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Research on Repeated Quantum Games with Public Goods under Strong Reciprocity

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Abstract: We construct a repeated quantum game with public goods by the quantum entanglement and strong reciprocity. By using the paradigm of quantum game analysis, we obtain that both the quantum entanglement and strong reciprocity are helpful to intensify cooperation and achieving Pareto optimality. In addition, under an example which is greenhouse green viable planting industry, we verify that the quantum entanglement and strong reciprocity have a positive role in cooperation.

Keywords: public goods; repeated game; strong reciprocity; quantum entanglement; Pareto

1. Introduction

1954, Samuelson considered public goods as goods and services that can be shared by members of society in *The Pure Theory of Public Expenditure*.

A large number of public goods game experimental data summarized by Cardenas and Carpenter [1] confirmed that human behavior deviated from the assumption of complete rationality. Chen [2] pointed out that individuals were not all rational and they had heterogeneous social preferences. In terms of the supply and use of public goods or public resources, cooperation among individuals is ubiquitous, such as environmental protection, rural residents jointly raising funds to repair water canals, and urban residents jointly purchasing cleaning services. These phenomena are formed through participants' repeated or multiple-stage games. Cooperation strategy makes the collective benefit greater than the sum of individual benefits. Therefore cooperation has synergy. In 1971, Friedman proved that each Nash equilibrium in which Pareto dominated the original game could be established in a perfect equilibrium of repeated games [3]. Whereafter, Aumann and Shapley proposed to replace Nash equilibrium with subgame perfect equilibrium. Theoreticians have explained the reasons for the emergence and maintenance of cooperation from various perspectives. Robert [4] believed that the repeated game of complete information was related to the evolution of the basic form of interactions between people, and proved that cooperation, altruism, revenge and threat in real life were the results of bounded rationality. For the expectation of future interests in an infinitely repeated game, the most appropriate measure to promote cooperation is to use the theory of strong reciprocity [5]. Therefore, in the repeated game of public goods, we can promote and maintain cooperation by constructing a reasonable strong reciprocity mechanism. The SantaFe Institute is the main creator of the theory of strong reciprocity, and after more than a decade of development, the influence of the theory of strong reciprocity has become increasingly powerful. Gintis [6] and Bowles [7] is a trait of agents that operate even when there is no expected benefit from doing so. Domestic scholars such as Wang Dingding, Ye Hang, and Huang Yong [8] mainly introduce the research results of the Santafei Institute, without conducting in-depth research on strong reciprocity. However, the papers by Gong Zhishui [9] and Wei Qian [10] review the new perspective of institutional evolution research on strong reciprocity in recent years. Therefore, strong reciprocity is a strong form of cooperation.

Since the end of the 1990s, expanding domestic demand has been an important goal that the Chinese government is committed to pursuing, and the government has done a lot of work for it. However, the problem of insufficient domestic demand has not been fundamentally solved. In the context of the US financial crisis and the European debt crisis, reversing the traditional export-oriented economy and establishing an economic structure dominated by domestic demand become one of the central tasks of China's economic work. Studying the supply mechanism of public goods to promote a more adequate and effective supply of public goods can promote the expansion of domestic demand. Changing an export-oriented economy into a domestic demand economy is an important issue.

The COVID-19 that broke out at the end of 2019 caused heavy losses to social welfare. The externalities of the epidemic itself determine that public finance plays an important role in combating the epidemic. However, the issue of public finance construction during the epidemic reminds us that public goods cannot be completely provided by the market, and we should increase the diversification of public goods supply. To some extent, it is necessary to encourage enterprises to participate in the supply of public goods, so that they can pay attention to these social issues from the perspective of mechanism design, and it is also another business opportunity for enterprise development.

So far, researches on games of public goods have mainly focused on theoretical analyses of market competitions based on Evolutionary Game more than two population [11], and the mechanism coexistence or supply chain mechanism [12,13]. Yu,Yang et al. did a large amount of research on the equilibrium of population game [14–18]. Taking a multi-player repeated game in eBay online bidding as an example. Khakzad studied repeated games for eco-friendly flushing in reservoir study interactions between multiple self-interested parties [19]. Escobara and Llanes studied cooperation dynamics in repeated games of adverse selection study cooperation dynamics in repeated games with Markovian private information. Many scholars have made a comparative analysis of the incentive effect of incentive contracts in different situations, and have drawn corresponding conclusions and inspiration. Cao et al [20] believed that the government's incentive contract for manufacturers to recycle and remanufacture could better motivate manufacturers and also improve the government's revenue to a certain extent. Tang et al. [21] pointed out that different discount rates would lead to different incentive effects of contracts. So the incentive effect of relationship contracts was gradually enhanced with the increase of discount rates.

In general, although some scholars have investigated and have discussed the incentive effects of different types of contracts, the binding force on game players is actually not strong.For example, free rider situations such as smart pig games still exist in public goods games.Therefore, it is necessary to study the impact of strong reciprocity mechanisms on game players' strategies.

As the cross field of quantum mechanics and classical information theory, quantum information theory has played a significant role in promoting the development of quantum computers and a series of scientific research progress. Quantum game theory is the product of the application of quantum information theory to the analysis framework of game theory, and is also one of the new expansion fields of game theory. It was proposed in Meyer's paper [22] on the quantum game of coin flipping . Eisert [23] and others further applied quantum game to the situation of prisoner's dilemma. Subsequent researchers in the field of physics and economics [24] proposed relevant theorems, further enriching the quantum game theory. Among them, Brandenburger [25] compared and analyzed the difference between classical game and quantum game, and pointed out that quantum strategy was not inferior to Nash equilibrium strategy. Iqbal [26] applied quantum games to the framework of evolutionary games and obtained an evolutionary stable strategy containing quantum strategies.

In recent years, scholars in information technology, computational mathematics, physics, electronic engineering and other disciplines have conducted very in-depth research on quantum games. Huang [27] investigated the quantized coward game and studied the influence of quantum decoherence on the Nash equilibrium solution of the quantum game.Professor Groisman [28] of Cambridge University proposed that quantum games could be regarded as classical extended games in some situations for quantized eagle pigeon games and prisoner's dilemma. At the same time,

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domestic scholars in related fields have also made a series of studies. Zheng Junjun and others studied the exit dilemma caused by the different views of heterogeneous bidders, and further solved the investment dilemma using quantum entanglement based on game theory. Wang [29] analyzed the quantum equilibrium quantities and quantum equilibrium profits of nonlinear quantum Cournot duopoly games by using the qualitative analysis. We believed in the advantages of quantum games that classical games could not achieve from the perspective of thinking form. Quantum games is a nonlinear, probabilistic, nondeterministic thinking mode. The research results showed that the greater the degree of entanglement of the game, the higher the overall maximum benefit. The above literature mainly focuses on the field of physics, but there are few articles on the economic background of applying quantum game to our production and life, and none on the application of quantum game to the situation of repeated supply of public goods.

Quantum games without entanglement have the same outcome as classical games, and this type of problem may usually be solved using classical games of corresponding contracts.

