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Metallic and Insulating Phases in Two-Dimensional Strongly-Coupled Field Theories With a Gravitational Dual

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Abstract: We consider a specific holographic model in the limit where the gauge field does not couple to the rest of of the holographic fields (gravity and scalar sector) and investigate a phase of matter at zero charge density, a realistic feature that may have implications for disordered strange metals. We then pick a specific form of the gauge coupling $Z(\alpha)$ with a certain disorder realization and argue that this provides a hard-gapped insulator with exponentially-suppressed conductivity by holographic methods. The limit is non-trivial: as there is backreaction at zero density amount, surviving in the coupled case.

1 Introduction

Our investigation has entailed a thorough examination of the electrical transport phenomena in strongly coupled holographic quantum field theories, specifically under the condition of zero charge density. By virtue of constructing explicit instances of perfect metals amidst substantial disorder, our findings bear significant ramifications for the practicability of more verisimilar models of highly disordered strange metals.

2 Conductivity

Let us consider a static, asymptotically anti-de Sitter space with a black hole horizon, sourced entirely by uncharged bulk matter, including a dynamical metric. Without loss of generality, we use diffeomorphism invariance to choose the bulk metric

$$\mathrm{d}s^2 = L^2\left[P\mathrm{d}r^2 - Q\mathrm{d}t^2 + G_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]. \tag{1}$$

i, j indices represent the spatial boundary directions, while M, N represent all dimensions, and L is the AdS radius. All functions in the metric are functions of r and $\mathbf{x}$. We further choose the bulk coordinate $0 < r < \infty$, with $r = 0$ the black hole horizon, and $r = \infty$ the AdS boundary. We do not need knowledge of what uncharged matter is required to set up this geometry, but do assume that all energy conditions are obeyed.

We add a U(1) gauge field to the bulk, so the action of our theory is

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\mathcal{L}_{\text{uncharged}} - \frac{Z}{4}F^2\right). \tag{2}$$

The function Z is a parameter of the (uncharged) scalar matter, but for our purposes now it is an arbitrary function of r and $\mathbf{x}$. The two-point functions of this gauge field correspond to calculations of current-current correlation functions in the boundary theory, such as the direct current electrical conductivity matrix σ^{ij}. The conductivity may be related, via the membrane paradigm [1], to data on the horizon of the black hole alone. In particular, the expected value of the boundary current is given by

$$J^i = \sigma^{ij}E_j = \mathbb{E}\left[Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right], \tag{3}$$

where E_j is the applied electric field, $\mathbb{E}[\cdots]$ denotes a uniform spatial average, $\gamma_{ij} = G_{ij}(r = 0)$ is the induced metric on the horizon, and α is the unique function which obeys the equation

$$0 = \partial_i\left(Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right) \tag{4}$$

with appropriate boundary conditions (for example, periodicity in compact boundary spatial directions). A proof is given in Appendix A. The membrane paradigm was used in holographic systems in [2], and similar computations appear in [3, 4, 5] for black holes with translational symmetry broken only in one direction. These results are special cases of this general formula. This formula may break down if the black hole horizon fragments and becomes disconnected, as was considered in [6, 7].

We can interpret (4) as a hydrostatic equation enforcing local charge conservation in an emergent horizon fluid. This is subtle – the local "electric current" in (4) is *not the same* as the expected value of the local current in the dual theory; only their spatial averages are equal. A powerful set of techniques has been recently developed to understand the qualitative behavior of transport in such fluids [8]; for example, it immediately follows from (4) that $\sigma_{ij} = \sigma_{ji}$.

In Appendix B we derive remarkable results for theories in $d = 2$. In particular, if $\sigma[Z;\gamma_{ij}]$ is the conductivity matrix with given Z and γ_{ij}:

$$\det\left(\sigma[Z;\gamma_{ij}]\right)\det\left(\sigma\left[\frac{1}{Z};\gamma_{ij}\right]\right) = \frac{1}{e^8}. \tag{5}$$

If we set $Z = 1$, (5) gives

$$\det(\sigma) = \frac{1}{e^4}. \tag{6}$$

If we expect that on average for a disordered sample, the conductivity matrix is isotropic ($\sigma^{ij} = \sigma\delta^{ij}$), that fixes the conductivity to be $\sigma = 1/e^2$, exactly the clean result!

