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Article

Reachable Set Estimation and Controller Design for Linear Time-Delayed Control System with Disturbances

Yongchun Jiang [1], Hongli Yang [1,2,*] and Ivan Ganchev Ivanov [3]

[1] College of Big Data, Qingdao Huanghai University, Qingdao 266427, China; jiangyc01@qdhhc.edu.cn
[2] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China
[3] Faculty of Economics and Business Administration, Sofia University "St. Kl. Ohridski", 125 Tzarigradsko Chaussee Blvd., Bl. 3, 1113 Sofia, Bulgaria; i_ivanov@feb.uni-sofia.bg
* Correspondence: yhlmath@126.com

Abstract: This paper investigates reachable set estimation and state-feedback controller design for linear time-delay control systems with bounded disturbances. By constructing an appropriate Lyapunov–Krasovskii functional, we obtain a delay-dependent condition, which determines the admissible bounding ellipsoid for the reachable set of the system we considered. Then, a sufficient condition in the form of liner matrix inequalities is given to solve the problem of controller design and reachable set estimation. Then, by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. A comparative numerical example is given to show the effectiveness of our result.

Keywords: time-delay; ellipsoid; Lyapunov–Krasovskii functional; reachable set; linear matrix inequalities

MSC: 34D45; 34E10; 34H05; 90C25

1. Introduction

The reachable set estimation of dynamic systems is an important research topic in control theory since it has a large number of applications in control systems with actuator saturation [1–3], peak-to-peak gain minimization [4], and aircraft collision avoidance [5]. The reachable set of a dynamic system with bounded peak input is defined as the set of system state vectors in the presence of all allowed input disturbances. Reachable set bounding was first considered in the late 1960s in the context of state estimation, and it later received a lot of attention in parameter estimation [6]. Boyd et al. studied the problem of reachable set estimation of linear systems without time delay and obtained an LMI condition for an ellipsoid that bounds the reachable set [7].

It is well known that time delays are extremely common in practice, such as in aircraft, chemical processes, long pipeline supply, belt transmission, extremely complex online analyzers in various industrial systems, and so on. Usually, the occurrence of time delay may lead to instability or performance degradation of dynamic systems [5,8–12]. Therefore, extensive research is devoted to the study of the reachable set estimation issue of dynamic systems with delay. During the past few decades, there have been some excellent results related to the reachable set estimation of time-delay systems [11,13–28].

In [14], based on the Lyapunov–Razumikhin functional method, Fridman and Shaked first investigated the reachable set estimation of a linear system with time-varying delay and obtained an LMI criteria of an ellipsoid bounding the reachable states set. Kim obtained an improved ellipsoidal bound of a reachable set [16] by using the Lyapunov–Krasovskii functional. Nam and Pathirana obtained a smaller reachable set bound [18] by employing the delay decomposition technique. Zuo et al. obtained a non-ellipsoidal bound of a reachable set of linear time-delayed systems through

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the maximal Lyapunov functionals and the Razumikhin method [26]. More recently, Zhang et al. investigated the reachable set estimation for uncertain nonlinear systems with time delay [27]. For more references and recent advances in reachable set estimation, one can refer to [29–32]. The key point of the reachable set estimation is the choice of Lyapunov functional. Motivated by the above idea, we study the reachable set estimation of linear time-delayed control systems with disturbances.

Contributions of this paper are listed below:

- We derived a sufficient condition that determines the admissible bounding ellipsoid for the reachable set related to the delay-dependent system; the condition is in the form of LMI;
- We propose a state feedback controller design method to find the minimum ellipsoidal bound so that the reachable set of the resulting closed-loop system is bounded by an ellipsoid, and the admissible ellipsoid should be as small as possible;
- We show that our conclusion is an extension of the available results in the paper [16].

