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Article

Interpretation of Entropy by Complex Fluid Dynamics

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Abstract: We consider that the relationship between entropy in statistical mechanics, which is the Boltzmann principle, and the complex velocity potential in complex fluid dynamics. We define the generalized complex entropy which expanded entropy from real space to complex space. We show that the complex entropy can be expressed by the composition of sources, sinks and laminar flows of the complex velocity potential. Therefore, the complex entropy is considered a special case of the complex fluid dynamics,that is, the complex velocity potential. In other words, we show that the complex entropy is expressed by the complex velocity potential. Moreover, we show that a complex number is expressed by the complex entropy. Thus, we show that the complex velocity potential is expressed by the complex entropy. Namely, we will expand entropy to complex space and see that the complex entropy is expressed by the complex velocity potential. Furthermore, we examine the possibility that it is an expansion of Boltzmann's principle and Planck distribution.

Keywords: Entropy; Complex Fluid Dynamics; Boltzmann principle; Planck law

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1. Introduction

We consider the relationship between entropy in statistical mechanics, which is the Boltzmann principle, and the complex velocity potential in complex fluid dynamics. We define the generalized Q-complex entropy $S_Q(z)$, which expand entropy from a real number x to a complex number z, where the function Q is a complex function. We show that the differential of the generalized Q-complex entropy $S'_Q(z)$ can be expressed by the composition of sources and sinks of the complex velocity potential. Therefore, the complex entropy is considered a special case of the complex fluid dynamics, that is, the complex velocity potential $w_Q(z)$. Moreover, we show that a complex number z is expressed by the generalized Q-complex entropy $S_Q(z)$. Thus, we show that the complex velocity potential $w_Q(z)$ is expressed by the generalized Q-complex entropy $S_Q(z)$.

1.1.

First, we introduce that the complex velocity potential $w(z)$, where z is a complex number. Furthermore, we consider sources and sinks of the complex velocity potential.

1.2.

Second, we introduce that the complex entropy $S(z)$. Furthermore, we discuss that the relationship between the complex velocity potential $w(z)$ and entropy S, where z is a complex number. Additionally, we show that the first derivative of entropy $S'(z)$ can be expressed by the composition of sources and sinks of the complex velocity potential $w_S(z)$. Therefore, the complex entropy $S(z)$ is considered a special case of the complex fluid dynamics $w_S(z)$. Moreover, the complex fluid dynamics $w_S(z)$ is expressed by the complex entropy $S(z)$.

1.3.

Third, let x be a positive real number. We consider the combination $W_{\pi_{f(x)},x}$ divided by the function approximating the number of primes $x/\log(x)$. We define the π_f-divided complex entropy $S_{\pi_f}(z)$ like the relationship of the Boltzmann. Additionally, we show that the first derivative of the π_f-divided complex entropy $S'_{\pi_f}(z)$ can be expressed by the composition of sources and sinks of the

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complex velocity potential $w_{\pi_f}(z)$. Therefore, the π_f-divided complex entropy $S_{\pi_f}(z)$ is considered a special case of the complex fluid dynamics $w_{\pi_f}(z)$. Moreover, the complex fluid dynamics $w_{\pi_f}(z)$ is expressed by the π_f-divided complex entropy $S_{\pi_f}(z)$.

1.4.

Fourth, we define the generalized complex entropy $S_Q(z)$, which expand entropy from a real number x to a complex number z. We consider the relationship between the generalized Q-complex entropy $S_Q(z)$ and the complex velocity potential $w_Q(z)$ in complex fluid dynamics. We see that the generalized Q-complex entropy $S_Q(z)$ can be thought of as the complex velocity potential $w_Q(z)$. Namely, we show that the first derivative of the generalized Q-complex entropy $S'_Q(z)$can be expressed by the composition of sources and sinks of the complex velocity potential $w_Q(z)$. Therefore, the generalized Q-complex entropy $S_Q(z)$ is considered a special case of the complex fluid dynamics $w_Q(z)$. Moreover, the complex fluid dynamics $w_Q(z)$ is expressed by the generalized Q-complex entropy $S_Q(z)$.

2. Complex Velocity Potential, Source and Sink

2.1. Complex Velocity Potential.

We define the velocity potential $\phi(x,y)$, the stream function $\psi(x,y)$ and the complex velocity potential $w(z)$ are expressed as follows.

Definition 1. *Let x and y are real numbers.*
Let $u(x,y)$ and $v(x,y)$ are real valued functions on two dimensions.
Let $\mathbf{v} = (u(x,y), v(x,y))$, the vector $\mathbf{v}$ is called velocity.
Let $z = x + iy$, where i is imaginary number.

Definition 2. *The definition of the velocity potential and the stream function:*
We define the velocity potential $\phi(x,y)$ and the stream function $\psi(x,y)$ are satisfied as follows:

$$u(x,y) = \frac{\partial\phi(x,y)}{\partial x} = -\frac{\partial\psi(x,y)}{\partial y}, \tag{2.1}$$

$$v(x,y) = \frac{\partial\phi(x,y)}{\partial y} = \frac{\partial\psi(x,y)}{\partial x}. \tag{2.2}$$

Definition 3. *The complex velocity potential :*
The complex velocity potential is defined as follows:

$$w(z) = \phi(x,y) + i\psi(x,y), \tag{2.3}$$

where $z = x + iy$ and i is imaginary number.

The derivative of the complex velocity potential $w(z)$ (2.3) is called the velocity field. The velocity field is defined as follows.

Definition 4. *The velocity field or the conjugate complex velocity is defined as follows:*

$$\frac{dw(z)}{dz} = u(x,y) - iv(x,y), \tag{2.4}$$

where $z = x + iy$ and i is imaginary number.

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2.2. Source and Sink and Laminar flow.