A simple way to understand this result is as follows: suppose that in local coordinates, the metric is given by

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \approx a_x^2\mathrm{d}x^2 + a_y^2\mathrm{d}y^2. \tag{7}$$

Then we expect "locally" $\sigma_{xx} \sim a_y/a_x$ and $\sigma_{yy} \sim a_x/a_y$ [9]. On average a_y and a_x should have identical distributions, and so we expect that, crudely speaking, σ_{xx} and $1/\sigma_{xx}$ have the same distributions. This implies $\sigma = 1/e^2$; analogous statements are known for random resistor lattices in $d = 2$ with analogous (e.g., log-normal) resistance distributions [?]. And more generally, if $\log Z$ is symmetrically distributed about 0, then in an isotropic theory, $\sigma = 1/e^2$ follows from (5) in the thermodynamic limit.

The robustness of σ in these strongly disordered $d = 2$ models is remarkable, and deserves further comments. In models where momentum dissipation is introduced through massive gravity [10] or "Q-lattice" axions [11], one finds the hydrodynamic result [12]

$$\sigma = \sigma_{\mathrm{Q}} + \frac{\mathcal{Q}^2\tau}{\epsilon + P}, \tag{8}$$

where $\mathcal{Q}$ is the charge density, ϵ the energy density, P the pressure, σ_{Q} the dissipative "quantum critical" conductivity without disorder, and τ a "momentum relaxation time", inversely related to the graviton mass. Before now, it was unclear whether the fact that (8) holds beyond the hydrodynamic limit was an unrealistic feature of massive gravity or similar theories. Our work confirms this is a sensible prediction of massive gravity for many systems at $\mathcal{Q} = 0$. (8) further implies another mechanism, $\tau \to 0$, by which the conductivity can reach its lower bound, σ_{Q}. The conductivity saturating this lower bound, at least qualitatively, is likely to occur at strong disorder [8]. Confirmation that strongly-disordered charged holographic models (with $Z = 1$) have a conductivity no smaller than $1/e^2$ in $d = 2$ would be a further non-trivial test of predictions of simple mean-field physics.

In $d \neq 2$, and/or if Z is distributed more generically, it is valuable to employ insight gained from the equivalence between Markov chains on lattices and the resistance of a resistor lattice [13]. For arbitrary Z, this analogy can be leveraged to find lower and upper bounds to σ, for a self-averaging disordered sample: [8]

$$\frac{L^{d-2}}{e^2}\mathbb{E}\left[\frac{\gamma_{ii}}{dZ\sqrt{\gamma}}\right]^{-1} \le \sigma \le \frac{L^{d-2}}{e^2}\frac{\mathbb{E}[Z\sqrt{\gamma}\gamma^{ii}]}{d}. \tag{9}$$

It is straightforward to test these results and bounds by numerically solving (4) for various disorder realizations. Good agreement with our exact analytic results and consistency with our bounds is obtained.

3 Conductor-Insulator Transition

(9) constrains σ to deviate from the clean result by the strength of fluctuations in Z and γ_{ij}. It is quite evident from (9) that if γ_{ij} and Z are finite at all points on the horizon, then the black hole necessarily

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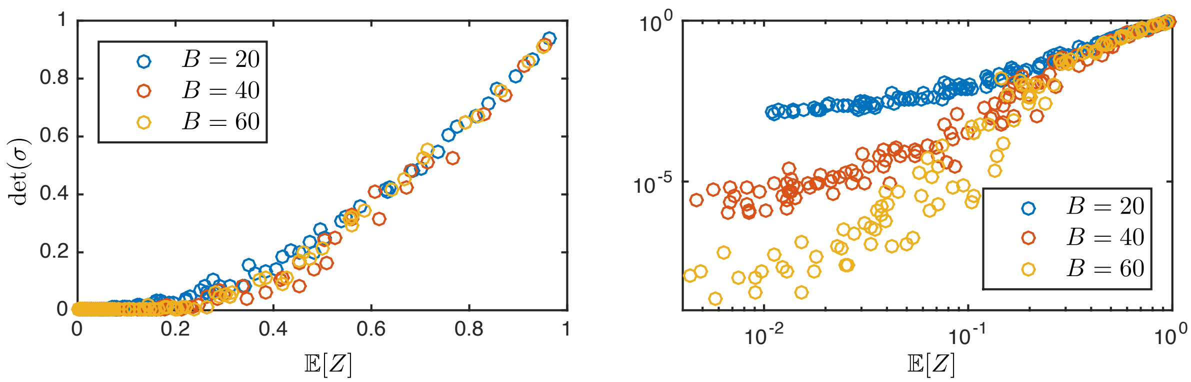

