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Approximate Closed-form Solutions for the Rabinovich System via the Optimal Auxiliary Functions Method

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Abstract

Based on some geometrical properties of the Rabinovich system the closed-form solutions of the equations has been established. More-over the Rabinovich system is reduced to a nonlinear differential equation depending on an auxiliary unknown function. The approxi-mate analytical solutions are built using the Optimal Auxiliary Func-tions Method (OAFM). A good agreement between the analytical and corresponding numerical results has been performed. The accuracy of the obtained results emphasizes that this procedure could be suc-cessfully applied for more dynamical systems with these geometrical properties. [1]

Key words: optimal auxiliary functions method; Rabinovich system; symmetries; Hamilton--Poisson realization; periodical orbits

1 Introduction

The Rabinovich system was first studied in [1] with the analysis of a concrete realization in a magnetoactive non-isothermal plasma. This system is a dynamical system of three resonantly coupled waves, parametrically excited [2].

The synchronization or optimization of nonlinear system performance, secure communications, and other applications in electrical engineering

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or medicine are based on the study of dynamical systems. In [3] was explored the stabilization of the T system via linear controls and in [4] was studied the Rikitake two-disk dynamic system and applied in modeling the reversals of the Earth's magnetic field [5], [6]. Some geometrical properties of the dynamical systems: the integral deformations, the equilibria points, Hamiltonian realization was analyzed in [7]-[38]. The symmetry represents an important geometrical property of the dynamical system. As it is well known, a dynamical system admits symmetry with respect to the origin point $O(0,0,0)$ or with the $Oz-$ axis or the plan $z=0$ if it is invariant under the transformation $(x,y,z) \to (-x,-y,-z)$, respectively $(x,y,z) \to (-x,-y,z)$ and $(x,y,z) \to (x,y,-z)$.

2 The Rabinovich system

2.1 Global analytic first integrals and Hamilton-Poisson realization

The Rabinovich system has the form (see [39], [7]):

$$\begin{cases} \dot{x} = yz - \alpha_1 x + \beta y \\ \dot{y} = -x\cdot z - \alpha_2 y + \beta x \\ \dot{z} = x\cdot y - \alpha_3 z \end{cases}, \tag{1}$$

where the unknown functions x, y and z depend on $t>0$, $(\beta,\ \alpha_1,\ \alpha_2,\ \alpha_3) \in \mathbf{R}^4$ and $\dot{x}$ denotes derivative of the function x with respect to t.

Remark 1. Is easy to see that the considered system admits a symmetry with respect to Oz- axis, for $\beta \neq 0$ and symmetries with respect to Oz, Ox, Oy axes, for $\beta = 0$, respectively.

In this section we also recall some geometrical properties of the system (1) [7].

The global analytic first integrals of the Rabinovich system are obtained in [39].

The considered system has a Hamilton-Poisson realization with the Hamiltonian and the Casimir given by $H(x,y,z) = \frac{1}{4}(x^2 - z^2)$ and $C(x,y,z) = \frac{1}{4}(x^2 + 2y^2 + z^2)$, respectively, for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$; $H(x,y,z) = -\frac{\beta}{2}x^2 + \frac{\beta}{2}y^2 + \beta z^2$ and $C(x,y,z) = -\frac{1}{4\beta}x^2 - \frac{1}{4\beta}y^2 + z$, for $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$.

There exist three isolated cases:

$H(x,y,z) = x^2 - z^2 - 2\beta z$, *for* $\beta \in \mathbf{R}$, $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_3 = 0$;
$H(x,y,z) = y^2 + z^2 - 2\beta z$, *for* $\beta \in \mathbf{R}$, $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\alpha_3 = 0$;
$H(x,y,z) = x^2 + y^2$, *for* $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$.

Remark 2. For the initial conditions

$$x(0) = x_0 \ , \quad y(0) = y_0 \ , \quad z(0) = z_0 \ , \tag{2}$$

the phase curves of dynamics (1) are the intersections of the surfaces $-\frac{\beta}{2}x^2 + \frac{\beta}{2}y^2 + \beta z^2 = -\frac{\beta}{2}x_0^2 + \frac{\beta}{2}y_0^2 + \beta z_0^2$ and $-\frac{1}{4\beta}x^2 - \frac{1}{4\beta}y^2 + z = -\frac{1}{4\beta}x_0^2 -$

$\frac{1}{4\beta}y_0^2 + z_0$, for $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$;
$x^2 - z^2 = x_0^2 - z_0^2$ and $x^2 + 2y^2 + z^2 = x_0^2 + 2y_0^2 + z_0^2$, for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, respectively.

2.2 Closed-form solutions

In this section we establish the closed-form solutions of the system Eq. (1) using previously results, taking into account of the real values for the physical parameters as β, a_1, a_2, a_3.

i) $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$.
Using the transformations:

$$\begin{cases} y(t) = \frac{R}{4\beta} \cdot \frac{2v(t)}{1+v^2(t)} \\ z(t) = \beta + \frac{R}{4\beta} \cdot \frac{1-v^2(t)}{1+v^2(t)} \end{cases} , \tag{3}$$

where $R = 4\sqrt{\beta \cdot (H_\beta - 2\beta^2 \cdot C_\beta + \beta^3)}$, $H_\beta = -\frac{\beta}{2}x_0^2 + \frac{\beta}{2}y_0^2 + \beta z_0^2$, $C_\beta = -\frac{1}{4\beta}x_0^2 - \frac{1}{4\beta}y_0^2 + z_0$, $v(t)$ is an unknown smooth function.
The third equation from Eq. (1) yields to

$$x(t) = -\frac{2\dot{v}(t)}{1+v^2(t)}. \tag{4}$$

Now, using the first equation from Eq. (1) we obtain:

$$\ddot{v}(t)\cdot(1+v^2(t)) - 2v(t)\cdot(\dot{v}(t))^2 + \frac{R^2}{16\beta^2}\cdot v(t)\cdot(1-v^2(t)) + \frac{R}{2}\cdot v(t)\cdot(1+v^2(t)) = 0 \,. \tag{5}$$

Using the initial conditions Eq. (2) and the relations Eqs. (3)-(4) the initial conditions $v(0)$ and $\dot{v}(0)$ become:

$$v(0) = \sqrt{\frac{1 - \frac{4\beta}{R}\cdot(z_0 - \beta)}{1 + \frac{4\beta}{R}\cdot(z_0 - \beta)}} \;, \quad \dot{v}(0) = -\frac{x_0}{2}\cdot\left(1 + \frac{1 - \frac{4\beta}{R}\cdot(z_0-\beta)}{1 + \frac{4\beta}{R}\cdot(z_0-\beta)}\right) \;. \tag{6}$$

Remark 3. If the function $v(t)$ is the *exact solution* of the problem given by Eqs. (5)-(6), then the relations Eqs. (3) and (4) give *closed-form solution* of the system Eq. (1). If the function $v(t)$ is an *analytic approximate solution* of the problem given by Eqs. (5)-(6), then the relations Eqs. (3) and (4) give *approximate closed-form solution* of the system Eq. (1).
ii) $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$.
For this particular case the system (1) reduces to

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -x\cdot z \\ \dot{z} = x\cdot y \end{cases} . \tag{7}$$

Making the transformations:

$$\begin{cases} x(t) = sign(x_0)\cdot R\cdot \frac{1}{\sqrt{1+u^2(t)}} \\ y(t) = sign(y_0)\cdot R\cdot \frac{u(t)}{\sqrt{1+u^2(t)}} \end{cases}, \tag{8}$$

where $R = \sqrt{x_0^2+y_0^2}$ and $u(t)$ is an unknown smooth function, then the second equation from Eq. (7) yields to

$$z(t) = -\frac{sign(y_0)}{sign(x_0)}\cdot\frac{\dot{u}(t)}{1+u^2(t)}. \tag{9}$$

Now, using the third equation from Eq. (7) we obtain:

$$\ddot{u}(t) - \frac{2u(t)}{1+u^2(t)}\cdot(\dot{u}(t))^2 + R^2\cdot u(t) = 0\ . \tag{10}$$

Using the initial conditions Eq. (2) and the relations Eqs. (3)-(4) the initial conditions $u(0)$ and $\dot{u}(0)$ become:

$$u(0) = \frac{sign(y_0)}{sign(x_0)}\cdot\frac{y_0}{x_0}\ \ ,\ \ \dot{u}(0) = -sign(z_0)\cdot z_0\cdot\left(1+u^2(0)\right)\ . \tag{11}$$

Remark 4. If the function $u(t)$ is the *exact solution* of the problem given by Eqs. (10)-(11), then the relations Eqs. (8) and (9) give *closed-form solution* of the system Eq. (7). If the function $v(t)$ is an *analytic approximate solution* of the problem given by Eqs. (5)-(6), then the relations Eqs. (3) and (4) give *approximate closed-form solution* of the system Eq. (7).

iii) $\beta\in\mathbf{R}$, $\alpha_1=0$, $\alpha_2\neq 0$, $\alpha_3=0$.

The closed-form solutions can be put in the following form:

$$\begin{cases} x(t) = R\cdot\frac{2\cdot u(t)}{1-u^2(t)} \\ z(t) = -\beta + R\cdot\frac{1+u^2(t)}{1-u^2(t)} \end{cases}, \tag{12}$$

where $R=\sqrt{(z_0+\beta)^2-x_0^2}$, for $(z_0+\beta)^2-x_0^2>0$, and

$$\begin{cases} x(t) = R\cdot\frac{1+u^2(t)}{1-u^2(t)} \\ z(t) = -\beta + R\cdot\frac{2\cdot u(t)}{1-u^2(t)} \end{cases}, \tag{13}$$

where $R=\sqrt{x_0^2-(z_0+\beta)^2}$, for $x_0^2-(z_0+\beta)^2>0$, respectively.