To sum up, under the new situation of the game of repeated supply of public goods, the article introduces the quantum strategy set and uses the analysis paradigm of quantum game to builds a public goods quantum game model with strong reciprocity mechanism by adding the strong reciprocity coefficient and discount factor.This article mainly makes two aspects of expansion. Firstly, it uses quantum game model with the continuous strategy to explore the strong reciprocity mechanism of public goods supply innovation, which solves the free riding phenomenon in the process of cooperation between the two parties and strong reciprocity are helpful to intensify cooperation and achieving Pareto optimality. Secondly, it analyzes the impact of expected returns on repeated game strategies using quantum entanglement.it constructs the "entanglement contract" in the cooperative innovation process of public goods supply to make up for the shortcomings of traditional cooperation contracts and to encourage the players to cooperate more actively.

2. Quantum game analysis of repeated supply of public goods under strong reciprocity

In the process of cooperation in the public goods industry, it is difficult to monitor the implicit level of cooperation efforts of game players. Therefore, industrial projects are not a "black or white" binary strategy game of "cooperation or no cooperation". In reality, industrial projects should consider the level of effort as a continuous variable, especially in the case of industrial cooperation projects with long-term and infinite repetition. Due to the existence of a state between "complete effort" and "complete lack of effort", which is very similar to the concept of superposition state in quantum mechanics, this article uses the analytical framework of quantum games to study the evolution process of industrial cooperation in the repeated supply of public goods.

The public goods supply market consists of two population, and the number of every population is infinite. The number of agents in the state-owned and private enterprises population are m and n, respectively. The private enterprise population is game player1, and the state-owned enterprise population is game player2. The supply strategy set of agents in each population is $\{C(\text{cooperate}), D(\text{betray})\}$. Then, The repeated game scenario of public goods supply is as following.

Firstly, their investment is assumed to be the degree of effort e_1 and e_2. And the investment risks of game population are $E_c(e_1) = \frac{\gamma_1}{2}e_1^2$, $G_c(e_2) = \frac{\gamma_2}{2}e_2^2$, where γ_1 and γ_2 are the cost parameters of the two population.

Secondly, since the mutual discount coefficient is strong and all population are the same after the game starts in the second stage, we set the total market income U of public goods is

$$U = be_1^{\alpha}e_2^{1-\alpha} + \varepsilon,$$

where b denotes the coefficient of outputs; α represents the weight of the state-owned enterprises population's cooperation degree $1-\alpha$; ε is a random perturbation on Cobb-Douglas.

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In this article,the income distribution contract will temporarily consider the ordinary linear form, where $E(U) = \beta U$ and β are the income distribution coefficients.

Further, let the degree of cooperation between private enterprise population and the state-owned enterprises population be $\theta_1 = 1 - e_1$ and $\theta_2 = 1 - e_2$,respectively, then $\theta_i = 0$ means full cooperation and $\theta_i = 1$ means no cooperation . From the perspective of quantum games, the two polarized states of complete effort and complete no effort correspond to $\theta_i = 0$ and $\theta_i = 1$, respectively, and correspond to the two polarized quantum states of $|0\rangle$ and $|1\rangle$ in quantum theory. The corresponding return matrices are shown in Table 1.

Table 1. Payoff matrix of supply with public goods.

Payoffs	The state-owned enterprises	
The private enterprises	$C(\|0\rangle)$	$D(\|1\rangle)$
$C(\|0\rangle)$	$(1-\beta)b - \frac{\gamma_1}{2}, \beta b - \frac{\gamma_2}{2}$	$-\frac{\gamma_1}{2}, 0$
$D(\|1\rangle)$	$0, -\frac{\gamma_2}{2}$	$0, 0$

Thus, the payoff function of the private enterprises is

$$EE_U = (1-\beta)be_1^{\alpha}e_2^{1-\alpha} - E_c(e_1) = (1-\beta)b(1-\theta_1)^{\alpha}(1-\theta_2)^{1-\alpha} - \frac{\gamma_1}{2}(1-\theta_1)^2,$$

where β represents the product factor of the return after the first game; and the payoff function of the state-owned enterprises is

$$EG_U = \beta be_1^{\alpha}e_2^{1-\alpha} - G_c(e_2) = \beta b(1-\theta_1)^{\alpha}(1-\theta_2)^{1-\alpha} - \frac{\gamma_2}{2}(1-\theta_2)^2.$$

Therefore, the benefits of pure strategies (0,0), (0,1), (1,0), and (1,1) for the private enterprises and the state-owned enterprises are:($(1-\beta)b - \frac{\gamma_1}{2}, \beta b - \frac{\gamma_2}{2}$), ($-\frac{\gamma_1}{2}$,0), (0,$-\frac{\gamma_2}{2}$), and (0,0).

2.1. Repeated game based on strong reciprocity public goods

In the repeated game including private enterprises and the state-owned enterprises, these agents repeatedly play a game with public goods, the payoff matrix of such a game is put in Table 1.

We further make the following assumptions.

(i) Game scenario strategy assumption: In the game process both the state-owned enterprises and the state-owned enterprises have only cooperation and betrayal strategies. After the first game, a tit for tat update mechanism is adopted in the repeated game. In the sub game, the game population only has two strategy choices, that is, hypothesis C-cooperation strategy and D-betrayal strategy. We will not consider the escape strategy of the game population for the time being.

(ii) Game process parameter assumption: Consider the strong reciprocal punishment that affects the game strategy in the repeated game process as $\delta(0 < \delta < 1)$. So when the player chooses the betrayal strategy he will pay δ as a cost of betrayal.The payoffs on betrayal decreases with the increase of strong reciprocity δ, which is reflected in the repeated quantum game payoffs B_E and C_G in 2.2 below.

(iii) Game result assumption: Consider the time value of returns in repeated game returns as $\rho(0 < \rho < 1)$. It is the discount factor and the probability of repeated games in the next stage. So $1 - \rho$ is the probability of game ending.

According to the above assumptions, after the public goods supply game population conducts the first stage of the game, there are four kinds of returns from the second stage of repeated game.

(1) In the second stage, we assume that the state owned enterprises and private enterprises always choose to cooperate except opponent choose to betray, then the payoffs of private enterprises

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are $A_1 \sum_{i=0}^{\infty} \rho^i = \frac{A_1}{1-\rho}, (A_1 = (1-\beta)b - \frac{\gamma_1}{2})$; and the payoffs of the state-owned enterprises are $A_2 \sum_{i=0}^{\infty} \rho^i = \frac{A_2}{1-\rho}, (A_2 = \beta b - \frac{\gamma_2}{2})$.

(2) In the second stage, we assume that the state owned enterprises with cooperation will always choose to betray in the future if the private enterprises respond with choosing to betray, then the payoffs of the private enterprises are 0; and the payoffs of the state-owned enterprises are $\frac{C_2(1-\rho)-\delta\rho}{1-\rho}, (C_2 = \frac{\gamma_2}{2})$.

(3) is similar to (2), the private enterprises with cooperation will always choose to betray in the future if the state owned enterprises respond with choosing to betray, then the payoffs of the private enterprises are $\frac{B_1(1-\rho)-\delta\rho}{1-\rho}, (B_1 = \frac{\gamma_1}{2})$. and the payoffs of the state-owned enterprises are 0.

