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The Duality Principle for Multidimensional Optional Semimartingales

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Abstract: In option pricing, we often deal with options whose payoffs depend on multiple factors such as foreign exchange rates, stocks, etc. Usually, it leads to a knowledge of the joint distributions and complicated integration procedures. The paper develops an alternative approach that converts the option pricing problem to a dual one and presents a solution to the problem in the optional semimartingale setting. The paper contains several examples which illustrate results in terms of the parameters of models and options.

Keywords: optional semimartingales; derivative pricing; duality relations

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1. Introduction

In this study, our objective is to build upon the exploration of the duality principle in option pricing for multidimensional optional semimartingales, as introduced in Eberlein et al. (2009). The options under consideration have payoffs dependent on various factors, such as foreign exchange rates, interest rates, and multiple stocks, including examples like swap options and quanto options.

The pricing of these complex options often necessitates a comprehensive understanding of joint probabilities and involves intricate integration procedures. A suggested alternative approach, as outlined in Eberlein et al. (2009), involves transforming the original problem into its dual option pricing problem, rather than directly solving it. The subsequent resolution of this dual problem provides valuable insights into the valuation of options characterized by intricate dependencies and multiple influencing factors.

As stated earlier, the model we employ to describe the evolution of asset price processes is in the context of multidimensional optional semimartingale models. The family of optional semimartingales constitutes a diverse class of stochastic processes that includes cadlag semimartingales as a subset. Generally, such processes lack cadlag modifications, i.e., they are not inherently right-continuous with finite left-limits. Recently, numerous articles have explored optional processes and their relevance in financial and energy markets. For a comprehensive discussion on their application in the energy market, particularly addressing spike-related issues, please refer to Abdelghani et al. (2022).

Applying the outcomes derived from our study, we can establish duality relationships among swap options, quanto options, and standard call and put options within the framework of optional semimartingale models. This revelation carries notable implications, primarily manifesting in a substantial reduction in the computational complexity associated with the valuation of these financial instruments.

By leveraging our findings, we unveil a practical and efficient means of interrelating swap and quanto options with their standard call and put counterparts in optional semimartingale models. This not only enhances the understanding of the intricate connections between these financial instruments but also provides a tangible advantage in terms of computational efficiency when determining their respective valuations.

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Extensive literature has delved into the duality principle of option pricing for multidimensional models. Notable contributions include studies by Margrabe (1978), Geman et al. (1995), and Gerber and Shiu (1996) for the Black-Scholes model, Eberlein and Papapantoleon (2005) for examining time-inhomogeneous Levy processes, Fajardo and Mordecki (2006) for Levy processes, and Schröder (2015) is focusing on cadlag semimartingales. In Eberlein et al. (2009) work, Eberlein, Papapantoleon, and Shiryaev establish the predictable characteristics triplet for one-dimensional semimartingales under the dual measure, employing the Esscher change of measure. On the contrary, limited attention has been given to optional semimartingales.

The paper is organized as follows. In Section 2, we will establish crucial definitions and notations that serve as the foundation for understanding the subsequent sections. Additionally, we will review pertinent results regarding the canonical decomposition of optional semimartingales. This includes an exploration of their quadruplet of predictable characteristics, and their the Laplace cumulant process. Section 3 contains details and lemmas tailored for Exponentially Optional Semimartingale models. We included the main result of this work in Section 4, where we present results concerning multidimensional optional semimartingales and the treatment of linear transformations applied to multidimensional semimartingales. In Section 5, we delve into the duality relation for optional processes, specifically examining the European, Margrabe, and quanto options. Alongside this, in section 6, we provide explicit examples elucidating the application of the duality principles examined in the previous section.

2. Canonical Decomposition of Optional Semimartingales

Let's begin by defining key terms and notations crucial for understanding the upcoming sections. Specifically, we need to clarify the concept of an *un*usual probability space. An *un*usual probability space is represented as $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, wherein the natural filtration $\mathcal{F}_t$ is characterized by the absence of both right- and left-continuity. The completeness of the space arises from the fact that $\mathcal{F}$ incorporates all its $\mathbf{P}$-null sets, and $\mathbf{P}$ serves as a completed probability measure. We define $\mathbf{F}_+ = (\mathcal{F}_{t^+})$, with $\mathcal{F}_{t^+} = \bigcap_{t\leq s} \mathcal{F}_s$. Now, considering $\mathbf{F}^* := \mathbf{F}_+^P$ as the standard filtration—both right-continuous and complete—the construction of $\mathbf{F}_+^P$ involves enriching $\mathbf{F}_+$ with $\mathbf{P}$-null sets.