Then the third equation from Eq. (1) yields to

$$y(t) = \frac{2\dot{u}(t)}{1-u^2(t)}\ . \tag{14}$$

The unknown smooth function $u(t)$ is solution of the nonlinear problem:

$$\begin{cases} \ddot{u}(t)\cdot(1-u^2(t))+2u(t)\cdot(\dot{u}(t))^2+R^2\cdot u(t)\cdot(1+u^2(t))- \\ -2\beta\cdot R\cdot u(t)\cdot(1-u^2(t))+\alpha_2\cdot(1-u^2(t))\cdot\dot{u}(t)=0 \\ u(0)=sign(x_0)\cdot\frac{x_0}{z_0+\beta+R} \quad , \quad \dot{u}(0)=\frac{y_0}{2}\cdot\left(1-u^2(0)\right) \ . \end{cases} \tag{15}$$

iv) $\beta\in\mathbf{R}$, $\alpha_1\neq 0$, $\alpha_2=0$, $\alpha_3=0$.
The closed-form solutions can be put in the following form:

$$\begin{cases} y(t)=R\cdot\frac{2\cdot u(t)}{1+u^2(t)} \\ z(t)=\beta+R\cdot\frac{1-u^2(t)}{1+u^2(t)} \end{cases} , \tag{16}$$

where $R=\sqrt{(z_0-\beta)^2+y_0^2}$. Then the third equation from Eq. (1) yields to

$$x(t)=-\frac{2\dot{u}(t)}{1+u^2(t)} \ . \tag{17}$$

The unknown smooth function $u(t)$ is solution of the nonlinear problem:

$$\begin{cases} \ddot{u}(t)\cdot(1+u^2(t))-2u(t)\cdot(\dot{u}(t))^2+R^2\cdot u(t)\cdot(1-u^2(t))+ \\ +2\beta\cdot R\cdot u(t)\cdot(1+u^2(t))+\alpha_1\cdot(1+u^2(t))\cdot\dot{u}(t)=0 \\ u(0)=\sqrt{\frac{R-(z_0-\beta)}{R+(z_0-\beta)}} \quad , \quad \dot{u}(0)=-\frac{x_0}{2}\cdot\left(1+u^2(0)\right) \ . \end{cases} \tag{18}$$

v) $\beta=0$, $\alpha_1=0$, $\alpha_2=0$, $\alpha_3\neq 0$.
The closed-form solutions can be put in the following form:

$$\begin{cases} x(t)=R\cdot\cos(u(t)) \\ y(t)=R\cdot\sin(u(t)) \end{cases} , \tag{19}$$

where $R=\sqrt{x_0^2+y_0^2}$. Then the first equation from Eq. (1) yields to

$$z(t)=-\dot{u}(t) \ . \tag{20}$$

The unknown smooth function $u(t)$ is solution of the nonlinear problem:

$$\begin{cases} \ddot{u}(t)+\alpha_3\cdot\dot{u}(t)+\frac{R^2}{2}\cdot\sin(2\cdot u(t))=0 \\ u(0)=\arctan\frac{y_0}{x_0} \quad , \quad \dot{u}(0)=-z_0 \ . \end{cases} \tag{21}$$

In literature there are several analytical methods for solving the nonlinear differential problem given by Eqs. (5)-(6), (10)-(11), (15), (18), (21) such as: the Optimal Homotopy Asymptotic Method (OHAM) [40], [41], [42], the Optimal Homotopy Perturbation Method (OHPM) [43], [44], the Optimal Variational Iteration Method (OVIM) [45], the Optimal Iteration Parametrization Method (OIPM) [46], the Polynomial Least Squares Method [47], the Least Squares Differential Quadrature Method [48], the Multiple Scales Technique [49], the Function Method [50], the Homotopy Perturbation Method (HPM) and the Homotopy Analysis Method (HAM) [51], the Variational Iteration Method (VIM) [52].

In this work the approximate analytic solutions of the nonlinear differential problem given by Eqs. (5)-(6), (10)-(11), (15), (18), (21) are analytically solved using the Optimal Auxiliary Functions Method (OAFM).

3 Basic ideas of the OAFM technique

By means of the operator theory, Eq. (5) with the initial conditions (6) can be written in a more general form as [42, 53]:

$$\mathcal{L}\Big[\Phi(t)\Big] + g(t) + \mathcal{N}\Big[\Phi(t)\Big] = 0, \tag{22}$$

where $\mathcal{L}$ is a linear operator, g is a known function and $\mathcal{N}$ is a given nonlinear operator, t denotes independent variable and $\Phi(t)$ is an unknown smooth function. The initial conditions are

$$\mathcal{B}\Big(\Phi(t), \frac{d\Phi(t)}{dt}\Big) = 0. \tag{23}$$

It is hard to find an exact solution for strongly nonlinear equation (22) with initial conditions (23). For obtain an approximate analytic solution of Eqs. (22) and (23), was proposed the approximate solution written in the form with just two components:

$$\overline{\Phi}(t) = \Phi_0(t) + \Phi_1(t, C_i), \quad i = 1,\ 2,\ ...,\ s, \tag{24}$$

where the initial approximation $\Phi_0(t)$ and the first approximation $\Phi_1(t, C_i)$ will be determined as follows. Replacing Eq. (24) into Eq. (22), it results in:

$$\begin{aligned} &\mathcal{L}\Big[\Phi_0(t)\Big] + \mathcal{L}\Big[\Phi_1(t, C_i)\Big] + g(t) + \\ &+\mathcal{N}\Big[\Phi_0(t) + \Phi_1(t, C_i)\Big] = 0. \end{aligned} \tag{25}$$

The initial approximation $\Phi_0(t)$ could be determined from the linear equation

$$\mathcal{L}\Big[\Phi_0(t)\Big] + g(t) = 0, \quad \mathcal{B}\Big(\Phi_0(t), \frac{d\Phi_0(t)}{dt}\Big) = 0\ . \tag{26}$$

The first approximation $\Phi_1(t, C_i)$ is obtained from the following equation

$$\begin{aligned} &\mathcal{L}\Big[\Phi_1(t, C_i)\Big] + N\Big[\Phi_0(t) + \Phi_1(t, C_i)\Big] = 0, \\ &\mathcal{B}\Big(\Phi_1(t, C_i), \frac{d\Phi_1(t, C_i)}{dt}\Big) = 0. \end{aligned} \tag{27}$$

Now, the nonlinear term from Eq. (27) is expanded in the form

$$\begin{aligned}&\mathcal{N}\Big[\Phi_0(t)+\Phi_1(t,C_i)\Big]=\\&=\mathcal{N}\Big[\Phi_0(t)\Big]+\sum_{k=1}^{\infty}\frac{\Phi_1^k(t,C_i)}{k!}\mathcal{N}^{(k)}\Big[\Phi_0(t)\Big].\end{aligned}\tag{28}$$

The difficulties that appears in solving of the nonlinear differential equation (27) could be eliminated by acceleration the rapid convergence of the first approximation $\Phi_1(t,C_i)$ and implicit of the approximate solution $\bar{\Phi}(t)$. Taking into account the Eq. (28), the last term from Eq. (27) is chosen, such that the nonlinear differential equation (27) can be written as a linear differential equation, , in the form:

$$\begin{aligned}&\mathcal{L}\Big[\Phi_1(t,C_i)\Big]+A_1\Big[\Phi_0(t),C_i\Big]\mathcal{N}\Big[\Phi_0(t)\Big]+\\&+A_2\Big[\Phi_0(t),C_j\Big]=0,\end{aligned}\tag{29}$$

$$\mathcal{B}\Big(\Phi_1(t,C_i),\frac{d\Phi_1(t,C_i)}{dt}\Big)=0,\tag{30}$$

where A_1 and A_2 are two arbitrary auxiliary functions depending on the initial approximation $\Phi_0(t)$ and several unknown parameters C_i and C_j, $i=1,\ 2,\ ...,\ p$, $j=p+1,\ p+2,\ ...,\ s$.

The auxiliary functions A_1 and A_2 (called optimal auxiliary functions) are not unique, and are of the same form like $\Phi_0(t)$ or $\mathcal{N}\Big[\Phi_0(t)\Big]$ or combinations of the forms of $\Phi_0(t)$ and $\mathcal{N}\Big[\Phi_0(t)\Big]$.

In all these sums, the coefficients of the polynomial, exponential, trigonometric and so on functions, are the parameters C_1, C_2, ..., C_s.

For special case $\mathcal{N}\Big[\Phi_0(t)\Big]=0$ it is clear that $\Phi_0(t)$ is an *exact solution* of Eqs. (22) and (23). The unknown parameters C_i and C_j can be optimally identified via different method such as: the least square method, the Ritz method, the Galerkin method, the collocation method, the Kantorovich method or by minimizing the square residual error, using:

$$J(C_i,C_j)=\int_a^b R^2(\beta,\alpha_1,\alpha_2,\alpha_3,t)\ dt,\tag{31}$$

where

$$\begin{aligned}&R(\beta,\alpha_1,\alpha_2,\alpha_3,t)=\mathcal{L}\Big[\bar{\Phi}(t,C_i,C_j)\Big]+g(t)+\\&+\mathcal{N}\Big[\bar{\Phi}(t,C_i,C_j)\Big],\\&i=1,\ 2,\ ...,p,\quad j=p+1,\ p+2,...,\ s,\end{aligned}\tag{32}$$

a and b are two values depending on the given problem.

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The unknown parameters C_i, C_j can be optimally identified from the equations

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=...=\frac{\partial J}{\partial C_p}=\frac{\partial J}{\partial C_{p+1}}=\;...\;=\frac{\partial J}{\partial C_s}=0. \tag{33}$$

Using this approach the approximate solution (24) is well determined.

An approximate analytic solution of the form Eq. (24) obtained via OAFM technique is called *OAFM solution*.

Therefore, it is validated that this technique is a powerful tool for solving nonlinear problems not depending on small or large parameters. It should be mentioned that this procedure contains the optimal auxiliary functions A_1 and A_2 which provides us with a simple way to adjust and control the convergence of the approximate solutions after only one iteration. Also, it is remarkable that a nonlinear differential problem is transformed into two linear differential problems.

4 Approximate analytic solutions via OAFM

We introduce the basic ideas of the OAFM by considering Eq. (5) with the initial conditions given by Eq. (6). The linear operator $\mathcal{L}$ could be chosen by the form [53]:

$$\mathcal{L}\Big(v(t)\Big)=\ddot{v}(t)+\omega_0^2 v(t), \tag{34}$$

where $\omega_0>0$ is an unknown parameter.

