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Article

Entropy and Its Application to Number Theory

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Abstract: In this paper, we propose an expansion of the Planck distribution function derived from the Boltzmann principle. Namely, we consider expanding Planck's law with a new distribution function, and discuss fine structure constant. Furthermore, using ideas applied to the expansion of the Planck distribution function, we show that Von Koch's inequality can be derived without using the Riemann Hypothesis, that is, the Riemann Hypothesis is true, and that the abc conjecture is negated. Furthermore, we define a generalization of Entropy and discuss that Entropy is relevant to dynamical systems described by logistic function models, such as the growth of bacteria or populations.

Keywords: entropy; Boltzmann's principle; Planck's law; dynamical system; logistic function; fine structure constant; Von Koch's inequality; Riemann hypothesis; abc conjecture

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1. Introduction

In this paper, we will explain in the following order.

1.1.

First, we explain the Boltzmann principle and the Planck distribution function to aid understanding. The Planck distribution function partitions particles P into resonators N and applies this partitioning method to Entropy S. Furthermore, Entropy S is made to correspond to the average energy of resonators U and an energy element ε. In addition, the Planck distribution function is derived by differentiating Entropy S with the average energy of resonators U.

1.2.

Second, we describe that the expansion of the Planck distribution function which is main contents of this article. We consider Entropy $S_{\pi_f}(x)$ which is the Boltzmann principle divided by function $x/f(x)$, where set the function $f(x)$ to $\log(x)$ and let x be a positive real variable. The function $x/\log(x)$ is an approximation of the number of prime numbers $\pi(x)$. The function $R^{\pm}_{\alpha}(x)$ is defined and describe the relation between $S_{\pi_f}(x)$ and $R^{\pm}_{\alpha}(x)$. Furthermore, we attempt to the possibility of expanding the Planck distribution function by using the function $R^{\pm}_{\alpha}(x)$. Besides, the relation between the constant α of $R^{\pm}_{\alpha}(x)$ and fine structure constant will be considered.

1.3.

Third, we consider applying the constant α of the function $R^{\pm}_{\alpha}(x)$ to number theory. Von Koch's inequality is derived without using the Riemann hypothesis. Namely, it proves the Riemann Hypothesis is correct. Additionally, we verify that the negation of the abc conjecture is true.

1.4.

Finally, we will describe some considerations and issues for the future. We generalize Entropy $S_{\pi_f}(x)$ to Entropy $S_D(x)$, where $D(x)$ is a function on x. Using Entropy $S_D(x)$, we will discuss that Entropy is related to dynamical systems described by logistic function models, such as bacterial and population growth.

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2. The Boltzmann principle and the Planck distribution function.

2.1. Introduction for Entropy S and the Planck distribution function.

To make it easier the understanding, we would first let us introduce the Boltzmann principle and the Planck distribution function as follows.

Definition 2.1. *We define symbols using on this article as follows :*

$$\begin{aligned} P &: The\ number\ of\ particles, \\ N &: The\ number\ of\ resonators, \\ U &: Average\ energy\ per\ a\ resonator, \\ U_N &: Total\ energy, \\ \varepsilon &: An\ element\ of\ energy, \\ \nu &: Frequency, \\ T &: Temperature, \\ k_B &: The\ Boltzmann\ constant, \\ h &: The\ Planck\ constant, \\ \beta &: Inverse\ temperature. \end{aligned} \tag{2.1}$$

□

Using the definitions above, the following equations are satisfied :

$$U_N = NU = P\varepsilon, \quad \frac{P}{N} = \frac{U}{\varepsilon}, \tag{2.2}$$

$$\beta = \frac{1}{k_B T}, \tag{2.3}$$

where the inequality $P > N$ is satisfied.

The concept of the Boltzmann principle is that the number of particles P is partitioned by the number of resonators N. Namely, we define the number of states $W_{N,P}$ and Entropy S (the Boltzmann Principle) such that the number of particles P is partitioned by the number of partitions $N-1$, where P and N can be regarded positive integer numbers as follows :

Definition 2.2. *Let the number of particles P and the number of resonators N be positive integers. The number of states and the Boltzmann Principle are defined as follows :*

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!}, \quad (the\ number\ of\ states,\ combination), \tag{2.4}$$

$$S_{N,P} = k_B \log W_{N,P}, \quad (Boltzmann\ Principle), \tag{2.5}$$

$$S = \frac{S_{N,P}}{N}, \quad (the\ average\ of\ S_{N,P}). \tag{2.6}$$

□

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Using Stirling's formula, for sufficiently large natural numbers P and N, the following formulas are satisfied :

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!} \approx \frac{(N+P)^{N+P}}{N^N P^P}. \tag{2.7}$$

Using the Boltzmann principle above, for sufficiently large the number of particles P and resonators N, we can obtain the following equations :

$$\begin{aligned} S_{N,P} &= k_B \log W_{N,P} \\ &= k_B\{(N+P)\log(N+P) - \log N^N - \log P^P\} \\ &= k_B N\{(1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log\frac{P}{N}\}. \end{aligned} \tag{2.8}$$

Using the definition the equality(2.2) and (2.6) above, the equality(2.8) is satisfied as follows :

$$S = k_B\Big\{(1+\frac{U}{\varepsilon})\log(1+\frac{U}{\varepsilon}) - \frac{U}{\varepsilon}\log\frac{U}{\varepsilon}\Big\}. \tag{2.9}$$

Differentiate both sides of the equation(2.9) above with respect to average energy per resonator U. Hence, the following equation is satisfied :

$$\frac{dS}{dU} = \frac{k_B}{\varepsilon}\Big\{\log(1+\frac{U}{\varepsilon}) - \log\frac{U}{\varepsilon}\Big\}. \tag{2.10}$$

Furthermore, differentiate both sides of the equation(2.10) with respect to average energy per resonator U, the following an equation is satisfied :

$$\frac{d^2S}{dU^2} = \frac{-k_B}{U(\varepsilon+U)}. \tag{2.11}$$

The rate of change of entropy at energy U ,that is dS/dU, is equal to the reciprocal of temperature T. Namely, the following relations between Entropy S, average energy per resonator U and temperature T is satisfied :

$$\frac{dS}{dU} = \frac{1}{T}. \tag{2.12}$$

Thus, using the equation(2.11) and (2.12), the following relation is satisfied :

$$\frac{d}{dU}\left(\frac{1}{T}\right) = \frac{-k_B}{U(\varepsilon+U)}. \tag{2.13}$$

Integrating both sides of the equation(2.13) with respect to average energy per resonator U, the following relation is satisfied :

$$U = \frac{\varepsilon}{\exp(\frac{\varepsilon}{k_B T}) - 1}. \tag{2.14}$$

Here, put ε as follows :

$$\varepsilon = h\nu. \tag{2.15}$$

Therefore, the following equations are obtained :

$$U = \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1} = \frac{h\nu}{\exp(h\nu\beta) - 1}, \quad (Planck's\ law). \tag{2.16}$$

The equation above(2.16) is determined the expression for the average energy of particles in a single mode of frequency ν in thermal equilibrium T, that is, named Planck's law. Besides, we define the distribution function $\bar{n}(\nu, \beta)$ as follows:

$$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}, \quad (Planck\ distribution\ function). \tag{2.17}$$

This is expressed the mean particle occupation number in thermal equilibrium T . This is named the Planck distribution function on this paper. Moreover, the equation(2.17) is transformed as follows :

$$\frac{\bar{n}(\nu, \beta)}{\bar{n}(\nu, \beta) + 1} = \exp(-h\nu\beta), \quad (Boltzmann\ factor). \tag{2.18}$$

The function $\exp(-h\nu\beta)$ is named the Boltzmann factor. Besides, let N_g and N_e be the mean number of atoms in the ground state and in the excited state. The following equation is satisfied :

$$\frac{N_e}{N_g} = \exp(-h\nu\beta). \tag{2.19}$$

Note: In this paper, Planck's radiation law refers to the following expression :

$$U_p = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp(h\nu\beta) - 1}. \quad (Planck's\ Radiation\ law) \tag{2.20}$$

where the constant c is the speed of light.

3. Expansion of the Planck distribution function.

We will continue the discussion with reference to ideas in subsection 2.1. We consider that the number of particles P is replaced to the positive real variable x, and the number of resonator N is replaced to the number of prime number $\pi(x)$. However the function $\pi(x)$ is not differentiable. Therefore, we consider to partition x by the function $x/\log(x)$, that is, divided a positive real variable x by the logarithm $\log(x)$. This function $x/\log(x)$ is an approximation of $\pi(x)$. Namely, the number of resonator N is replaced to the number of prime number $x/\log(x)$ Thus, we show that the function $R^{\pm}_{\alpha}(x)$ and expansion of the Planck distribution function are derived as follows.

3.1. Entropy S_{π_f} that partitioned by approximation of $\pi(x)$.

First, we start with the definition of Entropy $S_{\pi_f}(x)$ on a positive real variable x.