We define functions of source $w_{so}(z)$, sink $w_{si}(z)$ and $w_{la}(z)$ as follows :

Definition 5. *Let $m > 0$ be a positive real number and z be a complex number. The function of source is defined as follows :*

$$w_{so}(z) = m \log z = -m \log(\frac{1}{z}), \quad (Source\ of\ the\ strength\ m) \tag{2.5}$$

The function of sink is defined as follows :

$$w_{si}(z) = -m \log z = m \log(\frac{1}{z}), \quad (sink\ of\ the\ strength\ m) \tag{2.6}$$

The function of laminar flow is defined as follows :

$$w_{la}(z) = Az, \quad where\ A\ is\ a\ complex\ number. \tag{2.7}$$

We consider that the function $w_{so}(z)$ (2.5) to have a source of the strength m centered at the origin coordinate $(0, 0)$. Similarly that the function $w_{si}(z)$ (2.6) to have a sink of the strength m centered at the origin coordinate $(0, 0)$.

3. Complex velocity potential and Entropy

3.1. The relationship of the complex velocity potential and Entropy S.

Applying the above, we show below that a special case of complex fluid dynamics can be considered the complex entropy. In other words, we show below that the complex entropy is considered to be the flow of the complex velocity potential. First, we consider the following simple case.

Let z and a be complex numbers. The complex velocity potential $w(z)$ is defined as follows :

$$\begin{aligned} w(z) &= \log(\frac{1}{z}) + \log(\frac{1}{z+a}) \\ &= \log \frac{1}{z(z+a)} \\ &= \log \frac{1}{a}(\frac{1}{z} - \frac{1}{z+a}). \end{aligned} \tag{3.1}$$

If set $a = 1$, then the following formulas are satisfied :

$$w(z) = \log(\frac{1}{z} - \frac{1}{z+1}). \tag{3.2}$$

The first term on the right side of the above formula (3.2) can be regarded as a sink(origin coordinates(0,0)) as follows :

$$\log(\frac{1}{z}) = -\log(z). \tag{3.3}$$

The second term on the right side of the above formula (3.2) can be regarded as a source at coordinate of a point $(-1, 0)$ as follows :

$$-\log(\frac{1}{z+1}) = \log(z+1). \tag{3.4}$$

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In the discussion of the following subsection 3.3, we show the relationship between the compation of sources, sinks and laminar flows in the above equation to the complex entropy $S(z)$ as follows:

$$S(z) = k_{B_C}\{(1+z)\log(1+z) - z\log(z)\}, \tag{3.5}$$

where k_{B_C} is the Boltzmann constant expanded to complex numbers.

Below section, we will expand entropy to complex space and see that entropy is expressed by a complex velocity potential. Furthermore, we will verify that it is an expansion of Boltzmann's principle and Planck's distribution. In other words, we see that by setting $z := x$ on above, the complex entropy $S(z)$ (3.5) is regarded as entropy $S(x)$. Namely, by changing a complex number z to a real number x, the complex entropy $S(z)$ become entropy $S(x)$.

To make the discussion easier to understand, we first explain the Boltzmann principle, entropy, and the Planck distribution function on next subsection 3.2.

3.2. Introduction for Entropy S and the Planck distribution function.

We examine to be apply statistical mechanics concept to natural numbers. To make it easier the understanding, we would first let us introduce the Boltzmann principle and the Planck distribution function as follows.

Definition 6. *We define some symbols using on this article as follows :*

$$\begin{aligned}
P &: The\ number\ of\ particles,\\
N &: The\ number\ of\ resonators,\\
U &: The\ average\ energy\ per\ a\ resonator,\\
U_N &: Total\ energy,\\
\varepsilon &: An\ element\ of\ energy,\\
\nu &: Frequency,\\
T &: Temperature,\\
k_B &: The\ Boltzmann\ constant,\\
h &: The\ Planck\ constant,\\
\beta &: Inverse\ temperature.
\end{aligned} \tag{3.6}$$

□

Using the definitions above, the following equations are satisfied :

$$U_N = NU = P\varepsilon, \tag{3.7}$$

$$\frac{P}{N} = \frac{U}{\varepsilon}, \tag{3.8}$$

$$\beta = \frac{1}{k_B T}, \tag{3.9}$$

where the inequality $P > N$ is satisfied.

The concept of the Boltzmann principle is that the number of particles P is partitioned by the number of resonators N. Namely, we define the number of states $W_{N,P}$ and Entropy (the Boltzmann Principle) S such that the number of particles P is partitioned by the number of partitions $N-1$, where P and N can be regarded positive integer numbers as follows :

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Definition 7. *Let the number of particles P and the number of resonators N be positive integer numbers* $(P, N \in \mathbb{N})$.

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!}, \quad (the\ number\ of\ states), \tag{3.10}$$
$$S_{N,P} = k_B \log W_{N,P}, \quad (the\ Boltzmann\ Principle), \tag{3.11}$$
$$S = \frac{S_{N,P}}{N}, \quad (the\ average\ of\ S_{N,P}). \tag{3.12}$$

□

Using the Stirling's formula, for sufficiently large natural number P and N, the following conditions are satisfied :

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!} \approx \frac{(N+P)^{N+P}}{N^N P^P}. \tag{3.13}$$

Using the Boltzmann principle above, for sufficiently large the number of particles P and resonators N, we can obtain the following equations :

$$\begin{aligned} S_{N,P} &= k_B \log W_{N,P} \\ &= k_B\{(N+P)\log(N+P) - \log N^N - \log P^P\} \\ &= k_B N\{(1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log\frac{P}{N}\}. \end{aligned} \tag{3.14}$$

Using the definition above, the equality(3.8) $P/N = U/\varepsilon$ and (3.12) $S = S_{N,P}/N$, the equality(3.14) is satisfied as follows :

$$S = k_B\Big\{(1+\frac{U}{\varepsilon})\log(1+\frac{U}{\varepsilon}) - \frac{U}{\varepsilon}\log\frac{U}{\varepsilon}\Big\}. \tag{3.15}$$