Eq. (26) becomes ($g(t)=0$):

$$\begin{aligned}&\ddot{v}(t)+\omega_0^2 v(t)=0,\quad v(0)=\sqrt{\frac{1-\frac{4\beta}{R}\cdot(z_0-\beta)}{1+\frac{4\beta}{R}\cdot(z_0-\beta)}}\,,\\ &\dot{v}(0)=-\frac{x_0}{2}\cdot\left(1+\frac{1-\frac{4\beta}{R}\cdot(z_0-\beta)}{1+\frac{4\beta}{R}\cdot(z_0-\beta)}\right)\,,\end{aligned} \tag{35}$$

with the solution

$$v_0(t)=v(0)\cdot\cos\omega_0 t+\frac{\dot{v}(0)}{\omega_0}\cdot\sin\omega_0 t. \tag{36}$$

The nonlinear operator $\mathcal{N}\Big(v(t)\Big)$ is obtained from Eqs. (5) and (34):

$$\mathcal{N}\Big(v(t)\Big)=-\omega_0^2 v(t)+\ddot{v}(t)\cdot v^2(t)-2v(t)\cdot(\dot{v}(t))^2+\frac{R^2}{16\beta^2}\cdot v(t)\cdot(1-v^2(t))+\frac{R}{2}\cdot v(t)\cdot(1+v^2(t))\,. \tag{37}$$

By means of the Eqs. (36) and (37) it is obtain

$$\mathcal{N}\Big(v_0(t)\Big)=M_1\cdot\cos\omega_0 t+N_1\cdot\sin\omega_0 t+M_2\cdot\cos 3\omega_0 t+N_2\cdot\sin 3\omega_0 t\;, \tag{38}$$

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where

$$\begin{aligned}
M_1 &= -\omega_0^2 M - \frac{5M\omega_0^2}{4}(M^2+N^2) + \frac{R^2 M}{64\beta^2}(4-3M^2-3N^2) + \frac{RM}{8}(4+3M^2+3N^2)\ ,\\
N_1 &= -\omega_0^2 N - \frac{5N\omega_0^2}{4}(M^2+N^2) + \frac{R^2 N}{64\beta^2}(4-3M^2-3N^2) + \frac{RM}{8}(4-3M^2+3N^2)\ ,\\
M_2 &= -\frac{M\omega_0^2}{4}(3N^2-M^2) + \frac{R^2 M}{64\beta^2}(3N^2-M^2) + \frac{RM}{8}(M^2-3N^2)\ ,\\
N_2 &= -\frac{N\omega_0^2}{4}(N^2-3M^2) + \frac{R^2 N}{64\beta^2}(N^2-3M^2) + \frac{RN}{8}(3M^2-N^2)\ ,\\
M &= v(0)\quad,\quad N = \frac{\dot{v}(0)}{\omega_0}\ .
\end{aligned} \tag{39}$$

Taking into account of the Eqs. (29), (34) and (38), the first approximation is obtained from the equation:

$$\begin{aligned}
&\ddot{v}_1 + \omega_0^2 v_1 + A_2(\cos\omega_0 t, \sin\omega_0 t, \cos 3\omega_0 t, \sin 3\omega_0 t, C_j) + A_1(\cos\omega_0 t, \sin\omega_0 t, \cos 3\omega_0 t, \sin 3\omega_0 t, C_i)\times\\
&\times(M_1\cdot\cos\omega_0 t + N_1\cdot\sin\omega_0 t + M_2\cdot\cos 3\omega_0 t + N_2\cdot\sin 3\omega_0 t) = 0,
\end{aligned} \tag{40}$$

with the initial conditions

$$v_1(0) = 0,\ \dot{v}_1(0) = 0\ . \tag{41}$$

There are opportunity to choose the optimal auxiliary functions A_1 and A_2 in the following forms:

$$A_1\Big[v_0(t), C_i\Big] = \sum_{k=1}^{N_{max}-1} a_k^{(1)}\cdot\cos(2k+1)\omega_0 t + b_k^{(1)}\cdot\sin(2k+1)\omega_0 t\ , \tag{42}$$

$$A_2\Big[v_0(t), D_j\Big] = \sum_{k=1}^{N_{max}} a_k^{(2)}\cdot\cos(2k+1)\omega_0 t + b_k^{(2)}\cdot\sin(2k+1)\omega_0 t\ , \tag{43}$$

where the convergence-control parameters $C_i \in \{a_k^{(1)}\Big|\ k = \overline{1,\ N_{max}-1}\}\cup\{b_k^{(1)}\Big|\ k = \overline{1,\ N_{max}-1}\}$,
$D_j \in \{a_k^{(2)}\Big|\ k = \overline{1,\ N_{max}}\}\cup\{b_k^{(2)}\Big|\ k = \overline{1,\ N_{max}}\}$, $N_{max} > 2$ is an arbitrary fixed integer number,
or

$$A_1\Big[v_0(t), C_i\Big] = 0,$$

$$A_2\Big[v_0(t), D_j\Big] = A_2\Big[v_0(t), D_j\Big] = \sum_{k=1}^{N_{max}} a_k^{(2)}\cdot\cos(2k+1)\omega_0 t + b_k^{(2)}\cdot\sin(2k+1)\omega_0 t\ ,$$

where the convergence-control parameters $D_j \in \{a_k^{(2)}\Big|\ k = \overline{1,\ N_{max}}\}\cup\{b_k^{(2)}\Big|\ k = \overline{1,\ N_{max}}\}$, or yet

$$A_1\Big[v_0(t), C_i\Big] = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t \ ,$$

$$A_2\Big[v_0(t), C_j\Big] = C_3 \cos 3\omega_0 t + C_4 \sin 3\omega_0 t,$$

and so on.

If the auxiliary functions A_1 and A_2 are given by Eqs. (42) and (43) then Eq. (29) becomes:

$$\ddot{v}_1 + \omega_0^2 v_1 = \sum_{k=1}^{N_{max}} a_k^{(3)} \cdot \cos(2k+1)\omega_0 t + b_k^{(3)} \cdot \sin(2k+1)\omega_0 t \ , \tag{44}$$

with the initial conditions given in Eq. (41), whose solution is:

$$v_1(t, C_i) = a_0 \cos\omega_0 t + b_0 \sin\omega_0 t + \sum_{k=1}^{N_{max}} a_k^{(4)} \cdot \cos(2k+1)\omega_0 t + b_k^{(4)} \cdot \sin(2k+1)\omega_0 t \ , \tag{45}$$

where

$$a_0 = -\sum_{k=1}^{N_{max}} a_k^{(4)} \ \ , \ b_0 = -\sum_{k=1}^{N_{max}} (2k+1) b_k^{(4)} \ ,$$

with the unknown parameters $a_k^{(3)}$, $b_k^{(3)}$, $a_k^{(4)}$, $b_k^{(4)}$ depending on the convergence-control parameters $a_k^{(1)}$, $b_k^{(1)}$, $a_k^{(2)}$, $b_k^{(2)}$, so will be optimally identified.

Finally, the approximate analytic solution is obtain from the Eq. (24) in the form:

$$\overline{v}_{OAFM}(t) = v_0(t) + v_1(t, C_i), \quad i = 1,\ 2,\ ...,\ s, \tag{46}$$

with $v_0(t)$ and $v_1(t, C_i)$ given by Eqs. (36) and (45), respectively.

Analogue, in the particular case $\beta = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the Eq. (10) could be rewrite in the following form:

$$\ddot{u}(t) \cdot (1 + u^2(t)) - 2u(t) \cdot (\dot{u}(t))^2 + R^2 \cdot u(t) \cdot (1 + u^2(t)) = 0 \ .$$

So, choosing the linear operator $\mathcal{L}\Big(u(t)\Big) = \ddot{u}(t) + \omega_0^2 u(t)$ and the nonlinear operator $\mathcal{N}\Big(u(t)\Big) = -\omega_0^2 u(t) + \ddot{u}(t) \cdot u^2(t) - 2u(t) \cdot (\dot{u}(t))^2 + R^2 \cdot u(t) \cdot (1 + u^2(t))$, the approximate analytic solution $\bar{u}(t)$ of the Eq. (10) with the initial conditions given by Eq. (11) can be obtain via OAFM technique in the form:

$$\begin{aligned} \overline{u}_{OAFM}(t) &= u(0) \cdot \cos\omega_0 t + \frac{\dot{u}(0)}{\omega_0} \cdot \sin\omega_0 t + \tilde{a}_0 \cos\omega_0 t + \tilde{b}_0 \sin\omega_0 t + \\ &+ \sum_{k=1}^{N_{max}} \tilde{a}_k^{(4)} \cdot \cos(2k+1)\omega_0 t + \tilde{b}_k^{(4)} \cdot \sin(2k+1)\omega_0 t \ , \end{aligned} \tag{47}$$

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where the convergence-control parameters ω_0, $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{a}_k^{(4)}$, $\tilde{b}_k^{(4)}$ will be optimally identified.

Similarly, for the cases $\alpha_1 \neq 0$, $\alpha_2 = \alpha_3 = 0$ or $\alpha_2 \neq 0$, $\alpha_1 = \alpha_3 = 0$ respectively, the linear operator can be

$$\mathcal{L}(u(t)) = \ddot{u} + \omega_0^2 \cdot u(t)\ .$$

Then, the corresponding nonlinear operator $\mathcal{N}(u(t))$ is obtained from Eqs. (15) and (18), respectively, as:

$$\mathcal{N}(u(t)) = -\omega_0^2 \cdot u(t) - \ddot{u}(t) \cdot u^2(t) + 2u(t) \cdot (\dot{u}(t))^2 + R^2 \cdot u(t) \cdot (1 + u^2(t)) - \\ -2\beta \cdot R \cdot u(t) \cdot (1 - u^2(t)) + \alpha_2 \cdot (1 - u^2(t)) \cdot \dot{u}(t)$$

and

$$\mathcal{N}(u(t)) = -\omega_0^2 \cdot u(t) + \ddot{u}(t) \cdot u^2(t) - 2u(t) \cdot (\dot{u}(t))^2 + R^2 \cdot u(t) \cdot (1 - u^2(t)) + \\ +2\beta \cdot R \cdot u(t) \cdot (1 + u^2(t)) + \alpha_1 \cdot (1 + u^2(t)) \cdot \dot{u}(t)\ ,$$

respectively.