Definition 3.1. *Let $x > 1$ be a positive real variable, and $f(x)$ be a positive real valued function on x.*

$$\pi(x) = \sum_{\substack{p \le x \\ p:prime}} 1\,, \tag{3.1}$$

$$\pi_f(x) = \frac{x}{f(x)}\,, \tag{3.2}$$

$$Q_f(x) = \frac{x}{\pi_f(x)}\,. \tag{3.3}$$

The function $\pi(x)$ is expressed that the number of prime numbers less than or equal to x. By the definition above, it is satisfied that $Q_f(x) = f(x)$. □

We define the number of states $W_{\pi_f,x}$. Entropy $S_{\pi_f,x}$ under $W_{\pi_f,x}$ is defined by the number of states $W_{\pi_f,x}$. Moreover, Entropy $S_{\pi_f}(x)$ under $\pi_f(x)$ is defined to devided by Entropy $S_{\pi_f,x}$ by $\pi_f(x)$ as follows:

Definition 3.2. *We define that the number of state $W_{\pi_f,x}$, Entropy $S_{\pi_f,x}$ and Entropy $S_{\pi_f}(x)$. Let $x > 1$ be a positive real variable.*

$$W_{\pi_f,x} = \frac{(\pi_f(x) + x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)} x^x}, \tag{3.4}$$

$$S_{\pi_f,x} = \log W_{\pi_f,x}, \tag{3.5}$$

$$S_{\pi_f}(x) = \frac{S_{\pi_f,x}}{\pi_f(x)}. \tag{3.6}$$

□

Note: Since the definition of Combination below formula(3.7) cannot define real values well, therefore, we adopted the definition of formula(3.4) using Stirling's approximation.

$$\frac{(\pi_f(x) + x - 1)!}{(\pi_f(x) - 1)!x!}. \tag{3.7}$$

In the following discussion, unless otherwise specified, let the function $f(x)$ set to $\log(x)$. Namely, the following is satisfied :

$$f(x) = \log(x). \tag{3.8}$$

Therefore, using definitions above and the prime number theorem (Refer to Narkiewicz [1]), the following conditions are satisfied :

$$Q_f(x) = Q_{\log}(x) = \frac{x}{\pi_{\log}(x)} = \log(x) \sim \frac{x}{\pi(x)}. \tag{3.9}$$

By the definition3.2, for $x > 1$, the following equations are satisfied :

$$\begin{aligned} S_{\pi_f,x} &= (\pi_f(x) + x)\log(\pi_f(x) + x) - \pi_f(x)\log(\pi_f(x)) - x\log(x) \\ &= \pi_f(x)\Big((1 + \frac{x}{\pi_f(x)})\log(1 + \frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)}\log(\frac{x}{\pi_f(x)})\Big), \end{aligned} \tag{3.10}$$

$$S_{\pi_f}(x) = (1 + \frac{x}{\pi_f(x)})\log(1 + \frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)}\log(\frac{x}{\pi_f(x)}). \tag{3.11}$$

Using the function $Q_f(x)$ above, the function $S_{\pi_f}(x)$ is expressed as follows :

$$S_{\pi_f}(x) = (1 + Q_f(x))\log(1 + Q_f(x)) - Q_f(x)\log Q_f(x). \tag{3.12}$$

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Differentiating Entropy $S_{\pi_f}(x)$, the following formulas are satisfied.

$$\begin{aligned} S'_{\pi_f}(x) &= (\frac{x}{\pi_f(x)})' \log(1+\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})' \\ &\quad - \Big((\frac{x}{\pi_f(x)})' \log(\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})' \Big) \\ &= (\frac{x}{\pi_f(x)})' \Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Big). \end{aligned} \tag{3.13}$$

Furthermore, differentiating $S'_{\pi_f}(x)$ as follows :

$$\begin{aligned} S''_{\pi_f}(x) &= (\frac{x}{\pi_f(x)})'' \Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Big) \\ &\quad + (\frac{x}{\pi_f(x)})' \Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Big)'. \end{aligned} \tag{3.14}$$

Therefore, the equations above is expressed by using $Q_f(x)$ as follows :

$$S'_{\pi_f}(x) = Q'_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big), \tag{3.15}$$

$$\begin{aligned} S''_{\pi_f}(x) &= Q''_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big) \\ &\quad + Q'_f(x) Q'_f(x) \Big(\frac{-1}{Q_f(x)(1+Q_f(x))} \Big). \end{aligned} \tag{3.16}$$

Repeating differential of the part of $Q''_f(x)(\log(1+Q_f(x)) - \log Q_f(x))$ on (3.16), the following conditions are satisfied :

$$\begin{aligned} &\Big(Q''_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big) \Big)' \\ &= Q^{(3)}_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big) \\ &\qquad + Q^{(2)}_f(x) Q'_f(x) \Big(\frac{-1}{Q_f(x)(1+Q_f(x))} \Big), \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\Big(Q^{(n)}_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big) \Big)' \\ &= Q^{(n+1)}_f(x) \Big(\log(1+Q_f(x)) - \log Q_f(x) \Big) \\ &\qquad + Q^{(n)}_f(x) Q'(x)_f \Big(\frac{-1}{Q_f(x)(1+Q_f(x))} \Big). \end{aligned} \tag{3.18}$$

Therefore, for all $x > 1$, the following conditions are satisfied :

$$\begin{aligned} &Case(1), n > 1 \text{ is even number :} \\ &\quad Q^{(n+1)}_f(x) = \frac{(-1)^n (n)!}{x^{n+1}} > \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &Case(2), n > 1 \text{ is odd number :} \\ &\quad Q^{(n+1)}_f(x) = \frac{(-1)^n (n)!}{x^{n+1}} < \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x). \end{aligned} \tag{3.20}$$

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Furthermore, for all $x > 1$, the following are satisfied :

$$Q_f^{(n)}(x) > Q_f^{(2)}(x), \tag{3.21}$$

$$|Q_f^{(n+1)}(x)| > |Q_f^{(n)}(x)|, \tag{3.22}$$

where n is a positive integer.

Next, we define some functions $k_f(x)$, $R_m^+(x)$ and $R_m^-(x)$ as follows :

Definition 3.3. *The definition of the function $k_f(x)$.*

Let $x > 1$ be a positive real variable, and $f(x)$ be a real valued function. The function $k_f(x)$ is defined as follows :

$$k_f(x) = S''_{\pi_f}(x)\left(\frac{-Q_f(x)(1+Q_f(x))}{Q'_f(x)}\right). \tag{3.23}$$

Therefore, the following equation is satisfied :

$$S''_{\pi_f}(x) = k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \tag{3.24}$$

□

The function $k_f(x)$ is named the Boltzmann k_f-function.

Definition 3.4. *The function $R_m^+(x)$ and $R_m^-(x)$ are defined as follows :*

$$R_m^+(x) = \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right|. \tag{3.25}$$

Therefore, the following equations are satisfied :

$$R_m^+(x) = \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right| = \sum_{n=1}^{m}\left|(\log(x))^{(n)}\right|. \tag{3.26}$$

Same as discussion, the following inequality are satisfied :

$$R_m^-(x) = \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n}. \tag{3.27}$$

Therefore, the following equations are satisfied :

$$R_m^-(x) = \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n} = \sum_{n=1}^{m}(\log(x))^{(n)}. \tag{3.28}$$

□

The function $R_m^+(x)$ is named an m-th absolute lower approximation of $k_f(x)$ or simply R_m^+-function. Similarly, the function $R_m^-(x)$ is named an m-th lower approximation of $k_f(x)$ or simply R_m^--function. Using the defintion above, the following inequality is satisfied :

$$R_m^+(x) \geq R_m^-(x), \tag{3.29}$$

where the function $(\log(x))^{(n)}$ represents the n-th differentiation of $\log(x)$.

Note: To be distinguish from the function $(\log(x))^{(n)}$, the n-th power $\log(x)$ is represented the function $(\log(x))^n$ and $\log^n(x)$.

Using the equation(3.24), for all $x > 1$, the following conditions are satisfied :

$$k_f(x) = -S''_{\pi_f}(x)\frac{Q_f(x)(1+Q_f(x))}{Q'_f(x)} \leq \frac{1}{x}(2+\log(x)). \tag{3.30}$$

where the function $Q_f(x)$ is $\log(x)$ by the definition. Because, by the equation(3.16),

$$\begin{aligned} S''_{\pi_f}(x) &= Q''_f(x)\Big(\log(1+Q_f(x))-\log Q_f(x)\Big) \\ &\quad + Q'_f(x)Q'_f(x)\left(\frac{-1}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.31}$$

Therefore, for $x > 1$, the following are satisfied :

$$\begin{aligned} k_f(x) &= \frac{1}{x^2}\log\left(1+\frac{1}{\log(x)}\right)x\log(x)(1+\log(x)) + \frac{1}{x} \\ &= \frac{1}{x}\log\left(1+\frac{1}{\log(x)}\right)^{\log(x)}(1+\log(x)) + \frac{1}{x} \\ &\leq \frac{1}{x}\log(e)\Big(1+\log(x)\Big) + \frac{1}{x} \qquad \because) \lim_{x\to\infty}\left(1+\frac{1}{\log(x)}\right)^{\log(x)} \to e \\ &= \frac{1}{x}\Big(1+\log(x)\Big) + \frac{1}{x} \\ &= \frac{1}{x}(2+\log(x)). \end{aligned} \tag{3.32}$$

Furthermore, there exist a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(|Q'_f(x)| + |Q''_f(x)| + \cdots + |Q^{(m)}_f(x)|\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^+_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.33}$$

Using the same discussion above, there is a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(Q'_f(x) + Q''_f(x) + \cdots + Q^{(m)}_f(x)\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^-_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.34}$$

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The first order differentiation of Entropy $S_{\pi_f}(x)$ is always positive values, that is $S'_{\pi_f}(x) > 0$. Moreover, the second order differentiation of Entropy $S_{\pi_f}(x)$ has always negative values, so that $S''_{\pi_f}(x) < 0$. Therefore, Entropy $S_{\pi_f}(x)$ has no inflection points.