Figure 1: $\det(\sigma)$ from a black hole horizon for a theory in $d=2$; we set $e=1$, and use periodic boundary conditions with $|x|,|y|\le\pi$, with a discretized spatial grid of 701^2 points. We take $\gamma_{ij}=\delta_{ij}$ and $Z=\exp[-B\mathcal{Z}/(1+2\mathcal{Z})]$, where $\mathcal{Z}=\sum_{j=1}^{N}\exp(-(\sin^2(\phi_{jx}+x/2)+\sin^2(\phi_{jy}+y/2))/2\xi^2)$, with ϕ_{jx} and ϕ_{jy} independent random phases, and $B>0$ is a random constant. We took various values of B and fixed $\xi=20\pi/701$. When $\mathbb{E}[Z]\gtrsim 0.28\equiv Z^*$, curves at different B approximately collapse, implying that current avoids the non-conducting bubbles; when $\mathbb{E}[Z]\lesssim Z^*$, the value of conductivity is sensitive to B. In the limit $B\to\infty$ and $\xi\to 0$, a metal-insulator transition appears at Z^*.

conducts electrical current, no matter how strong the disorder. This is a remarkable result. In contrast, in non-interacting quantum field theory, a conductor-insulator transition occurs at a finite disorder strength [14] in $d>2$, and at arbitrarily small disorder in $d\le 2$ [15]. This transition relates to the destructive interference of matter waves scattering off of the disorder. Apparently, bulk fluctuations of the gauge field in holographic theories do not suffer from such interference. While it is known [16, 17] that metal-insulator transitions occur at a finite disorder strength in an interacting quantum system, even such systems ultimately succumb to (many-body) localization at strong disorder. Perhaps holographic models have taken the "coupling $\to\infty$" limit first, rendering such a transition impossible.

Realizing a holographic conductor-insulator transition takes more care. A "helical lattice" approach has generated such a transition in [18, 19], but there is no satisfying physical interpretation. However, even in these papers, the conductivity in the insulating phase only decays as algebraically in T as $T\to 0$, in contrast to canonical insulators.

Henceforth, we focus on the case $d=2$, though our discussion readily generalizes. We will also assume a probe limit where the geometry is described by AdS-Schwarzchild, so $\gamma_{ij}\sim\delta_{ij}$, though again, this probably captures the appropriate qualitative physics in many models. (9) implies that to obtain an insulator with vanishing conductivity, we need $\mathbb{E}[1/Z]$ to be parametrically large. A substantial fraction of the horizon must have $Z\to 0$, as in these regions charge cannot effectively be transported. In fact, the $Z\to 0$ "bubbles" must percolate across the black hole horizon – this is because otherwise, electrical current can simply flow around these bubbles. When regions of space where Z is finite become disconnected from each other, charge transport is no longer possible. The classical percolation transition of these bubbles is a disorder-driven holographic metal-insulator transition. "Metal-insulator" transitions, similarly driven by the percolation of regions where charge cannot propagate, in a simple random resistor lattice are well-known [20].

A simple test of this proposal is to simply write down an ansatz for Z where "bubbles" where $Z\to 0$ percolate across the horizon, and to numerically compute the conductivity. Our numerics support this picture: see Figures 1 and 2.

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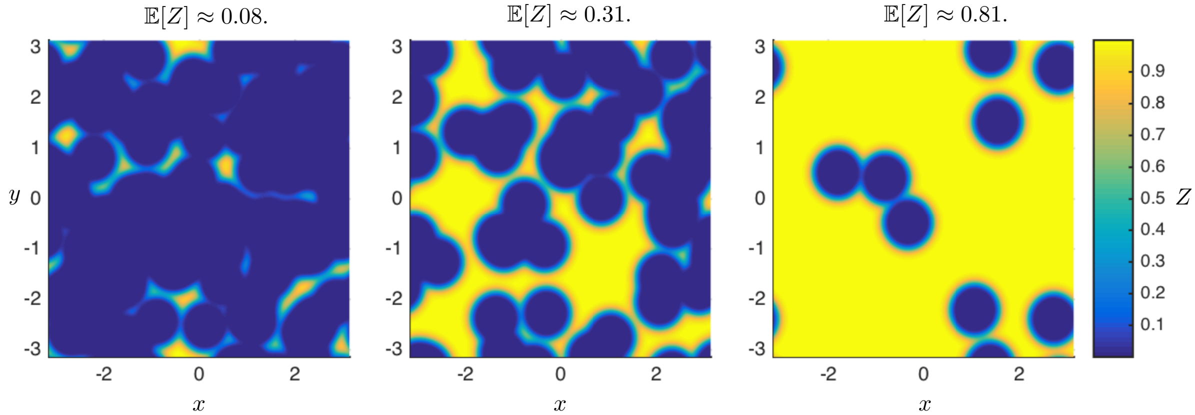

Figure 2: Surface plots of $Z(x, y)$ for various bubble densities. Depending on whether regions of high or low Z percolate across the horizon determines whether we are in the metallic or insulating phase, as is clear upon comparing with Figure 1.