Differentiate both sides of the equation(3.15) with respect to average energy per resonator U. Hence, the following equation is satisfied :

$$\frac{dS}{dU} = \frac{k_B}{\varepsilon}\Big\{\log(1+\frac{U}{\varepsilon}) - \log\frac{U}{\varepsilon}\Big\}. \tag{3.16}$$

Furthermore, differentiate both sides of the equation(3.16) with respect to U, the following an equation is satisfied :

$$\frac{d^2S}{dU^2} = \frac{-k_B}{U(\varepsilon+U)}. \tag{3.17}$$

The change of entropy dS is the multiplication of the change of a energy dU and the reciprocal of a temperature T. Namely, the following relation between entropy S, energy U and a temperature T are satisfied :

$$\frac{dS}{dU} = \frac{1}{T}. \tag{3.18}$$

Thus, using the equation(3.17) and (3.18), the following relation is satisfied :

$$\frac{d}{dU}\left(\frac{1}{T}\right) = \frac{-k_B}{U(\varepsilon+U)}. \tag{3.19}$$

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Integrating both sides of the equation(3.19) with respect to U, the following relation is satisfied :

$$U = \frac{\varepsilon}{\exp(\frac{\varepsilon}{k_B T}) - 1}. \tag{3.20}$$

Here, put ε as follows :

$$\varepsilon = h\nu. \tag{3.21}$$

Therefore, the following equations is satisfied :

$$U = \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1} = \frac{h\nu}{\exp(h\nu\beta) - 1}, \quad (Planck's\ law). \tag{3.22}$$

The equation above (3.22) is determined the expression for the average energy of particles in a single mode of frequency ν in thermal equilibrium T, that is, called Planck's law. Besides, we define the distribution function $\bar{n}(\nu, \beta)$ as follows:

$$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}, \quad (the\ Planck\ distribution\ function). \tag{3.23}$$

This is expressed the mean particle occupation number in the thermal equilibrium T. This is called the Planck distribution function on this paper. Moreover, the equation(3.23) is transformed as follows :

$$\frac{\bar{n}(\nu, \beta)}{\bar{n}(\nu, \beta) + 1} = \exp(-h\nu\beta), \quad (the\ Boltzmann\ factor). \tag{3.24}$$

The function $\exp(-h\nu\beta)$ is called the Boltzmann factor. Besides, let N_g and N_e be the mean number of atoms in the ground state and in the excited state. the following equation also satisfied :

$$\frac{N_e}{N_g} = \exp(-h\nu\beta). \tag{3.25}$$

Note: In this paper, Planck's Radiation law refers to the following expression :

$$U = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\exp(h\nu\beta) - 1}, \quad (Planck's\ Radiation\ law) \tag{3.26}$$

where the constant c is the speed of light.

Here, we define the function $w(U, \varepsilon)$ as follow :

$$w(U, \varepsilon) = \log\frac{1}{\varepsilon}\left(\frac{1}{U} - \frac{1}{\varepsilon + U}\right). \tag{3.27}$$

Therefore, it satisfied as follows:

$$w(U, \varepsilon) = \log\left(-\frac{1}{k_B}\frac{d^2S}{dU^2}\right), \tag{3.28}$$

$$U = \frac{1}{\exp(\frac{\varepsilon}{k_B}\frac{dS}{dU}) - 1}, \tag{3.29}$$

$$S = -\int\int k_B \exp(w(U, \varepsilon)) + C, \tag{3.30}$$

where C is a constant.

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Therefore, there are the relationship as shown in the following figure :

$$\begin{array}{ccc} & U & \\ \nearrow & \circlearrowleft & \searrow \\ w(U,\varepsilon) & \longleftarrow & S \end{array} \tag{3.31}$$

On the below discussion, we show that the expansion of entropy to complex numbers is related to the complex velocity potential using the above discussion.

3.3. The complex velocity potential and Entropy.

We consider entropy expanded to complex numbers. In the above subsection 3.2, by replacing U/ε in entropy S with complex number z, the complex entropy $S(z)$ is defined as follows:

Definition 8. *The complex entropy $S(z)$.*
Let z be a complex number. We define the complex entropy $S(z)$ expanded to complex numbers as follows:

$$S(z) = k_{B_C}\{(1+z)\log(1+z) - z\log(z)\}, \tag{3.32}$$

where the constant k_{B_C} is the Boltzmann constant expanded to complex number. □

We can see that the equation (3.32) itself is a composition of complex velocity potentials. Namely, we can interpret the equation (3.32) as follows:

Definition 9. *Let x and y be real numbers, i be imaginary number. The complex number z is satisfied $z = x + iy$. It is defined and focus on the following formulas :*

The source w_{so+1} at point $(-1, 0)$:

$$w_{so+1} = \log(1+z). \tag{3.33}$$

The product of a laminar and a source $w_{z,so+1}$ at point $(-1, 0)$:

$$w_{z,so+1} = z\log(1+z). \tag{3.34}$$

The product of a laminar and a sink $w_{z,si}$ at point $(0, 0)$:

$$w_{z,si} = -z\log(z). \tag{3.35}$$

□

Therefore, the complex entropy $S(z)$, that is (3.32), is expressed as a composition of sources sinks and laminar on the above (3.33), (3.34) and (3.35). Namely, it is satisfied as follows :

$$\begin{aligned} S(z) &= k_{B_C}\{(1+z)\log(1+z) - z\log(z)\} \\ &= k_{B_C}(w_{so+1} + w_{z,so+1} + w_{z,si}). \end{aligned} \tag{3.36}$$

By setting $z := x$ on above, the complex entropy $S(z)$ (3.32) is regarded as entropy $S(x)$. Namely, by changing a complex number z to a real number x, Therefore, the complex entropy $S(z)$ expanded on the complex space is regarded as the expansion of entropy $S(x)$ on the real space.