3.2. Derivation of the functions $R^{\pm}_{\alpha}$.

Next, the function $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ are derived as follows :

Definition 3.5. *$R^{+}_{\alpha}(x)$, $R^{-}_{\alpha}(x)$ and $R^{\pm}_{\alpha}(x)$*
Let the constant $\alpha > 0$ be a positive real number. For all positive real variable $x > 1$, the function $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ are defined as follows :

$$R^{\pm}_{\alpha}(x) = \frac{\sqrt{2\pi\alpha}}{ex \pm 1}. \tag{3.35}$$

Therefore, the following conditions are satisfied :

$$xR^{\pm}_{\alpha}(x) = \frac{\sqrt{2\pi\alpha x}}{ex \pm 1}, \tag{3.36}$$

$$\frac{1}{xR^{\pm}_{\alpha}(x)} = \frac{e}{\sqrt{2\pi\alpha}}\left(1 \pm \frac{1}{ex}\right). \tag{3.37}$$

□

This function $R^{\pm}_{\alpha}(x)$ is named $R^{\pm}_{\alpha}$ lower approximation by $k_f(x)$ and α or simply $R^{\pm}_{\alpha}$-function. The relations of functions $R^{\pm}_{\alpha}(x)$ are satisfied as follows :

Lemma 3.6. *The relation $R^{\pm}_{m}(x) \geq R^{\pm}_{\alpha}(x)$.*
Let the constant $\alpha > 0$ be a positive real number. There exist an integer $m \geq 1$ such that for sufficiently large real variable $x > 1$, the following inequality is satisfied :

$$R^{\pm}_{m}(x) \geq R^{\pm}_{\alpha}(x) = \frac{\sqrt{2\pi\alpha}}{ex \pm 1} \tag{3.38}$$

where a positive real number α are satisfied as follows :

$$x \geq \frac{\mp}{e - \sqrt{2\pi\alpha}}, \tag{3.39}$$

that is, satisfied as follows:

$$\frac{e}{\sqrt{2\pi}}(1 \pm \frac{1}{x}) \geq \alpha. \tag{3.40}$$

Proof. The proof of Lemma 3.6 are described the following the subsection 6.1. □

Consequently, for all real variable $x > 1$, a real valued function $f(x)$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$S''_{\pi_f}(x) \geq R^{\pm}_{m}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}, \tag{3.41}$$

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Namely, the following inequality is satisfied :

$$S''_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}. \tag{3.42}$$

The second order differentiation of Entropy $S''_{\pi_f}(x)$ is suppressed from the bottom side by formula(3.42). Besides, the function $R^{\pm}_{\alpha}(x)$ is suppressed from the upper side by formula as follows:

Lemma 3.7. *The upper upper approximation of $R^{\pm}_{m}(x)$*
For sufficiently large real variable $x > 1$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$\frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \geq R^{\pm}_{m}(x) \geq R^{\pm}_{\alpha}(x) = \frac{\sqrt{2\pi\alpha}}{ex \pm 1}. \tag{3.43}$$

Proof. The proof of Lemma 3.7 are described the following the subsection 6.2. □

Note: The function $R_{\alpha}\pm$ and the constant α are derived by dividing the entropy space x by the approximation $x/\log(x)$ of $\pi(x)$ and by applying Starling's formula to the series obtained by the n-th differentiation of $Q_f(x)$. These are reasons that is defined as the set of irrational numbers $\sqrt{2}$, e, and π.

On the next subsection, we discuss the meaning of inequalities in inequality(3.42).

3.3. *Expanded Planck distribution function $n^{\pm}(x,\alpha)$.*

Next, we examine to define the expanded distribution functions $n^{\pm}(x,\alpha)$ by using $R^{\pm}_{\alpha}(x)$. Integrating the inequality(3.41) by a variable x, the following formulas are satisfied :

$$\int S''_{\pi_f}(x)dx \geq \int R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{3.44}$$

Beside, the following are satisfied :

$$S'_{\pi_f}(x) = \int S''_{\pi_f}(x)dx + C, \tag{3.45}$$

$$Q'_f(x) \geq R^{\pm}_{\alpha}(x). \tag{3.46}$$

Therefore, the following formulas are satisfied :

$$\begin{aligned} S'_{\pi_f}(x) &= Q'_f(x)(\log(1+Q_f(x)) - \log Q_f(x)) + C \\ &\geq R^{\pm}_{\alpha}(x)(\log(1+Q_f(x)) - \log Q_f(x)) + C \\ &= R^{\pm}_{\alpha}(x)\log(1+\frac{1}{Q_f(x)}) + C. \end{aligned} \tag{3.47}$$

where the constant C is a positive real number.

Here, for all sufficiently large $Q_f(x) > 0$, the following equation is satisfied :

$$\log(1+\frac{1}{Q_f(x)}) = 0. \tag{3.48}$$

Hence, the first order differentiation $S'_{\pi_f}(x)$ is satisfied as follows :

$$S'_{\pi_f}(x) = Q'_f(x)\log(1+\frac{1}{Q_f(x)}) = 0. \tag{3.49}$$

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Thus, the constant C is satisfied as follows :

$$C = 0. \tag{3.50}$$

Therefore, the inequality(3.47) is satisfied as follows :

$$S'_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\log(1 + \frac{1}{Q_f(x)}). \tag{3.51}$$

For positive real variable $x > 1$, the function $S'_{\pi_f}(x)$ are satisfied as follows :

$$\frac{1}{x} \geq S'_{\pi_f}(x) = \frac{1}{x}\log(1 + \frac{1}{\log(x)}). \tag{3.52}$$

According to inequalities(3.51) and (3.52), the following is satisfied:

$$\frac{1}{xR^{\pm}_{\alpha}(x)} \geq \log(1 + \frac{1}{Q_f(x)}). \tag{3.53}$$

Therefore, by (3.53) the following inequality is derived :

$$Q_f(x) \geq \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1}. \tag{3.54}$$

Focusing on the equality part of the inequality (3.54), we define new distribution function $n^{\pm}(x,\alpha)$ as follows:

Definition 3.8. *Expanded Planck distribution function $n^{\pm}(x,\alpha)$.*
Expanded distribution functions $n^{\pm}(x,\alpha)$ of the Planck distribution function $\bar{n}(\nu,\beta)$ are defined as follows:

$$n^{\pm}(x,\alpha) = \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1}, \tag{3.55}$$

where $\alpha > 0$. □

This distribute function $n^{\pm}(x,\alpha)$ is named Expanded Planck distribution function. The definition above (3.55) is transformed as follows :

$$\frac{n^{\pm}(x,\alpha)}{n^{\pm}(x,\alpha)+1} = \exp(\frac{-1}{xR^{\pm}_{\alpha}(x)}), \quad (\alpha > 0). \tag{3.56}$$

Thus, this function $n^{\pm}(x,\alpha)$ can be regarded as one of the distribution functions. Therefore, the expanded distribution functions $n^{\pm}(x,\alpha)$ are regards as the approximate density of $x/\pi(x)$ until the number x. Furthermore, this function $n^{\pm}(x,\alpha)$ is seems to be expanded the Planck distribution function $\bar{n}(\nu,\beta)$. According to imitate the Boltzmann factor, the following function

$$\exp(\frac{-1}{xR^{\pm}_{\alpha}(x)}) \tag{3.57}$$

is named Expanded Boltzmann factor or $R^{\pm}_{\alpha}$ factors. We will consider the further relationship in the next subsection.

3.4. Correspondence the Planck distribution function $\bar{n}(\nu,\beta)$ and Expanded Planck distribution function $n^{\pm}(x,\alpha)$.

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We examine to correspond the Planck distribution function $\bar{n}(\nu,\beta)$ and expanded distribution function $n^{\pm}(x,\alpha)$ as follows :

$$\bar{n}(\nu,\beta) = \frac{1}{\exp(h\nu\beta)-1}, \ (Planck\ distribution\ function) \tag{3.58}$$

where h: the Planck constant , ν: Frequency, β: Inverse temperature.