3.1 Holographic Realizations

We now ask whether percolation mechanism proposed above for a disorder-driven metal-insulator transition can occur in a "realistic" holographic model: a bottom-up Einstein-Maxwell-dilaton (Φ)-axion (α) theory with action

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} - \mathcal{M}^d\left[\frac{1}{2}(\partial\Phi)^2 + \frac{\mathrm{e}^{-\Phi}}{2}(\partial\alpha)^2 - \frac{V(\alpha)+U(\Phi)}{L^2}\right] - \frac{Z(\alpha)}{4e^2}F^2\right). \tag{10}$$

Here $\mathcal{M}$ is a mass scale, whose precise value is unimportant – we choose it so that Φ is strictly dimensionless, for simplicity, and

$$\Lambda = -\frac{d(d+1)}{2L^2}. \tag{11}$$

We are in a probe limit, so $G \to 0$. It is possible to generalize the choices above and obtain similar results, but (10) contains the essential ingredients. The scalar kinetic terms are that of the axio-dilaton which naturally appears in type IIB supergravity compactifications [?]. Cosine potentials for $Z(\alpha)$ – which may be suitable for our purposes – can arise in an effective action due to instanton effects, as in quantum chromodynamics [?], though we emphasize that in our holographic model, $Z(\alpha)$ is not suppressed by G/L^d (the scale of quantum corrections in the bulk).

The most important ingredients for obtaining our conductor-insulator transitions are that $V(\alpha)$ has (at least) two minima, α_{c} and α_{i}, with $Z(\alpha_{\mathrm{c}}) > 0$ and $Z(\alpha_{\mathrm{i}}) = 0$. α will drive the conductor-insulator transition; Φ is necessary to stabilize this transition down to low temperatures, though theories with finite Lifshitz or hyperscaling-violating exponents [21] may also prove satisfactory for this purpose. The heuristic idea is that is that if "bubbles" of $\alpha = \alpha_{\mathrm{i}}$ percolate across the horizon, then we obtain an insulator; if not, then we obtain a conductor. So our goal is to show how such bubbles can form and persist to low temperatures. For this purpose, a simple choice of potentials, though certainly not the only one, is

$$U(\Phi) = \frac{7\lambda^2}{2} - 3\lambda\Phi - 4\lambda^2\mathrm{e}^{-\Phi/\lambda} + \frac{\lambda^2}{2}\mathrm{e}^{-2\Phi/\lambda}, \tag{12a}$$

$$V(\alpha) = -\alpha^2 + \frac{\alpha^4}{2\alpha_0^2}, \tag{12b}$$

$$Z(\alpha) = \left(1 - \frac{\alpha}{\alpha_0}\right)^2. \tag{12c}$$

Here $\lambda > 2$ is a dimensionless constant. Note that $\alpha_{\mathrm{i}} = \alpha_0$, $\alpha_{\mathrm{c}} = -\alpha_0$. A similar choice of Z was used to obtain a holographic insulator in [22]. We take $\alpha_0 \to 0$, so that the axion backreaction on the dilaton is negligible. For details on this choice, and on the calculations that follow, see Appendix C. In some sense, this choice of $V(\alpha)$ is less than ideal – the dual operator is marginal according to the Harris criterion [21], which means that we cannot source disordered modes of all wavelengths without the geometry backreacting in the UV.

Let us begin by sourcing the dilaton with (positive) δ-like sources on the AdS boundary – analogous to point-like impurities in the dual theory. Each impurity produces an expanding bubble which becomes insulating; the width of the "bubbles" of α is $\sim 1/T$. If the density of the impurities is n, then the bubbles percolate across the horizon when $T \lesssim \sqrt{n}$. Within each bubble, $\alpha \to \alpha_0$, and thus at low temperatures we obtain an insulator – see Figure **??**.

A second mechanism for obtaining the transition is as follows: suppose that α.