In this paper, we intend to design a state feedback controller so that the reachable set of the resulting closed-loop system is contained in an admissible ellipsoid, and the admissible ellipsoid should be as small as possible. The rest of this paper is organized as follows: In Section 2, in order to obtain the main result, some useful lemmas and preliminary knowledge are given. In Section 3, by constructing an appropriate Lyapunov–Krasovskii functional, we obtain a condition related to delay-dependent, which determines the admissible bounding ellipsoid for the reachable set of the system we considered. Then, a sufficient condition in the form of liner matrix inequalities is given to solve the problem of controller design with reachable set estimation. Finally, by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. Section 4 presents a comparative numerical example to show the effectiveness of the proposed methods.

Notation

Throughout this paper, the notations are standard. $\mathbb{R}^n$ is the vector of real numbers, $\mathbb{R}^{n\times m}$ is the $n \times m$ real matrix, I is the identity matrix, 0 is the zero matrix, and A^T presents the transpose of A. For a matrix P, $P > 0$ denotes P as a symmetric positive definite matrix; also, $x_t(\theta) = x(t+\theta), \theta \in [-h, 0]$, and symbol $(\star)$ in a matrix represents the symmetric part.

2. Problem Statement and Preliminaries

Consider the following linear time-delay control system with bounded disturbances:

$$\begin{cases} \dot{x}(t) = Ax(t) + Dx(t-d(t)) + Bu(t) + Ew(t), \\ x(t) \equiv 0, t \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t)$ is the control vector, $A \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$, and $E \in \mathbb{R}^{n\times m}$, A, D, B, and E are constant matrices, $w(t) \in \mathbb{R}^m$ is the disturbance, satisfying

$$w^T(t)w(t) \le w_m^2 \tag{2}$$

and $d(t)$ is a time-varying delay, satisfying

$$0 \le d(t) \le h, \qquad |\dot{d}(t)| \le u \le 1 \tag{3}$$

where w_m, d, and u are constants.

In this paper, based on the modified Lyapunov–Krasovskii functional, which is used for exponential stability analysis in [33,34], we intend to design a state feedback controller K, G, that is $u(t) = Kx(t) + Gx(t-d(t))$, such that the reachable set of the closed-loop system

$$\dot{x}(t) = (A+BK)x(t) + (D+BG)x(t-d(t)) + Ew(t) \tag{4}$$

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is bounded by an ellipsoid $\varepsilon(P,1)$:

$$\varepsilon(P,1) = \left\{x \in \mathbb{R}^n : x^T P x \le 1, P > 0\right\} \tag{5}$$

The reachable set of system (4) is denoted as follows:

$$R_x = \{x(t)\,|x(t) \text{ and } w(t) \text{ satisfy } (2) \text{ and } (3), t \ge 0\}$$

The following three useful lemmas are given to derive the main results.

Lemma 1 ([35]). *The following relation is known as the Leibniz rule*

$$\frac{d}{dt}\int_{b(t)}^{a(t)} f(t,s)\,ds = \dot{a}(t)f[t,a(t)] - \dot{b}(t)f[t,b(t)] + \int_{b(t)}^{a(t)} \frac{\delta}{\delta t} f(t,s)\,ds.$$

Lemma 2 ([36]). *For any constant matrix $Q = Q^T > 0$, we have*

$$-\int_{t-d(t)}^{t} \dot{x}^T(s)Q\dot{x}^T(s)ds \le [x^T(t), x^T(t-d(t))]\begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix}[x^T(t), x^T(t-d(t))]^T.$$

Lemma 3 ([7]). *Let Q be a symmetric positive definite matrix. For any matrices P, S with appropriate dimensions, where $P = P^T$, then*

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0$$

if and only if $P - SQ^{-1}S^T > 0$.