3.4. The complex velocity potential and derivative of entropy.

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Next, we consider the differentiate of the complex entropy $S(z)$, that is (3.32), with respect to a complex number z as follows:

$$S'(z) = k_{B_C}\{\log(1+z) - \log(z)\}, \tag{3.37}$$

$$S''(z) = -k_{B_C}\left(\frac{1}{z} - \frac{1}{1+z}\right), \tag{3.38}$$

where k_{B_C} is the Boltzmann constant of the expanded to complex numbers.

Here, we put the complex velocity potential (3.2) as follows:

$$w_S(z) = w(z) = \log\left(\frac{1}{z} - \frac{1}{z+1}\right). \tag{3.39}$$

Comparing formulas (3.39) and (3.38), we can see the following relationship are satisfied:

$$w_S(z) = \log\left(-\frac{S''(z)}{k_{B_C}}\right), \tag{3.40}$$

$$\exp(w_S(z)) = -\frac{S''(z)}{k_{B_C}}. \tag{3.41}$$

Integrating the above formulas (3.41) gives the following equation :

$$S(z) = -k_{B_C}\int\int \exp(w(z)) + C, \tag{3.42}$$

where the constant C is a constant. Therefore, we can describe as follows:

Theorem 1. *Let z be a complex number. The complex velocity potential $w_S(z)$ and the complex entropy $S(z)$ are defined as follows:*

$$w_S(z) = \log\left(\frac{1}{z}\right) + \log\left(\frac{1}{z+1}\right), \tag{3.43}$$

$$S(z) = k_{B_C}\{(1+z)\log(1+z) - z\log(z)\}. \tag{3.44}$$

In this case, the following equation is satisfied :

$$S(z) = -\int\int k_{B_C}\exp(w_S(z)) + C, \tag{3.45}$$

where $k_{B_C}$and C is constants. Namely, the complex entropy $S(z)$ is expressed by the complex velocity potential $w_S(z)$. □

From the above discussion so far, the complex entropy $S(z)$ expanded to complex numbers can be expressed by the complex velocity potential $w_S(z)$. If we describe like the equation(3.20), then by transforming the equation (3.42) and (3.37) we obtain the equation as follows:

$$z = \frac{1}{\exp\left(\frac{S'(z)}{k_{B_C}}\right) - 1}. \tag{3.46}$$

Namely, a complex number z is expressed by the first derivative of the complex entropy $S'(z)$.

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Corollary 1. *Let z be a complex number, the complex entropy $S(z)$ and the constant k_{B_C}. The following equation is satisfied :*

$$z = \frac{1}{\exp\left(\frac{S'(z)}{k_{B_C}}\right) - 1}. \tag{3.47}$$

A complex number z is expressed by the first derivative of the complex entropy $S'(z)$. □

Namely, a complex number z is expressed by the complex entropy $S(z)$. The complex entropy $S(z)$ is expressed by the complex velocity potential $w_S(z)$. Moreover, for any the complex velocity potential $w(z)$ is expressed by complex numbers z. Therefore, there is the relationship as shown in the following figure :

$$\begin{array}{ccc} & z & \\ \nearrow & \circlearrowleft & \searrow \\ w_S(z) & \longleftarrow & S(z) \end{array} \tag{3.48}$$

Namely, the following corollary is satisfied :

Corollary 2. *Let z be a complex number, the complex entropy $S(z)$ and the constant k_{B_C}. The complex velocity potential $w_S(z)$, $w_S(z)$ is expressed by $S(z)$. Furthermore, according to corollary 1, for any complex velocity potential $w(z)$, $w(z)$ is expressed by $S(z)$.* □

The complex entropy is expressed by the complex velocity potential. Additionally, the complex velocity potential is expressed by the complex entropy. Moreover, transforming the equation(3.37), the following equation is satisfied :

$$\frac{z}{z+1} = \exp\left(-\frac{S'(z)}{k_{B_C}}\right). \tag{3.49}$$

Namely, it is thought that a complex number z corresponds to a energy U, expanded the Boltzmann constant k_{B_C} corresponds to the Boltzmann constant k_B, and the first derivative of expanded the complex entropy $S'(z)$ corresponds to a temperature T as follows :

$$\begin{aligned} \frac{U}{\varepsilon} &\longleftrightarrow z, \\ k_B &\longleftrightarrow k_{B_C}, \\ \frac{\varepsilon}{T} &\longleftrightarrow S'(z). \end{aligned} \tag{3.50}$$

4. The relationship of the π_f-divided complex entropy $S_{\pi_f}(z)$ and Complex Velocity Potential $w_{\pi_f}(z)$.

Next, we consider the following complex velocity potential $w_{\pi_f}(z)$. First, we consider the following the expanded entropy $S_{\pi_f,x}$ and the π_f-divided complex entropy $S_{\pi_f}(x)$.

4.1. The Boltzmann Principle $S_{n,x}$.

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Let n be a positive integer and x be a positive integer. We define the Boltzmann principle $S_{n,x}$ as follows:

$$W_{n,x} = \frac{(n+x-1)!}{(n-1)!x!}, \tag{4.1}$$

$$S_{n,x} = \log W_{n,x}, \tag{4.2}$$

where $n < x$.

The number of states $W_{n,x}$ represents a combination that divides x into n. We examine whether or not we can give a non-linear change (divide) to the method of dividing the part with the number of states $W_{n,x}$. The Boltzmann principle $S_{n,x}$ gives entropy of the number of states $W_{n,x}$.

4.2. The Boltzmann principle by dividing approximation the number of primes $S_{\pi_f,x}$.

We consider the expanded entropy $S_{\pi,x}$, that is, divided the number of states $W_{\pi,x}$ by the number of primes $\pi(x)$ less than or equal to a integer $x > 0$.

$$W_{\pi,x} = \frac{(\pi(x)+x-1)!}{(\pi(x)-1)!x!}, \tag{4.3}$$

$$S_{\pi,x} = \log W_{\pi,x}, \tag{4.4}$$

where $\pi(x) < x$.