As $T \to 0$ in the insulating phase, we predict:

$$\sigma(T) \sim \exp\left[-\frac{8}{\lambda}\left(\frac{\zeta}{T}\right)^{\lambda/2}\right] \tag{13}$$

4 Outlook

We have studied electrical transport in strongly coupled holographic quantum field theories at zero charge density. In particular, we reduced the computation of σ^{ij} to solving linear differential equations on the black hole horizon. We found analytic bounds on the resulting conductivity matrix, and proposed a disorder-driven holographic metal-insulator transition. There are recent models [23, 24, 25, 26] of (quasi-2d) strange metals where momentum is not conserved past microscopic time scales. We have explicitly constructed examples of perfect conductors in the presence of strong disorder, and we predict that deforming such models by a finite charge density does not decrease the conductivity. We encourage the extension of our holographic approach to charged black holes, and searching for non-holographic field theories where σ_{Q} is immune to disorder.

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A Membrane Paradigm

In this appendix we derive (3) and (4). To compute the conductivity, we must solve a linear response problem for the gauge field, in the black hole background (1). As the (uncharged) matter and gravity sectors will only be sourced at second order in the gauge field, no matter or metric perturbations will be sourced. We need only solve the bulk Maxwell's equations:

$$\nabla_M \left(ZF^{MN}\right) = \frac{1}{\sqrt{-g}}\partial_M \left(Z\sqrt{-g}F^{MN}\right) = 0. \tag{14}$$

Without loss of generality, the asymptotics on the metric ansatz (1) are:

$$P(\mathbf{x}, r \to 0) = \frac{S(\mathbf{x})}{4\pi T r} + \cdots, \tag{15a}$$
$$Q(\mathbf{x}, r \to 0) = -4\pi T r S(\mathbf{x}) + \cdots, \tag{15b}$$
$$G_{ij}(\mathbf{x}, r \to 0) = \gamma_{ij}(\mathbf{x}) + \cdots, \tag{15c}$$
$$P(\mathbf{x}, r \to \infty) = \frac{1}{r^2} + \cdots, \tag{15d}$$
$$Q(\mathbf{x}, r \to \infty) = r^2 + \cdots \tag{15e}$$
$$G_{ij}(\mathbf{x}, r \to \infty) = \delta_{ij} r^2 + \cdots. \tag{15f}$$

Here T is the Hawking temperature of the black hole, and the temperature of the dual field theory. The boundary conditions we impose on A are analogous to [4]. We write

$$A = a(\mathbf{x}, r) - E_j t \mathrm{d}x^j, \tag{16}$$

and the boundary conditions on the form a are:

$$a_r(\mathbf{x}, r \to \infty) = a_t(\mathbf{x}, r \to \infty) = 0, \tag{17a}$$
$$a_i(\mathbf{x}, r \to \infty) = -E_i t, \tag{17b}$$
$$4\pi T r a_r(\mathbf{x}, r \to 0) = a_t(\mathbf{x}, r \to 0) = \alpha(\mathbf{x}), \tag{17c}$$
$$a_i(\mathbf{x}, r \to 0) = -\frac{E_i}{4\pi T} \log(4\pi T r) + \text{finite}. \tag{17d}$$

The function $\alpha(\mathbf{x})$ is undetermined, and so (17c) demonstrates the proper asymptotics. These boundary conditions serve to produce a constant electric field $\mathbf{E}$ in the boundary theory, and are in-falling at the horizon. The direct conductivity matrix is found by the standard holographic dictionary [27], by computing

$$\sigma^{ij} \mathbb{E}[\partial_t A^j (r \to \infty)] = -\sigma^{ij} E_j = \frac{L^{d-2}}{e^2} \mathbb{E}[r^d \partial_r A^i (r \to \infty)]. \tag{18}$$

We also assume that $Z(r = 0) = 1$, which is generically the case, as in the models described in the main text.

The holographic membrane paradigm states that σ^{ij} can be computed by finding a quantity that is independent of bulk radius r, which equals the conductivity as $r \to \infty$. One then evaluates this quantity at the horizon and is able to use the boundary conditions to uniquely fix σ^{ij} in terms of horizon data – for us, the metric at $r = 0$, and the dilaton coupling. It is easy to find such a quantity: plugging the ansatz (16) into (14) we obtain the equations (r and i components respectively):

$$\partial_i \left(Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right) = 0, \tag{19a}$$
$$\partial_r \left(Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right) + \partial_k \left(Z\sqrt{-g} g^{kl} g^{ij} (\partial_l a_j - \partial_j a_l) \right) = 0. \tag{19b}$$

Using the second of these equations we see that

$$J^i \equiv \frac{1}{e^2} \mathbb{E} \left[-Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right]. \tag{20}$$

is independent of r. And as $r \to \infty$, J^i simplifies to

$$J^i = \frac{L^{d-2}}{e^2} \mathbb{E} \left[-r^d \partial_r a_j (r \to \infty) \right], \tag{21}$$

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and so we recognize this as the expected value of the spatially averaged current operator in the field theory.