Lemma 4 ([7]). *Let $V(x(0)) = 0$ and $w^T(t)w(t) \le w_m^2$, if $\dot{V}(x_t) + \alpha V(x_t) - \beta w^T(t)w(t) \le 0, \alpha > 0, \beta > 0$, then we have $V(x_t) \le \frac{\beta}{\alpha} w_m^2, \forall t > 0$.*

3. Main Results

Theorem 1. *For given scalars $h, u > 0$, if there exist matrices $L, H \in \mathbb{R}^{1\times n}, \overline{M}, \tilde{P}, \tilde{R}, \tilde{S}, \tilde{W}, \tilde{X}, \tilde{Y}, \tilde{Z}, \hat{R}, \hat{Z}, \check{Z} \in \mathbb{R}^{n\times n}$ with $\overline{M}, \tilde{R}, \tilde{S}, \tilde{W}, \tilde{X} > 0$ and a scalar $\alpha > 0$ such that they satisfy the following matrix inequalities:*

$$\begin{bmatrix} \overline{\Phi}_{11} & D\overline{M} + BH - \tilde{Y} & \overline{M}A^T + L^T B^T + \check{Z} + \alpha \overline{M} & E & \tilde{Y} \\ \star & -(1-u)e^{-\alpha h}\tilde{S} + u^2\tilde{W} & \overline{M}D + H^T B^T - \check{Z} & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}\hat{R} + \alpha\hat{Z} & E & \check{Z}^T \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -\tilde{W} \end{bmatrix} \le 0, \tag{6}$$

$$\begin{bmatrix} \overline{M} - \tilde{P} & \tilde{Y} \\ \star & \tilde{Z} + \frac{1}{h}e^{-\alpha h}\tilde{S} \end{bmatrix} \ge 0, \tag{7}$$

where $\overline{\Phi}_{11} = A\overline{M} + BL + \overline{M}A^T + L^T B^T + \alpha\overline{M} + \tilde{Y} + \tilde{Y}^T + \tilde{S} + h\tilde{R}$, then the reachable sets of the system (4) are bounded by an ellipsoid $\varepsilon(P,1)$ defined in (5). At this point, the state feedback gain is $K = L\overline{M}^{-1}, G = H\overline{M}^{-1}$.

Proof. To prove this theorem, let us consider the following Lyapunov–Krasovskii function:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{8}$$

where

$$V_1(x_t) = x^T(t)Px(t),$$
$$V_2(x_t) = \int_{t-d(t)}^{t} e^{-\alpha(s-t)}[x^T(s)Sx(s) + (h-t+s)x^T(s)Rx(s)]ds,$$
$$V_3(x_t) = \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix},$$
$$\eta(t) = \int_{t-d(t)}^{t} x(s)ds,$$

where P, S, R, X, Y, Z are symmetric matrices with appropriate dimensions. First, we prove that $V(x_t)$ in (8) is a good Lyapunov–Krasovskii functional candidate. For $t - d(t) \le s \le t$ and $0 \le d(t) \le h$, we can get $e^{-h} \le e^{-d(t)} \le e^{s-t} \le 1$ and $0 \le h - d(t) \le h - t + s \le h$. In the light of the Lamma 2, we have

$$V_2(x_t) \ge \int_{t-d(t)}^{t} e^{-\alpha(s-t)} x^T(s)Sx(s)ds \ge \frac{1}{h} e^{-\alpha h} \eta^T(t)S\eta(t),$$

then

$$V_2(x_t) + V_3(x_t) \ge \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix} \begin{bmatrix} X & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}.$$

If

$$\begin{bmatrix} X & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \ge 0, \tag{9}$$

then we have $V_2(x_t) + V_3(x_t) \ge 0$.

Hence, we have

$$\begin{cases} V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \ge V_1(x_t) = x^T(t)Px(t), \\ V(x_t) = 0, \ when \ \ x(s) = 0, \forall s \in [t - d(t), t], \end{cases} \tag{10}$$

which shows $V(x_t)$ in (8) is an L–K functional.

