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Calculation of Sommerfeld Integrals in Dipole Radiation Problems

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Abstract: The article proposes asymptotic methods for calculating Sommerfeld integrals, which enable us to calculate the integral using the expansion of a function into an infinite power series at the saddle point, where the role of a rapidly oscillating function under the integral can be fulfilled either by an exponential or by its product by the Hankel function. The proposed types of Sommerfeld integrals are generalized on the basis of integral representations of the Hertz radiator fields in the form of the inverse Hankel transform with the subsequent replacement of the Bessel function by the Hankel function. It is shown that the numerical values of the saddle point are complex. During integration, auxiliary or so-called standard integrals, which contain the main features of the integrand function, were used. As a demonstration of the accuracy of the technique, a previously known asymptotic formula for the Hankel functions was obtained in the form of an infinite series. The proposed method for calculating Sommerfeld integrals can be useful in solving the half-space Sommerfeld problem. An example is the expression in the form of an infinite series for the magnetic field of reflected waves, obtained directly through the Sommerfeld integral (SI).

Keywords: maxwell's equations; convolution; green's function; electromagnetic wave scattering; hertz dipole; sommerfeld integrals

MSC: 28A10; 28A25; 30B10; 30D10; 41A60; 78A02; 78A40

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1. Introduction

In 1896 A. Sommerfeld first proposed integrals for describing the diffraction of a plane wave in a problem on a half-line with ideal boundary conditions, which later received his name [1]. Later, at the beginning of the 20th century, Sommerfeld used the apparatus of SIs to solve the problem of diffraction of a plane wave at angles [2]. This also includes the exact solution of the problem of diffraction on an ideally reflecting half-plane, as well as his famous work on wave propagation in wireless telegraphy, the so-called Sommerfeld problem, involving two half-spaces filled with air and a lossy earth [3,4]

The mathematical apparatus of SI was further developed by Carslaw [5], and by G.D. Malyuzhinets [6], whose method enables us to solve diffraction problems in angular regions under various boundary conditions regardless of the angle.

From, the voluminous amount of published data, one may refer to the works of Fock [7], Van der Pol [8] and many others.

The calculation of SIs has both practical and theoretical significance since they can be used as Green's functions in the frame of integral equation formulations. In recent years, reports on combining analytical approximations and numerical computations have been published [9–12].

The article further considers two asymptotic methods for calculating SIs in the form of a sum of an infinite power series, when under the integral as a rapidly oscillating function there can, as a rule, be an exponential function or its product by the HF.

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2. General Solutions of Maxwell's Equations

Maxwell's equations for stationary electromagnetic fields with wave number k_0 equations $k_0 = \omega\sqrt{\mu_0\mu\varepsilon_0\varepsilon}$ can be reduced to the Helmholtz equations

$$(k_0^2 + \triangle)H = -\nabla \times j, \tag{1}$$

$$(k_0^2 + \triangle)E = -i\omega\mu_0\mu\, j + \frac{1}{\varepsilon_0\varepsilon}\nabla \cdot \rho. \tag{2}$$

The charge density ρ can be removed from the equation (2) by expressing it through the current density j using the charge conservation law $\rho = \frac{1}{i\omega}\nabla \cdot j$.

We will assume that all dynamic quantities in the above equations depend on time according to the harmonic law $\exp(-i\omega t)$. The dipole radiates time-harmonic electromagnetic (EM) waves at angular frequency ω [13] ($e^{-i\omega t}$ time dependence is assumed).

By performing the convolution operation on both sides of the equations (1), (2) and taking into account the equation for the Green's function ψ

$$(k_0^2 + \triangle)\psi = \delta(r),$$

we obtain general form of solutions to the Helmholtz equations

$$H = -\nabla \times (j * \psi), \tag{3}$$

$$E = \frac{1}{i\varepsilon_0\varepsilon\omega}(\nabla\nabla \cdot + k_0^2)(j * \psi) = \frac{1}{i\varepsilon_0\varepsilon\omega}\Big(\nabla \times \nabla \times (j * \psi) + j\Big). \tag{4}$$