Here, we consider to correspond the internal parameter of the Boltzmann factor $\exp(-h\nu\beta)$

$$h\nu\beta \tag{3.59}$$

and the internal function of $R_{\alpha}^{\pm}$ factor $\exp(\frac{-1}{xR_{\alpha}^{\pm}(x)})$

$$\frac{1}{xR_{\alpha}^{\pm}(x)} \quad \left(= \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex})\right). \tag{3.60}$$

Namely, we suppose the correspondence as follows:

$$h\nu\beta = \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex}). \tag{3.61}$$

Furthermore, we can consider by separating the correspondence between and the variable parts and the constant parts as follows :

$$\nu\beta = (1 \pm \frac{1}{ex}), \ \ (variable\ parts) \tag{3.62}$$

$$h = \frac{e}{\sqrt{2\pi\alpha}}. \ \ (constant\ parts) \tag{3.63}$$

The relationship diagram between $S_{N,P}$ and $S_{\pi_f,x}$ is shown below :

$$\begin{array}{ccc}
W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!}, & & S_{N,P} = k_B \log W_{N,P}, \\
\exp(-h\nu\beta) & \xrightarrow{\log} & -h\nu\beta \\
{\scriptstyle N:=\pi_f(x) \atop P:=x} \downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi}h} \atop x:=\frac{\mp 1}{e(1-\nu\beta)}} & & {\scriptstyle N:=\pi_f(x) \atop P:=x} \downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi}h} \atop x:=\frac{\mp 1}{e(1-\nu\beta)}} \\
W_{\pi_f,x} = \frac{(\pi_f(x)+x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)}x^x}, & & S_{\pi_f,x} = \log W_{\pi_f,x}, \\
\exp(\frac{-1}{xR_{\alpha}^{\pm}(x)}) & \xrightarrow{\log} & \frac{-1}{xR_{\alpha}^{\pm}(x)}
\end{array} \tag{3.64}$$

Corresponding the above, the expanded distribution function $n^{\pm}(x,\alpha)$ becomes to expand the Planck distribution function$\bar{n}(\nu,\beta)$. Namely, the following conditions are satisfied :

Suggestion 3.9. *The expansion of the Planck distribution $\bar{n}(\nu,\beta)$.*
Let the constant $\alpha > 0$ be a positive real number. For all real variable $x > 1$ the following equation is satisfied :

$$\bar{n}(\nu,\beta) = n^{\pm}(x,\alpha), \tag{3.65}$$

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where

$$\begin{aligned} x &= \frac{\mp 1}{e(1-\nu\beta)}, \quad \textit{that is, } \nu\beta = (1 \pm \frac{1}{ex}), \\ \alpha &= \frac{e}{\sqrt{2\pi h}}, \quad \textit{that is, } h = \frac{e}{\sqrt{2\pi\alpha}}. \end{aligned} \tag{3.66}$$

Namely, the distribution function $n^{\pm}(x,\alpha)$ can be regarded as representing an expansion of the Planck distribution function $\bar{n}(\nu,\beta)$. □

For sufficiently large $x > 1$, the correspondence of equation(3.62) is satisfied as follows:

$$\nu\beta = \lim_{x\to\infty}(1 \pm \frac{1}{ex}) = 1. \tag{3.67}$$

Moreover, according to the method to divide each S and $S_{\pi_f}(x)$, we remember that the following corresponds :

1. The number of particles P is replaced to the positive real variable x.
2. The number of resonators N is replaced to the approximate of $\pi(x)$, that is, $\frac{x}{\log(x)}$.

We remember that the following conditions are satisfied :

$$\frac{U}{\varepsilon} = \frac{P}{N} \tag{3.68}$$

$$Q_f(x) = \frac{x}{\frac{x}{\log(x)}} = \frac{1}{\frac{1}{\log(x)}}(\sim \frac{x}{\pi(x)}). \tag{3.69}$$

Namely, we suppose the correspondence as follows :

$$\begin{aligned} U &\longleftrightarrow x \longleftrightarrow 1, \\ \varepsilon &\longleftrightarrow \frac{x}{\log(x)} \longleftrightarrow \frac{1}{\log(x)}. \end{aligned} \tag{3.70}$$

Thereby, we define the following function $U^{\pm}_{x,\alpha}$ corresponding to Planck's law U.

Definition 3.10. *The function $U^{\pm}_{x,\alpha}$ as the expansion of Planck's law U.*
Let the constant $\alpha > 0$ be a positive real number. For sufficiently large real variable $x > 1$, the real valued function $U^{\pm}_{x,\alpha}$ is defined as follows :

$$U^{\pm}_{x,\alpha} = n^{\pm}(x,\alpha)\frac{1}{\log(x)} = \frac{1}{\log(x)(\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1)} \tag{3.71}$$

□

Using suggestion 3.9 and definition 3.10, the following suggestion is obtained:

Suggestion 3.11. *The expansion of Planck's law U.*
Let $h > 0$, $\nu > 0$ and $\beta > 0$ be real numbers. Each values h, ν and β means the Planck constant, frequency and inverse temperature.
There exists a real variable $x > 1$ and a constant $\alpha > 0$ such that the following equality is satisfied :

$$U = U^{\pm}_{x,\alpha}, \tag{3.72}$$

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where the following conditions are satisfied :

$$\begin{aligned} &h\nu = \frac{1}{\log(x)}, \quad \nu = \frac{1}{h\log(x)}, \quad h = \frac{e}{\sqrt{2\pi}\alpha}, \\ &\beta = h\log(x)(1 \pm \frac{1}{ex}). \end{aligned} \tag{3.73}$$

Proof. The above proposal is satisfied by setting as follows. For any $\nu > 0$,

$$x = \exp(\frac{1}{h\nu}) \fallingdotseq \exp(\frac{1}{6.626 \times 10^{-34}\nu}) > 1, \tag{3.74}$$

where the Planck constant $h \fallingdotseq 6.626 \times 10^{-34} m^2 kg/s$. □

Namely, the real valued function $U^{\pm}_{x,\alpha}$ can be regarded as representing the expansion of Planck's law U. The function $U^{\pm}_{x,\alpha}$ is named expanded Planck's law.

As a result of suggestion 3.11 above, energy $h\nu$ is put $1/\log(x)$. Namely, Planck's law U is seems to has an spectrum such that an reciprocal of logarithm. It is possible that the discrete values of an element of energy are related to the distribution of prime numbers. The relationship diagram between Planck's law U and expanded Planck's law $U^{\pm}(x,\alpha)$ is shown below :

$$\begin{array}{ccc} S_{N,P} = k_B \log W_{N,P}, & & U = \frac{h\nu}{\exp(\frac{h\nu}{k_B T})-1}, \\ \textit{Boltzmann Principle} & \xrightarrow{\frac{U}{\varepsilon},\, S''} & \textit{Planck's law} \\ {\scriptstyle N:=\pi_f(x) \atop P:=x} \Big\downarrow \Big\uparrow {\scriptstyle \alpha := \frac{e}{\sqrt{2\pi}h} \atop x := \frac{\mp 1}{e(1-\nu\beta)}} & & {\scriptstyle N:=\pi_f(x) \atop P:=x} \Big\downarrow \Big\uparrow {\scriptstyle \alpha := \frac{e}{\sqrt{2\pi}h} \atop h\nu := \frac{1}{\log(x)}} \\ S_{\pi_f,x} = \log W_{\pi_f,x}, & & U^{\pm}(x,\alpha) = \frac{1}{\log(x)} \frac{1}{\exp(\frac{1}{R^{\pm}_{\alpha}(x)})-1}, \\ S_{\pi_f,x}\textit{-Entropy} & \xrightarrow{\frac{1}{\log(x)},\, S''_{\pi}(x)} & \textit{Expanded Planck's law} \end{array} \tag{3.75}$$

3.5. fine structure constants.

The above section, expanded distribution $n^{\pm}(x,\alpha)$ and the Planck distribution $\bar{n}(\nu,\beta)$ was associated by a constant α. In this section, we discuss that the constant $\alpha = e/\sqrt{2\pi}h$ is thought like the fine structure constant.

Let a constant α set as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}h}. \tag{3.76}$$

For all sufficiently large $x > 1$, the following inequalities are satisfied :

$$\frac{e}{\sqrt{2\pi}}(2 + \log(x)) \geq \alpha, \tag{3.77}$$

that is, take $x > 1$ as follows:

$$x \geq \exp(\frac{1}{h} - 2) > 1. \tag{3.78}$$

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According to the Prime numbers theorem, the following relation is satisfied :

$$\pi(x) \sim \frac{x}{\log(x)} \sim \frac{x}{\log(x)+2}. \tag{3.79}$$

Thus, the constant α is satisfied such that

$$\frac{e}{\sqrt{2\pi}} \frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0). \tag{3.80}$$

Hence, the positive real number α can be regard as fine structure constant by $\sqrt{2}, \pi, e$ and $\pi(x)$. Furthermore, the following inequality is satisfied :

Suggestion 3.12. *The relation between the Planck constant h and $\pi(x)$.*
There exists a positive real number $\alpha > 0$ such that for sufficiently large $x > 1$, the following formulas are satisfied :

$$h = \frac{e}{\sqrt{2\pi}\alpha} \geq \frac{1}{2+\log(x)} \sim \frac{\pi(x)}{x}. \tag{3.81}$$

Namely, for sufficiently large $x > 1$, the Planck constant h is bigger than the ratio of a positive real number x and $\pi(x)$. □

Therefore, it is considered that the constant α is associated between the Planck distribution function

$$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}, \tag{3.82}$$

and expanded Planck distribution function

$$n^{\pm}(x, \alpha) = \frac{1}{\exp(\frac{1}{xR_{\alpha}^{\pm}(x)}) - 1}. \tag{3.83}$$

Specially, suppose the constant h_α and α_h are decided by α, e and π as follows :

$$h_\alpha = \frac{e}{\sqrt{2\pi}\alpha}, \tag{3.84}$$

$$\alpha_h = \frac{e}{\sqrt{2\pi}h}. \tag{3.85}$$

Therefore, by suggestion 3.9, Modern physics may be a special case that satisfy the following conditions :

$$\alpha_h = \alpha = \frac{e}{\sqrt{2\pi}h}, \tag{3.86}$$

$$h_\alpha = h = \frac{e}{\sqrt{2\pi}\alpha}. \tag{3.87}$$

In other words, the following suggestion is stated :

Suggestion 3.13. *Let the constant $\alpha > 0$ be a positive real number. The constant h_α can be selected as follows :*

$$h_\alpha = \frac{e}{\sqrt{2\pi}\alpha}, \tag{3.88}$$

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where the following inequality is satisfied :

$$\frac{e}{\sqrt{2\pi}}\frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0). \tag{3.89}$$