Next, from Lemma 1, we obtain the following time derivatives:

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$$\frac{d}{dt}V_1(x_t) = 2x^T(t)P[(A+BK)x(t) + (D+BG)x(t-d(t)) + Ew(t)], \tag{11}$$

$$\begin{aligned}
&\frac{d}{dt}V_2(x_t) = x^T(t)(S+hR)x(t) - (1-\dot{d}(t))e^{-\alpha d(t)}x^T(t-d(t))Sx(t-d(t)) \\
&- (1-\dot{d}(t))e^{-\alpha d(t)}(h-d(t))x^T(t-d(t))Rx(t-d(t)) \\
&- \int_{t-d(t)}^{t} e^{-\alpha(s-t)}x^T(s)Rx(s)\,ds - \alpha V_2(x_t) \\
&\leq x^T(t)(S+hR)x(t) - (1-u)e^{-\alpha h}x^T(t-d(t))Sx(t-d(t)) \\
&- \frac{1}{h}e^{-\alpha h}\eta^T(t)R\eta(t) - \alpha V_2(x_t),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\frac{d}{dt}V_3(x_t) = 2\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix}\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ (x(t)-x(t-d(t)) \end{bmatrix} \\
&+ 2\dot{d}\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix} x(t-d(t)) \\
&\leq 2\begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix}\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix}\begin{bmatrix} (A+BK)x(t) + (D+BG)x(t-d(t)) + Ew(t) \\ x(t)-x(t-d(t)) \end{bmatrix} \\
&+ \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix} W^{-1}\begin{bmatrix} Y^T & Z^T \end{bmatrix}\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + u^2x^T(t-d(t))Wx(t-d(t)),
\end{aligned} \tag{13}$$

where we used the relation that $2a^Tb \leq a^TW^{-1}a + b^TWb, W > 0$ and the constraints (2) and (3) in the derivation of the inequality (13).

Through (9) and (11)–(13), we obtain:

$$\begin{aligned}
&\dot{V}(x_t) + \alpha V(x_t) - \beta w^T(t)w(t) \\
&\leq \xi_t^T\left(\Omega + \begin{bmatrix} Y^T \\ 0 \\ Z^T \\ 0 \end{bmatrix} W^{-1}\begin{bmatrix} Y^T & 0 & Z^T & 0 \end{bmatrix}\right)\xi_t \\
&:= \xi_t^T\Psi\xi_t,
\end{aligned}$$

where

$$\xi_t^T = [x^T(t) \;\; x^T(t-d(t)) \;\; \eta^T(t) \;\; w^T(t)],$$

$$\Omega = \begin{bmatrix} \phi_{11} & (P+X)(D+BG)-Y & (A+BK)^TY+Z+\alpha Y & (P+X)E \\ \star & -(1-u)e^{-\alpha h}S+u^2W & (D+BG)^TY-Z & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}+\alpha Z & Y^TE \\ \star & \star & \star & \frac{-\alpha}{w_m^2} \end{bmatrix}$$

and

$\phi_{11} = (P+X)(A+BK) + (A+BK)^T(P+X) + \alpha(P+X) + Y + Y^T + S + hR.$

If $\Psi \leq 0$, by virtue of Lamma 3, we get

$$\begin{bmatrix} \Phi_{11} & M(D+BG)-Y & (A+BK)^TY+Z+\alpha Y & ME & Y \\ \star & -(1-u)e^{-\alpha h}S+u^2W & (D+BG)^TY-Z & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}+\alpha Z & Y^TE & Z \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -W \end{bmatrix} \leq 0, \tag{14}$$