Therefore, applying the same procedure it obtain that the expression $\mathcal{N}(u_0(t))$ is a combination of the elementary functions $\cos(\omega_0 t)$, $\sin(\omega_0 t)$, $\cos(3\omega_0 t)$, $\sin(3\omega_0 t)$ in the both cases. So, the first approximation $u_1(t)$ has the form by Eq. (45) and the first-order approximate analytic solution $\bar{u}(t)$ has the form by Eq. (46).

In the case $\alpha_3 \neq 0$, $\beta = \alpha_1 = \alpha_2 = 0$, the linear operator is $\mathcal{L}\Big(u(t)\Big) = \ddot{u}(t) + \omega_0^2 u(t)$ and the nonlinear operator is deduced from Eq. (21) as $\mathcal{N}\Big(u(t)\Big) = -\omega_0^2 u(t) + \alpha_3 \cdot \dot{u}(t) + \dfrac{R^2}{2} \cdot \sin(2 \cdot u(t))$. The initial approximation is $u_0(t) = u(0) \cdot \cos(\omega_0 t) + \frac{\dot{u}(0)}{\omega_0} \cdot \sin(\omega_0 t)$, solution of the equation $\mathcal{L}\Big(u(t)\Big) = 0$, with initial conditions given by Eq. (21). Then, the expression $\mathcal{N}\Big(u_0(t)\Big)$ contain a combination of the elementary functions $\cos(2\omega_0 t)$, $\sin(2\omega_0 t)$, $\cos(4\omega_0 t)$, $\sin(4\omega_0 t)$. So, the first approximation $u_1(t)$ has the form by

$$\overline{u}_{OAFM}(t) = u(0) \cdot \cos\omega_0 t + \frac{\dot{u}(0)}{\omega_0} \cdot \sin\omega_0 t + \tilde{a}_0 \cos\omega_0 t + \tilde{b}_0 \sin\omega_0 t + \\ + \sum_{k=1}^{N_{max}} \tilde{a}_k^{(5)} \cdot \cos(2k\ \omega_0 t) + \tilde{b}_k^{(5)} \cdot \sin(2k\ \omega_0 t)\ , \tag{48}$$

where the convergence-control parameters ω_0, $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{a}_k^{(5)}$, $\tilde{b}_k^{(5)}$ will be optimally identified.

In this way the approximate analytic solutions of the nonlinear problems Eqs. (15), (18), (21), can be constructed, via OAFM method.

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5 Numerical results and Discussions

In this section, we discuss the accuracy of the OAFM method by taking into consideration the first-order approximate solutions given by Eqs. (46), (47), where the index $N_{max} \in \{10,\ 15,\ 25,\ 35\}$ is an arbitrary fixed positive integer number.

By means of the Eqs. (3), (4), (46), for $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, the Eqs. (8), (9), (47), for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$,

the Eqs. (12), (14), (46), for $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_3 = 0$, the Eqs. (16), (17), (46), for $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, and the Eqs. (19), (20), (48), for $\alpha_1 = 0$, $\beta = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$, respectively, the approximate closed-form solutions of the Rabinovich system are well-determined, via OAFM technique.

The accuracy of the obtained results is shown in the Figs. 1 - 2 (for $\beta = 0.25 \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$), the Figs. 4 - 5 (for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$) respectively, by comparison of the above obtained approximate solutions with the corresponding numerical integration results, computed by means of the fourth-order Runge-Kutta method using Wolfram Mathematica 9.0 software. On the other hand, the cases $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$ are depicted in Figs. 7-12. The convergence-control parameters $C_1 = a_0 + v(0)$, $C_i = a^{(4)}_{k-1}$, $B_1 = b_0 + \frac{\dot{v}(0)}{\omega_0}$, $B_i = b^{(4)}_{k-1}$, $i = 2,\ 3,\ \cdots\ N_{max}$, which appear in Eqs. (46), (47), (48) are optimally identified by the least square method for different values of the known parameter N_{max}. As it could be observed in the figures there are the symmetry with respect to the Oz- axis, for $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ and are the symmetry with respect to the all coordinate axes, for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$. The Figs. 3, 6 highlight the symmetry of the 3D trajectory.

The convergence-control parameters are presented in the section Appendices.

The influence of the index number N_{max} on the values of the relative errors is examined in Tables 1-2. The better approximate analytical solution corresponds to the value $N_{max} = 25$ for $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, and $N_{max} = 35$ for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, respectively. This values were chosen for the efficiency of the solutions shown in Tables 3-5.

6 Conclusions

In the present paper, some geometrical properties of the Rabinovich system are emphasized and the approximate analytic solutions were established. A good agree-

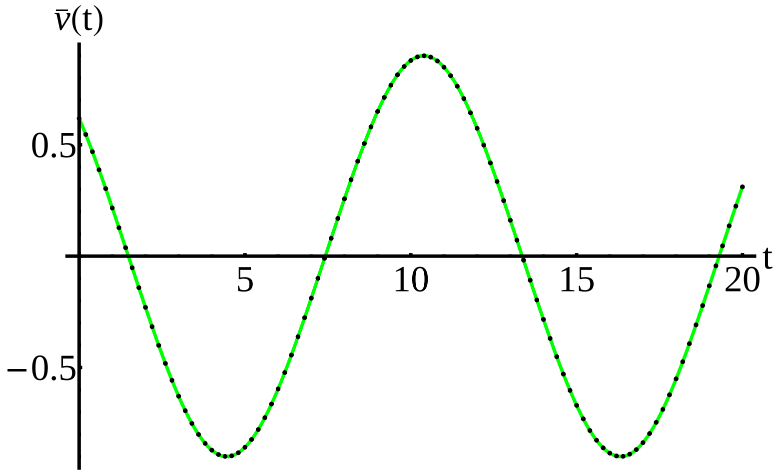

Figure 1: The auxiliary function $\bar{v}(t)$ given by Eqs. (46), (51) using the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

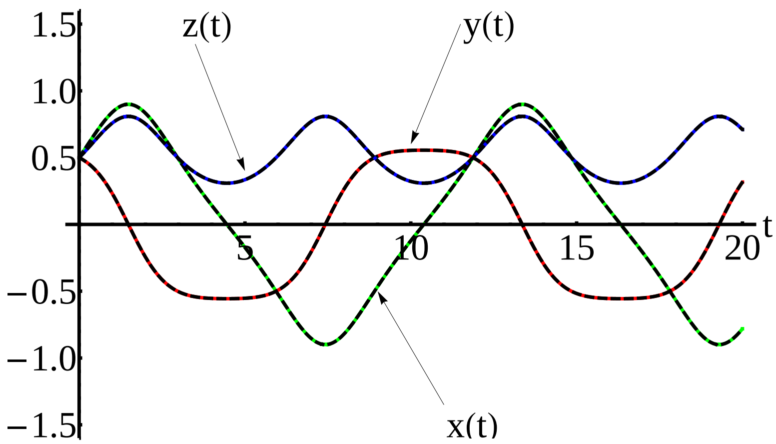

Figure 2: The set of solutions $x(t)$, $y(t)$, $z(t)$ given by Eqs. (3), (4) using Eqs. (46), (51) with the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

ment between the approximate analytic solutions (using OAFM) and corresponding numerical solutions (using the fourth-order Runge-Kutta method) was found for symmetric solutions with respect to the coordinate planes. These obtained solu-

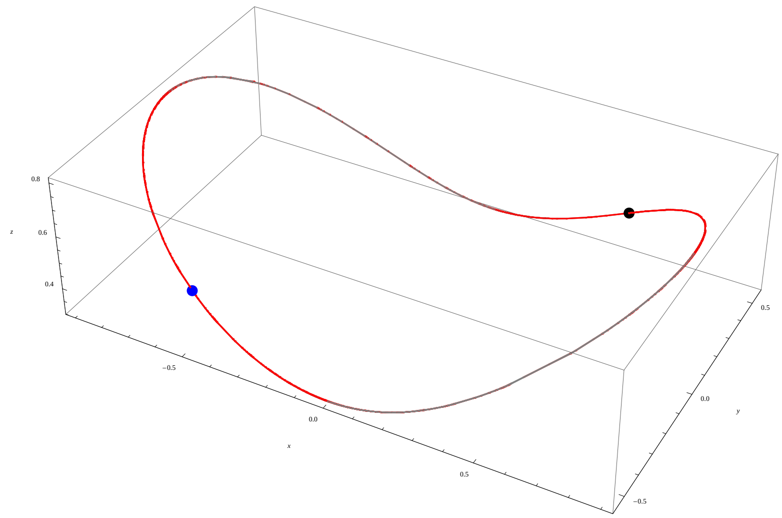

Figure 3: The points $(0.5, 0.5, 0.5)$ (black), $(-0.5, -0.5, 0.5)$ (red) and the parametric 3D curve $x = x(t)$, $y = y(t)$, $z = z(t)$ given by Eqs. (3), (4) using Eqs. (46), (51) with the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with gray line) and numerical solution (dashing red line), respectively.