(4) In this repeated game, a tit for tat update strategy was adopted. Once a player chooses a betrayal strategy in the current stage, the other player will choose a betrayal (never cooperate) strategy in the following stages. That is to say, both sides of the game have chosen a betrayal strategy, and the payoffs on the game is 0.

2.2. *Public goods repeated quantum game*

In Sun et al. [30], the Pareto optimal state is achieved by designing a reasonable mechanism for strong reciprocity coefficient δ and discount factor ρ, but the achievement of the Pareto optimal state is roundabout. Fortunately, this problem is solved by some quantum schemes.

According to quantum game theory, the two polarization states $\theta_i = 0$ (complete cooperation) and $\theta_i = 1$ (complete betrayal) correspond to the polarized quantum states $|0\rangle$ and $|1\rangle$, respectively. Then the corresponding payoff matrix is shown in Table 2.

Table 2. Payoff matrix of repeated supply of public goods based on quantum game.

Payoffs	The state-owned enterprises	
The private enterprises	$C(\|0\rangle)$	$D(\|1\rangle)$
$C(\|0\rangle)$	$\frac{A_1}{1-\rho}, \frac{A_2}{1-\rho}$	$-B_1 - \frac{\delta\rho}{1-\rho}, 0$
$D(\|1\rangle)$	$0, -C_2 - \frac{\delta\rho}{1-\rho}$	$0, 0$

The payoff matrix of Table 2 implies that the payoffs reach the Pareto optimal state only when the private enterprises and the state-owned enterprises choose the strategy of full effort. Moreover, if one cooperates completely and the other one betrays at all, then the fully cooperative one will bear both the cost of the cooperation and no profit. It means the risk of the cooperation aggravated by the increase of the public goods supply investment (E_c, G_c). However, the Pareto optimal state is not the unique Nash Equilibrium.

According to the two extreme states $|0\rangle$ and $|1\rangle$, we set the initial quantum state of both parties is $|00\rangle$, where the first digit represents the private enterprises, the second digit represents the state-owned enterprises, and $|00\rangle = |0\rangle \bigotimes |1\rangle$. Then, let that the entanglement matrix be

$$\hat{J} = \exp(i\frac{w}{2}\sigma_x \bigotimes \sigma_x) = \cos\frac{w}{2} \cdot I + i\sin\frac{w}{2} \cdot H.$$

where

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

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I denotes 4×4 identity matrix, and w represents the degree of entanglement. If $\omega = \frac{\pi}{2}$, namely, the degree of entanglement is the maximum, then the anti entanglement matrix is

$$\hat{J}^{+} = \cos\frac{w}{2} \cdot I - i\sin\frac{w}{2} \cdot H;$$

the strategy matrix of the private enterprises is

$$U_1(\theta_1, \varphi_1) = \begin{pmatrix} e^{i\varphi_1 \cos\frac{\theta_1}{2}} & \sin\frac{\theta_1}{2} \\ -\sin\frac{\theta_1}{2} & e^{-i\varphi_1 \cos\frac{\theta_1}{2}} \end{pmatrix};$$

and the strategy matrix of the state-owned enterprises is

$$U_2(\theta_2, \varphi_2) = \begin{pmatrix} e^{i\varphi_2 \cos\frac{\theta_2}{2}} & \sin\frac{\theta_2}{2} \\ -\sin\frac{\theta_2}{2} & e^{-i\varphi_2 \cos\frac{\theta_2}{2}} \end{pmatrix};$$

where $\theta_1, \theta_2 \in [0, \pi]$, $\varphi_1, \varphi_2 \in [0, \frac{\pi}{2}]$. The general process of quantum game is shown in Figure 1.

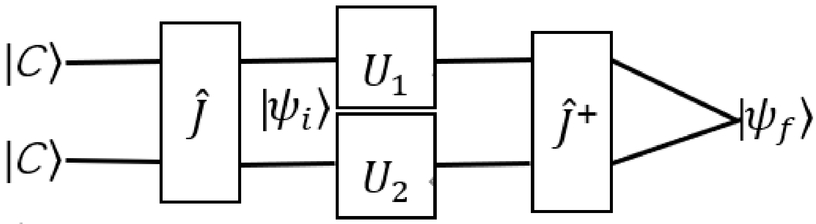

Figure 1. Schematic diagram of the general process of quantum games.

If $\theta_1 = 0$, $\varphi_1 = 0$, namely, both the private enterprises and the state-owned enterprise choose complete cooperation, then

$$U_1(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\theta_1 = \pi$, $\varphi_1 = 0$, namely, the private enterprises choose complete betrayal and the state-owned enterprise choose complete cooperation, then

$$U_1(\pi,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the simplest entanglement free situation, that is $\widehat{J} = I$, we have

$$|\psi_f\rangle = \hat{J}^{+}(U_1 \bigotimes U_2)\widehat{J}|00\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -|10\rangle.$$

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This degenerates to the situation of player 1's betrayal in prisoner's dilemma game. When the entanglement of states is considered, namely,
$\widehat{J} = \exp(i\frac{w}{2}\sigma_x \otimes \sigma_x)$, it follows that

$$
\begin{aligned}
|\psi_f\rangle &= \hat{J}^{+} \cdot [U_1(\theta_1, \varphi_1) \bigotimes U_2(\theta_2, \varphi_2)] \cdot \widehat{J}|00\rangle \\
&= [\cos(\varphi_1 + \varphi_2) - i \cdot \sin(\varphi_1 + \varphi_2)] \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2}|00\rangle \\
&+ [\cos\varphi_1 - i \cdot \sin\varphi_1 \cdot \cos w] \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2}|01\rangle \\
&+ [\sin w \cdot \sin\varphi_2] \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2}|01\rangle \\
&+ [\cos\varphi_2 - i \cdot \sin\varphi_2 \cdot \cos w] sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2}|10\rangle \\
&+ [\sin w \cdot \sin\varphi_1] \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2}|10\rangle \\
&+ [\sin w \cdot \sin(\varphi_1 + \varphi_2)] \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2}|11\rangle + \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}|11\rangle.
\end{aligned}
$$

Thus, the probability of each quantum state is

$$
P(Q) = \begin{cases}
P_{00} &= [\cos^2(\varphi_1+\varphi_2) + \sin^2(\varphi_1+\varphi_2)\cos^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\
P_{01} &= [\cos^2\varphi_1 + \sin^2\varphi_1\cos^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\
&+ [\sin^2\varphi_2\sin^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\
P_{10} &= [\sin^2\varphi_1\sin^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\
&+ [\cos^2\varphi_2 + \sin^2\varphi_2\cos^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\
P_{11} &= [\sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}
\end{cases}.
$$

Obviously, since $P_{00} + P_{01} + P_{10} + P_{11} = 1$, the expected payoffs of the private enterprises are obtained as follows:

$$
\begin{aligned}
EE_U &= A_E P_{00} + B_E P_{01} + 0P_{10} + 0P_{11} \\
&= A_E[1 - \sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\
&+ B_E[\cos^2\varphi_1 + \sin^2\varphi_1 \cdot \cos^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\
&+ B_E[\sin^2\varphi_2 \cdot \sin^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2};
\end{aligned}
$$

and the expected payoffs of the state-owned enterprises are obtained as follows:

$$
\begin{aligned}
EG_U &= A_G P_{00} + 0P_{01} + C_G P_{10} + 0P_{11} \\
&= A_G[1 - \sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\
&+ C_G[\sin^2\varphi_1 \cdot \sin^2 w]\cos^{\frac{\theta_1}{2}}\sin^2\frac{\theta_2}{2} \\
&+ C_G[\cos^2\varphi_2 + \sin^2\varphi_2 \cdot \cos^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2};
\end{aligned}
$$

where $A_E = \frac{A_1}{1-\rho}$, $B_G = \frac{A_2}{1-\rho}$, $B_E = -B_1 - \frac{\delta\rho}{1-\rho}$, $C_G = -C_2 - \frac{\delta\rho}{1-\rho}$.