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where

$M = P + X, \Phi_{11} = M(A + BK) + (A + BK)^T M + \alpha M + Y + Y^T + S + hR.$

By defining $N_1 = diag(M^{-1}; M^{-1}; Y^{-1}; I; M^{-1})$, pre- and post-multiplying the inequality (14) by N_1 and N_1^T, and defining $\overline{M} = M^{-1}, L = KM^{-1}, H = GM^{-1}, \tilde{X} = M^{-1}XM^{-1}$, $\tilde{Y} = M^{-1}YM^{-1}, \tilde{R} = M^{-1}RM^{-1}, \tilde{Z} = M^{-1}ZM^{-1}, \tilde{S} = M^{-1}SM^{-1}, \tilde{W} = M^{-1}WM^{-1}$, $\check{Z} = M^{-1}ZY^{-1}, \hat{R} = Y^{-1}RY^{-1}, \hat{Z} = Y^{-1}ZY^{-1}$, the following inequality is derived:

$$\begin{bmatrix} \overline{\Phi}_{11} & D\overline{M} + BH - \tilde{Y} & \overline{M}A^T + L^T B^T + \check{Z} + \alpha\overline{M} & E & \tilde{Y} \\ \star & -(1-u)e^{-\alpha h}\tilde{S} + u^2\tilde{W} & \overline{M}D + H^T B^T - \check{Z} & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}\hat{R} + \alpha\hat{Z} & E & \check{Z}^T \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -\tilde{W} \end{bmatrix} \leq 0, \quad (15)$$

where $\overline{M} = (P + X)^{-1}, \overline{\Phi}_{11} = A\overline{M} + BL + \overline{M}A^T + L^T B^T + \alpha\overline{M} + \tilde{Y} + \tilde{Y}^T + \tilde{S} + h\tilde{R}$. Thus, if inequality (15) holds, we have

$$\dot{V}(x_t) + \alpha V(x_t) - \frac{\alpha}{w_m^2} w^T(t)w(t) \leq 0.$$

which means, by virtue of Lemma 4, that $V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \leq 1$. Since $V_2(x_t) + V_3(x_t) \geq 0$, from inequality (9), we get $V_1(x_t) = x^T(t)Px(t) \leq 1$. Following a similar line, we need to convert condition (9) into

$$\begin{bmatrix} M - P & Y \\ \star & Z + \frac{1}{h}e^{-\alpha h}S \end{bmatrix} \geq 0 \quad (16)$$

By defining $N_2 = diag(M^{-1}; M^{-1})$ and pre- and post-multiplying the inequality (16) by N_2 and N_2^T, the following inequality is derived:

$$\begin{bmatrix} \overline{M} - \tilde{P} & \tilde{Y} \\ \star & \tilde{Z} + \frac{1}{h}e^{-\alpha h}\tilde{S} \end{bmatrix} \geq 0 \quad (17)$$

which completes the proof. This implies that the reachable sets of the closed-loop system in (4) are bounded by the ellipsoid $\varepsilon(P, 1)$ defined in (5), and the desired state-feedback controller can be obtained as $K = L\overline{M}^{-1}, G = H\overline{M}^{-1}$. □

Remark 1. *The L–K functional (8) is a function of the system state and control input, with certain positive definite and sub-positive definite conditions. With this function, the stability of time-delay systems can be determined, error bounds can be estimated, and control strategies can be designed to ensure system stability.*

Remark 2. *In order to obtain the 'smallest' possible bound for the reachable set, we introduce the method in [14,16]. That is, maximize δ subject to $\delta I \leq P$, which can be transformed to the following optimization problem for a scalar $\delta > 0$:*

$$\begin{cases} min & \bar{\delta}\,,\ (\bar{\delta} = \frac{1}{\delta}) \\ s.t & \begin{bmatrix} \bar{\delta} & I \\ I & P \end{bmatrix} \geq 0. \end{cases} \quad (18)$$

Then, by defining $N_3 = diag(I; \overline{M})$, pre- and post-multiplying inequality (18) by N_3 and N_3^T, and defining $\tilde{P} = \overline{M}P\overline{M}$, the following optimization is derived:

$$\begin{cases} min & \bar{\delta}\,,\ (\bar{\delta} = \frac{1}{\delta}) \\ s.t & \begin{bmatrix} \bar{\delta} & \overline{M} \\ \overline{M} & \tilde{P} \end{bmatrix} \geq 0. \end{cases} \quad (19)$$