Namely, when the condition is satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi h}}, \tag{3.90}$$

Therefore, the constant h_α becomes the Planck constant h. □

Note:
Let me mention here for your attention as follows: The fine structure constant is a physical constant α and is originally expressed as following(3.91) using the Planck constant. In this paper, we describe it as original the fine structure constant α_p to distinguish it from the constant α. Besides, describe it as the elementary charge e_p to distinguish it from Napier's number e. original the fine structure constant α_p is follows:

$$\alpha_p = \frac{e_p^2}{2h\varepsilon_0 c} = \frac{\mu_0 e_p^2 c}{2h}, . \tag{3.91}$$

where

$$\begin{aligned} h &: the\ Planck\ constant, \\ \varepsilon_0 &: the\ electric\ constant, \\ \mu_0 &: the\ magnetic\ constant, \\ e_p &: the\ elementary\ charge, \\ c &: the\ speed\ of\ light. \end{aligned} \tag{3.92}$$

Therefore, the relation original the fine structure constant α_p and the constant α is satisfied as follows :

$$\frac{\alpha_p}{\alpha} = \sqrt{\frac{\pi}{2e^2}}\frac{e_p^2}{\varepsilon_0 c} = \sqrt{\frac{\pi}{2e^2}}e_p^2\mu_0 c. \tag{3.93}$$

The suggestion 3.13 is seems obvious. However, on the following section, we show that some examples such that the constant $\alpha \neq \frac{e}{\sqrt{2\pi h}}$ as follows :

$$\alpha = \frac{1}{\sqrt{2\pi}}, \quad \alpha = \frac{2}{\sqrt{2\pi}}, \quad \alpha = \frac{e}{\sqrt{2\pi}}, \quad \alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{3.94}$$

4. Application the function $R_\alpha^\pm$ to Number Theory.

4.1. Examples using the function R_α^+ for deriving Von Koch's inequality.

We derive Von Koch's inequality using the above the constant α and the function $R_\alpha^+(x)$. (Refer to Fujino [24] for the proof)

Note : In this paper, we consider an element of energy ε and an arbitrary real number ϵ separately.

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Theorem 4.1. *Inequalities for evaluating the number of prime numbers (1). Let $\alpha > 0$ be a positive real number (constant). There exist a positive real number $C > 1$ such that for all sufficiently large real variable $x \geq 2$, the following conditions are satisfied :*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{\sqrt{2\pi\alpha}}{48})^{\frac{1}{4}} \exp(\frac{e}{\sqrt{2\pi\alpha}}) x^{\frac{1}{\sqrt{2\pi\alpha}}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}), \tag{4.1}$$

where the positive real number $\alpha > 0$ are satisfied as follows :

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp(\frac{e}{\sqrt{2\pi\alpha}}), \tag{4.2}$$

$$\frac{1}{\sqrt{2\pi}} \leq \alpha \leq C\frac{e}{\sqrt{2\pi}}, \tag{4.3}$$

$$\exp(\frac{e}{\sqrt{2\pi\alpha}}) = \lim_{x\to\infty} \exp(\frac{1}{xR_\alpha^+(x)}). \tag{4.4}$$

Proof. See [24] for the proof. □

Corollary 4.2. *Inequalities for evaluating the number of prime numbers (2).*
There exist a positive real number $C > 1$ such that for all $\epsilon > 0$ and for all sufficiently large $x \geq 2$, the following conditions is satisfied:

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{48})^{\frac{1}{4}} \exp(e) x (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.5}$$

Proof. Using Theorem 4.1, put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{1}{\sqrt{2\pi}}. \tag{4.6}$$

Therefore, the inequality(4.5) is satisfied.
□

The result of Corollary 4.1 is similar to the following result:(Refer to Narkiewicz [1])

$$(\exists C > 0)|\pi(x) - \mathrm{li}(x)| \leq O(x\exp(-C\sqrt{log(x)})). \tag{4.7}$$

Comparing inequalities(4.5) and (4.7), the following condition is satisfied :

$$O\left(x(\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)})\right) \leq O(x\exp(-C\sqrt{log(x)})). \tag{4.8}$$

Namely, the asymptotic of (4.5) gives better than that of (4.7).
Therefore, let $\alpha = 2/\sqrt{2\pi}$, $(h_\alpha = e/2)$. the theorem 4.1 is satisfied as follows :

Corollary 4.3. *Inequalities for evaluating the number of prime numbers (3).*
There exist a positive real number $C > 1$ such that for all $\epsilon > 0$ and for all sufficiently large $x \geq 2$, the following condition is satisfied:

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp(\frac{e}{2}) x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.9}$$

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Proof. Using Theorem(4.1) and the following conditions is satisfied:

$$1 \leq \frac{1}{\sqrt{2\pi}\alpha} \exp\left(\frac{e}{\sqrt{2\pi}\alpha}\right). \tag{4.10}$$

Put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{2}{\sqrt{2\pi}} (\geq \frac{1}{\sqrt{2\pi}}). \tag{4.11}$$

Hence, the following inequalities is satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi}\alpha} \exp\left(\frac{e}{\sqrt{2\pi}\alpha}\right) \\ &= \frac{1}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}} \exp\left(\frac{e}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}}\right) \quad (\because \alpha = \frac{2}{\sqrt{2\pi}}) \\ &= \frac{1}{2} \exp\left(\frac{e}{2}\right) \quad (= 1.946424 \cdots). \end{aligned} \tag{4.12}$$

Thus, the positive real number $\alpha > 0$ is satisfied conditions of (4.10) and (4.11). Therefore, there exist a positive real number $C > 1$ such that for all sufficiently large $x \geq 2$ the following condition is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp\left(\frac{e}{2}\right) x^{\frac{1}{2}} \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right). \tag{4.13}$$

□

Corollary 4.4. *Von Koch's inequality.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{2}} \log(x), \tag{4.14}$$

where C, ϵ and x are real numbers. Namely,

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{2}} \log(x)). \tag{4.15}$$

Proof. Fix $\epsilon > 0$. For all sufficient large $x \geq 2$, the following conditions are satisfied :

$$\left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right) < \log(x) < x^{\epsilon}. \tag{4.16}$$

Therefore, there exist a positive real number $C > 0$ such that for all sufficiently large $x \geq 2$, the following inequalities are satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{1}{24})^{\frac{1}{4}} \exp\left(\frac{e}{2}\right) x^{\frac{1}{2}} \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right) \\ &\leq C x^{\frac{1}{2}} \log(x). \end{aligned} \tag{4.17}$$

□

As well known, Corollary 4.4 is equivalent to the Riemann Hypothesis. (Refer to Narkiewicz [1]) Therefore, the Riemann Hypothesis is true.

Furthermore, let $\alpha = e/\sqrt{2\pi}$, $(h_\alpha = 1)$. The following inequality is satisfied :

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Corollary 4.5. *The reduction of the upper limit of the inequality that evaluate the number of prime number.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq Cx^{\frac{1}{e}} \log(x), \tag{4.18}$$

where C, ϵ and x are real numbers. Namely,

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{e}} \log(x)). \tag{4.19}$$

Proof. Using theorem 4.1 , put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}}(\geq \frac{e}{\sqrt{2\pi}}). \tag{4.20}$$

Thus, the following conditions are satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) \\ &= \frac{1}{\sqrt{2\pi\frac{e}{\sqrt{2\pi}}}} \exp\left(\frac{e}{\sqrt{2\pi\frac{e}{\sqrt{2\pi}}}}\right) \quad (\because \alpha = \frac{e}{\sqrt{2\pi}}) \\ &= \frac{1}{e} \exp(1) \quad (= 1). \end{aligned} \tag{4.21}$$

Therefore the following condition is satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{e}{48})^{\frac{1}{4}} \exp(1) x^{\frac{1}{e}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\ &\leq Cx^{\frac{1}{e}} \log(x). \end{aligned} \tag{4.22}$$

□

We have a question that the upper limit of the real number d that satisfies the formula $|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/d} \log(x))$ is Napier's constant e correct or not. Namely, the following problem can be considered.

Problem 4.6. *The upper limit of the inequality that evaluate the number of prime number.*

$$e = \sup\{d > 1 \mid (\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq Cx^{\frac{1}{d}} \log(x)\}, \tag{4.23}$$

where C, ϵ and x are real numbers. Namely,

$$e = \sup\{d > 1 \mid |\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{d}} \log(x))\}. \tag{4.24}$$

□

We expect problem 4.6 to be correct. In future, we attempt to solve this problem.

4.2. Example using the function R_α^- for the abc conjecture.

We derive the weak abc conjecture and the strong abc conjecture using the constant α. Namely, using the constant α, the following theorems are satisfied : (Refer to Fujino [25] for the proof)

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Theorem 4.7. *Let $\alpha > 0$ be a positive real number. For all real number $\epsilon > 0$ and the constant $K_\epsilon \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :*

$$K_\epsilon \mathrm{rad}(abc) < c^{\exp(\frac{e}{\sqrt{2\pi\alpha}})-1} \tag{4.25}$$

where the following equation is satisfied :

$$\exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) = \lim_{x\to\infty} \exp\left(\frac{1}{xR_\alpha^-(x)}\right). \tag{4.26}$$

□

Set the constant α as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{4.27}$$

Therefore, the following is satisfied :

$$h_\alpha = \log\left(\frac{\epsilon+2}{\epsilon+1}\right). \tag{4.28}$$

Therefore, the negation of the weak abc conjecture is satisfied as follows :

Theorem 4.8. *The negation of the weak abc conjecture.*
For all real number $\epsilon > 0$ and constant $\bar{K}_\epsilon \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :

$$\bar{K}_\epsilon \mathrm{rad}(abc)^{1+\epsilon} < c. \tag{4.29}$$

Namely, There exist a counter-example in the weak abc conjecture. Therefore, the weak abc conjecture is not true.
□

Furthermore, let $\epsilon = 1$ and $\bar{K}_\epsilon = 1$. The negative of the strong abc conjecture is satisfied as follows :

Theorem 4.9. *The negation of the strong abc conjecture.*
There exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :

$$\mathrm{rad}(abc)^2 < c. \tag{4.30}$$

Namely, the strong abc conjecture is not true. □

The function $R_\alpha^\pm(x)$ was used the above discussions of Von Koch's inequality and the abc conjecture. Entropy and Number theory are thought to be closely related.