Using the Fourier operator, solutions to the equations (3), (4) can be written as

$$H = -\nabla \times \mathbf{F}^{-1}[\tilde{j}\tilde{\psi}], \tag{5}$$

$$E = \frac{1}{i\varepsilon_0\varepsilon\omega}(\nabla\nabla \cdot + k_0^2)\mathbf{F}^{-1}[\tilde{j}\tilde{\psi}], \tag{6}$$

where

$$\psi = -\frac{1}{4\pi}\frac{e^{ik_0 r}}{r}, \quad \tilde{\psi} = \mathbf{F}^{-1}[\psi] = \frac{1}{k_0^2 - k^2} \tag{7}$$

are the Green function and its Fourier transform respectively, j is the current density, $\tilde{j} = \mathbf{F}^{-1}[j]$ is its Fourier transform, $*$ is the convolution symbol through all coordinates. Note that when performing the Fourier transform into (3) and (4) it is convenient to use the formal substitution

$$\nabla \to ik.$$

3. Field of a Hertz Point Radiator in Integral Representation

Prior to calculating the SIs, it is necessary to get integral representations of the fields of a Hertz point emitter, as the dipole current density is determined quite simply.

A vertical point (Hertzian) dipole, characterized by dipole moment $p = p\,e_z$, $p = \text{const}$, is directed to the positive z axis, at altitude z_0. The dipole moment p corresponds to the current density

$$j_d = -i\omega p\delta(x)\delta(y)\delta(z - z_0) \tag{8}$$

and its Fourier image

$$\tilde{j}_d = \mathbf{F}[j_d] = e_z\tilde{j}_d, \quad \tilde{j}_d = -i\omega p e^{-ik_z z_0}. \tag{9}$$

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Fourier inversion (5) in a cylindrical coordinate system

$$\mathbf{F}^{-1}\left[\tilde{j}_d\tilde{\psi}\right] = -\frac{i\omega p}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi} e_z e^{i(k_\rho\rho\cos k_\alpha + k_z(z-z_0))}\frac{k_\rho}{k_0^2-k^2}dk_\rho dk_z dk_\alpha = -\frac{p\omega}{4\pi}\int_0^\infty e_z e^{i\varkappa|z-z_0|}\mathrm{J}_0(k_\rho\rho)\frac{k_\rho}{\varkappa}dk_\rho \quad (10)$$

can be easily calculated using the integral representation of the Bessel function

$$\mathrm{J}_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos k_\alpha}dk_\alpha, \quad (11)$$

and the residue on the complex k_z plane.

Substituting the expression (10) into (5) and keeping in mind the relation $\nabla\times e_z f = -e_\alpha\partial f/\partial\rho$, we finally obtain the integral representation of the magnetic field of the emitter (in the form of the so-called inverse Hankel transform)

$$H_d = -e_\alpha\frac{p\omega}{4\pi}\frac{\partial}{\partial\rho}\int_0^\infty e^{i\varkappa|z-z_0|}\mathrm{J}_0(k_\rho\rho)\frac{k_\rho}{\varkappa}dk_\rho = e_\alpha\frac{p\omega}{4\pi}\int_0^\infty e^{i\varkappa|z-z_0|}\frac{k_\rho}{\varkappa}\mathrm{J}_1(k_\rho\rho)k_\rho dk_\rho, \quad (12)$$

$$\varkappa = \sqrt{k_0^2-k_\rho^2}.$$

Making similar calculations in (6), we find the integral representation for the electric field

$$E_d = \frac{ip}{4\pi\varepsilon_0\varepsilon_1}\int_{-\infty}^\infty e^{i\varkappa|z-z_0|}\left(e_\rho\mathrm{sgn}(z-z_0)i\frac{\partial}{\partial\rho}\mathrm{J}_0 k_\rho\rho) + e_z\frac{k_\rho^2}{\varkappa}\mathrm{J}_0(k_\rho\rho)\right)k_\rho dk_\rho = \frac{-ip}{4\pi\varepsilon_0\varepsilon_1}\int_{-\infty}^\infty e^{i\varkappa|z-z_0|}\left(e_\rho\mathrm{sgn}(z-z_0)i\mathrm{J}_1(k_\rho\rho) - e_z\frac{k_\rho}{\varkappa}\mathrm{J}_0(k_\rho\rho)\right)k_\rho^2 dk_\rho. \quad (13)$$