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Therefore, we can obtain the 'smallest' possible bound for the reachable set of the system (4) by solving the following optimization problem for a scalar $\delta > 0$:

$$\begin{aligned} &min \;\; \bar{\delta}\;, \;(\bar{\delta} = \tfrac{1}{\delta}) \\ &s.t \begin{cases} (a) & \begin{bmatrix} \bar{\delta} & \overline{M} \\ \overline{M} & \tilde{P} \end{bmatrix} \geq 0, \\ (b) & (6),(7) or (15),(17). \end{cases} \end{aligned} \tag{20}$$

Remark 3. *In [16], for given scalars $h, u > 0$, if there exist matrices $P, S, R, W, X, Y, Z \in \mathbb{R}^{n\times n}$ with $P, S, R, W > 0$ and a scalar $\alpha > 0$, they satisfy the following matrix inequalities:*

$$\begin{bmatrix} \Phi_{11} & (P+X)D-Y & A^TY+Z+\alpha Y & (P+X)E & Y \\ \star & -(1-u)e^{-\alpha h}S+u^2W & D^TY-Z & 0 & 0 \\ \star & \star & -\frac{1}{h}e^{-\alpha h}+\alpha Z & Y^TE & Z \\ \star & \star & \star & \frac{-\alpha}{w_m^2} & 0 \\ \star & \star & \star & \star & -W \end{bmatrix} \leq 0, \tag{21}$$

$$\begin{bmatrix} X & Y \\ \star & Z+\frac{1}{h}e^{-\alpha h}S \end{bmatrix} \geq 0, \tag{22}$$

where $\Phi_{11} = (P+X)A + A^T(P+X) + \alpha(P+X) + Y + Y^T + S + hR.$

To find the 'smallest' bound for the reachable set, one may propose a simple optimization problem. That is, maximize δ subject to $\delta I \leq P$, which can be transformed to the following optimization problem for a scalar $\delta > 0$:

$$\begin{aligned} &min \;\; \bar{\delta}\;, \;(\bar{\delta} = \tfrac{1}{\delta}) \\ &s.t \begin{cases} (a) & \begin{bmatrix} \bar{\delta} & I \\ I & P \end{bmatrix} \geq 0, \\ (b) & (21),(22). \end{cases} \end{aligned} \tag{23}$$

Remark 4. *If $B = 0, K = 0$, and $G = 0$ in (6) and (7) of Theorem 1, the condition becomes the condition in [16], and (20) also becomes (23) in Remark 2; in this respect, the conclusion can be seen as an extension of [16].*

4. Numerical Example

An example is presented to illustrate our proposed method. The simulation is performed on Matlab and by using the LMI toolbox, a package for specifying and solving linear matrix inequalities.

Consider the linear state-delayed control system (1) with the following parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, w_m = 1.$$

By solving optimization problem (20), we get the sizes of the ellipsoidal bound of a reachable set for various u when $h = 0.70$ and $h = 0.75$. These results are summarized in Tables 1 and 2; figures are also provided in Figures 1 and 2. The results of our method and the method in [16] are compared.

Table 1. Computed $\bar{\delta}$'s in Example for $0 \leq d(t) \leq 0.7, |\dot{d}(t)| \leq u \leq 1$.

	u						
Method	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
[16]	2.2586	2.4970	2.8497	3.4355	4.5384	7.0915	16.8263
Theorem 1	1.4571	1.6372	1.8905	2.2702	2.9171	4.2496	8.1427

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As can be seen in Table 1, when $h = 0.7, u = 0.6$, our results greatly reduce the size of the ellipsoid. At this point, the state feedback gain is $K = [-0.5209 \quad -1.3698]$, $G = [0.9414 \quad -0.4495]$.

