Definition A5 (Fractional order velocity). *Define the fractional velocity of fractional order β as the limit*

$$v_{\pm}^{\beta}f(x) := \lim_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{\pm}[f](x)}{|\epsilon|^{\beta}} . \tag{A2}$$

where $0 < \beta \leq 1$ are real parameters and $f(x)$ is real-valued function. A function for which at least one of $v_{\pm}^{\beta}f(x)$ exists finitely will be called β-differentiable at the point x.

In the above definition we do not require upfront equality of left and right β-velocities. This amounts to not demanding continuity of the β-velocities in advance. Instead, continuity is a property, which is fulfilled under certain conditions.

Definition A6. *The set of points where the fractional velocity exists finitely and $v_{\pm}^{\beta}f(x) \neq 0$ will be denoted as the **set of change** $\chi_{\pm}^{\beta}(f) := \left\{ x : v_{\pm}^{\beta}f(x) \neq 0 \right\}$.*

Since the set of change $\chi_{+}^{\alpha}(f)$ is totally disconnected [18] some of the useful properties of ordinary derivatives, notably the continuity and the semi-group composition property, are lost.

Definition A7. *β-Regularized derivative of a function is defined as:*

$$\frac{d^{\beta\pm}}{dx}f(x) := \lim_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{\pm}[f](x) - v_{+}^{\beta}f(x)\epsilon^{\beta}}{\epsilon}$$

We will require as usual that the forward and backward regularized derivatives be equal for a uniformly continuous function.

In this section we assume that the functions are BVC in the neighborhood of the point of interest. Under this assumption we have

- Product rule

$$v_+^\beta [f\,g]\,(x) = v_+^\beta f\,(x)\,g(x) + v_+^\beta g\,(x)\,f(x) + [f,g]_\beta^+\,(x)$$

$$v_-^\beta [f\,g]\,(x) = v_-^\beta f\,(x)\,g(x) + v_-^\beta g\,(x)\,f(x) - [f,g]_\beta^-\,(x)$$

- Quotient rule

$$v_+^\beta [f/g]\,(x) = \frac{v_+^\beta f\,(x)\,g(x) - v_+^\beta g\,(x)\,f(x) - [f,g]_\beta^+}{g^2(x)}$$

$$v_-^\beta [f/g]\,(x) = \frac{v_-^\beta f\,(x)\,g(x) - v_-^\beta g\,(x)\,f(x) + [f,g]_\beta^-}{g^2(x)}$$

where

$$[f,g]_\beta^\pm\,(x) := \lim_{\epsilon\to 0} v_{\epsilon\pm}^\gamma\,[f]\,(x)\,v_{\epsilon\pm}^{\beta-\gamma}\,[g]\,(x)$$

wherever $[f,g]_\beta^\pm\,(x) \neq 0$.

For compositions of functions

- $f \in \mathbb{H}^\beta$ and $g \in \mathbb{C}^1$

$$v_+^\beta f \circ g\,(x) = v_+^\beta f\,(g)\,(g'(x))^\beta$$

$$v_-^\beta f \circ g\,(x) = v_-^\beta f\,(g)\,(g'(x))^\beta$$

- $f \in \mathbb{C}^1$ and $g \in \mathbb{H}^\beta$

$$v_+^\beta f \circ g\,(x) = f'(g)\,v_+^\beta g\,(x)$$

$$v_-^\beta f \circ g\,(x) = f'(g)\,v_-^\beta g\,(x)$$

Basic evaluation formula

$$v_\pm^\beta f\,(x) = \frac{1}{\beta} \lim_{\epsilon\to 0} \epsilon^{1-\beta} f'(x \pm \epsilon)$$

Derivative regularization

Let $f(t,w) \in \mathbb{C}^2$ be composition with $w(x)$, a $1/q$-differentiable function at x, then

$$\frac{d^\pm}{dx} f(x,w) = \frac{\partial f}{\partial x} + \frac{d^\pm}{dx} w(x) \cdot \frac{\partial f}{\partial w} \pm \frac{1}{q!} [w^q]^\pm \cdot \frac{\partial^q f}{\partial w^q} \tag{A3}$$

where

$$[w^q]^\pm = \left(v_\pm^{1/q} w\,(x)\right)^q$$

is the fractal q-adic (co-)variation.