4.3. Conclusion and Application to Number Theory.

The above discussion, we attempted to proceed through applying the Boltzmann principle and the Planck distribution function to prime number theory. We considered dividing natural number x by an approximation of $\pi(x)$, that is, $x/\log(x)$. Thereby, we obtained that the function $R_\alpha^\pm(x)$. Furthermore, using the function $R_\alpha^\pm(x)$, we derived and define new distribution function $n^\pm(x, \alpha)$.

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As mentioned above, modern physics is considered to be the special condition that the real number α be satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi h}}, \text{ that is, } h_\alpha = h, \tag{4.31}$$

where the constant h is Planck constant.

Furthermore, using expanded Planck distribution function $R_\alpha^\pm$, we evaluated Von Koch's inequality that equivalent to the Riemann Hypothesis and the abc conjecture. Namely, we considered the system different from modern physics such that Von Koch's inequality is satisfied as follows :

$$\alpha = \frac{2}{\sqrt{2\pi}}, \text{ that is, } h_\alpha = \frac{e}{2}, \tag{4.32}$$

and, the abc conjecture is satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}, \text{ that is, } h_\alpha = \log(\frac{\epsilon+2}{\epsilon+1}). \tag{4.33}$$

Namely, we consider that there exist the relation between expanded Planck distribution function $n^\pm(x,\alpha)$ and the Boltzmann principle, furthermore the Planck distribution function. Furthermore, it is meaningful that there exist the relation between statistical mechanics and number theory.

In the future, based on the above discussion, It may be possible that there exists new constant h_α and non-constant $k_f(x)$ that are different from the constants of modern physics, such as Planck's constant and Boltzmann's constant.

5. Generalization and application to dynamical systems.

We would describe some future issues on this section. Namely, we consider that that Entropy is related to dynamical systems described by logistic function models, such as bacterial and population growth.

5.1. *What are $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$?*

We would like to consider what $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ are (Refer to Nicolis, Prigogine [18], [19]). Let $x > 1$ be a real variable. These functions $S_{\pi_f}(x)$, $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ above are regarded as follows:

$$\begin{aligned} &S_{\pi_f}(x) : \textit{Entropy partitioned by } \pi_f(x), \\ &S'_{\pi_f}(x) : \textit{Entropy velocity } S_{\pi_f}(x), \qquad (\textit{Entropy generation}) \\ &S''_{\pi_f}(x) : \textit{Entropy acceleration } S_{\pi_f}(x). \end{aligned} \tag{5.1}$$

where these functions are satisfied as follows:

$$\begin{aligned} &f(x) = \log(x), \\ &Q_f(x) = \log(x), \\ &\pi_f(x) = \frac{x}{f(x)} = \frac{x}{\log(x)}. \end{aligned} \tag{5.2}$$

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The first order differentiation of function $S_{\pi_f}(x)$, that is, $S'_{\pi_f}(x)$ and the second order differentiation of function $S_{\pi_f}(x)$, that is, $S''_{\pi_f}(x)$ can be describe as follows :

$$\begin{aligned} S'_{\pi_f}(x) &= Q'_f(x)\log\left(1+\frac{1}{Q_f(x)}\right), \\ S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right), \end{aligned} \tag{5.3}$$

where the function $k_f(x)$ is regard as a function decided by a real variable x and the function $\pi_f(x)$. The function $Q_f(x)$ can be regard as the position divided a real variable x by $Q_f(x)$. The first order differentiation of $Q_f(x)$, that is, $Q'_f(x)$ can be regard as the slope of the function $Q_f(x)$ (the change $Q'_f(x)$ of the position $f(x)$, the charge or potential $Q_f(x)$ of the position $f(x)$). Entropy velocity can be regraded as Entropy generation (Refer to Nicolis, Prigogine [18], [19]). We consider the generalization below subsection.

5.2. Generalize of the function $S''_D(x)$.

We generalize the equation above (3.44) as follows. Let $x > 1$ be real variable and $\xi > 0$ be a constant. The function $D(x)$ be a positive real valued function such that $D(x) \le x$. The function $D(x)$ can be thought of as a division of x. Therefore, the above $S_D(x)$, $S'_D(x)$ and $S''_D(x)$ are regarded and defined as follows:

$$S_D(x) = (1+\frac{Q_D(x)}{\xi})\log(1+\frac{Q_D(x)}{\xi}) - \frac{Q_D(x)}{\xi}\log\frac{Q_D(x)}{\xi}, \tag{5.4}$$

$$S'_D(x) = \frac{Q'_D(x)}{\xi}\log\left(1+\frac{\xi}{Q_D(x)}\right), \tag{5.5}$$

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(\xi+Q_D(x))}\right), \tag{5.6}$$

where the relation between the functions $D(x)$ and $Q_D(x)$ are satisfied as follows:

$$D(x) = \frac{\xi x}{Q_D(x)}. \tag{5.7}$$

Moreover, the function $k_D(x)$ can be regard as the function decided by x, ξ and $D(x)$. The function $Q_D(x)$ can be regard as the position divided a real value ξx by $Q_D(x)$. The first order differentiation of $Q_D(x)$, that is, $Q'_D(x)$ can be regard as the change of the position by x and ξ. Each functions above are real valued functions.

5.3. New distribution function $R^{\pm}_{\alpha}(x)$.

We examine that the correspondence between generalized the equation(5.6) of $S''_D(x)$ and the equation(3.44) of $S''_{\pi_f}(x)$. We put as follows:

$$\begin{aligned} S''_D(x) &:= S''_{\pi_f}(x)\ (<0), \\ Q_D(x) &:= Q_f(x), \\ Q'_D(x) &:= Q'_f(x). \\ k_D(x) &:= R^{\pm}_{\alpha}(x), \\ \xi &:= 1. \end{aligned} \tag{5.8}$$

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Therefore, we can obtain the following equation :

$$S''_{\pi_f}(x)dx = R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{5.9}$$

This equation(5.9), that is, (3.44), can be regarded as an application of the equation(5.6). In the subsections below, we examine some laws can be regarded as the accelerations of $S_D(x)$, that is, (5.6).

5.4. Logistic function of the bacterial and the population growth.

We examine the relationship between generalized the function $S''_D(x)$ of equation(5.6) and the logistic function such that the bacterial and the population growth. (Refer to R. May [20], N. Bacaër [21], M,Yamaguchi [22])

Let r and K be positive integer constants. For positive real number $t > 0$, let $N(t)$ be a positive real valued function. We put to set as follows:

$$\begin{aligned} S''_D(x) &:= -r(<0), \\ Q_D(x) &:= -\frac{N(t)}{K}, \\ Q'_D(x) &:= -\frac{1}{K}\frac{dN(t)}{dt}, \\ k_D(x) &:= 1, \\ x &:= t, \\ \xi &:= 1, \end{aligned} \tag{5.10}$$

where parameters r, K, t and the function $N(x)$ mean as follows :

$$\begin{aligned} r &: \ \textit{the growth rate}, \\ N(t) &: \ \textit{the bacterial or the population growth} \\ K &: \ \textit{carrying capacity}, \\ t &: \ \textit{time or step}. \end{aligned} \tag{5.11}$$

Thus, the equation(5.6) becomes the equation of the dynamical system as follows:

$$\int -rdt \geq \int \frac{-(\frac{-1}{K})}{-\frac{N(t)}{K}(1-\frac{N(t)}{K})}\frac{dN(t)}{dt}dt. \tag{5.12}$$

Thus, transforming the formula above as follows :

$$\int rdt \geq \int \frac{K}{N(t)(K-N(t))}dN(t). \tag{5.13}$$

Therefore, the following equation is obtained :

$$\frac{dN(t)}{dt} = rN(t)\frac{(K-N(t))}{K}. \tag{5.14}$$

The solution of logistic equation(5.14) above are satisfied as follows:

$$N(t) = \frac{K}{1+\exp(-(rt+C))}, \tag{5.15}$$

where C is a constant.

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The equation(5.14) above is the logistic function of dynamics. In other words, the equation(5.14) derived from the equation(5.6) can be regarded as an application of the dynamical system. Namely,, we considered the following possibilities :
For sufficiently large $x > 1$ and a constant $\xi > 0$, the equation(5.6), that is,

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(\xi + Q_D(x))}\right),$$

is regarded as a generalized expression and the approximate representation of the equation(5.14), that is,

$$\frac{dN(t)}{dt} = rN(t)\frac{(K - N(t))}{K},$$

where the function $D(x)$ needs to be chosen appropriately.