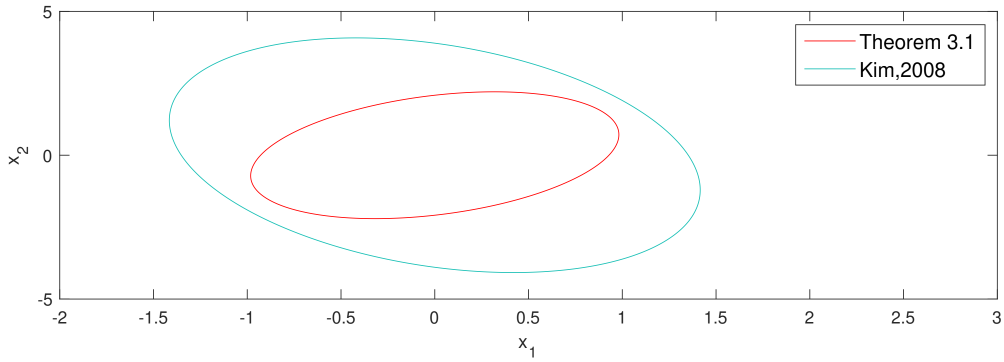

Figure 1. The bounding ellipsoids ε for $h = 0.7$ and $\mu = 0.6$.

Table 2. Computed $\bar{\delta}$'s in Example for $0 \le d(t) \le 0.75, |\dot{d}(t)| \le u \le 1$.

Method	**u**						
	0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**
[16]	2.5077	2.8071	3.2462	3.9935	5.4419	8.9945	25.1048
Theorem 1	1.6222	1.8417	2.1363	2.5992	3.4134	5.1046	10.4056

It is obvious that the results of this paper are better than the autonomous systems in [16], which shows the effectiveness of our method.

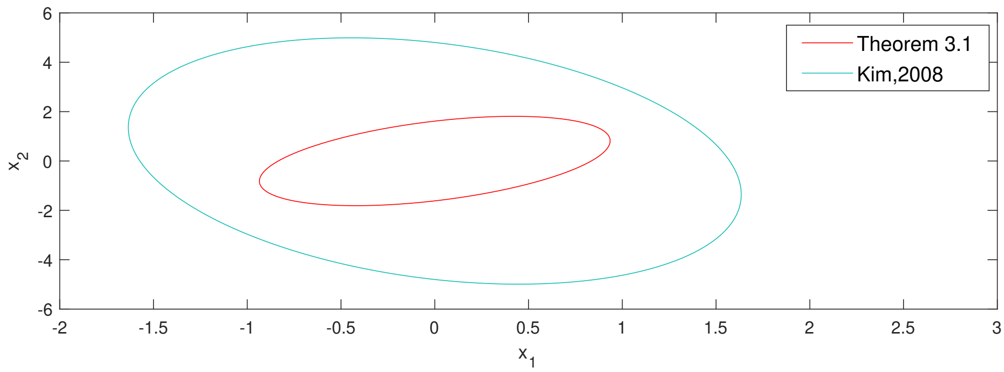

Figure 2. The bounding ellipsoids ε for $h = 0.75$ and $\mu = 0.6$.

5. Conclusions

In this paper, we deal with the problem of reachable set estimation and state-feedback controller design for linear time-delay control systems with bounded disturbances. Firstly, by constructing an appropriate L–K functional, we obtained a delay-dependent condition, which determines the admissible bounding ellipsoid for the reachable set of the system we considered. Secondly, a sufficient condition in form of liner matrix inequalities is given to solve the problem of controller design with reachable set estimation. Finally, by minimizing the volume of the ellipsoid and solving the liner matrix inequality, we obtain the desired ellipsoid and controller gain. The numerical example shows that the results of the method in this paper are better than that in [16] for the autonomous systems in [16], which shows the effectiveness of the proposed method.

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