However, the number of primes $\pi(x)$ can not be differentiated. Therefore, we consider the combination $W_{\pi_f,x}$ divided by the function $x/\log(x)$ that approximation of the number of primes. Thus, we consider the expanded entropy $S_{\pi_f,x}$ which is its logarithm. Here we set the function $\pi_f(x) = x/\log(x)$. In other words, we consider dividing the number of states $W_{\pi_f,x}$ by the function $\pi_f(x)$.

$$\pi_f(x) = \frac{x}{\log(x)}, \tag{4.5}$$

$$W_{\pi_f,x} = \frac{(\pi_f(x)+x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)}x^x}, \tag{4.6}$$

$$S_{\pi_f,x} = \log W_{\pi_f,x}, \tag{4.7}$$

where $\pi_f(x) < x$.

Note: Since the definition of Combination below formula(4.3) cannot define real values well, therefore, we adopted the definition of formula(4.6) using Stirling's approximation.

We consider in the following subsection 4.3 that entropy can be explained by complex hydrodynamics. (Refer to Fujino [11–13])

4.3. The π_f-divided entropy $S_{\pi_f}(x)$.

We continue the discussion with reference to ideas in subsection 3.2. The number of particles P is replaced to the positive real number x. The number of resonator N is replaced to the number of primes number $\pi(x)$. We consider to divide the positive real number x by approximation the number of prime numbers $\pi(x)$, that is, the function $x/\log(x)$. First, we start with some definitions.

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Definition 10. *Let $x > 1$ be a positive real number $(x \in \mathbb{R})$ and $f(x)$ be a positive real valued function on a positive real number x.*

$$\pi(x) = \sum_{\substack{p \le x \\ p:prime}} 1 \,, \qquad (4.8)$$

$$\text{The number of prime numbers less than or equal to } x.$$

$$\pi_f(x) = \frac{x}{f(x)} \,, \qquad (4.9)$$

$$Q_f(x) = \frac{x}{\pi_f(x)} \,. \qquad (4.10)$$

By the definition above, it is satisfied that the equation $Q_f(x) = f(x)$. □

We define the number of states $W_{\pi_f,x}$. Therefore, the $\pi_{f,x}$-entropy $S_{\pi_f,x}$ under $W_{\pi_f,x}$ is defined by the number of states $W_{\pi_f,x}$. Moreover, the π_f-divided entropy $S_{\pi_f}(x)$ is defined dividing the $\pi_{f,x}$-entropy $S_{\pi_f,x}$ by the function $\pi_f(x)$ as follows :

Definition 11. *The π_f-divided entropy $S_{\pi_f}(x)$ divided by $\pi_f(x)$. Let $x > 1$ be a positive real number $(x \in \mathbb{R})$.*

$$W_{\pi_f,x} = \frac{(\pi_f(x) + x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)} x^x}, \qquad (4.11)$$

$$S_{\pi_f,x} = \log W_{\pi_f,x}, \quad (\pi_{f,x}\text{-entropy}) \qquad (4.12)$$

$$S_{\pi_f}(x) = \frac{S_{\pi_f,x}}{\pi_f(x)}. \quad (\pi_f\text{-divided entropy}) \qquad (4.13)$$

□

In discussion below, unless otherwise specified, let the function $f(x)$ set to the function $\log(x)$. Namely, the following is satisfied :

$$f(x) = \log(x). \qquad (4.14)$$

Therefore, using definitions above, the following conditions are satisfied :

$$Q_f(x) = Q_{\log}(x) = \frac{x}{\pi_{\log}(x)} = \log(x). \qquad (4.15)$$

Therefore, for $x > 0$, the following equations are satisfied :

$$\begin{aligned} S_{\pi_f,x} &= (\pi_f(x) + x)\log(\pi_f(x) + x) - \pi_f(x)\log(\pi_f(x)) - x\log(x) \\ &= \pi_f(x)\Big((1 + \frac{x}{\pi_f(x)})\log(1 + \frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)}\log(\frac{x}{\pi_f(x)})\Big), \end{aligned} \qquad (4.16)$$

$$S_{\pi_f}(x) = \Big(1 + \frac{x}{\pi_f(x)}\Big)\log\Big(1 + \frac{x}{\pi_f(x)}\Big) - \frac{x}{\pi_f(x)}\log\Big(\frac{x}{\pi_f(x)}\Big). \qquad (4.17)$$

Using the function $Q_f(x)$ above, the π_f-divided entropy $S_{\pi_f}(x)$ is expressed as follows :

$$S_{\pi_f}(x) = (1 + Q_f(x))\log(1 + Q_f(x)) - Q_f(x)\log Q_f(x). \qquad (4.18)$$

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Therefore, it is satisfied as follows :

$$S_{\pi_f}(x) = (1+\log(x))\Big(\log(1+\log(x))\Big) - \log(x)\Big(\log\log(x)\Big). \tag{4.19}$$

4.4. The π_f-complex velocity potential $w_{\pi_f}(z)$.

Applying the above discussion, we consider and define the π_f-complex velocity potential $w_{\pi_f}(z)$ by having sources and sinks in complex fluid dynamics as follows:

$$\begin{aligned} w_{\pi_f}(z) &= \log\Big(\frac{1}{z}\Big) + \log\Big(\frac{1}{\log(z)}\Big) + \log\Big(\frac{1}{\log(z)+1}\Big) \\ &= \log(\log(z))') + \log\Big(\frac{1}{\log(z)}\Big) + \log\Big(\frac{1}{\log(z)+1}\Big) \\ &= \log\Big(\frac{(\log(z))'}{\log(z)(\log(z)+1)}\Big) \\ &= \log\Big(\frac{(\log(z))'}{\log(z)} - \frac{(\log(z))'}{\log(z)+1}\Big). \end{aligned} \tag{4.20}$$

This equation can be interpreted as compositions of sink and source as follows:

$$Sink : -\log(z), \tag{4.21}$$

$$Apply\ a\ sink\ to\ a\ source : -\log(\log(z)), \tag{4.22}$$

$$Apply\ a\ sink\ to\ source\ shifted\ by\ (-1,0) : -\log(\log(z)+1). \tag{4.23}$$

4.5. The relationship between the π_f-divided complex entropy $S_{\pi_f}(z)$ and the complex velocity potential $w_{\pi_f}(z)$.