Therefore, we consider that Entropy and dynamical system are closely related and are studying these applications.

5.5. Complicated Dynamical System.

We introduce the relationship between the logistic function(5.14) above and complicated dynamics system as follows. (Refer to R.M May [20], N.Bacaër [21], M.Yamaguchi [22])

Definition 5.1. *Let the constant a be a real number that satisfies $4 \geq a > 0$ and x_n be real numbers between [0,1] where $n \geq 0$ is positive integers. $(x_n \in [0,1])$ The discrete dynamical system is defined as follows :*

$$x_{n+1} = ax_n(1 - x_n). \tag{5.16}$$

□

As isknown, the equation(5.14) can be expressed in the equation(5.16) as follows. Namely, the equation(5.14) is rewrite as a difference equation as follows:

$$\frac{N(t + \Delta t) - N(t))}{\Delta t} = rN(t)\frac{(K - N(t))}{K}. \tag{5.17}$$

We put as follows:

$$N_n = N(n\Delta t). \tag{5.18}$$

The difference equation of (5.18) above is expressed in the difference equation of N_n as follows:

$$N_{n+1} = \left((1 + r\Delta t) - \frac{r\Delta t}{K}N_n\right)N_n. \tag{5.19}$$

Therefore, the following is satisfied :

$$\frac{r\Delta t}{K(1 + r\Delta t)}N_{n+1} = (1 + r\Delta t)\left(1 - \frac{r\Delta t}{K(1 + r\Delta t)}N_n\right)\frac{r\Delta t}{K(1 + r\Delta t)}N_n. \tag{5.20}$$

Furthermore, we interpret and correspond as follows :

$$\begin{aligned} x_n &:= \frac{r\Delta t}{K(1 + r\Delta t)}N_n, \\ a &:= (1 + r\Delta t). \end{aligned} \tag{5.21}$$

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Using interpretation of the equation(5.21), we can transform the equation(5.20) to the following equation :

$$x_{n+1} = ax_n(1 - x_n). \tag{5.22}$$

This equation(5.22) above is same as the equation(5.16) in the definition 5.1. According to the research results of Lee, Yorke and R.M.May et al. (Refer to [20], [23]), depending on how the number $4 \geq a > 0$ is taken, a complicated orbit, that is, chaos appear. The discussion so far can be summarized as follows :

1. Logistic function(5.14) is derived from Entropy acceleration $S''_D(x)$ of (5.6),
2. Complicated dynamics(5.22) is derived from Logistic function(5.14),
3. Chaos is derived from Logistic function(5.22).

Logistic function and Dynamic systems are special cases of Entropy acceleration. Logistic function produces periodic order depending on how its conditions are taken. On the other hand, Dynamic systems create chaos depending on how their conditions are set. In other words, Entropy not only increases, but also has the potential to create chaos and order.

5.6. Conclusion.

Entropy was related to statistical quantum mechanics by Boltzmann and Planck. By partitioning the average energy of a resonator by its energy component (or partitioning the number of particles by the number of resonators), the Planck distribution and Planck's law are derived and these are expressed the actual physical phenomenon that describes the spectrum of black-body radiation.

We defined Entropy $S_{\pi_f}(x)$ by partitioning the Boltzmann principle by an approximate of $\pi(x)$ (that is, $x/\log(x)$). Thereby, we discussed that expanded Planck distribution function, expanded Planck' law and fine structure constants. Although no examples of physical phenomena expressing the extended Planck distribution or extended Planck's law have been found, is it possible that there exists a constant that satisfies a constant other than Planck's constant?

In addition, we applied to number theory by using the constant α and the function $R^{\pm}_{\alpha}(x)$. Namely, by using the constant $\alpha = 2/\sqrt{2\pi}$, R^{+}_{α}-factor and $\exp(1/R^{+}_{\alpha}(x))$, we derive Van Koch's inequality that is equivalent to the Riemann hypothesis. In addition, by using the constant $\alpha = e/\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})$, R^{-}_{α}-factor and $\exp(\frac{1}{R^{-}_{\alpha}(x)})$, it could be applied to be derived the abc conjecture.

These examples of number theory are not actual physical phenomena, but the constants α and $R^{\pm}_{\alpha}$-factor that satisfy extended Planck distribution and extended Planck's law exists in number theory created by the human mind. If we can allow number theory created by the human mind as part of physical phenomena, then it may be thought that it expresses a physical phenomenon that is "a kind of radiation that comes out from within humans." In any case, we hope to find examples of physical phenomena on each fine structure constants..

Furthermore, Entropy was associated with complex systems by Prigogine and Nicolis, et al. We attempted generalization of Entropy $S_D(x)$, and discussed that Entropy acceleration $S''_D(x)$ (the second order differentiation of Entropy $S_D(x)$) can be related to dynamical systems described by logistic function models such as the bacterial and the population growth.

We are sure there are other potentially useful applications. We hope that it is worth continuing to explore and research. and that this paper will serve as a bridge to further research and that Entropy will be further studied and many things will develop in the future.

Increasing Entropy (the second law) does not mean becoming disordered. On the contrary, the second law has the potential to cause movement and order in phenomena. (Refer to Fujino [26])

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Thus, it is thought that there exists the following relationship:

$$\begin{array}{c} \text{(Complex fluid mechanics [26])} \\ \uparrow \\ \text{Statistical mechanics} \longleftarrow \text{Entropy} \longrightarrow \text{Number theory} \\ \downarrow \\ \text{Logistic function} \\ \downarrow \\ \text{Complicated dynamics} \\ \downarrow \\ \text{Chaos} \end{array}$$

6. The proof of Lemma3.6 and Lemma3.7

6.1. The proof of Lemma3.6

Proof. Let $n \geq 1$ a positive integer. For all sufficiently large positive real variable $x > 0$, the following conditions are satisfied :

$$\begin{aligned} |(Q_f(x))^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\ &= \frac{(n-1)!}{x^n} \\ &\geq \frac{\sqrt{2\pi}(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\ &(\because \text{Stirling's formula}: \ n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\ &= \left(\frac{\sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}}{e^{(n-1)}x^n}\right) \\ &= \sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (*2). \end{aligned} \tag{6.1}$$

Therefore, dividing the end of the formula(6.1), that is (*2), by the number $n^{n-(\frac{1}{2})}$, for sufficiently large $x > 1$, the following condition are satisfied :
Case 1) $x > n$: Because $x \geq x - (\frac{1}{2})$, therefore

$$\begin{aligned} (*2) &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \\ &\geq \sqrt{2\pi}(\frac{x-1}{x})^{x-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (\because x > n) \\ &\geq \sqrt{2\pi}(\frac{x-1}{x})^{x}\frac{1}{e^{(n-1)}x^n} \\ &(\because x \geq x-(\frac{1}{2}) \text{ and } (\frac{x-1}{x})^x \geq \lim_{x\to\infty}(\frac{x-1}{x})^x = e^{-1}) \\ &\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}. \end{aligned} \tag{6.2}$$

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Case 2) $n \geq x$:

$$\begin{aligned}(*2) &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \\ &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n}\frac{1}{e^{(n-1)}x^n} \\ &\quad (\because n \geq n-(\frac{1}{2}) \text{ and } (\frac{n-1}{n})^n \geq \lim_{n\to\infty}(\frac{n-1}{n})^n = e^{-1}) \\ &\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}.\end{aligned} \tag{6.3}$$

Therefore, using Case 1) and Case 2) above, for all sufficiently large $x > 0$, the following inequality is satisfied :

$$|(Q_f(x))^{(n)}| \geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}. \tag{6.4}$$

Therefore, the following conditions are satisfied :

$$\begin{aligned}R_m^+(x) &\geq \lim_{N\to\infty}\sum_{n=1}^{N}\left|(Q_f(x))^{(n)}\right| \\ &\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\left|\frac{(-1)^{n-1}}{e^{(n-1)}x^n}\right| \\ &\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n}.\end{aligned} \tag{6.5}$$

Put the function $A(x)$ as follows:

$$A(x) := \sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n}. \tag{6.6}$$

Therefore,

$$A(x) = \frac{1}{e^0x^1} - \frac{1}{e^1x^2} + \frac{1}{e^2x^3} - \cdots, \tag{6.7}$$

$$\frac{1}{ex}A(x) = \frac{1}{e^1x^2} - \frac{1}{e^2x^3} + \frac{1}{e^3x^4} - \cdots + \frac{(-1)^N}{e^Nx^{(N+1)}}. \tag{6.8}$$

Add the equation(6.7) and (6.8). Hence, the following conditions are satisfied :

$$\left(1+\frac{1}{ex}\right)A(x) = \lim_{N\to\infty}\left(\frac{1}{x} + \frac{(-1)^N}{e^Nx^{(N+1)}}\right) = \frac{1}{x}. \tag{6.9}$$

Therefore,

$$\lim_{N\to\infty}A(x) = \frac{1}{x}\left(\frac{1}{1+\frac{1}{ex}}\right) = \frac{e}{ex+1}. \tag{6.10}$$

Therefore, for integer $m > 1$, the function R_m^+ is approximated as follows:

$$R_m^+(x) \geq \sqrt{2\pi}e^{-1}\lim_{N\to\infty}A(x) = \frac{\sqrt{2\pi}}{ex+1}. \tag{6.11}$$

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Here, the following conditions are satisfied :

$$R_m^+(x) > \frac{1}{x} > \frac{\sqrt{2\pi}\alpha}{ex+1} = R_\alpha^+(x). \tag{6.12}$$