When evaluating J^i in the limit $r \to 0$, the only nonvanishing terms are

$$J^i = \frac{L^{d-2}}{e^2}\mathbb{E}\left[-ZS\sqrt{\gamma}\frac{4\pi Tr}{S}\gamma^{ij}\left(-\frac{E_j}{4\pi Tr} - \frac{\partial_j \alpha}{4\pi Tr}\right)\right] = \frac{L^{d-2}}{e^2}\mathbb{E}\left[Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right]. \tag{22}$$

Thus, we have recovered (3). (4) follows straightforwardly from (19a):

$$0 = \partial_i\left(ZS\sqrt{\gamma}\frac{4\pi Tr}{S}\gamma^{ij}\left(-\frac{E_j}{4\pi Tr} - \frac{\partial_j\alpha}{4\pi Tr}\right)\right) = -\frac{1}{\sqrt{\gamma}}\partial_i\left(Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right). \tag{23}$$

If we compactify our spatial directions, then up to a constant shift, there is a unique solution to this equation – on the infinite plane, we expect there is only one solution (up to a constant) which is well-behaved and finite. Physically this constant is simply a gauge redundancy.

The above derivation is valid for any black hole, with any translational symmetry breaking, regardless of how strong. There is, however, one exception which we note: if the horizon is disconnected, then it may be impossible to perform our membrane paradigm inspired calculation without pushing J^i through an event horizon of a "floating black hole" – see Figure **??**. Scenarios where this may occur are described in [6, 7].

It is also worth noting that

$$\mathcal{J}^i(\mathbf{x}) = \frac{L^{d-2}}{e^2}Z\sqrt{\gamma}\gamma^{ij}(E_j + \partial_j\alpha) \tag{24}$$

is also conserved: $\partial_i\mathcal{J}^i = 0$. Though $\mathcal{J}^i \neq J^i$, $\mathbb{E}[\mathcal{J}^i] = \mathbb{E}[J^i]$. As discussed in [8], this implies that one can think of an emergent fluid living on the black hole horizon, and so (23) can be thought of as the hydrostatic response of this horizon fluid.

B Two Dimensions

In this appendix we analyze the consequences of (22) and (23) in $d = 2$. For simplicity we set $e = 1$, which removes some clutter. The key observation is as follows: there must exist a differential form "$\partial_j\mathit{\Omega}$" such that

$$Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right) = \epsilon^{ij}\partial_j\mathit{\Omega} \tag{25}$$

where $\epsilon^{xy} = -\epsilon^{yx} = 1$ is the Levi-Civita "tensor" without any metric pre-factors.

$$\partial_i\mathit{\Omega} = \psi_i + \partial_i\mathit{\Psi}(\mathbf{x}), \tag{26}$$

with $\mathit{\Psi}$ a single-valued function and ψ_i a set of two constants, the elements of the non-trivial cohomology group of the torus.

Some simple manipulations, along with the fact that

$$\sqrt{\gamma}\gamma^{ij} = -\epsilon^{ik}\frac{\gamma_{kl}}{\sqrt{\gamma}}\epsilon^{lj}, \tag{27}$$

lead to the following "dual" equation:

$$-\epsilon^{ij}(E_j + \partial_j\alpha) = \frac{1}{Z}\sqrt{\gamma}\gamma^{ij}\left(\psi_j + \partial_j\mathit{\Psi}\right). \tag{28}$$

We immediately obtain

$$0 = \frac{1}{\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}}{Z}\gamma^{ij}\left(\psi_j + \partial_j\mathit{\Psi}\right)\right). \tag{29}$$

This equation is identical to (23), up to the factor $Z(\mathbf{x}) \to 1/Z(\mathbf{x})$. Since these equations are all linear in E_j, we may also write

$$\alpha \equiv \alpha^j E_j, \tag{30a}$$
$$\psi_i \equiv \psi_i^j E_j, \tag{30b}$$
$$\Psi \equiv \Psi^i \psi_i^j E_j. \tag{30c}$$

and solve equations for α^j, Ψ^j and ψ_i^j. It follows from (22) that

$$\epsilon^{ik}\psi_k^j = \sigma^{ij}[Z, \gamma_{ij}]. \tag{31}$$

Plugging (30) into (25), we obtain

$$Z\sqrt{\gamma}\gamma^{ij}\left(\delta_j^k + \partial_j\alpha^k\right) = \epsilon^{ij}\left(\psi_j^k + \psi_l^k\partial_j\Psi^l\right) = \epsilon^{ij}\left(\delta_j^l + \partial_j\Psi^l\right)\psi_l^k \tag{32}$$