3. Entanglement of quantum states

Since the game process is uniformly affected by the entanglement ω, we only need to discuss the entanglement with or without states.

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3.1. The entanglement without states

When the entanglement of states is not considered, i.e. $\omega = 0,\ \hat{J}^{+} = \widehat{J} = I$, we have

$$\begin{aligned}|\psi_f\rangle &= \hat{J}^{+}\cdot[U_1(\theta_1,\varphi_1)\bigotimes U_2(\theta_2,\varphi_2)]\cdot\widehat{J}|00\rangle\\ &= e^{i(\varphi_1+\varphi_2)}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle - e^{i\varphi_1}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle\\ &- e^{i\varphi_2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle;\end{aligned}$$

the payoffs of the private enterprises are

$$\begin{aligned}EE_U &= A_E P_{00} + B_E P_{01} + 0P_{10} + 0P_{11}\\ &= A_E cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + B_E\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2};\end{aligned}$$

and the payoffs of the state-owned enterprises are

$$\begin{aligned}EG_U &= A_G P_{00} + 0P_{01} + C_G P_{10} + 0P_{11}\\ &= A_G\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + C_G\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}.\end{aligned}$$

Theorem 1. *For the entanglement without states,*

(i) *the private enterprises' payoffs* EE_U *rise the increase of effort degree* e_1 *if and only if* $A_E\cos^2\frac{\theta_2}{2} + B_E\sin^2\frac{\theta_2}{2} > 0$;

(ii) *the state-owned enterprises' payoffs* EG_U *fall the increase of effort degree* e_2 *if and only if* $A_G\cos^2\frac{\theta_1}{2} + C_G\sin^2\frac{\theta_1}{2} > 0$.

Proof. Since proof of the state-owned enterprises is similar to the private enterprises', we only verify the case of the private enterprises. By

$$\begin{aligned}EE_U &= A_E P_{00} + B_E P_{01} + 0P_{10} + 0P_{11}\\ &= A_E\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + B_E\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\\ &= \cos^2\frac{\theta_1}{2}(A_E\cos^2\frac{\theta_2}{2} + B_E\sin^2\frac{\theta_2}{2}),\end{aligned}$$

it follows that the private enterprises' payoffs EE_U is positive if and only if $A_E\cos^2\frac{\theta_2}{2} + B_E\sin^2\frac{\theta_2}{2} > 0$. Hence, the private enterprises' payoffs EE_U rise the increase of effort degree e_1.

So Theorem 1 is proved. □

In this game, the benefits of the players are equally affected by strong reciprocity and quantum entanglement. When conducting numerical simulation analysis, we only consider the impact of private enterprise profits. In order to better reflect the impact of strong reciprocity δ on the payoffs of the private enterprises, we first give the parameters $\beta = 0.4, b = 1.5, \gamma_1 = 0.2, \rho = 0.5$, and then take $\delta = 0, \delta = 0.4, \delta = 0.8$ for numerical simulation.

The numerical simulation diagrams are as follows:

Figure 2(1–3). Among them: (a) is three-dimensional image of the enterprises' payoffs and effort level without quantum entanglement, (b) is EE_U and θ_2 corresponds to projection on θ_1, (c) is EE_U and θ_1 corresponds to projection on θ_2.

From Figure 2(1a,2a,3a), we find that the closer θ_2 is to 0, the more obvious the decreasing trend of the private enterprises' payoffs EE_U are with the increase of θ_1. And as θ_2 is close to π, it is difficult

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to judge whether the private enterprises' payoffs EE_U will increase or decrease with θ_1, because $A_E \cos^2 \frac{\theta_2}{2} + B_E \sin^2 \frac{\theta_2}{2} > 0$.

However, from the projections Figure 2(1b,2b,3b), we can see that when the strong reciprocity parameter takes different values of δ , the payoffs EE_U of private enterprises increases as θ_1 increases from 0 to a certain threshold and θ_2 increases, but after reaching the threshold, it decreases as θ_1 increases from threshold to π and θ_2 increases.

At the same time, the threshold of θ_2 is obtained by taking different values of δ for the positive and negative payoffs EE_U of private enterprises. As mentioned earlier, when δ is 0, 0.4 and 0.8, respectively, the thresholds for θ_2 are:$arcsin\sqrt{\frac{8}{11}}$, $arcsin\sqrt{\frac{8}{13}}$, and $arcsin\sqrt{\frac{8}{15}}$.Therefore, it can be seen that the payoffs of the game party decreases with the increase of the value of the strong reciprocity parameter δ, and this change is very clear in the numerical simulation of the threshold point θ_2 in Figure 2(1c,2c,3c).

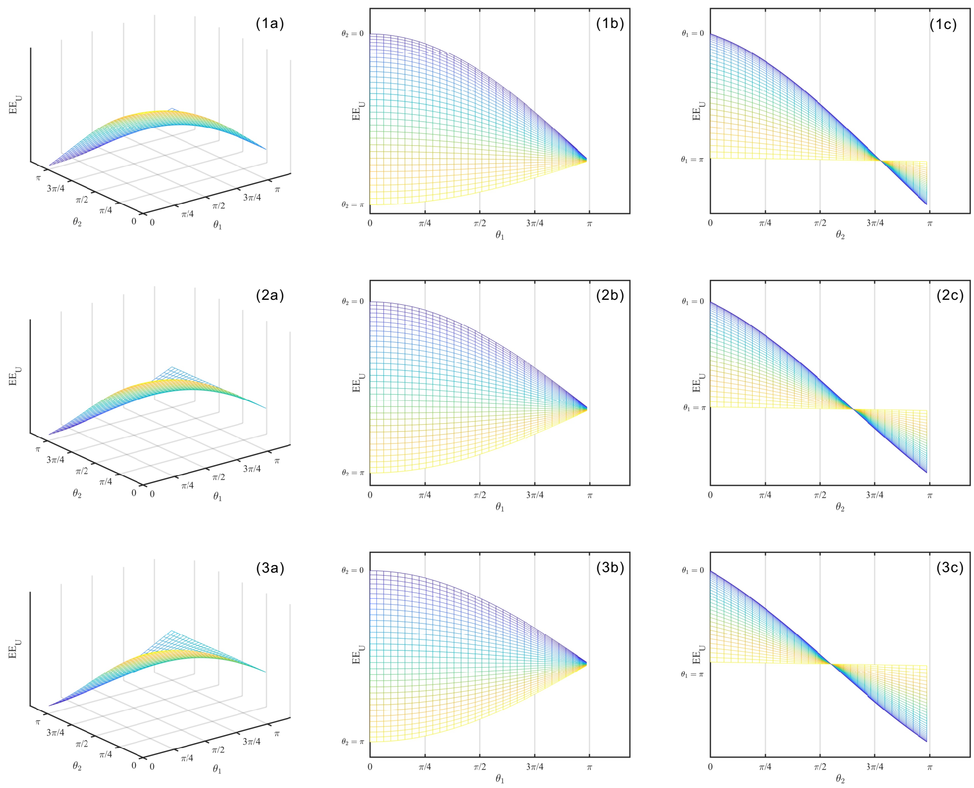

Figure 2. Three-dimensional image of the enterprises' payoffs EE_U [(**a**)]. EE_U and θ_2 corresponds to projection on θ_1 [(**b**)]; EE_U and θ_1 corresponds to projection on θ_2 [(**c**)].