Let $\mathbb{R}^d$ represent the d-dimensional Euclidean space. In this space, the scalar product between two vectors u and v, both elements of $\mathbb{R}^d$, is denoted as $\langle u, v\rangle = u'v$, where u' denotes the transpose of the vector (or matrix) u.

Moving forward, the next step involves defining optional processes, which serve as the foundational processes under consideration in this paper. All definitions in this regard are drawn from Abdelghani and Melnikov (2020). To formulate the definition of optional processes, it is essential to establish the concept of an optional σ-algebra. In this context, we designate the σ-algebra $\mathcal{O}(F)$ on the interval $[0,\infty) \times \Omega$ as optional if it is generated by all right-continuous $\mathcal{F}_t$-adapted processes with left-hand limits.

Definition 1. *A random process $X = (X_t)$, $t \in [0,\infty)$, is said to be optional if it is $\mathcal{O}(F)$-measurable. Optional processes are progressively measurable, and thus clearly measurable. In general, optional processes have right and left limits but are not necessarily continuous on the right, left, or otherwise.*

Additionally, when it comes to optional stochastic processes, we can introduce the following definitions. The left process, denoted as $X_- = (X_{t_-})_{t\geq 0}$, where $X_{0_-} = X_0$ and $X_{t_-} = \lim_{s\uparrow t} X_s$. Similarly, we have the right processes $X_+ = (X_{t_+})_{t\geq 0}$. Also, the regular differential process $\Delta X = (\Delta X_t)_{t\geq 0}$ where $\Delta X_t = X_t - X_{t-}$ and the forward differential process $\Delta^+ X = (\Delta^+ X_t)_{t\geq 0}$ where $\Delta^+ X_t = X_{t+} - X_t$.

It is also crucial to understand the classification of stopping times, a topic that is thoroughly discussed in Chapter 2 of Abdelghani and Melnikov (2020). A solid understanding of these definitions is paramount for delving into the subsequent sections. Here, we would like to briefly define stopping

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(or Markov) times. Then we will restate two important categories of this random time which is important in the rest of the paper.

Definition 2. *Let $(\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, P)$. A random variable T on Ω with values in $R_+ \cup \{+\infty\}$ is called a stopping time or optional time of the filtration $(\mathcal{F}_t)$ if, for all $t \in R_+$, the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$ and belongs to the trivial filtration. The set of all stopping times is denoted by $\mathcal{T}$.*

Furthermore, Stopping times are also called Markov time. The simplest examples of them are $1_{T \leq t}$ and every positive constant.

Definition 3. *T is called a wide sense stopping time if for all t,*

$$\{T < t\} \in F_t \tag{1}$$

Also, we can say that T as a wide sense stopping time is optional in the wide sense if and only if it is optional to the family $\mathbf{F}_+$. Another characterization of wide sense stopping times is, for all $t > 0$, $T \wedge t$ is F_t-measurable.

We also require another classification of these stopping times, specifically, the concept of totally inaccessible stopping times, which will prove beneficial later on.

Definition 4. *$T > 0$ is totally inaccessible if for every sequence $(T_n)_{n=1,2,\ldots}$ of stopping times that increases to T the event $\{\lim_{n\uparrow\infty} T_n = T, T < \infty\}$ has zero probability*

Understanding the definition of a predictable process and σ-algebra is also valuable. We designate the σ-algebra $\mathcal{P}(\mathcal{F})$ on $\Omega \times [0, \infty)$ as predictable if it is generated by all left-continuous $\mathcal{F}_t$-adapted processes with right-hand limits or by sets $\{(\omega, t) : S(\omega) < t \leq T(\omega)\}$, where S, T vary across all Markov times.

Definition 5. *A random process $Y = (Y_t)$, $t \in [0, \infty)$, whose trajectories have limits from the right is said to be predictable if $P(\mathcal{F})$-measurable. Furthermore, it is termed strongly predictable if $Y_t \in P(\mathcal{F})$ and $Y_t^+ \in \mathcal{O}(\mathcal{F})$ holds for all t.*

Predictable processes are also optional, i.e., $\mathcal{P}(\mathcal{F}) \subseteq \mathcal{O}(\mathcal{F})$. Moreover, the definition of strongly predictable processes implies that for every stopping time T, the random variables $X_{T^+} 1_{T<\infty}$ and $X_{T^-} 1_{T<\infty}$ are $\mathcal{F}_{T^+}$-measurable and $\mathcal{F}_{T^-}$-measurable, respectively. The set of strongly predictable processes is denoted by $P_s(\mathcal{F})$ or P_s.