Now, using the transformation (A1) (see Appendix A), we can replace the Bessel function in the expression (12) for the magnetic field by the HF

$$H_d = -e_\alpha\frac{p\omega}{8\pi}\frac{\partial}{\partial\rho}\int_{-\infty}^\infty e^{i\varkappa|z-z_0|}\mathrm{H}_0^{(1)}(k_\rho\rho)\frac{k_\rho}{\varkappa}dk_\rho = e_\alpha\frac{p\omega}{8\pi}\int_{-\infty}^\infty e^{i\varkappa|z-z_0|}\mathrm{H}_1^{(1)}(k_\rho\rho)\frac{k_\rho^2}{\varkappa}dk_\rho. \quad (14)$$

This can be done similarly for the electric field in (13)

$$E_d = \frac{-ip}{8\pi\varepsilon_0\varepsilon}\int_{-\infty}^\infty e^{i\varkappa|z-z_0|}\left(e_\rho\mathrm{sgn}(z-z_0)i\mathrm{H}_1^{(1)}(k_\rho\rho) - e_z\frac{k_\rho}{\varkappa}\mathrm{H}_0^{(1)}(k_\rho\rho)\right)k_\rho^2 dk_\rho. \quad (15)$$

Note that the integral (14) can be calculated accurately

$$H_d = e_\alpha\frac{ip\omega}{4\pi}\frac{d}{d\rho}\frac{e^{ik_0 r}}{r} = -e_\alpha\frac{\omega p k_0}{4\pi}\frac{e^{i\kappa}}{r}\left(1+\frac{i}{\kappa}\right)\frac{\rho}{r}, \quad (16)$$

$$\kappa = k_0 r,\ \rho = r\sin\theta, \quad (17)$$

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using the integral representation of the spherical wave

$$\frac{e^{i\kappa}}{r} = -\frac{1}{2i}\int_{-\infty}^{\infty} e^{i\varkappa|z-z_0|}\mathrm{H}_0^{(1)}(k_\rho \rho)\frac{k_\rho}{\varkappa}\,dk_\rho, \tag{18}$$
$$r = \sqrt{\rho^2 + |z-z_0|^2}.$$

The same result can be easily obtained from (3) by calculating the convolution directly

$$H_d = ip\omega\nabla\times e_z\psi = e_\alpha \frac{ip\omega}{4\pi}\frac{d}{d\rho}\frac{e^{ikr}}{r}. \tag{19}$$

It should be noted that the exact expression for the integrals, as in (14), (18), can serve as an auxiliary integral, called the standard integral [14].

It is easier to obtain an exact expression for the electric field by calculating the curl of the expression (16) in a cylindrical coordinate system, according to the equations (4) and (3)

$$E_d = \frac{i}{\varepsilon_0\varepsilon\omega}\nabla\times H_d = -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\rho \mathrm{sgn}(z-z_0)\frac{\rho}{r}\frac{(z-z_0)}{r}\Big(1+3\big(i\kappa - 1\kappa^2\big)\Big) - e_z\Big(\frac{\rho^2}{r^2} - \Big(2\frac{(z-z_0)^2}{r^2} - \frac{\rho^2}{r^2}\Big)\big(i\kappa - 1\kappa^2\big)\Big)\Big\}. \tag{20}$$

In a spherical coordinate system it will take the form

$$E_d = -\frac{pk_0^2}{4\pi\varepsilon_0\varepsilon}\frac{e^{i\kappa}}{r}\Big\{e_\theta \sin\theta\Big(1+i\kappa - 1\kappa^2\Big) - e_r 2\cos\theta\Big(i\kappa - 1\kappa^2\Big)\Big\}, \tag{21}$$

due to transformations of basis vectors

$$\begin{cases} e_\rho = e_r \sin\theta + e_\theta \cos\theta, \\ e_z = e_r\cos\theta - e_\theta \sin\theta. \end{cases} \tag{22}$$

4. Generalized Asymptotic Formula for SI

Let us represent SI in the general form

$$I = \int_S G(\theta)e^{i\Psi(\theta)}d\theta, \tag{23}$$

where $G(\theta)$ is an analytical function, and the phase function $\Psi(\theta)$ is rapidly oscillating, which, according to the condition $d\Psi(\theta^*)/d\theta = 0$, has a saddle point $\theta = \theta^*$.