But we can also write

$$\frac{1}{Z}\sqrt{\gamma}\gamma^{ij}\left(\delta_j^k + \partial_j\Psi^k\right) = -\epsilon^{ij}\left(\delta_j^m + \partial_j\alpha^m\right)\left(\psi^{-1}\right)_m^k = -\epsilon^{ij}\left(\delta_j^m + \partial_j\alpha^m\right)\left(\sigma^{-1}\right)^{mn}\epsilon^{nk}. \tag{33}$$

The constant term on the right hand side of (33) is the conductivity

$$\sigma^{ik}\left[\frac{1}{Z};\gamma\right] = -\epsilon^{ij}\left(\sigma[Z,\gamma]^{-1}\right)^{jn}\epsilon^{nk} = \frac{\sigma^{ik}[Z;\gamma]}{\det(\sigma[Z;\gamma])}, \tag{34}$$

from which (5) follows, when $Z = 1$.

Furthermore, both E_j and ψ_j are constant sources, and α and Ψ are single-valued functions. Uniqueness theorems for Poisson's equation [28] thus guarantee that

$$\Psi^i = \psi_j^i\alpha^j. \tag{35}$$

We thus obtain an overdetermined set of equations for α^i:

$$\begin{pmatrix} \sqrt{\gamma}\gamma^{xx} & 0 & \sqrt{\gamma}\gamma^{xy} - \sigma_{\mathrm{dc}}^{yx} & \sigma_{\mathrm{dc}}^{xx} \\ 0 & \sqrt{\gamma}\gamma^{xx} & -\sigma_{\mathrm{dc}}^{yy} & \sqrt{\gamma}\gamma^{xy} + \sigma_{\mathrm{dc}}^{xy} \\ \sqrt{\gamma}\gamma^{xy} + \sigma_{\mathrm{dc}}^{yx} & -\sigma_{\mathrm{dc}}^{xx} & \sqrt{\gamma}\gamma^{yy} & 0 \\ \sigma_{\mathrm{dc}}^{yy} & \sqrt{\gamma}\gamma^{xy} - \sigma_{\mathrm{dc}}^{xy} & 0 & \sqrt{\gamma}\gamma^{yy} \end{pmatrix}\begin{pmatrix} \partial_x\alpha^x \\ \partial_x\alpha^y \\ \partial_y\alpha^x \\ \partial_y\alpha^y \end{pmatrix} = \begin{pmatrix} -\sigma_{\mathrm{dc}}^{xx} - \sqrt{\gamma}\gamma^{xx} \\ -\sigma_{\mathrm{dc}}^{xy} - \sqrt{\gamma}\gamma^{xy} \\ -\sigma_{\mathrm{dc}}^{yx} - \sqrt{\gamma}\gamma^{yx} \\ -\sigma_{\mathrm{dc}}^{yy} - \sqrt{\gamma}\gamma^{yy} \end{pmatrix}. \tag{36}$$

A particular solution to this linear set of equations is, in general, $\partial_i\alpha^j = -\delta_i^j$. However, we know that the function α^i must be single-valued. Evidently, the matrix in the above equation must have a non-trivial null space. Thus the determinant of this matrix must vanish everywhere. This happens when

$$\sigma_{\mathrm{dc}}^{xx}\sigma_{\mathrm{dc}}^{yy} - \sigma_{\mathrm{dc}}^{yx}\sigma_{\mathrm{dc}}^{xy} = 1. \tag{37}$$

C Axion-Dilaton Theories

In this appendix, we discuss details of analytic calculations involving the axio-dilaton theory described in the main text.

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C.1 Background Dilaton and Axion Profiles

Let us begin by briefly discussing the choice of $U(\Phi)$. For reasons which will be clear shortly, it is sensible to request the dilaton profile

$$\Phi(z) = \lambda \log\left(1+\zeta z\right), \tag{38}$$

where ζ has the dimensions of temperature T, and is related to the UV boundary conditions on Φ. In the absence of axion disorder, it is straightforward to check that (38) is a solution to the dilaton equation of motion with dilaton potential (12a), at $T=0$. Any potential U which leads to qualitatively similar Φ should be just as good for our purposes.