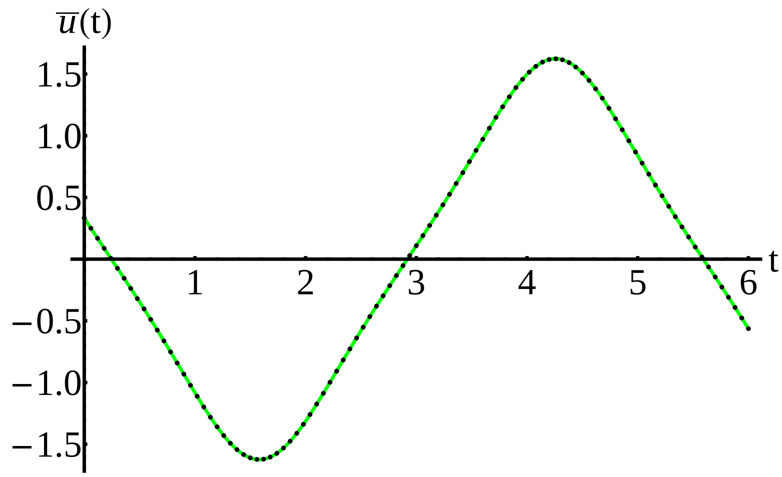

Figure 4: The auxiliary function $\bar{u}(t)$ given by Eqs. (47), (54) using the initial conditions $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 35$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

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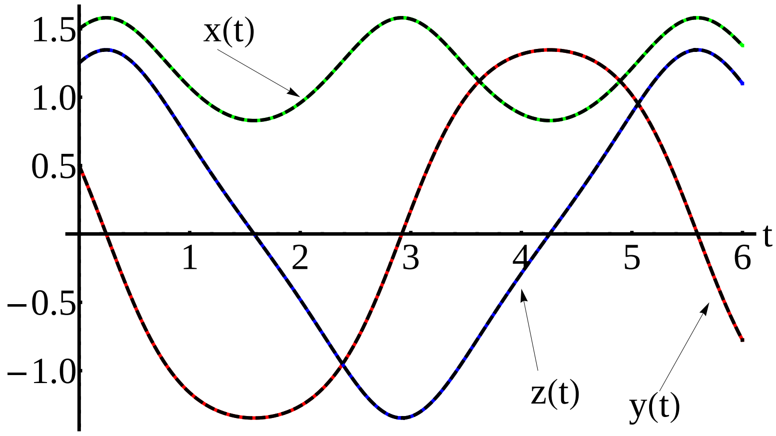

Figure 5: The set of solutions $x(t)$, $y(t)$, $z(t)$ given by Eqs. (8), (9) using Eqs. (47), (54) with the initial conditions $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 35$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

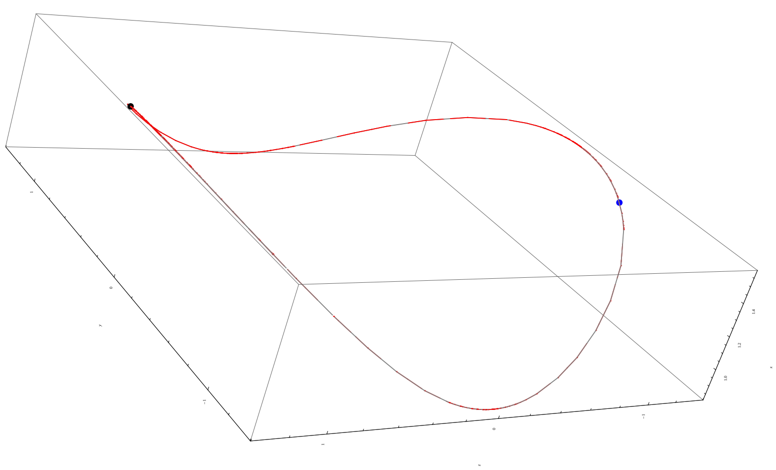

Figure 6: The points $(1.5, 0.5, 1.25)$ (black), $(1.5, -0.5, -1.25)$ (red) and the parametric 3D curve $x = x(t)$, $y = y(t)$, $z = z(t)$ given by Eqs. (8), (9) using Eqs. (47), (54) with the initial conditions $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 35$:
OAFM solution (with gray line) and numerical solution (dashing red line), respectively.

tions can be usefully in many applications of technological interest.

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Appendices

In the following we will present just the values of the convergence-control parameters that appear in Eqs. (46), (47) and (48), respectively.

The case $\beta \neq 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

Example 1. The initial conditions are $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$.

a) for Eq. (46) with $N_{max} = 10$:

$$\begin{aligned}
&\omega_0 = 0.0842084063,\ B_1 = -7.7850095373,\ B_2 = 1.1935759266,\\
&B_3 = 10.9766341996,\ B_4 = 1.0576315879,\ B_5 = -5.5540946245,\\
&B_6 = -1.3916077665,\ B_7 = 1.2287912091,\ B_8 = 0.3774505937,\\
&B_9 = -0.0856590499,\ B_{10} = -0.0177125387,\ C_1 = 1.2647924166,\\
&C_2 = 9.8753733548,\ C_3 = -0.2632330456,\ C_4 = -8.4861481647,\\
&C_5 = -1.5913245793,\ C_6 = 2.9461688979,\ C_7 = 0.8568908158,\\
&C_8 = -0.3803812033,\ C_9 = -0.1212724374,\ C_{10} = 0.0080186284;
\end{aligned} \tag{49}$$

b) for Eq. (46) with $N_{max} = 15$:

$$\begin{aligned}
&\omega_0 = 0.0842084063,\ B_1 = -7.8470508367,\ B_2 = 2.2499897809,\\
&B_3 = 12.9280127794,\ B_4 = -0.4712356520,\ B_5 = -9.7259134278,\\
&B_6 = -1.3690509281,\ B_7 = 4.4434866499,\ B_8 = 1.1492580188,\\
&B_9 = -1.1762526193,\ B_{10} = -0.3861053828,\ B_{11} = 0.1603359895,\\
&B_{12} = 0.0563660451,\ B_{13} = -0.0090989642,\ B_{14} = -0.0027442264,\\
&B_{15} = 2.773923 \cdot 10^{-6},\ C_1 = 1.6353827930,\ C_2 = 10.5668121613,\\
&C_3 = -1.8492817431,\ C_4 = -11.8272674286,\ C_5 = -0.7138152947,\\
&C_6 = 7.0224266784,\ C_7 = 1.4476285900,\ C_8 = -2.4512890639,\\
&C_9 = -0.7451197226,\ C_{10} = 0.4745313849,\ C_{11} = 0.1651164438,\\
&C_{12} = -0.0436858802,\ C_{13} = -0.0147119639,\ C_{14} = 0.0012846915,\\
&C_{15} = 0.0004718731;
\end{aligned} \tag{50}$$

c) for Eq. (46) with $N_{max} = 25$:

$$
\begin{aligned}
&\omega_0 = 0.0842084063,\ B_1 = 113.1906313266,\ B_2 = 700.5143533516,\\
&B_3 = -63.8042262780,\ B_4 = -1726.7156941065,\ B_5 = -727.2541799033,\\
&B_6 = 1850.2472856376,\ B_7 = 1458.7090868963,\ B_8 = -1023.3252919624,\\
&B_9 = -1371.45280473155,\ B_{10} = 192.8265190738,\ B_{11} = 763.9333733724,\\
&B_{12} = 118.7239396963,\ B_{13} = -261.2640026833,\ B_{14} = -98.9226990064,\\
&B_{15} = 51.1015243215,\ B_{16} = 33.0902447676,\ B_{17} = -3.9323646180,\\
&B_{18} = -5.8957249378,\ B_{19} = -0.3704710427,\ B_{20} = 0.5357572940,\\
&B_{21} = 0.0886533581,\ B_{22} = -0.0195324516,\ B_{23} = -0.0045186018,\\
&B_{24} = 0.0001056260,\ B_{25} = 0.0000356013,\ C_1 = 237.0082862136,\\
&C_2 = -181.1626337930,\ C_3 = -1260.1800896233,\ C_4 = -266.1579780903,\\
&C_5 = 1943.0958639398,\ C_6 = 1170.9065813163,\ C_7 = -1501.3433857085,\\
&C_8 = -1520.5635275053,\ C_9 = 554.5650629146,\ C_{10} = 1087.6826289541,\\
&C_{11} = 26.0903663084,\ C_{12} = -475.6102111380,\ C_{13} = -128.1837256078,\\
&C_{14} = 125.1740070213,\ C_{15} = 62.1706696066,\ C_{16} = -16.9073998070,\\
&C_{17} = -15.0951778584,\ C_{18} = 0.2064815885,\ C_{19} = 1.9508776536,\\
&C_{20} = 0.2353541523,\ C_{21} = -0.1178869685,\ C_{22} = -0.0237146201,\\
&C_{23} = 0.0021457461,\ C_{24} = 0.0005641598,\ C_{25} = 3.317757 \cdot 10^{-6}.
\end{aligned}
\tag{51}
$$

Now, for the initial conditions $x_0 = -0.5$, $y_0 = -0.5$, $z_0 = 0.5$ and $N_{max} = 25$, $\beta = 0.25$ the convergence-control parameters for the symmetric solution (with respect to the Oz-axis) given by Eq. (46) are given in Eq. (51).

The remarkable case $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

Example 2. The initial conditions are $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$.

a) for Eq. (47) with $N_{max} = 15$:

$$
\begin{aligned}
&\omega_0 = 0.1869876739,\ B_1 = -554.4037761129,\ B_2 = 226.7457730999,\\
&B_3 = 782.3721160745,\ B_4 = -48.5974967462,\ B_5 = 49.4030334587,\\
&B_6 = -957.7117203144,\ B_7 = 283.2006105162,\ B_8 = -159.1917408848,\\
&B_9 = 674.7077918425,\ B_{10} = 23.5125177445,\ B_{11} = -495.1189399730,\\
&B_{12} = 141.3482748307,\ B_{13} = 50.5506946689,\ B_{14} = -16.7536940267,\\
&B_{15} = -0.0634441781,\ C_1 = -93.4671025566,\ C_2 = 1047.0086776914,\\
&C_3 = -495.9436000281,\ C_4 = -133.5434805609,\ C_5 = -746.4077270771,\\
&C_6 = 236.7019711439,\ C_7 = 421.4165739752,\ C_8 = 177.1891806165,\\
&C_9 = 254.4730564674,\ C_{10} = -758.0078082782,\ C_{11} = 147.8047599915,\\
&C_{12} = 204.0625242583,\ C_{13} = -66.0059654899,\ C_{14} = -6.0287025308,\\
&C_{15} = 1.8659955242;
\end{aligned}
\tag{52}
$$

b) for Eq. (47) with $N_{max} = 25$:

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$$
\begin{aligned}
&\omega_0 = 0.1869876739,\ B_1 = 516.1242386938,\ B_2 = -370.2739755607,\\
&B_3 = -438.3744282729,\ B_4 =\ 232.6309328533,\ B_5 = -282.3256743841,\\
&B_6 = 387.5617017331,\ B_7 = -47.8049293623,\ B_8 = 162.7251241075,\\
&B_9 = -117.3367937705,\ B_{10} = 59.7546990652,\ B_{11} = -111.8077188524,\\
&B_{12} = 71.8237012001,\ B_{13} = -149.4299059907,\ B_{14} = 60.2531991051,\\
&B_{15} = -76.3055734495,\ B_{16} = 113.6663413024,\ B_{17} = -16.3900394802,\\
&B_{18} = 108.9340003143,\ B_{19} = -91.5339247140,\ B_{20} = -72.0679666456,\\
&B_{21} = 67.2469830900,\ B_{22} = -0.6349607714,\ B_{23} = -7.3149658364,\\
&B_{24} = 0.8138266745,\ B_{25} = 0.0661089513,\ C_1 = 54.6954626367,\\
&C_2 = -889.6642684770,\ C_3 = 678.0047311577,\ C_4 = -235.8685894546,\\
&C_5 = 401.5470307526,\ C_6 = -43.0268550584,\ C_7 = 40.6366633629,\\
&C_8 = -169.6938622093,\ C_9 = 23.4686692840,\ C_{10} = -115.4282122040,\\
&C_{11} = 73.1279637248,\ C_{12} = -92.7226387177,\ C_{13} = 55.3435319845,\\
&C_{14} = 12.4804646698,\ C_{15} = 67.0157693986,\ C_{16} = 29.4365871710,\\
&C_{17} = -9.0367780737,\ C_{18} = -25.8911752609,\ C_{19} = -126.8226917481,\\
&C_{20} = 103.3382634787,\ C_{21} = 20.0135321168,\ C_{22} = -27.9178547423,\\
&C_{23} = 2.5193238481,\ C_{24} = 1.0883135883,\ C_{25} = -0.0948041290;
\end{aligned}
\tag{53}
$$

c) for Eq. (47) with $N_{max} = 35$:

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$$\begin{aligned}
&\omega_0 = 0.1869876739,\ B_1 = 27.1589347487,\ B_2 = 12.0827987658,\\
&B_3 = -31.2924026686,\ B_4 = -6.2126489550,\ B_5 = -8.4925801994,\\
&B_6 = -13.9924569422,\ B_7 = 6.0373789169,\ B_8 = 19.7307983361,\\
&B_9 = -1.4098763264,\ B_{10} = -23.1824059776,\ B_{11} = 19.5047102382,\\
&B_{12} = 18.0959242465,\ B_{13} = -20.3437872078,\ B_{14} = -2.3657500207,\\
&B_{15} = 8.0506645554,\ B_{16} = 2.1157429592,\ B_{17} = 8.2577036885,\\
&B_{18} = -24.5293827842,\ B_{19} = 9.7431739692,\ B_{20} = -13.3366939523,\\
&B_{21} = 21.0972405836,\ B_{22} = 1.3733966515,\ B_{23} = -10.4215638468,\\
&B_{24} = -0.6346613181,\ B_{25} = 3.7660664321,\ B_{26} = -6.3989961662,\\
&B_{27} = 14.1834202800,\ B_{28} = -8.4270149394,\ B_{29} = -5.0794721575,\\
&B_{30} = 6.5727871915,\ B_{31} = -1.2212175488,\ B_{32} = -0.6570547398,\\
&B_{33} = 0.2354677108,\ B_{34} = -0.0056214482,\ B_{35} = -0.0026220746,\\
&C_1 = 19.7054442097,\ C_2 = -54.4346582382,\ C_3 = -0.8103617202,\\
&C_4 = -1.8697297809,\ C_5 = 5.5410468189,\ C_6 = 4.1456744599,\\
&C_7 = 25.1576973040,\ C_8 = -4.3010254027,\ C_9 = -18.3207242611,\\
&C_{10} = 9.5312211886,\ C_{11} = 21.8263082280,\ C_{12} = -20.0001654379,\\
&C_{13} = -14.6586935042,\ C_{14} = 21.2392716691,\ C_{15} = -9.2030700586,\\
&C_{16} = 10.5290971616,\ C_{17} = -24.7946288753,\ C_{18} = 8.9376532008,\\
&C_{19} = 1.1913707498,\ C_{20} = 8.9900769507,\ C_{21} = 8.2043548306,\\
&C_{22} = -16.8096509190,\ C_{23} = -1.8558514640,\ C_{24} = 8.7834639788,\\
&C_{25} = -5.1597513330,\ C_{26} = 6.9589549263,\ C_{27} = -0.6370483091,\\
&C_{28} = -12.0432828958,\ C_{29} = 10.0288355195,\ C_{30} = 0.0799417017,\\
&C_{31} = -2.6893238533,\ C_{32} = 0.7569825071,\ C_{33} = 0.0695529173,\\
&C_{34} = -0.0388832605,\ C_{35} = 0.0017247410.
\end{aligned} \tag{54}$$

Now, for the initial conditions: $x_0 = -1.5$, $y_0 = -0.5$, $z_0 = 1.25$ (symmetry with respect to the Oz-axis) and $N_{max} = 35$, $\beta = 0$, $x_0 = 1.5$, $y_0 = -0.5$, $z_0 = -1.25$ (symmetry with respect to the Ox-axis) and $N_{max} = 35$, $\beta = 0$, $x_0 = -1.5$, $y_0 = 0.5$, $z_0 = -1.25$ (symmetry with respect to the Oy-axis) and $N_{max} = 35$, $\beta = 0$, the convergence-control parameters for the symmetric solution (with respect to the Oz-axis) given by Eq. (47) are given in Eq. (54).

The case $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0.05$, $\alpha_3 = 0$

Example 3. The initial conditions are $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $N_{max} = 25$. The convergence-control parameters for the approximate analytic solution $\bar{u}(t)$ given by Eq. (46) are:

$$\begin{aligned}
&\omega_0 = 0.0694543429,\ B_1 = -69.5705751030,\ B_2 = -120.8993887304,\\
&B_3 = 134.8567394227,\ B_4 = 80.2715486367,\ B_5 = 133.5091139810,\\
&B_6 = -1.6079535511,\ B_7 = -107.8296608169,\ B_8 = -89.0630671934,\\
&B_9 = -153.5628496068,\ B_{10} = 103.8865887516,\ B_{11} = 81.7020914138,\\
&B_{12} = 124.0154016728,\ B_{13} = 71.0190829276,\ B_{14} = -279.0140436547,\\
&B_{15} = -47.4832381804,\ B_{16} = 186.4926850959,\ B_{17} = -1.6034548489,\\
&B_{18} = -58.0109625378,\ B_{19} = 5.9577818217,\ B_{20} = 8.6271212497,\\
&B_{21} = -1.2371738255,\ B_{22} = -0.5383432616,\ B_{23} = 0.0738473761,\\
&B_{24} = 0.0093636835,\ B_{25} = -0.0006547226,\ C_1 = -65.0212060551,\\
&C_2 = 146.9430296120,\ C_3 = 122.0164695356,\ C_4 = -12.7416766751,\\
&C_5 = -35.4755424788,\ C_6 = -188.0064243933,\ C_7 = -37.3549977788,\\
&C_8 = -42.6527153638,\ C_9 = 125.7222674988,\ C_{10} = 160.3454312782,\\
&C_{11} = -9.0512355641,\ C_{12} = 14.4067469635,\ C_{13} = -234.7064480557,\\
&C_{14} = -74.8990251987,\ C_{15} = 253.4010415854,\ C_{16} = 16.8710869660,\\
&C_{17} = -113.8065189746,\ C_{18} = 7.2254321395,\ C_{19} = 24.6401183173,\\
&C_{20} = -3.1759631665,\ C_{21} = -2.4394846206,\ C_{22} = 0.3573106958,\\
&C_{23} = 0.0874588482,\ C_{24} = -0.0098968427,\ C_{25} = -0.0005010425\ .
\end{aligned} \tag{55}$$

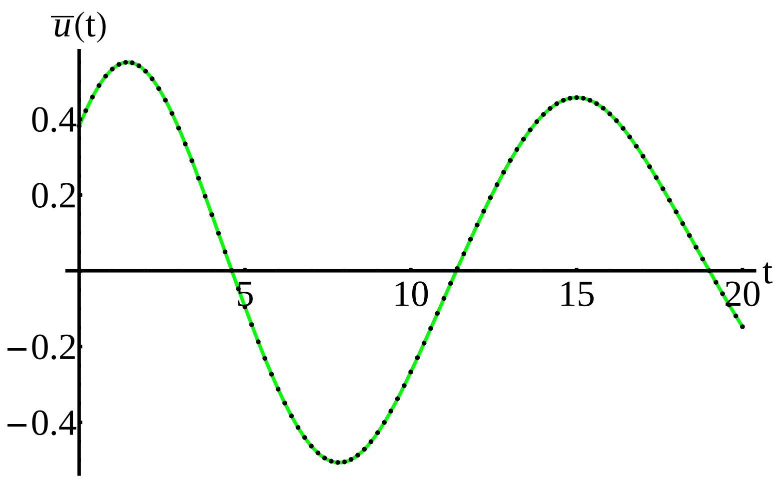

Figure 7: The auxiliary function $\bar{u}(t)$ given by Eqs. (46), (55) using the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0.05$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

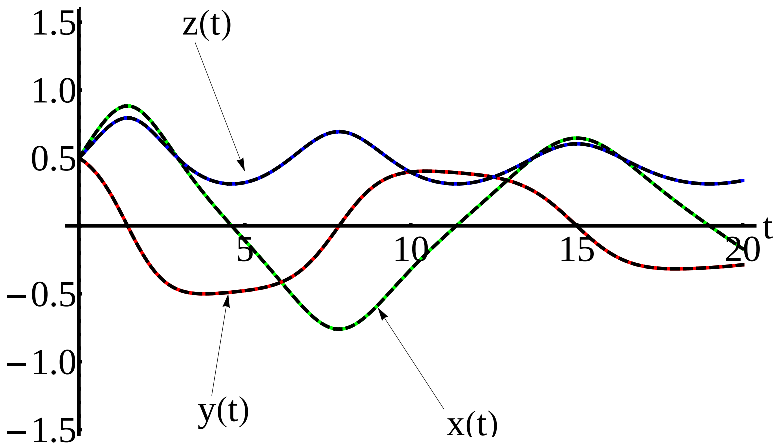

Figure 8: The set of solutions $x(t)$, $y(t)$, $z(t)$ given by Eqs. (12), (14) using Eqs. (46), (55) with the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0.05$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