We see below the relationship between the above equation (4.20) and the π_f-divided complex entropy $S_{\pi_f}(z)$. The π_f-divided complex entropy $S_{\pi_f}(z)$ is satisfied as follows:

$$S_{\pi_f}(z) = (1+\log(z))\Big(\log(1+\log(z))\Big) - \log(z)\Big(\log\log(z)\Big), \tag{4.24}$$

$$\begin{aligned} S'_{\pi_f}(z) &= \frac{1}{z}\log\Big(1+\frac{1}{\log(z)}\Big) \\ &= Q'_f(z)\Big(\log(1+Q_f(z)) - \log(Q_f(z))\Big), \end{aligned} \tag{4.25}$$

$$\begin{aligned} S''_{\pi_f}(z) &= k_f(z)\Big(\frac{-(\log(z))'}{\log(z)(1+\log(z))}\Big) \\ &= -k_f(z)\Big(\frac{(\log(z))'}{\log(z)} - \frac{(\log(z))'}{1+\log(z)}\Big), \end{aligned} \tag{4.26}$$

where the function $k_f(z)$ is satisfied as follows :

$$k_f(z) = -S''_{\pi_f}(z)\Big(\frac{\log(z)(1+\log(z))}{(\log(z))'}\Big), \tag{4.27}$$

$$Q_f(z) = f(z) = \log(z). \tag{4.28}$$

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Next, we show that the π_f-divided complex entropy $S_{\pi_f}(z)$ can be expressed using the complex velocity potential $w_{\pi_f}(z)$. It is satisfied as follows :

$$\begin{aligned} w_{\pi_f}(z) &= \log\left(\frac{Q'_f(z)}{Q_f(z)(Q_f(z)+1)}\right) \\ &= \log\left(\frac{Q'_f(z)}{Q_f(z)} - \frac{Q'_f(z)}{Q_f(z)+1}\right). \end{aligned} \tag{4.29}$$

Therefore, using equations (4.26) and (4.29) the following are satisfied :

$$\begin{aligned} \exp(w_{\pi_f}(z)) &= \left(\frac{Q'_f(z)}{Q_f(z)} - \frac{Q'_f(z)}{Q_f(z)+1}\right) \\ &= -\frac{S''_{\pi_f}(z)}{k_f(z)}. \end{aligned} \tag{4.30}$$

Transforming this formula, the following is satisfied :

$$w_{\pi_f}(z) = \log\left(-\frac{S''_{\pi_f}(z)}{k_f(z)}\right). \tag{4.31}$$

In other words, this is an equation expressing the complex velocity potential $w_{\pi_f}(z)$ by the second derivative of entropy. The following are satisfied :

$$\exp(w_{\pi_f}(z)) = -\frac{S''_{\pi_f}(z)}{k_f(z)}, \tag{4.32}$$

$$S''_{\pi_f}(z) = -k_f(z)\exp(w_{\pi_f}(z)). \tag{4.33}$$

Therefore, the π_f-divided complex entropy can also be expressed as follows :

$$S_{\pi_f}(z) = -\int\int k_f(z)\exp(w_{\pi_f}(z)) + C, \tag{4.34}$$

where C is a constant. Therefore, we can describe as follows:

Theorem 2. *Let z be a complex number. For the complex real function $\pi_f(z)$ and $Q_f(z) = log(z)$, the complex velocity potential w_{π_f} and the π_f-divided complex entropy $S_{\pi_f}(z)$ is defined as follows:*

$$w_{\pi_f}(z) = \log Q'_f(z) + \log\left(\frac{1}{Q_f(z)}\right) + \log\left(\frac{1}{Q_f(z)+1}\right), \tag{4.35}$$

$$S_{\pi_f}(z) = (1+Q_f(z))\log(1+Q_f(z)) - Q_f(z)\log Q_f(z). \tag{4.36}$$

In the above case, the following is satisfied :

$$S_{\pi_f}(z) = -\int\int k_f(z)\exp(w_{\pi_f}(z)) + C, \tag{4.37}$$

where C is a constant. □

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From the above discussion so far, the π_f-divided complex entropy $S_{\pi_f}(z)$ to complex numbers can be expressed by a π_f-complex velocity potential $w_{\pi_f}(z)$. If we describe like the equation(3.20), then by transforming the equation(4.25) we obtain the equation as follows:

$$Q_f(z) = \frac{1}{\exp(zS'_{\pi_f}(z)) - 1}. \tag{4.38}$$

Because $Q_f(z) = \log(z)$, the following equation is satisfied :

$$z = \exp\left(\frac{1}{\exp(zS'_{\pi_f}(z)) - 1}\right). \tag{4.39}$$

Namely, a complex number z is expressed by the first derivative of the π_f-divided complex entropy $S'_{\pi_f}(z)$.