Let us expand the function at the saddle point into a Taylor series

$$G(\theta)e^{i\Psi(\theta)}e^{-i\left(\Psi(\theta^*)+\frac{1}{2}\Psi''(\theta^*)(\theta-\theta^*)^2\right)} = \sum_{m=0}^{\infty}\frac{a_m}{m!}(\theta-\theta^*)^m \tag{24}$$

by series coefficients

$$a_m = e^{-i\Psi(\theta^*)}\frac{d^m}{d\theta^m}G(\theta)e^{i\left(\Psi(\theta)-\frac{1}{2}\Psi''(\theta^*)(\theta-\theta^*)^2\right)}\Big|_{\theta=\theta^*}. \tag{25}$$

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Then SI (23) can be reduced to the auxiliary integral (A4) and represented as an infinite series

$$I = e^{i\Psi(\theta^*)} \sum_{m=0}^{\infty} \frac{a_m}{m!} \int_S (\theta - \theta^*)^m e^{\frac{i}{2}\Psi''(\theta^*)(\theta-\theta^*)^2} d\theta =$$
$$e^{i\Psi(\theta^*)} \sum_{m=0}^{\infty} \frac{a_{2m}}{(2m)!} \left(\frac{2i}{\Psi''(\theta^*)}\right)^{m+\frac{1}{2}} \Gamma(m + \tfrac{1}{2}). \tag{26}$$

4.1. Calculation of HF Asymptotics

Consider as the integral I in (23) SI representation of HF

$$\mathrm{H}_\nu^{(1)}(z) = \frac{1}{\pi} \int_{S_z} e^{i\nu(\theta - \frac{\pi}{2})} e^{iz\cos\theta} d\theta. \tag{27}$$

In order to obtain the asymptotic formula for the HF in the form of a power series, it is sufficient to make the following substitutions in (23):

$$I = \mathrm{H}_\nu^{(1)}(z),\ G(\theta) = \tfrac{1}{\pi} e^{i\nu(\theta - \pi/2)},\ \Psi(\theta) = z\cos\theta, \theta^* = 0,\ \Psi''(\theta^*) = -z \tag{28}$$

and calculate the initial coefficients of the series

$$\{a_{2m}\} = e^{-i\nu\pi/2}\Big(1; -\nu^2; \nu^4 + iz; -(\nu^6 + i15\nu^2 z + iz); \nu^8 +$$
$$i(70\nu^4 + 28\nu^2 + 1)z - 35z^2); \ldots\Big). \tag{29}$$

As a result, we get the well-known asymptotic expansion of HF of the first type for $z \to \infty$ (see [15])

$$\mathrm{H}_\nu^{(1)}(z) = \frac{e^{iz}}{\pi} \sum_{m=0}^{\infty} \frac{a_{2m}}{(2m)!} \left(\frac{2}{iz}\right)^{m+\frac{1}{2}} \Gamma(m + \tfrac{1}{2}) = \sqrt{\frac{2}{i\pi z}}$$
$$e^{i(z - \nu\frac{\pi}{2})} \left(1 + i\frac{(4\nu^2 - 1)}{8z} - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!\,(8z)^2} + \ldots\right). \tag{30}$$

The main term of the series is ($m = 0$)

$$\mathrm{H}_\nu^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} e^{i(z - \nu\frac{\pi}{2} - \frac{\pi}{4})}. \tag{31}$$