Let us argue that domain walls are stabilized at low temperatures, so long as $\lambda > 2$. Here, low T means $T \ll \zeta$, so that $\Phi(z) \approx \lambda \log(\zeta z)$. As in [21], the approximate profile of α on the horizon can be found qualitatively by solving the equations of motion in AdS ($T=0$), and then reading off $\alpha(z = 3/4\pi T)$. Recall that we work in a limit where Φ is given by (38), independent of α.

To argue that domain wall solutions exist, let us consider the case where the thickness of the domain wall is much smaller than the size of the overall "bubble" in α. Let the domain wall separate a region of $\alpha \approx -\alpha_0$ for $x<0$, to a region of $\alpha \approx \alpha_0$ for $x>0$. Make the "similarity" ansatz

$$\alpha(z) \approx A\left(xz^b\right), \tag{39}$$

where we wish that $b>0$ be a fixed, positive coefficient, and $A(\pm\infty) = \pm\alpha_0$. We wish to determine under what conditions this is an asymptotically good solution to the axion equations of motion, which (on this ansatz) read:

$$V'(f(X)) = \left[z^{2+b-\lambda}A''(X) + X^2 z^{-\lambda}A'' - (d+\lambda+1-b)Xz^{-\lambda}A'(X)\right]\zeta^{-\lambda} \tag{40}$$

where we have defined $X \equiv xz^b$. If we take

$$b = \frac{\lambda}{2} - 1, \tag{41}$$

then in the limit $z\to\infty$, $A(X)$ solves an ordinary differential equation independent of z, so indeed our similarity ansatz is consistent. By interpreting the simple second order equation for A as analogous to Newton's Law, we obtain that

$$\frac{A'(X)^2}{2\zeta^\lambda} - V(A(X)) = \frac{\alpha_0^2}{2}. \tag{42}$$

The constant on the right hand side is the maximum of $-V$, and is fixed by the requirement that $A\to\pm\alpha_0$ as $X\to\pm\infty$. For very small $X\to-\infty$, we analytically determine

$$A(X) \approx -\alpha_0 + \alpha_1 \mathrm{e}^{2\zeta^{\lambda/2}X} \tag{43}$$

with $\alpha_1 > 0$ an undetermined constant; a similar argument holds for $X\to\infty$. We conclude that α approaches $\pm\alpha_0$ exponentially fast, away from the domain walls, which have a thickness of $\zeta^{-\lambda/2}z^{1-\lambda/2}$ as $z\to\infty$.

On the horizon, we find that $z\sim 1/T$, so we conclude that at low T, domain walls are stable, and shrink – they have a thickness of $(T/\zeta)^{\lambda/2}/T \to 0$. Of course, this argument is not entirely rigorous, as we have assumed in this argument that $\alpha = \pm\alpha_0$ far from the domain wall, whereas in reality α is not quite $\pm\alpha_0$. But this is a sensible argument.

C.2 Temperature Scaling of the Conductivity

Let us discuss the T scaling of the conductivity. In the conducting phase, it is obvious from our choice of Z that $\sigma \sim T^0$, so we focus on the insulating regime. The scaling in the insulating regime can be estimated from studying the T-scaling of σ in a homogeneous system. In this case, the axion equation of motion is given by

$$z^{d+2}\partial_z\left(z^{-d-\lambda}\zeta^{-\lambda}\partial_z\alpha\right) = -2\alpha\left(1-\frac{\alpha^2}{\alpha_0^2}\right) \approx 4(\alpha-\alpha_0). \quad (44)$$

In the second equality, we have approximated that α is close to α_0, which will occur at low T. Make the ansatz (valid at low T)

$$\alpha \approx \alpha_0 - \alpha_2 \mathrm{e}^{-\mu z^c} \quad (45)$$

with μ and c positive constants to be determined. Plugging this ansatz in we obtain

$$-\alpha_2(\mu c)^2 z^{2c-\lambda}\zeta^{-\lambda}\mathrm{e}^{-\mu z^c} + \alpha_2\frac{d+a+1-c}{\zeta^\lambda}z^{c-\lambda}\mathrm{e}^{-\mu z^c} \approx -\alpha_2(\mu c)^2 z^{2c-\lambda}\zeta^{-\lambda}\mathrm{e}^{-\mu z^c} = -4\alpha_2\mathrm{e}^{-\mu z^c}, \quad (46)$$

which is consistent as $z \to \infty$ when

$$c = \frac{\lambda}{2} \quad (47)$$

and

$$\mu = \frac{4}{\lambda}\zeta^{\lambda/2}. \quad (48)$$

(13) follows straightforwardly. The two linearly independent solutions to this differential equation are of the form (45), with μ either positive or negative, and of the magnitude given in (48).

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