Corollary 3. *Let z be a complex number, the π_f-divided complex entropy $S_{\pi_f}(z)$. The following equation is satisfied :*

$$z = \exp\left(\frac{1}{\exp(zS'_{\pi_f}(z)) - 1}\right). \tag{4.40}$$

A complex number z is expressed by the first derivative of entropy $S'_{\pi_f}(z)$. □

A complex number z is expressed by the π_f-complex entropy $S_{\pi_f}(z)$. The π_f-complex entropy $S_{\pi_f}(z)$ is expressed by the complex velocity potential $w_s(z)$. Furthermore, the complex velocity potential $w_{\pi_f}(z)$ is expressed by a complex number z and some function etc. Therefore, there is the relationship as shown in the following figure :

$$\begin{array}{ccc} & z & \\ \nearrow & \circlearrowleft & \searrow \\ w_{\pi_f}(z) & \longleftarrow & S_{\pi_f}(z) \end{array} \tag{4.41}$$

Namely, the following corollary is satisfied :

Corollary 4. *Let z be a complex number and the π_f-complex entropy $S_{\pi_f}(z)$. For any π_f-complex velocity potential $w_{\pi_f}(z)$, the π_f-complex velocity potential $w_{\pi_f}(z)$ is expressed by the π_f-complex entropy $S_{\pi_f}(z)$. Furthermore, according to corollary 3, for any complex velocity potential $w(z)$, $w(z)$ is expressed by $S_{\pi_f}(z)$.* □

The π_f-complex entropy $S_{\pi_f}(z)$ is expressed by the π_f-complex velocity potential $w_{\pi_f}(z)$. Additionally, the π_f-complex velocity potential $w_{\pi_f}(z)$ is expressed by the π_f-complex entropy $S_{\pi_f}(z)$. Moreover, transforming the equation(4.39), the following equation is satisfied :

$$\frac{Q_f(z)}{Q_f(z) + 1} = \exp(-zS'_{\pi_f}(z)). \tag{4.42}$$

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Namely, it is thought that a complex number z corresponds to a energy U, expanded the Boltzmann constant k_{B_C} corresponds to the Boltzmann constant k_B, and the first derivative of the π_f-complex entropy $S'_{\pi_f}(z)$ corresponds to a temperature T as follows :

$$\begin{aligned} \frac{U}{\varepsilon} &\longleftrightarrow Q_f(z), \\ \frac{\varepsilon}{k_B T} &\longleftrightarrow z S'_{\pi_f}(z). \end{aligned} \tag{4.43}$$

5. The generalized Q-complex entropy S_Q and the Q-complex velocity potential w_Q.

Based on these ideas, in order to try to generalize it, we consider a formula in which the complex number z in the formula (3.1) is replaced by the complex real function $Q(z)$, or consider the formula (4.20) with $\log(z)$ replaced by the complex real function $Q(z)$.

Let z be a complex number and $Q(z)$ be the complex function. We define the Q-complex velocity potential $w_Q(z)$ as follows :

$$\begin{aligned} w_Q(z) &= \log Q'(z) + \log\left(\frac{1}{Q(z)}\right) + \log\left(\frac{1}{Q(z)+1}\right) \\ &= \log\left(\frac{Q'(z)}{Q(z)(Q(z)+1)}\right) \\ &= \log\left(\frac{Q'(z)}{Q(z)} - \frac{Q'(z)}{Q(z)+1}\right). \end{aligned} \tag{5.1}$$

This equation can also be interpreted as a combination of suction as follows :

$$\textit{Source} : \log(Q'(z)), \tag{5.2}$$

$$\textit{Sink} : -\log(Q(z)), \tag{5.3}$$

$$\textit{Sink} : -\log(Q(z)+1). \tag{5.4}$$

Here, we consider the generalized Q-complex entropy $S_{Q(z)}$ as follows.

$$S_Q(z) = (1+Q(z))\log(1+Q(z)) - Q(z)\log Q(z). \tag{5.5}$$

$$\begin{aligned} S'_Q(z) &= Q'(z)(\log(1+Q(z)) - \log Q(z)) \\ &= Q'(z)\log\left(1+\frac{1}{Q(z)}\right). \end{aligned} \tag{5.6}$$

$$\begin{aligned} S''_Q(z) &= k_Q(z)\left(\frac{-Q'(z)}{Q(z)(1+Q(z))}\right) \\ &= -k_Q(z)\left(\frac{Q'(z)}{Q(z)} - \frac{Q'(z)}{1+Q(z)}\right). \end{aligned} \tag{5.7}$$

where the complex function $k_Q(z)$ is defined as follows :

$$k_Q(z) = -S''_Q(z)\left(\frac{Q(z)(1+Q(z))}{Q'(z)}\right). \tag{5.8}$$

Source : $\log(1+Q(z))$ multiplied by $1+Q(z)$ is expressed as follows:

$$(1+Q(z))\Big(\log(1+Q(z))\Big). \tag{5.9}$$

Sink : $-\log Q(z)$ multiplied by $Q(z)$ is expressed as follows:

$$-Q(z)\Big(\log Q(z)\Big). \tag{5.10}$$

By compositing the above two equations, we obtain the following the generalized Q-complex entropy $S_Q(z)$:

$$S_Q(z) = (1+Q(z))\Big(\log(1+Q(z))\Big) - Q(z)\Big(\log Q(z)\Big). \tag{5.11}$$

In other words, the generalized Q-complex entropy $S_Q(z)$ can be expressed in terms of complex hydrodynamics :

$$w_Q(z) = \log\left(\frac{Q'(z)}{Q(z)} - \frac{Q'(z)}{Q(z)+1}\right). \tag{5.12}$$

Therefore, using (5.7) and (5.12), it is satisfied as follows :

$$\begin{aligned} \exp(w_Q(z)) &= \left(\frac{Q'(z)}{Q(z)} - \frac{Q'(z)}{Q(z)+1}\right) \\ &= -\frac{S''(z)}{k_Q(z)}. \end{aligned} \tag{5.13}$$

By transforming the above formula, the following formula is satisfied.

$$w_Q(z) = \log\left(-\frac{S''_Q(z)}{k_Q(z)}\right). \tag{5.14}$$

In other words, this is an equation expressing the Q-complex velocity potential $w_Q(z)$ by the second derivative of the generalized Q-complex entropy $S''_Q(z)$. Another way to write, the following are satisfied :

$$\exp(w_Q(z)) = -\frac{S''_Q(z)}{k_Q(z)}, \tag{5.15}$$

$$S''_Q(z) = -k_Q(z)\exp(w_Q(z)). \tag{5.16}$$

In other words, the generalized Q complex entropy $S_Q(z)$ can be expressed as follows :

$$S_Q(z) = -\int\int k_Q(z)\exp(w_Q(z)) + C, \tag{5.17}$$

where C is a constant.