5. Asymptotic Expansion of the Integral Transformation Whose the Kernel is the Product of the Exponential and HF

Consider an integral of the type

$$F(r) = \int_{-\infty}^{\infty} R(k_\rho) \frac{e^{i\varkappa|z|}}{\varkappa} \mathrm{H}_0^{(1)}(k_\rho \rho) k_\rho dk_\rho, \tag{32}$$

where $R(k_\rho)$ is a complex-valued function,

$$r = \sqrt{\rho^2 + z^2},\ \varkappa = \sqrt{k_0^2 - k_\rho^2}.$$

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Here we have to use the auxiliary (standard) integral

$$\Phi(\rho, z) = \int_{-\infty}^{\infty} \frac{e^{i\varkappa|z|}}{\varkappa} \mathrm{H}_0^{(1)}(k_\rho \rho) k_\rho dk_\rho = -2i \frac{e^{ik_0 r}}{r}, \tag{33}$$

which follows from (32) (in the case of $R = 1$) and can be directly calculated using the expression (18).

Note that the integrand function F has a saddle point k_ρ^*, which satisfies the condition

$$\frac{d}{dk_\rho} \Psi(k_\rho) = z \frac{ik_\rho}{\sqrt{k_0^2 - k_\rho^2}} - \rho \frac{\mathrm{H}_1^{(1)}(k_\rho \rho)}{\mathrm{H}_0^{(1)}(k_\rho \rho)} = 0, \tag{34}$$

where

$$\Psi(k_\rho) = iz\sqrt{k_0^2 - k_\rho^2} + \ln \mathrm{H}_0^{(1)}(k_\rho \rho).$$

The root of the equation (34) for large values of the argument asymptotically approaches a real value

$$k_\rho^* \to \tilde{k}_\rho = k_0 \, \rho / r, \tag{35}$$

due to the asymptotics of HFs in (31) ($k_0 \, \rho \gg 1$).

Table 1 shows the numerical solutions of the equation (34), as well as their approximate values, according to the asymptotics in (35) for $k_0 = 70$, $z = 2$.

Table 1. Exact and approximate values of saddle points ($k_0 = 70$, $z = 2$).

Saddle point k_ρ^*	Approx. $\tilde{k}_\rho$	ρ	Calcul. error k_ρ^* (in %)	Error ratio $\tilde{k}_\rho$(in %)
49,4+i 9.5	49.5	2	0	19
62.7+i 4.9	62.6	4	0	8
68.0+i 2.4	66.4	6	$4 \cdot 10^{-10}$	4
69.4+i 1.4	67.9	8	$1 \cdot 10^{-8}$	3
69.8+i 0.9	68.6	10	$7 \cdot 10^{-8}$	2
69.9+i 0.6	69.0	12	$3 \cdot 10^{-7}$	1.5

To calculate the integral (32), we first expand the function in a Taylor series

$$R(k_\rho) = \sum_{m=0}^{\infty} \frac{(\varkappa - \varkappa^*)^n}{n!} \frac{d^n}{d\varkappa^n} R\left(\sqrt{k_0^2 - \varkappa^2}\right)\Big|_{\varkappa = \varkappa^*}, \quad \varkappa^{*2} = \sqrt{k_0^2 - k_\rho^{*2}} \tag{36}$$

in the saddle point k_ρ^*.

Substituting the function $R(k_\rho)$ in (32), the integral

$$F(r) = \sum_{m=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\varkappa^n} R\left(\sqrt{k_0^2 - \varkappa^{*2}}\right) \int_{-\infty}^{\infty} (\varkappa - \varkappa^*)^n \frac{e^{i\varkappa|z|}}{\varkappa} \mathrm{H}_0^{(1)}(k_\rho \rho) k_\rho dk_\rho \tag{37}$$

can be reduced to a simple integral, which can be calculated by n multiple differentiation of the integral (33) with respect to the parameter z

$$(-i)^n e^{i\varkappa^* z} \frac{d^n}{dz^n} (e^{-i\varkappa^* z} \Phi).$$

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Thus, the integral (32) is represented as an infinite series

$$F(r) = \sum_{m=0}^{\infty} \frac{(-i)^n}{n!} \frac{d^n}{d\varkappa^n} R\left(\sqrt{k_0^2 - \varkappa^{*2}}\right) e^{iz\varkappa^*} \frac{d^n}{dz^n}\left(\Phi(\rho, z) e^{-iz\varkappa^*}\right). \tag{38}$$