Put the positive real number $\alpha > 0$ such that as follows:

$$x \geq \frac{-1}{e - \sqrt{2\pi}\alpha}. \tag{6.13}$$

Consequently, the following conditions are satisfied :

$$R_m^+(x) \geq \sqrt{2\pi}e^{-1}\alpha \lim_{N\to\infty} A(x) = \frac{\sqrt{2\pi}\alpha}{ex+1} = R_\alpha^+(x), \tag{6.14}$$

$$x \geq \frac{-1}{e - \sqrt{2\pi}\alpha}, \quad \text{that is,} \quad \frac{ex+1}{\sqrt{2\pi}x} \geq \alpha. \tag{6.15}$$

Furthermore, for all sufficiently large $x > 1$, the following conditions are satisfied:

$$\frac{1}{x}(2 + \log(x)) \geq k_f(x) \geq R_\alpha^+(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{6.16}$$

$$\frac{e}{\sqrt{2\pi}}(2 + \log(x)) \geq \frac{ex+1}{\sqrt{2\pi}x} \geq \alpha > 0, \tag{6.17}$$

where $\alpha > 0$ is a real number. (The end of proof of $R_m^+(x)$ on Lemma3.6)

By the same method, for integer $m > 1$, the function R_m^- is approximated as follows:

$$\sqrt{2\pi}e^{-1}\alpha \lim_{N\to\infty} A(x) = \frac{\sqrt{2\pi}\alpha}{ex-1} = R_\alpha^-(x), \tag{6.18}$$

$$x \geq \frac{1}{e - \sqrt{2\pi}\alpha}. \tag{6.19}$$

For all $x > 1$, the following conditions are satisfied :

$$\frac{1}{x}(2 + \log(x)) \geq k_f(x) \geq R_\alpha^-(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}, \tag{6.20}$$

$$\frac{e}{\sqrt{2\pi}}(2 + \log(x)) \geq \frac{ex-1}{\sqrt{2\pi}x} \geq \alpha > 0. \tag{6.21}$$

(The end of proof of $R_m^-(x)$ on Lemma3.6). □

6.2. The proof of Lemma 3.7

Proof. Let $n \geq 1$, and $x > 0$ is sufficiently large.

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$$
\begin{aligned}
|(Q_f(x))^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\
&= \frac{(n-1)!}{x^n} \\
&\leq \frac{e(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\
&(\because \text{Stirling's formula: } ee^{-n}n^{n+\frac{1}{2}} \geq n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\
&= \left(\frac{e(n-1)^{n-(1/2)}}{e^{(n-1)}x^n}\right) \\
&= e(n-1)^{n-(1/2)}\frac{1}{e^{(n-1)}x^n} \\
&= e\left(\frac{n-1}{x}\right)^n\frac{(n-1)^{-1/2}}{e^{(n-1)}} \\
&= e\left(\frac{x-1}{x}\right)^x\frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (x \gg n) \\
&\leq e\left(\frac{x-1}{x}\right)^x\frac{1}{e^{(n-1)}(x-1)^{1/2}} \\
&\leq ee^{-1}\frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (\because \lim_{x\to\infty}(\frac{x-1}{x})^x = \frac{1}{e}) \\
&\leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}.
\end{aligned} \tag{6.22}
$$

Therefore, the following inequality are satisfied :

$$|(Q_f(x))^{(n)}| \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.23}$$

Furthermore, the inequality

$$|(Q_f(x))^{(n)}| > |(Q_f(x))^{(n+1)}|. \tag{6.24}$$

is satisfied. Besides, the following relation are satisfied :

$$\text{if } n \text{ is even} \quad : \quad 0 < (Q_f(x))^{(n)} \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}, \tag{6.25}$$

$$\text{if } n \text{ is odd} \quad : \quad 0 > (Q_f(x))^{(n)} \geq \frac{-1}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.26}$$

The discussion above, the following are satisfied :

$$R_m^+(x) = \lim_{N\to\infty}\sum_{n=1}^{N}\left|(Q_f(x))^{(n)}\right| \leq \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.27}$$

Put the function $A_2(x)$ as follows :

$$A_2(x) = \sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.28}$$

Hence, we can describe as follows :

$$A_2(x) = \frac{1}{e^0(x-1)^{1/2}} - \frac{1}{e^1(x-1)^{1/2}} + \frac{1}{e^2(x-1)^{1/2}} - \cdots, \tag{6.29}$$

$$\frac{1}{e}A_2(x) = \frac{1}{e^1(x-1)^{1/2}} - \frac{1}{e^2(x-1)^{1/2}} + \cdots + \frac{(-1)^N}{e^N(x-1)^{1/2}} \tag{6.30}$$

Add the equivalent(6.29) and (6.30), the following equivalent are satisfied :

$$\left(1+\frac{1}{e}\right)A_2(x) = \lim_{N\to\infty}\left(\frac{1}{(x-1)^{1/2}} + \frac{(-1)^N}{e^N(x-1)^{1/2}}\right) = \frac{1}{(x-1)^{1/2}}. \tag{6.31}$$

Therefore,

$$\lim_{N\to\infty} A_2(x) = \frac{1}{(x-1)^{1/2}}\left(\frac{1}{1+\frac{1}{e}}\right) = \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.32}$$

Therefore, the function $R_m^+(x)$ is satisfied as follows :

$$R_m^+(x) \leq \lim_{N\to\infty} A_2(x) = \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.33}$$

where $m > 1$.

For all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) \leq \frac{1}{x}(2+\log(x)) \leq \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.34}$$

By the same method, R_m^+ is approximated as follows :

$$R_m^+(x) \leq \frac{1}{(x-1)^{1/2}}\frac{e}{e-1}. \tag{6.35}$$

where $m > 1$.

For all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) \leq \frac{1}{x}(2+\log(x)) \leq \frac{1}{(x-1)^{1/2}}\frac{e}{e-1}. \tag{6.36}$$

The proof of $R_m^-(x)$ is similar. □

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References

1. Wladyslaw Narkiewicz , The Development of Prime Number Theory, Springer, 2000
2. P.L.Chebychev. Mémoire sur les nombres premiers. J. de Math. Pures Appl., 17:366–390, 1852
3. P.L.Chebychev. Sur la totalité des nombres premiers inf érieurs á une limite donneé, J. de Math. Pures Appl., 17:341–365, 1852
4. Max Planck, Vorlesungen über die Theorie der Wärmestrahlung, J.A Barth, 1906
5. G. H. Hardy and J. E. Littlewood, Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes, Acta Math. 41: 119-196 (1916)
6. Feynman et.al, The Feynman, Lecture on Physics,Volume III, Quantum Mechanics ,1963
7. André LeClair, Riemann Hypothesis and Random Walks: The Zeta Case ,Symmetry 2021, 13, 2014
8. Von Koch H, Sur la distribution des nombres premiers, Acta Mathematica, 24 159–182, 1901

 doi:10.20944/preprints202203.0371.v4

9. Entropy of Riemann zeta zero sequence, Advanced Modeling and Optimization, Volume 15, Number 2,O.Shanker, 2013
10. Ioannis Kontoyiannis, Counting the Primes Using Entropy, The 2008 IEEE Information Theory Workshop, Porto, Portugal, Lecture given on Thursday, May 8 2008
11. Arturo Ortiz, Hans Henrik Stleum. Studies of entropy measures concerning the gaps of prime numbers, arXiv:1606.08293 [math.GM], 2016.
12. Michael Atiyah. The Riemann Hypothesis. 2018
13. Michael Atiyah. The Fine Structure Constant. 2019
14. Michel Waldschmidt, On the abc Conjecture and some of its consequences,Universit*é* P. et M. Curie (Paris VI), 2018
15. Michel Waldschmidt, Lecture on the abc conjecture and some of its consequences, Springer Proceedings in Mathematics and Statistics 98 , 211-230. ,(2015)
16. Masreliez C.J., The Pioneer Anomaly-A cosmological explanation. preprint (2005) Ap&SS, v.299, no.1, pp.83–108
17. Alec Misra B.A.(Hons) M.A.(Oxon), Entropy and Prime Number Distribution; (a Non-heuristic Approach)., Feb 6, 2006
18. Gregoire Nicolis, Ilya Prigogine, Exploring Complexity: An Introduction, W.H.Freeman, 1989.
19. Gregoire Nicolis, Ilya Prigogine, Self-Organization in Nonequilibrium Systems: From Dissipative Structures to Order Through Fluctuations, Wiley, 1977 (A Wiley-Interscience publication)
20. Robert M May, Simple Mathematical Models With Very Complicated Dynamics, Nature. July, 1976.
21. Nicolis Bacaër, A Short History of Mathematical Population Dynamics, Springer-Verlag London Limited 2011
22. Masaya Yamaguchi, Chaos and Fractal, KODANSHA Bule Backs Japan 1989.
23. Tien-Yien Li, James A. Yorke, Period Three Implies Chaos,The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975), pp. 985-992.
24. Seiji Fujino, Deriving Von Koch's inequality without using the Riemann Hypothesis, Preprints.org, Dec. 2021, (DOI: 10.20944/preprints202112.0074.v2).
25. Seiji Fujino, Examination the abc conjecture using some functions, Preprints.org, Feb.2022, (DOI: 10.20944/preprints202202.0021.v3).
26. Seiji Fujino, Interpretation of Entropy by Complex Fluid Dynamics. Preprints.org May.2023,(DOI:10.20944/preprints202305.0066.v1).