Table 3 provides the payoff matrix of four special strategies for the private enterprises and the state-owned enterprises. From Table 3, we easily find the value of the parameter φ_1 and φ_2 do not affect the payoffs. Besides, the payoffs of the private enterprises will be reduced to B_E if the private enterprises fully cooperates $(\theta_1 = 0)$ and state-owned enterprises do not choose to cooperate $(\theta_2 = 0)$. Namely, the cost of "betrayal" of the state-owned enterprises will be borne by the effortful one.

Next, let $q \equiv \cos^2 \frac{\theta_2}{2}$, then $1 - q = \sin^2 \frac{\theta_2}{2}$. Noticeably, the enterprises chooses the strategy of full effort the sufficient and necessary conditions can be rewritten by $A_E q + B_E(1 - q) > 0$. That is, the conclusion of quantum game model is consistent with that of the deterministic evolutionary game model when the entanglement of states is not considered, which agents choose the complete cooperation strategy only when the expectation of the cooperation degree of the other ones is higher than a certain threshold.

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Table 3. Payoff Matrix under Four Strategies Without Entanglement.

Payoff	The state owned enterprises			
The private enterprises	$\theta_2=0,\varphi_2=0$	$\theta_2=0,\varphi_2=\frac{\pi}{2}$	$\theta_2=\pi,\varphi_2=0$	$\theta_2=\pi,\varphi_2=\frac{\pi}{2}$
$\theta_1=0,\varphi_1=0$	A_E,A_G	A_E,A_G	$B_E,0$	$B_E,0$
$\theta_1=0,\varphi_1=\frac{\pi}{2}$	A_E,A_G	A_E,A_G	$B_E,0$	$B_E,0$
$\theta_1=\pi,\varphi_1=0$	$0,C_G$	$0,C_G$	$0,0$	$0,0$
$\theta_1=\pi,\varphi_1=\frac{\pi}{2}$	$0,C_G$	$0,C_G$	$0,0$	$0,0$

3.2. *The entanglement of states*

Quantum game extra possesses state entanglement different from traditional game, this state entanglement have a positive effect on equilibria. For every the state entanglement $\omega\in(0,\frac{\pi}{2})$, the probabilities of each quantum state are as follows:

$$\begin{cases} P_{00}=[\cos^2(\varphi_1+\varphi_2)+\sin^2(\varphi_1+\varphi_2)\cos^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ P_{01}=[\cos^2\varphi_1+\sin^2\varphi_1\cos^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}+[\sin^2\varphi_2\sin^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ P_{10}=[\sin^2\varphi_1\sin^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}+[\cos^2\varphi_2+\sin^2\varphi_2\cos^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ P_{11}=[\sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}+\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \end{cases}.$$

Then, the expected payoffs of the private enterprises are

$$\begin{aligned} EE_U &= A_E P_{00}+B_E P_{01}+0P_{10}+0P_{11} \\ &= A_E[1-\sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ B_E[\cos^2\varphi_1+\sin^2\varphi_1\cdot\cos^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\ &+ B_E[\sin^2\varphi_2\cdot\sin^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}; \end{aligned}$$

and the expected payoffs of the state-owned enterprises are

$$\begin{aligned} EG_U &= A_G P_{00}+0P_{01}+C_G P_{10}+0P_{11} \\ &= A_G[1-\sin^2(\varphi_1+\varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ C_G[\sin^2\varphi_1\cdot\sin^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\ &+ C_G[\cos^2\varphi_2+\sin^2\varphi_2\cdot\cos^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}. \end{aligned}$$

We only consider the case of $\omega=\frac{\pi}{2}$ due to the case of $0<\omega<\frac{\pi}{2}$ is similar. Without special instructions, the "entanglement of the considered state" in this paper refers to $\omega=\frac{\pi}{2}$.

Theorem 2. *For the entanglement of states $\omega=\frac{\pi}{2}$,*

(*i*) *when the private enterprises adopts maximal quantum strategy ($\varphi_1=\frac{\pi}{2}$) and $\sin^2\varphi_2\cos^2\frac{\theta_2}{2}>0$, the private enterprises' payoffs EE_U rise the increase of effort degree θ_1, δ;*
(*ii*) *when the private enterprises adopts maximal non quantum strategy ($\varphi_2=\frac{\pi}{2}$) and $\sin^2\varphi_1\cos^2\frac{\theta_1}{2}>0$, the state-owned enterprises' payoffs EG_U rise the increase of effort degree θ_2, δ.*

Proof. We only consider the private enterprises due to the likeness of the state-owned enterprises.

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Let $\varphi_1 = \frac{\pi}{2}$, then the private enterprises' payoffs EE_U is

$$\begin{aligned} EE_U = &[A_E \cos^2(\varphi_1 + \varphi_2) \cos^2 \frac{\theta_2}{2} \\ &+ B_E \cos^2 \varphi_1 \sin^2 \frac{\theta_2}{2}] \cdot \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \varphi_2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ &= [A_E \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \frac{\theta_1}{2}] \sin^2 \varphi_2 \cos^2 \frac{\theta_2}{2}. \end{aligned}$$

Obviously, since $A_E \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \frac{\theta_1}{2}$ decreases with the increase of θ_1, $\sin^2 \varphi_2 \cos^2 \frac{\theta_2}{2} > 0$ implies that EE_U decreases with the increase of θ_1.

So Theorem 2 is proved. □

Noteworthy, the achievement of Theorem 2 have to join a third party to determine the strong reciprocal punishment and the time value of returns observable and quantifiable performance indicators and sign an "entanglement contract."

Theorem 3. *For the entanglement of states $\omega = \frac{\pi}{2}$,*

(i) when the private enterprises adopts a non quantum strategy ($\varphi_1 = 0$) and $A_E \cos^2 \varphi_2 \cos^2 \frac{\theta_2}{2} - B_E \sin^2 \frac{\theta_2}{2} \geq 0$, the private enterprises' payoffs EE_U rise the increase of effort degree θ_1, δ;
(ii) when the state-owned enterprises adopts the non quantum strategy ($\varphi_2 = 0$) and $A_G \sin^2 \varphi_1 \cos^2 \frac{\theta_1}{2} + C_G \sin^2 \frac{\theta_1}{2} > 0$, the the state-owned enterprises' payoffs EE_U rise the increase of effort degree θ_2, δ.