Here the main term of the asymptotic series is

$$F(r)_0 = -2i \frac{e^{ik_0 r}}{r} R(k_\rho^*). \tag{39}$$

Note that the traditional calculation of the integral (32) using the saddle point method along the Sommerfeld contour on the complex plane θ gives the same result as in (39)

$$F(r) \simeq -2i \frac{e^{ik_0 r}}{r} R(k_\rho^*) \quad \left(\sin\theta = \frac{\rho}{r}\right), \tag{40}$$

where integration is carried out over the angular integration parameter θ, according to the transformation $k_\rho = k_0 \sin\theta$.

6. Discussion

SI is represented as an infinite power series. Such an asymptotic series (26) allows one to calculate SIs with a given accuracy if G and Ψ are analytic functions. Using the example of the asymptotic expansion of HF (30) in subsection 4.1, we demonstrate an estimate of the accuracy of calculating SI, where G and Ψ are chosen as integer functions. However, it should be noted that the expression (26) loses its validity when the functions G and Ψ have singularities near the saddle point.

Obviously, the function , as a rapidly oscillating part of the integrand, affects the accuracy of the asymptotics of SIs. Therefore, in Section 5, a special integration technique is proposed. It should be noted that the value of the saddle point, because HF $\mathrm{H}_0^{(1)}(k_\rho \rho)$ is complex, and its imaginary part becomes vanishing at large values of the function argument.

It is interesting to note that the wave attenuation does not depend on the imaginary values of k_ρ^* and $\varkappa^*$. In fact, the exponent of the initial two terms of the series

$$F(r) \simeq -2i \frac{e^{ik_0 r}}{r} \left(R(k_\rho^*) + 2\frac{\varkappa*}{k_\rho^*} \frac{dR(k_\rho^*)}{dk_\rho} \left(\varkappa^* - k_0 \frac{z}{r} - \frac{i}{r}\frac{z}{r}\right)\right) \tag{41}$$

does not contain k_ρ^* or $\varkappa^*$, although, on the other hand, they are components of the wave vector $(k_0^2 = k_\rho^{*2} + \varkappa^{*2})$.

It should be noted that the integral in (32) can be effectively applied in solving the Sommerfeld problem in a half-space [16]. For the magnetic field of reflected waves

$$H(z, \rho) = -e_\alpha \frac{p\omega}{4\pi} \frac{\partial}{\partial\rho} F(r) \simeq H_d(z, \rho) R(k_\rho^*), \tag{42}$$

where the main term of the series is expressed through the magnetic field of the emitter (16) and $R(k_\rho^*)$. Here the function R has the meaning of the Fresnel reflection coefficient. It is convenient to calculate the electric field E through the rotor of the magnetic field H, similar to (20).

Further, we plan to take into account the case when the meromorphic function G has singularities on the complex plane k_ρ in the form of a pole and a branch point.

7. Conclusions

A generalized asymptotic formula for SI is obtained in the form of a complete power series (26) under the condition that G and Ψ are analytic functions. The proposed method for integrating SI involves a rapidly oscillating function of an exponential type, therefore, in solving practical problems,

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the asymptotics of HFs can often be used as an exponent. The integration technique was tested on the example of finding the asymptotics of HFs in the form of a series, where it (30), as expected, turned out to be identical to the previously known formula. The inverse Hankel transformation is reduced to the form of an integral along the real axis (32), where the rapidly oscillating part of the integrand, in addition to the exponential, contains HF. Moreover, the integrand function (32) has a singularity in the denominator in the form of a branch point.

The integral is represented as the sum of the asymptotic series (38) at the saddle point k_ρ^*. It is shown that the value at the saddle point is complex, whereas far from this point it tends to a real value.

In the quasi-static approximation ($n = 1$) the formula (41) is obtained, which leads to the interesting conclusion that the imaginary value of the saddle point, as can be seen from the formula, practically does not participate in the attenuation of waves, despite the fact that k_ρ^* is actually a component of the wave vector of the beam. It is shown that the leading term of the infinite series (39) and SI along a contour on the complex plane θ calculated by the saddle-point method are, as one would expect, identical. The integral F in (32) can be effectively used in solving the Sommerfeld problem for the magnetic field of reflected waves (42). In this case, the electric field is more easily determined through the rotor of the magnetic field.