Appendix C. The Stochastic Variation Problem

The study of stochastic Lagrangian variational principles has been motivated initially by quantum mechanics and also by optimal control problems. This section gives only sketch for the treatment of

the problem. The reader is directed to [14,20,21]. In the simplest form this is the minimization of the regularized functional

$$S_\alpha(t_0, T) := \lim_{N\to\infty} \mathbb{E}\left((\mathcal{P}_N) \sum_{t=t_0}^{t=T} \frac{1}{2} \frac{(\Delta X_k)^2}{\Delta t_k} - \left(\alpha - \frac{1}{2}\right) b^2 \middle| X(t_k) = x(\alpha t_k + (1-\alpha)t_{k+1}) \right)$$

for the partition $\mathcal{P}_N$. So suppose that $\alpha = 1$. Then the increments can be interpreted as Itô integrals so that by the Itô isometry since finite summation and integration commute

$$\mathbb{E}\left(\frac{1}{2\Delta t_k}(\Delta X_k)^2 - \frac{1}{2}b^2 \middle| X(t_k) = x(t_k) \right) =$$

$$\frac{1}{2\Delta t_k}\left(\int_{t_k}^{t_{k+1}} a\,ds \right)^2 + \frac{1}{\Delta t_k}\left(\int_{t_k}^{t_{k+1}} a\,ds \right) \mathbb{E}\left(\int_{t_k}^{t_{k+1}} b\,dw \right) + \frac{1}{2\Delta t_k}\mathbb{E}\left(\int_{t_k}^{t_{k+1}} b\,dw \right)^2 - \frac{1}{2}b^2 =$$

$$\frac{1}{2\Delta t_k}\left(\int_{t_k}^{t_{k+1}} a\,ds \right)^2 + \frac{1}{2\Delta t_k}\int_{t_k}^{t_{k+1}} b^2\,ds - \frac{1}{2}b^2 = |a|\Delta t_k + \mathcal{O}(\Delta t_k)$$

Therefore, $S_\alpha(t_0, T)$ is minimal if the drift vanishes on $\mathcal{P}_N$. Suppose that X_t is varied by small smooth function $\lambda\phi(t, x)$ then the Itô lemma should be applied so that $\mathbb{E}(d\delta X_t | \mathcal{F}) = 0$ on the difference process $\delta X_t = \lambda\phi(t, x)dt + b dW_t$. Therefore,

$$\mathbb{E}(d\phi|\mathcal{F}) = \lambda dt \left(\frac{\partial}{\partial t}\phi + \phi\frac{\partial}{\partial x}\phi + \frac{b^2}{2}\frac{\partial^2}{\partial x^2}\phi \right) = 0 \tag{A4}$$

should hold. The same calculation can be performed for any $\alpha \in [0,1]$.

Remark A1. *The treatment of Pavon uses the symmetrized functional $S = S_0 + S_1$ together with a constraint on anti-symmetrized functional $S_0 - S_1$ in the present notation.*

Appendix D. Simulation script

Listing 1: Exact simulation Matlab code

```matlab
1    clear all;
2    close all;
3    dt = 1/2^10;
4    st=sqrt(dt);
5    sig=1;
6    T=10;
7
8    t = 0:dt:T-dt;              % Time vector
9    % Set initial condition
10   rand('state',200);              % Set random seed
11   nsim=1000;
12   N=length(t);
13
14   r= randn(N,nsim);
15
16   x0=T*(rand(nsim,1)-0.5)/2;
17
18   %% SDE
19   % Euler- Maruyama
20   y = zeros(N,nsim);
21   y(1,:)=x0';
22   for i = 1 :N-1
23           y(i+1,:) = y(i,:)+ dt*y(i,:)./(t(i)+T) + st*r(i,:);
24   end
25
```

```
26          %%
27          % Exact simluation
28          tau=t'+T;
29          w= zeros(N,nsim);
30          z= zeros(N,nsim);
31          for i=1:nsim
32          z=  r(:,i)./tau;
33              w(:,i)=  tau.*( cumsum(z)*st);
34              w(:,i)=[x0(i); w(1:end-1,i)+x0(i)*tau(1:end-1)/T];
35          end
```

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