Proof. It is similar to the proof of Theorem 3, we only consider the private enterprises.

Let $\varphi_1 = 0$, then the private enterprises' payoffs EE_U is

$$\begin{aligned} EE_U = &[A_E \cos^2(\varphi_1 + \varphi_2) \cos^2 \frac{\theta_2}{2} \\ &+ B_E \cos^2 \varphi_1 \sin^2 \frac{\theta_2}{2}] \cdot \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \varphi_2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ &= [A_E \cos^2 \varphi_2 \cos^2 \frac{\theta_2}{2} + B_E \sin^2 \frac{\theta_2}{2}] \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \varphi_2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}. \end{aligned}$$

Obviously, $A_E \cos^2 \varphi_2 \cos^2 \frac{\theta_2}{2} + B_E \sin^2 \frac{\theta_2}{2} > 0$ decreases with the increase of θ_1, and $\sin^2 \varphi_2 \cos^2 \frac{\theta_2}{2} > 0$ decreases with the increase of θ_1. Notably, for $A_E \cos^2 \varphi_2 \cos^2 \frac{\theta_2}{2} + B_E \sin^2 \frac{\theta_2}{2} \geq 0$ and $\sin^2 \varphi_2 \cos^2 \frac{\theta_2}{2} \geq 0$, if "=" is not taken at the same time, EE_U decreases with the increase of θ_1.

Hence, Theorem 3 is proved. □

Theorem 4. *For the entanglement of states $\omega = \frac{\pi}{2}$,*

(i) when the private enterprises adopts a quantum strategy ($\varphi_1 \in (0, \frac{\pi}{2})$) and $A_E \cos^2(\varphi_1 + \varphi_2) \cos^2 \frac{\theta_2}{2} + B_E \cos^2 \varphi_1 \sin^2 \frac{\theta_2}{2} \geq 0$, the private enterprises' payoffs EE_U rise the increase of effort degree θ_1, δ;
(ii) when the state-owned enterprises adopts a quantum strategy ($\varphi_2 \in (0, \frac{\pi}{2})$) and $A_G \cos^2(\varphi_1 + \varphi_2) \cos^2 \frac{\theta_1}{2} + C_G \cos^2 \varphi_2 \sin^2 \frac{\theta_1}{2} > 0$, the the state-owned enterprises' payoffs EG_U rise the increase of effort degree θ_2, δ.

Proof. Similarly, we consider the private enterprises.

From the private enterprises' payoffs

$$\begin{aligned} EE_U = &[A_E \cos^2(\varphi_1 + \varphi_2) \cos^2 \frac{\theta_2}{2} \\ &+ B_E \cos^2 \varphi_1 \sin^2 \frac{\theta_2}{2}] \cdot \cos^2 \frac{\theta_1}{2} + B_E \sin^2 \varphi_2 \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}, \end{aligned}$$

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It follows that the first term on the right of EE_U decreases with the increase of θ_1 as $A_E\cos^2(\varphi_1+\varphi_2)\cos^2\frac{\theta_2}{2}+B_E\cos^2\varphi_1\sin^2\frac{\theta_2}{2}>0$; and the second term on the right of F decreases with the increase of θ_1 as $\sin^2\varphi_2\cos^2\frac{\theta_2}{2}>0$. Notably, for $A_E\cos^2(\varphi_1+\varphi_2)\cos^2\frac{\theta_2}{2}+B_E\cos^2\varphi_1\sin^2\frac{\theta_2}{2}\geq 0$ and $\sin^2\varphi_2\cos^2\frac{\theta_2}{2}\geq 0$, if "=" is not taken at the same time, EE_U decreases with the increase of θ_1.

Thereby, Theorem 4 is proved. □

Next, we verify that Theorems 2–4 by simulation. The parameters of Theorem 1 remain unchanged ,then we still take $\delta=0,\delta=0.4,\delta=0.8$. To compare the changes in payoffs EE_U of private enterprises using two quantum states, It shows in Fig.3(1) and(2) of $\varphi_1=\frac{\pi}{2}$ and $\varphi_2=0$ and $\varphi_1=\frac{\pi}{2}$ and $\varphi_2=\frac{\pi}{2}$.

Firstly, from Figure 3(1a,2a), we can clearly see that when the strong reciprocity parameter δ takes different values, there is a significant change in the payoffs EE_U of private enterprises. When θ_1 and θ_2 are closer to π, the payoffs EE_U decreasing with the increase of the strong reciprocity parameter δ.

Secondly, from Figure 3b,c, $\varphi_1=\frac{\pi}{2}$, $\varphi_2=0$ and $\varphi_1=\frac{\pi}{2}$, $\varphi_2=\frac{\pi}{2}$, we find that if θ_1 , $\theta_2\in(\frac{\pi}{4}$, $\frac{3\pi}{4})$, that the more obvious the trend is that the payoffs EE_U of the private enterprise decreases with the increase of θ_2 or θ_1. Similarly, we find that if θ_2 , $\theta_1\in(0,\frac{\pi}{4})$ or θ_2 , $\theta_1\in(\frac{3\pi}{4},\pi)$ it is difficult to judge the impact of the payoffs EE_U of the private enterprises changing with θ_1 or θ_2. It implies that the private enterprise still tends to cooperate even if the payoffs of the private enterprises decline due to the decline in the degree of cooperation between the state-owned enterprises.

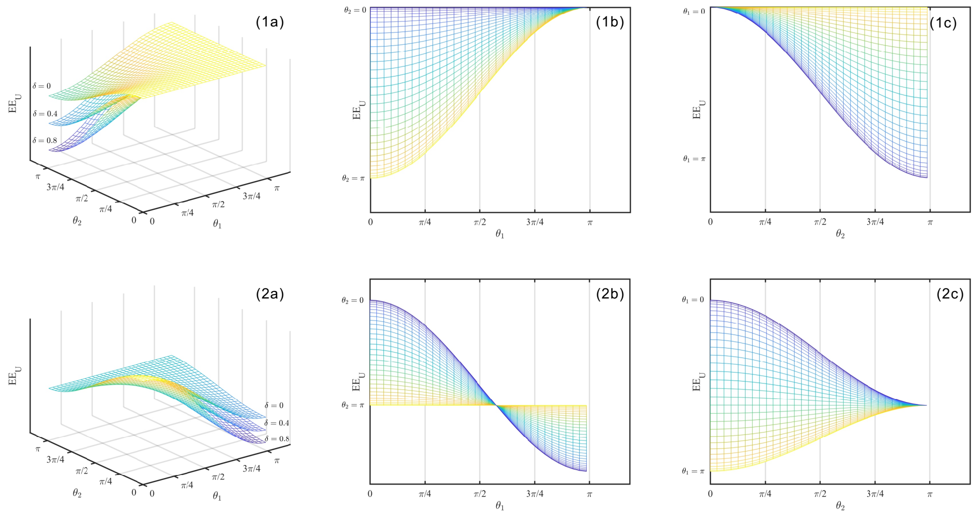

Figure 3. Three-dimensional image of EE_U When taking $\varphi_1=\frac{\pi}{2}$ and $\varphi_2=0$ [(**1a**)] or $\varphi_1=\frac{\pi}{2}$ and $\varphi_2=\frac{\pi}{2}$ [(**2a**)]; EE_U and θ_2 corresponds to Projection on θ_2 [(**1b**,**2b**)]; EE_U and θ_1 corresponds to Projection on θ_2 [(**1c**,**2c**)].