The case $\beta = 0.25$**,** $\alpha_1 = 0.05$**,** $\alpha_2 = 0$**,** $\alpha_3 = 0$

Example 4. The initial conditions are $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $N_{max} = 25$. The convergence-control parameters for the approximate analytic solution $\bar{u}(t)$ given by Eq. (46) are:

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$$
\begin{aligned}
&\omega_0 = 0.0979970641,\ B_1 = -7.86946701639623`,\ B_2 = -6.4646829651,\\
&B_3 = 8.7356943964,\ B_4 = 16.3328135131,\ B_5 = 5.1958788407,\\
&B_6 = -10.8702566280,\ B_7 = -12.6215405753,\ B_8 = -1.5426658313,\\
&B_9 = 6.8534549134,\ B_{10} = 5.3338415271,\ B_{11} = -0.1190016122,\\
&B_{12} = -2.4166370753,\ B_{13} = -1.2668179431,\ B_{14} = 0.1849844809,\\
&B_{15} = 0.4668487148,\ B_{16} = 0.1558661112,\ B_{17} = -0.0410984637,\\
&B_{18} = -0.0439753213,\ B_{19} = -0.0079302657,\ B_{20} = 0.0031640290,\\
&B_{21} = 0.0015152397,\ B_{22} = 0.0000782713,\ B_{23} = -0.0000586884,\\
&B_{24} = -7.908883 \cdot 10^{-6},\ B_{25} = 2.571457 \cdot 10^{-7},\ C_1 = -1.7097438990,\\
&C_2 = 10.0351472420,\ C_3 = 12.4760001691,\ C_4 = -2.2874375788,\\
&C_5 = -15.6734001521,\ C_6 = -10.8070637066,\ C_7 = 4.2334673452,\\
&C_8 = 10.7552729215,\ C_9 = 4.7764382238,\ C_{10} = -2.8811116370,\\
&C_{11} = -4.1426512477,\ C_{12} = -1.1302932517,\ C_{13} = 0.9994302741,\\
&C_{14} = 0.8951531888,\ C_{15} = 0.1235578694,\ C_{16} = -0.1781342705,\\
&C_{17} = -0.0986544376,\ C_{18} = -0.0025652300,\ C_{19} = 0.0143255850,\\
&C_{20} = 0.0043772770,\ C_{21} = -0.0003038516,\ C_{22} = -0.0003624506,\\
&C_{23} = -0.0000398691,\ C_{24} = 5.795421 \cdot 10^{-6},\ C_{25} = 7.292742 \cdot 10^{-7}\ .
\end{aligned}
\tag{56}
$$

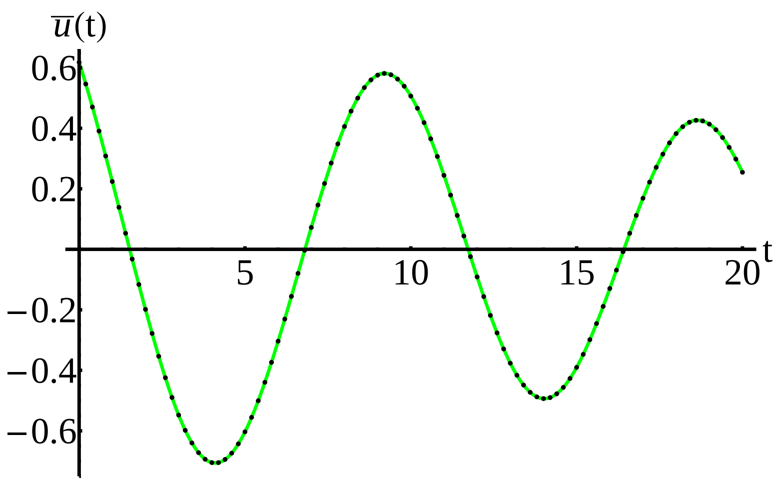

Figure 9: The auxiliary function $\bar{u}(t)$ given by Eqs. (46), (56) using the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0.05$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

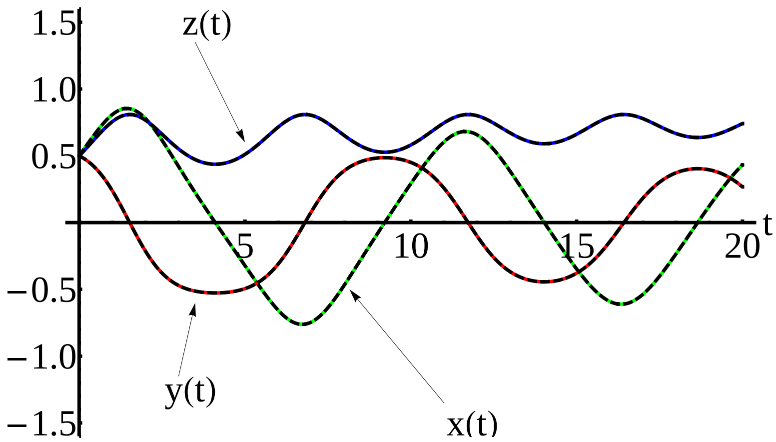

Figure 10: The set of solutions $x(t)$, $y(t)$, $z(t)$ given by Eqs. (16), (17) using Eqs. (46), (56) with the initial conditions $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and $\beta = 0.25$, $\alpha_1 = 0.05$, $\alpha_2 = 0$, $\alpha_3 = 0$ for $N_{max} = 25$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

The case $\beta = 0$**,** $\alpha_1 = 0$**,** $\alpha_2 = 0$**,** $\alpha_3 = 0.15$

Example 5. The initial conditions are $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $N_{max} = 35$. The convergence-control parameters for the approximate analytic solution $\bar{u}(t)$ given by Eq. (48) are:

$$
\begin{aligned}
&\omega_0 = 0.2172006104,\ B_1 = -7.3251070555,\ B_2 = -0.1087849966,\\
&B_3 = 4.1113264263,\ B_4 = 4.21373739429,\ B_5 = 1.0687156558,\\
&B_6 = -1.5599721276,\ B_7 = -1.43202796290,\ B_8 = 0.7373618607,\\
&B_9 = 2.3909438155,\ B_{10} = 1.77937304336,\ B_{11} = -0.5409997087,\\
&B_{12} = -2.5317090264,\ B_{13} = -2.67997580834,\ B_{14} = -1.1492629171,\\
&B_{15} = 0.6922268586,\ B_{16} = 1.59791789658,\ B_{17} = 1.3153031289,\\
&B_{18} = 0.4470104143,\ B_{19} = -0.2651316628,\ B_{20} = -0.4881158033,\\
&B_{21} = -0.3386938091,\ B_{22} = -0.0985227111,\ B_{23} = 0.0470955155,\\
&B_{24} = 0.0741216873,\ B_{25} = 0.0431448916,\ B_{26} = 0.0106926238,\\
&B_{27} = -0.0035675339,\ B_{28} = -0.0047137818,\ B_{29} = -0.0021456860,\\
&B_{30} = -0.0004138394,\ B_{31} = 0.0000775657,\ B_{32} = 0.0000738535,\\
&B_{33} = 0.0000199500,\ B_{34} = 1.929570 \cdot 10^{-8},\ B_{35} = -8.095036 \cdot 10^{-8},\\
&C_1 = 4.8320147836,\ C_2 = 6.1545928815,\ C_3 = 2.7172376117,\\
&C_4 = -0.9374420135,\ C_5 = -2.9359918166,\ C_6 = -1.5185091234,\\
&C_7 = 0.9997803324,\ C_8 = 1.8049281860,\ C_9 = 0.1894325635,\\
&C_{10} = -2.1155718115,\ C_{11} = -2.9032537901,\ C_{12} = -1.5215478343,\\
&C_{13} = 0.7804328110,\ C_{14} = 2.2301227152,\ C_{15} = 2.0399788427,\\
&C_{16} = 0.7655400005,\ C_{17} = -0.4739199152,\ C_{18} = -0.9603870342,\\
&C_{19} = -0.7236971824,\ C_{20} = -0.2267309158,\ C_{21} = 0.1229722256,\\
&C_{22} = 0.2088434889,\ C_{23} = 0.1332553039,\ C_{24} = 0.0359490437,\\
&C_{25} = -0.0146330134,\ C_{26} = -0.0212551967,\ C_{27} = -0.0111045215,\\
&C_{28} = -0.0024764806,\ C_{29} = 0.0006442642,\ C_{30} = 0.0007496569,\\
&C_{31} = 0.0002808121,\ C_{32} = 0.0000431036,\ C_{33} = -4.960082 \cdot 10^{-6},\\
&C_{34} = -3.110982 \cdot 10^{-6},\ C_{35} = -3.647933 \cdot 10^{-7}\ .
\end{aligned} \tag{57}
$$

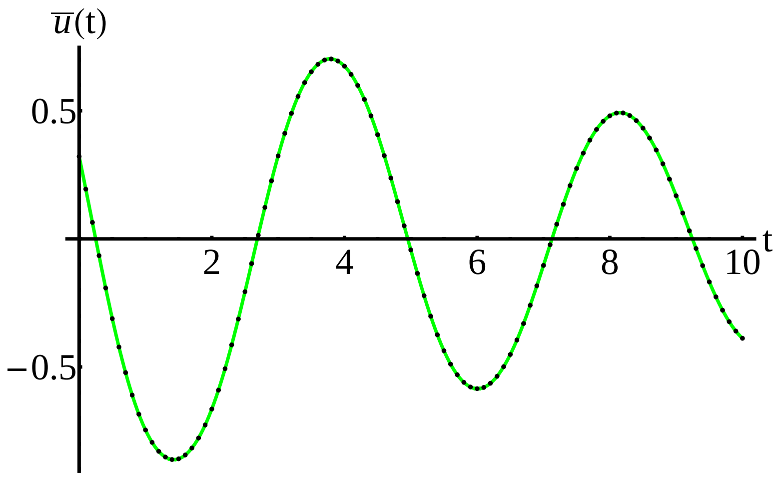

Figure 11: The auxiliary function $\bar{u}(t)$ given by Eqs. (48), (57) using the initial conditions $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0.15$ for $N_{max} = 35$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