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Abbreviations

The following abbreviations are used in this manuscript:

SI	Sommerfeld integral
HF	Hankel function

Appendix A. Replacing the Bessel Function by HF in the Inverse Hankel Transformations

Let an arbitrary function $f(k_\rho)$ satisfy the condition

$$f(k_\rho) = e^{i\pi\nu} f(e^{-i\pi} k_\rho),$$

then there is a representation of the inverse Hankel transform of order ν of the function $f(k_\rho)$

$$\int_0^\infty f(k_\rho) \mathrm{J}_\nu(k_\rho \rho) k_\rho dk_\rho = \frac{1}{2} \int_{-\infty+i0}^{\infty} f(k_\rho) \mathrm{H}_\nu^{(1)}(k_\rho \rho) k_\rho dk_\rho. \tag{A1}$$

Proof. Using the representation of the Bessel function

$$\mathrm{J}_\nu(k_\rho \rho) = \frac{1}{2}\left(\mathrm{H}_\nu^{(1)}(k_\rho \rho) + \mathrm{H}_\nu^{(2)}(k_\rho \rho)\right), \tag{A2}$$

and the analytical continuation of HF of the second kind

$$\mathrm{H}_\nu^{(2)}(k_\rho \rho) = -e^{i\pi\nu} \mathrm{H}_\nu^{(1)}(k_\rho \rho\, e^{i\pi}), \tag{A3}$$

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the integral along the negative real semi-axis can be expressed through HF of the first kind. Then the integrals are combined into one, which is contained on the right side of the expression (A1), based on the equality of the integrands and the above condition.

It should be kept in mind that HF has a cut along the negative real semi-axis. Therefore, the integration path must run parallel to the cut from above to a vanishingly small imaginary value $+i0$, which will be omitted in further presentation.

In particular, according to the above condition, the function $f(k_\rho)$ must be even with respect to the function $\mathrm{J}_0(k_\rho\rho)$ or $\mathrm{H}_0^{(1)}(k_\rho\rho)$.

Appendix B. Auxiliary Integral for Calculating SIs

Asymptotic calculations of SIs can be performed using the following auxiliary integral

$$\int\limits_{S_z} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = \left(\frac{2}{i\kappa}\right)^{m+\frac{1}{2}} \Gamma\left(m+\tfrac{1}{2}\right). \tag{A4}$$

In order to calculate the auxiliary integral, we first deform the contour S_z to the imaginary axis on the complex plane, which passes from top to bottom and slightly deviates from it by an infinitesimal real value to ensure the convergence of the integral. Then the integral can be represented successively as

$$\int\limits_{i\infty-0}^{-i\infty+0} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = -2 \int\limits_{0}^{i\infty-0} \theta^{2m} e^{-i\kappa\theta^2/2} d\theta = 2\sqrt{\pi}\,(2i)^{m-\frac{1}{2}} \frac{d^m}{d\kappa^m} \kappa^{-\frac{1}{2}}, \tag{A5}$$

where the last integral can be found by m-fold calculation of the derivative with respect to the parameter κ on both sides of the equal sign of the expression for the integral

$$\int\limits_{0}^{i\infty-0} e^{-i\kappa\theta^2/2} d\theta = -\sqrt{\frac{\pi}{2\kappa}} e^{-i\pi/4}, \tag{A6}$$

taking into account the representation of the gamma-function

$$\Gamma\left(m+\tfrac{1}{2}\right) = \sqrt{\pi}\,\frac{1\cdot 3\cdot \cdots \cdot (2m-1)}{2^m}.$$

The last integral (A6) can be easily obtained

$$\mathrm{C}(\infty) + i\,\mathrm{S}(\infty) = \int\limits_0^\infty e^{i\pi t^2/2} dt,$$

using Fresnel integrals [15] and the substitution $t = -i\sqrt{\kappa/\pi}\,\theta$.

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