Similar to Table 3, we also give the payoff matrix of private enterprises and the state-owned enterprises of four special strategies with the entanglement of states as shown in Table 4.

Table 4. Payoff matrix of repeated game for special strategic public goods.

Payoffs	The state-owned enterprises			
The private enterprises	$\theta_2=0,\varphi_2=0$	$\theta_2=0,\varphi_2=\frac{\pi}{2}$	$\theta_2=\pi,\varphi_2=0$	$\theta_2=\pi,\varphi_2=\frac{\pi}{2}$
$\theta_1=0,\varphi_1=0$	A_E,A_G	0,0	$B_E,0$	$B_E,0$
$\theta_1=0,\varphi_1=\frac{\pi}{2}$	0,0	A_E,A_G	$0,C_G$	$0,C_G$
$\theta_1=\pi,\varphi_1=0$	$0,C_G$	$B_E,0$	0,0	0,0
$\theta_1=\pi,\varphi_1=\frac{\pi}{2}$	$0,C_G$	$B_E,0$	0,0	0,0

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According to Table 4, we easily find the "full quantum strategy of maximum cooperation" ($\theta_1 = 0,\ \varphi_1 = \frac{\pi}{2},\ \theta_2 = 0,\ \varphi_2 = \frac{\pi}{2}$) achieves Pareto optimality and avoids the risk of betrayal of the other one. Therefore, once private enterprises choose the "maximum effort cooperation strategy", the "betrayal" of state-owned enterprises will result in their payoffs being damaged. If the state-owned enterprise industry does not adopt a quantum strategy, its profits will decrease to 0. If state-owned enterprises adopt a "no effort strategy", their profits will decrease to C_G, and they will bear the risk of "betrayal". In the other three "betrayal" situations, if the private enterprise's payoffs only decrease to 0, then the private enterprise does not need to bear the risk of betrayal. Therefore, after considering the entanglement of states, it is only necessary to adopt a "maximum effort complete quantum strategy" to achieve both self maximization and Pareto optimality, without the risk of the other party's "betrayal".

To sum up, it is necessary to sign the "entanglement contract" in the game of repeated supply of public goods. Not only can the payoffs of both the state owned enterprises and the private enterprises be closely linked to form a "revenue community", but also the measurement of explicit and quantifiable performance indicators can be shifted from the invisible "degree of cooperation". It can not only enable the state-owned enterprises and private enterprises to "tell the truth" and restrain the behavior of exaggerating performance, but also send signals to both players of the game. Therefore it can make the game information more open and transparent, which will greatly improve the efficiency of repeated supply of public goods.

4. Example numerical analysis

This section uses the analysis paradigm of quantum game to further analyze the game case of repeated supply of public goods, vividly shows the theoretical analysis in Sections 3 and 4, and further explains the way to promote the maximum cooperation between the state-owned enterprises and the private enterprises in the process of repeated supply of public goods.

Case: A private enterprise signed an agreement with the state-owned enterprise to jointly develop the distribution of the rural revitalization project "Greenhouse green vegetable planting industry in a certain place". The agreement stated that the state owned enterprises was responsible for the land rent 35 needed for the "Greenhouse green vegetable planting industry", and the private enterprises was responsible for the planting equipment, road, irrigation and other infrastructure construction, as well as marketing. The total human cost is about 55. However, due to the existence of bilateral credit risk in the context of the greenhouse green vegetable planting industry, some hidden inputs will decline with the decline of the degree of cooperation between the two players. Suppose that the discount amount of the state owned enterprises is between 30 and 60, and that of the private enterprises is between 40 and 80. Since the specific investment amount of both players cannot be observed by the other players, how can we promote long-term and repeated cooperation between both palyers?

Case analysis: This case is abstracted into a quantum game model of repeated supply of public goods. Suppose that the strategy set of the private enterprises is $[40, 80]$, namely, $40 \leq e_1 \leq 80$, where e_1 is the cost of the enterprises. Similarly, let the strategy set of the state-owned enterprises be $[30, 60]$, then the cost of the state-owned enterprises $e_2 \in [30, 60]$. Let the final output meets $U = be_1e_2$ and the distribution coefficient is $\beta = 0.4$, then the relationship between strategy selection and effort level $\theta_1 = 2\pi - \frac{\pi}{40}e_1,\ \theta_2 = 2\pi - \frac{\pi}{30}e_2$.

This section mainly analyzes the operation and application of the public goods rural revitalization project "greenhouse green vegetable planting industry", mainly involving the concepts of quantum strategy and quantum entanglement. The difference between quantum strategy and classical strategy is that the imaginary unit i is introduced, which is another dimension perpendicular to the real number axis in the coordinate system. In the application scenario of repeated supply of public goods, it is regarded as a measurable and quantifiable performance indicator in the rural revitalization project, such as total working hours, project construction costs, production costs, sales costs, etc. These observable indicators reflect the degree of cooperation of the players. Quantum strategies reflect both the degree of quantization φ_i and the degree of cooperation θ_i , which respectively represent

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various quantifiable performance indicators and the unquantifiable degree of cooperation of the agents. However, due to incompletely positively correlated, the quantifiable performance indicators can not fully reflect the degree of cooperation and relevant hidden investment in the project. So it is necessary to strike an appropriate balance between the formulation of strong reciprocity policy and discount policy.

Quantum entanglement is to "bundle" the strategy of the agents together. It makes the degree of correlation between the investment of agents is improved. A "entanglement contract" stating the project performance value with stronger reciprocal binding force should be signed before signing a contract for the repeated supply of public goods. Everyone sets a "discount fund" according to the stated performance value, and the amount of the "discount fund" is linked to the high or low performance target value. On the assessment date specified in the contract, the agents who fail to reach the performance target value in the assessment will be fined three times the difference in the "discount fund". So the investment of both agents will be "tied". And the previously stated performance target value can also convey some unobservable information. It greatly reduces the probability of credit risk behavior.

4.1. Regardless of entanglement of states

In the first case, before the launch of the "Greenhouse green viable planting industry in a certain place" project, no relevant performance indicators were specified, that is, quantum strategy φ_i and quantum entanglement ω were not considered. From the analysis of case 1 in Section 3.1, it is known that the expected payoffs of the private enterprises are

$$EE_U = A_E \cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) + B_E \cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2);$$

and the expected payoffs of the state-owned enterprises are

$$EG_U = A_G \cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) + C_G \sin^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2).$$

Let's first discuss the payoffs of private enterprises. According to the exported payment function in Part 2, if the values of parameters $\beta = 0.4, b = 1.5, \gamma_1 = 0.2$ remain unchanged, then

$$EE_U = \frac{0.9e_1e_2 - 0.05e_1^2}{1-\rho}\cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) + (e_1 - \frac{\delta\rho}{1-\rho})\cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2).$$

Under the model assumption in Section 3.1, we set $\rho = 0.5$, then

$$EE_U = (1.8e_1e_2 - 0.1e_1^2)\cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) + (e_1 - \delta)\cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2)$$

Obviously, When $0 < \delta < 1$, that is to say, the game party has signed a strong reciprocity agreement, $e_1\cos^2(\pi - \frac{\pi}{80}e_1)$ is an increasing function of e_1.