Theorem 3. *Let z be a complex number. For any complex function $Q(z)$, the Q-complex velocity potential w_Q and the generalized Q-complex entropy $S_Q(z)$ is defined as follows:*

$$w_Q(z) = \log Q'(z) + \log\left(\frac{1}{Q(z)}\right) + \log\left(\frac{1}{Q(z)+1}\right), \tag{5.18}$$

$$S_Q(z) = (1+Q(z))\log(1+Q(z)) - Q(z)\log Q(z). \tag{5.19}$$

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In the above case, the following is satisfied :

$$S_Q(z) = -\int\int k_Q(x)\exp(w_Q(z)) + C, \tag{5.20}$$

where C is a constant. □

From the above discussion so far, the generalized Q-complex entropy $S_Q(z)$ to complex numbers can be expressed by a Q-complex velocity potential $w_Q(z)$. If we describe like the equation (3.20), then by transforming equation (5.6) we obtain the equation as follows:

$$Q(z) = \frac{1}{\exp\left(\frac{S'_Q(z)}{Q'(z)}\right) - 1}. \tag{5.21}$$

Suppose the complex function Q exists reverse function Q^{-1}, the following equation is satisfied :

$$z = Q^{-1}\left(\frac{1}{\exp\left(\frac{S'_Q(z)}{Q'(z)}\right) - 1}\right). \tag{5.22}$$

Namely, a complex number z is expressed by the first derivative of the generalized Q-complex entropy $S'_Q(z)$.

Corollary 5. *Let z be a complex number, Q be a complex function and the generalized Q-complex entropy $S_Q(z)$. Suppose the complex function Q exists reverse function Q^{-1} and the first order differentiable, Let z be a complex number, the generalized Q-complex entropy $S_Q(z)$. The following equation is satisfied :*

$$z = Q^{-1}\left(\frac{1}{\exp\left(\frac{S'_Q(z)}{Q'(z)}\right) - 1}\right). \tag{5.23}$$

A complex number z is expressed by the first derivative of the generalized Q-complex entropy $S'_Q(z)$. □

A complex number z is expressed by the generalized Q-complex entropy $S_Q(z)$. The generalized Q-complex entropy $S_Q(z)$ is expressed by the complex velocity potential $w_Q(z)$. Furthermore, the Q-complex velocity potential $w_Q(z)$ is expressed by a complex number z and some function. Therefore, there is the relationship as shown in the following figure :

$$\begin{array}{ccc} & z & \\ \nearrow & \circlearrowleft & \searrow \\ w_Q(z) & \longleftarrow & S_Q(z) \end{array} \tag{5.24}$$

Namely, the following corollary is satisfied :

Corollary 6. *Let z be a complex number, Q be a complex function and $S_Q(z)$ be the generalized Q-complex entropy. Suppose there exists reverse function Q^{-1} of the complex function Q and the complex function Q is the first order differentiable. For any complex velocity potential $w_Q(z)$, $w_Q(z)$ is expressed by $S_Q(z)$. Furthermore, according to corollary 5, for any complex velocity potential $w(z)$, $w(z)$ is expressed by $S_Q(z)$.* □

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The generalized Q-complex entropy $S_Q(z)$ is expressed by the Q-complex velocity potential $w_Q(z)$. Additionally, the Q-complex velocity potential $w_Q(z)$ is expressed by the generalized Q-complex entropy $S_Q(z)$. Moreover, transforming the equation(5.21), the following equation is satisfied :

$$\frac{Q(z)}{Q(z)+1} = \exp\left(-\frac{S'_Q(z)}{Q'(z)}\right). \tag{5.25}$$

Namely, it is thought that a complex number z corresponds to a energy U, expanded the Boltzmann constant k_{B_C} corresponds to the Boltzmann constant k_B, and the generalized Q-complex entropy $S_Q(z)$ corresponds to a temperature T as follows :

$$\begin{aligned} \frac{U}{\varepsilon} &\longleftrightarrow Q(z), \\ \frac{\varepsilon}{k_B T} &\longleftrightarrow \frac{S'_Q(z)}{Q'(z)}. \end{aligned} \tag{5.26}$$

6. Conclusion

So far, we have considered the complex entropy $S(z)$ and the π_f-divided complex entropy $S_{\pi_f}(z)$, which is Boltzmann's principle, that is, entropy expanded to the complex space. We showed that the complex entropy $S(z)$ and the π_f-divided complex entropy $S_{\pi_f}(z)$ can be expressed by the complex velocity potential $w_S(z)$ and $w_{\pi_f}(z)$, respectively. Moreover, we showed that a complex number z is expressed by the complex entropy $S(z)$ or the π_f-divided complex entropy $S_{\pi_f}(z)$. Furthermore, We showed that the complex velocity potential $w_S(z)$ and $w_{\pi_f}(z)$ can be expressed by the complex entropy $S(z)$ and the π_f-divided complex entropy $S_{\pi_f}(z)$, respectively.

Same as thinking in this way, the generalized Q-complex entropy $S_Q(z)$ can be thought of as a special case of the complex hydrodynamics $w_Q(z)$. Therefore, the complex entropy is one flow in complex fluid dynamics. Furthermore, by choosing various partitioning methods, it is possible that many changes can be expressed as entropy depending on the partitioning method Q. Moreover, we showed that a complex number z is expressed by the generalized Q-complex entropy $S_Q(z)$ and the complex function $Q(z)$. Thus, we showed that the Q-complex velocity potential $w_Q(z)$ is expressed by the generalized Q-complex entropy $S_Q(z)$.

As a consequence, the complex velocity potential can be thought of as the complex entropy. On the contrary, the complex entropy can be thought of as the complex velocity potential. It is believed that entropy can be considered as a special case of the complex velocity potential in fluid mechanics. It is thought that entropy not only increases, but also creates flows such that sinks and sources. (Refer to Fujino [13])

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