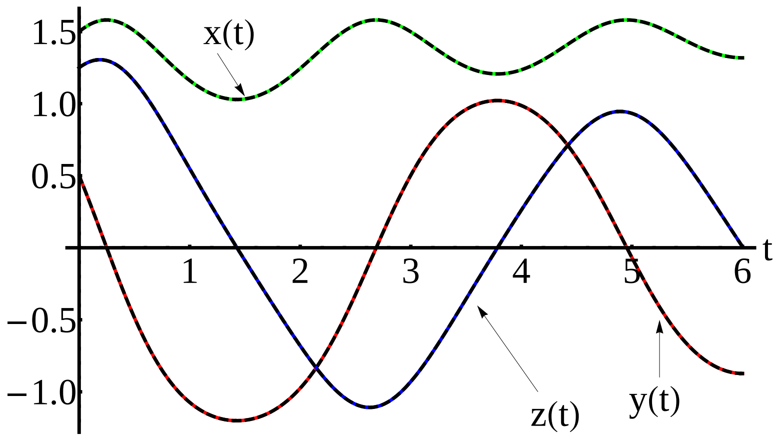

Figure 12: The set of solutions $x(t)$, $y(t)$, $z(t)$ given by Eqs. (19), (20) using Eqs. (48), (57) with the initial conditions $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0.15$ for $N_{max} = 35$:
OAFM solution (with lines) and numerical solution (dashing lines), respectively.

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Table 1: Comparison between the relative errors: $\epsilon_v = |v_{numerical} - \bar{v}_{OAFM}|$ for $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and different values of the index N_{max};
$\bar{v}_{OAFM}$ obtained from Eqs. (46), (49), (50), (51)

t	$N_{max} = 10$	$N_{max} = 15$	$N_{max} = 25$
0	$1.332267 \cdot 10^{-15}$	$4.440892 \cdot 10^{-16}$	$2.646771 \cdot 10^{-13}$
7/5	0.0002311701	$9.690649 \cdot 10^{-7}$	$5.424820 \cdot 10^{-10}$
14/5	0.0001494743	$9.806902 \cdot 10^{-7}$	$3.389437 \cdot 10^{-10}$
21/5	0.0001987102	$1.243573 \cdot 10^{-6}$	$1.842952 \cdot 10^{-10}$
28/5	0.0000961699	$5.341956 \cdot 10^{-8}$	$6.126734 \cdot 10^{-10}$
7	0.0001210484	$2.545193 \cdot 10^{-6}$	$4.273881 \cdot 10^{-10}$
42/5	0.0000661653	$1.815027 \cdot 10^{-6}$	$2.335903 \cdot 10^{-10}$
49/5	$9.306109 \cdot 10^{-6}$	$2.151637 \cdot 10^{-6}$	$5.521745 \cdot 10^{-10}$
56/5	0.0000211790	$2.055369 \cdot 10^{-6}$	$4.816658 \cdot 10^{-10}$
63/5	0.0001510944	$2.318730 \cdot 10^{-7}$	$7.166223 \cdot 10^{-11}$
14	0.0001919623	$1.595892 \cdot 10^{-6}$	$1.900378 \cdot 10^{-10}$

Table 2: Comparison between the relative errors: $\epsilon_u = |u_{numerical} - \bar{u}_{OAFM}|$ for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and different values of the index N_{max};
$\bar{u}_{OAFM}$ obtained from Eqs. (47), (52), (53), (54)

t	$N_{max} = 15$	$N_{max} = 25$	$N_{max} = 35$
0	$9.475753 \cdot 10^{-14}$	$1.587063 \cdot 10^{-13}$	$7.716050 \cdot 10^{-15}$
3/5	$3.504900 \cdot 10^{-4}$	$3.316779 \cdot 10^{-5}$	$8.194535 \cdot 10^{-8}$
6/5	$2.914220 \cdot 10^{-4}$	$2.904368 \cdot 10^{-5}$	$7.775160 \cdot 10^{-8}$
9/5	$4.067788 \cdot 10^{-4}$	$3.306752 \cdot 10^{-5}$	$1.002242 \cdot 10^{-7}$
12/5	$5.020959 \cdot 10^{-4}$	$3.350227 \cdot 10^{-5}$	$1.013316 \cdot 10^{-7}$
3	$2.399299 \cdot 10^{-4}$	$3.095774 \cdot 10^{-5}$	$6.350594 \cdot 10^{-8}$
18/5	$7.499806 \cdot 10^{-5}$	$3.033164 \cdot 10^{-5}$	$9.363366 \cdot 10^{-8}$
21/5	$2.634217 \cdot 10^{-4}$	$3.698857 \cdot 10^{-5}$	$7.855941 \cdot 10^{-8}$
24/5	$1.023441 \cdot 10^{-4}$	$3.459891 \cdot 10^{-5}$	$2.951037 \cdot 10^{-8}$
27/5	$1.061241 \cdot 10^{-4}$	$3.200782 \cdot 10^{-5}$	$4.558553 \cdot 10^{-8}$
6	$1.528191 \cdot 10^{-4}$	$3.492756 \cdot 10^{-5}$	$5.671638 \cdot 10^{-8}$

Table 3: Comparison between the approximate analytic solutions $\bar{u}_{OAFM}$ given by Eq. (46) and corresponding numerical solution for $\beta = 0.25$, $\alpha_1 = 0$, $\alpha_2 = 0.05$, $\alpha_3 = 0$, $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and the index $N_{max} = 25$;
(relative errors: $\epsilon_u = |u_{numerical} - \bar{u}_{OAFM}|$)

t	$u_{numerical}$	$\bar{u}_{OAFM}$	ϵ_u
0	0.3819660112	0.3819660112	$6.566969 \cdot 10^{-14}$
8/5	0.5487198876	0.5487198871	$4.569626 \cdot 10^{-10}$
16/5	0.3347566620	0.3347566624	$4.373337 \cdot 10^{-10}$
24/5	-0.0474300084	-0.0474300074	$9.474402 \cdot 10^{-10}$
32/5	-0.3828045448	-0.3828045450	$2.316530 \cdot 10^{-10}$
8	-0.5042437854	-0.5042437838	$1.624801 \cdot 10^{-9}$
48/5	-0.3375790298	-0.3375790299	$8.105455 \cdot 10^{-11}$
56/5	-0.0332606638	-0.0332606646	$8.140353 \cdot 10^{-10}$
64/5	0.2599740718	0.2599740722	$3.694533 \cdot 10^{-10}$
72/5	0.4407383794	0.4407383798	$4.701539 \cdot 10^{-10}$
16	0.4140215615	0.4140215610	$5.571915 \cdot 10^{-10}$

Table 4: Comparison between the approximate analytic solutions $\bar{u}_{OAFM}$ given by Eq. (46) and corresponding numerical solution for $\beta = 0.25$, $\alpha_1 = 0.05$, $\alpha_2 = 0$, $\alpha_3 = 0$, $x_0 = 0.5$, $y_0 = 0.5$, $z_0 = 0.5$ and the index $N_{max} = 25$;
(relative errors: $\epsilon_u = |u_{numerical} - \bar{u}_{OAFM}|$)

t	$u_{numerical}$	$\bar{u}_{OAFM}$	ϵ_u
0	0.6180339887	0.6180339887	$8.881784 \cdot 10^{-16}$
8/5	-0.0321978118	-0.0321978110	$7.381383 \cdot 10^{-10}$
16/5	-0.5983010647	-0.5983010645	$2.040423 \cdot 10^{-10}$
24/5	-0.6430063264	-0.6430063265	$1.829493 \cdot 10^{-10}$
32/5	-0.1556677170	-0.1556677166	$3.093710 \cdot 10^{-10}$
8	0.4057018841	0.4057018832	$9.304779 \cdot 10^{-10}$
48/5	0.5626679938	0.5626679941	$3.481130 \cdot 10^{-10}$
56/5	0.1791865300	0.1791865294	$6.380252 \cdot 10^{-10}$
64/5	-0.3292856584	-0.3292856577	$6.622721 \cdot 10^{-10}$
72/5	-0.4774573136	-0.4774573135	$1.349091 \cdot 10^{-10}$
16	-0.1296948250	-0.1296948251	$8.001704 \cdot 10^{-11}$

Table 5: Comparison between the approximate analytic solutions $\bar{u}_{OAFM}$ given by Eq. (48) and corresponding numerical solution for $\beta = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0.15$, $x_0 = 1.5$, $y_0 = 0.5$, $z_0 = 1.25$ and the index $N_{max} = 35$;
(relative errors: $\epsilon_u = |u_{numerical} - \bar{u}_{OAFM}|$)

t	$u_{numerical} = 15$	$\bar{u}_{OAFM}$	ϵ_u
0	0.3217505543	0.3217505543	$1.665334 \cdot 10^{-16}$
3/5	-0.4222547682	-0.4222561402	$1.371977 \cdot 10^{-6}$
6/5	-0.8296437373	-0.8296451162	$1.378897 \cdot 10^{-6}$
9/5	-0.7776543854	-0.7776557673	$1.381888 \cdot 10^{-6}$
12/5	-0.3129872607	-0.3129886423	$1.381552 \cdot 10^{-6}$
3	0.3239044822	0.3239031031	$1.379085 \cdot 10^{-6}$
18/5	0.6815903593	0.6815889752	$1.384109 \cdot 10^{-6}$
21/5	0.5989387285	0.5989373473	$1.381160 \cdot 10^{-6}$
24/5	0.1453366672	0.1453352855	$1.381609 \cdot 10^{-6}$
27/5	-0.3742949616	-0.3742963398	$1.378268 \cdot 10^{-6}$
6	-0.5850269138	-0.5850282851	$1.371267 \cdot 10^{-6}$