When $\delta = 0$, the game becomes a one-time classic game. then

$$EE_U = (1.8e_1e_2 - 0.1e_1^2)\cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) + e_1\cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2)$$

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$e_1 \cos^2(\pi - \frac{\pi}{80}e_1)$ is an increasing function of e_1.

Obviously, Since $F(e_2) = (1.8e_1e_2 - 0.1e_1^2)\cos^2(\pi - \frac{\pi}{60}e_2) + e_1 \sin^2(\pi - \frac{\pi}{60}e_2)$ is an increasing function of e_2, the increase or decrease of EE_U is determined by the positive or negative of $F(e_2)$. Assume that the zero point of $F(e_2)$ is e_2^*, then,

1) when $30 \le e_2 \le e_2^*$, $F(e_2)$ is negative, and EE_U is a monotone decreasing function of e_1, the optimal choice of the private enterprises is $e_1^* = 40$, which is the minimum investment;
2) when $e_2^* \le e_2 \le 60$, $F(e_2)$ is positive, and EE_U is a monotone increasing function of e_1, the optimal choice of the private enterprises is $e_1^* = 80$, which is the maximum investment.

Similar to Theorem 1, only when the strategy of state-owned enterprises reaches a certain threshold of $e_2^* \approx 42$, the optimal choice of private enterprises tends towards maximum cooperation.

4.2. Considering entanglement of states

Now we consider the entanglement of states. Before the launch of "Greenhouse green viable planting industry in a certain place" project, both parties entrusted to establish relevant measurable and quantifiable performance indicators and signed a binding entanglement contract. According to the analysis of Case 2 in Section 3.2, when the entanglement of states being considered, the expected payoffs of the private enterprises are

$$\begin{aligned} EE_U &= A_E[1 - \sin^2(\varphi_1 + \varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ B_E[\cos^2\varphi_1 + \sin^2\varphi_1 \cdot \cos^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\ &+ B_E[\sin^2\varphi_2 \cdot \sin^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}; \end{aligned}$$

and the expected payoffs of the state-owned enterprises are

$$\begin{aligned} EG_U &= A_G[1 - \sin^2(\varphi_1 + \varphi_2)\sin^2 w]\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ C_G[\sin^2\varphi_1 \cdot \sin^2 w]\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\ &+ C_G[\cos^2\varphi_2 + \sin^2\varphi_2 \cdot \cos^2 w]\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}. \end{aligned}$$

We keep the parameter values in 4.1 unchanged and consider the maximum value of entanglement ($\omega = \frac{\pi}{2}$), we obtain the payoffs of the private enterprises are

$$\begin{aligned} EE_U &= (1.8e_1e_2 - 0.1e_1^2)\cos^2(\varphi_1 + \varphi_2)\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ (e_1 - \delta)\cos^2\varphi_1\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + (e_1 - \delta)\sin^2\varphi_2\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}; \end{aligned}$$

and the expected payoffs of the state-owned enterprises are

$$\begin{aligned} EG_U &= (1.2e_1e_2 - 0.1e_1^2)\cos^2(\varphi_1 + \varphi_2)\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} \\ &+ (e_1 - \delta)\sin^2\varphi_1\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + (e_1 - \delta)\cos^2\varphi_2\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}. \end{aligned}$$

Here, it is assumed that the performance system of the private enterprises and the state-owned enterprises is determined by the total investment cost of project implementation of the two population. It is stipulated that the total investment of the private enterprises is 80, the total investment of the state-owned enterprises is 60, and the "incentive fund" for project performance is 50. Before signing the project contract both parties signed a performance "entanglement contract", and agreed that once

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one party fails to reach the investment amount, he will be fined three times the difference in the "incentive fund". Now the quantization degree is corresponding to the total investment as shown in the following formula, $\varphi_1 = \frac{\pi}{2} \cdot \frac{E_c}{80}$ and $\varphi_2 = \frac{\pi}{2} \cdot \frac{G_c}{60}$.

Under the model assumption in Section 2.1, we set $\rho = 0.5$, $0 < \delta < 1$, then the expected payoffs of the private enterprises are

$$\begin{aligned} EE_U = &(1.8e_1e_2 - 0.1e_1^2)\cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) \\ &+ e_1\sin^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) \\ &- \delta(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2); \end{aligned}$$

and the expected payoffs of the state-owned enterprises are

$$\begin{aligned} EG_U = &(1.2e_1e_2 - 0.1e_2^2)\cos^2(\pi - \frac{\pi}{80}e_1)\cos^2(\pi - \frac{\pi}{60}e_2) \\ &+ e_2\cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2) \\ &- \delta\cos^2(\pi - \frac{\pi}{80}e_1)\sin^2(\pi - \frac{\pi}{60}e_2). \end{aligned}$$

Therefore, we look at the payoffs of private enterprises: as long as $e_2 > \frac{5}{9}e_1$, and $cos^2(\pi - \frac{\pi}{60}e_2) \neq 0$, δ remains constant, the expected payoffs EE_U of the private enterprises will increase with the increase of the degree of cooperation e_1. Similarly, we look at the payoffs of state-owned enterprises: as long as $e_1 > \frac{5}{6}e_2$, and $cos^2(\pi - \frac{\pi}{80}e_2) \neq 0$, δ remains constant,the expected payoffs EG_U of the state-owned enterprises will increase with the increase of the degree of cooperation e_2. However, due to the existence of a strong reciprocity coefficient δ, it is easy to see that the expected payoffs of the game players will decrease with the increase of δ. So in the game of repeated supply of public goods, the strong reciprocity coefficient can promote the game population to repeat the supply of public goods.In the case, as long as the other party does not choose not to cooperate at all, the revenue of the game population will increase with the increase of the cooperation degree, instead of bearing the loss of the other party's "betrayal". The signing of the "entanglement contract" can increase the constraints on both sides of the game and reduce the occurrence of free riding and other smart pig game phenomena.

5. Conclusions

Focusing on the quantum entanglement and strong reciprocity, we investigate uantum game model and explore the strong reciprocity mechanism of public goods supply innovation, which solves the free riding phenomenon in the process of cooperation between the two partiesIn the quantum game of repeated supply of public goods, including private and state-owned enterprises, quantum entanglement helps to strengthen cooperation and achieve Pareto optimality. Specially, we provide a scenario on greenhouse green viable planting industry as an application. From our conclusions, we suggest that

1. by designing different strong reciprocity parameter values through contracts, the guidance of indicators ensure that the project can be implemented efficiently;
2. by signing "entanglement contracts", the benefits of agents are linked, it helps agents to work hard.

We futurely consider evolution of cooperation under the quantum entanglement and strong reciprocity in order to achievement and stabilization of cooperation.

Data Availability Statement: All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest: The authors declare that they have no conflict of interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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