Let's revisit the definition of optional martingales, highlighting the difference from the definition of regular martingales on $\mathbf{F}^*$ satisfying the usual conditions. For an adapted process $X = (X_t)$ with $t \in \mathbb{R}_+$, it qualifies as a martingale (or alternatively, a supermartingale or submartingale) concerning $\mathbf{F}^*$ if, for every $t \in \mathbb{R}_+$, X_t is integrable, and the conditional expectation $X_s = \mathbb{E}[X_t|\mathbf{F}_s^*]$ (or $X_s \geq \mathbb{E}[X_t|\mathbf{F}_s^*]$, $X_s \leq \mathbb{E}[X_t|\mathbf{F}_s^*]$) holds almost surely for all $s \leq t$. Referring to the existence and uniqueness of optional martingales in Chapter 5 of Abdelghani and Melnikov (2020), we outline the definitions of optional martingale and optional local martingale here.

Definition 6. *We define $M = (M_t)_{t\in\mathbb{R}_+}$, as an optional martingale (or alternatively, an optional supermartingale or an optional submartingale) if;*

(a) M is an optional process (i.e., $M \in O(\mathcal{F})$),
(b) The random variable $M_T \cdot 1_{\{T<\infty\}}$ is integrable for any $T \in \mathcal{T}$,
(c) There exists an integrable random variable $\hat{M}$ such that $M_T = \mathbb{E}[\hat{M}|\mathcal{F}_T]$ (or, $M_T \geq \mathbb{E}[\hat{M}|\mathcal{F}_T]$, $M_T \leq \mathbb{E}[\hat{M}|\mathcal{F}_T]$) almost surely on $\{T < \infty\}$ for any $T \in \mathcal{T}$.

Definition 7. *A process $M = (M_t)_{t\geq 0}$ is called an optional local martingale if there exists a sequence $(R_n, M^{(n)}), n \in \mathbb{N}$, where R_n are wide sense stopping times, $R_n \uparrow \infty$ a.s. and $M^{(n)}$ is an optional martingale, such that $M1_{[0,R_n]} = M^{(n)}1_{[0,R_n]}$ and the random variable $M_{R_{n+}}$ is integrable for any $n \in \mathbb{N}$.*

Consider $\mathcal{M}^O(P)$ ($\mathcal{M}^O_{loc}(P)$), representing the set of optional (local) martingales in relation to the probability measure P. Let $\mathcal{V}$ denote the collection of all **F**-adapted processes with a finite variation where the variation for an optional process $A = (A_t)$, is given by

$$\mathrm{Var}(A)_t = \sum_{0\leq s\leq t} |A_{s^+} - A_s| + \int_{0^+}^{t} |dA^r| < \infty \tag{2}$$

where A^r is a right-continuous finite-variation process, and the series is absolutely convergent. This discussion lays the groundwork for the exploration of optional semimartingales and their canonical decomposition, which is the next crucial step in this study.

Definition 8. *The stochastic process X is called an optional semimartingale denoted by the set $\mathcal{S}$, if*

$$X = X_0 + M + A,$$

where $M \in \mathcal{M}^O_{loc}$, $A \in \mathcal{V}$, $A_0 = M_0 = 0$ and X_0 is an $\mathcal{F}_0$-measurable finite random variable.
If this decomposition holds with a strongly predictable process A of locally integrable variation, then the optional semimartingale X attains the special designation of a special optional semimartingale, denoted by the set $\mathcal{S}_s$.

The canonical and component representation of semimartingales is also essential to our analysis of optional semimartingales. The canonical and component representation of optional semimartingales can be seen as a natural consequence of the decomposition

$$X = X_0 + X^c + X^d + X^g$$

where X^c is a continuous optional semimartingale with the decomposition $X^c = a + m$, where a is continuous and strongly predictable with locally integrable variation, and m is a continuous local martingale. The discrete optional semimartingale parts, $X^d = a^d + m^d$ and $X^g = a^g + m^g$, are expressible in terms of some underlying measures of right and left jumps, respectively.

Let's proceed to define the characteristics of an optional semimartingale. Consider a d-dimensional optional semimartingale denoted as $Y = (Y_t)_{0\leq t\leq T}$, where $Y = (Y_1, \ldots, Y_d)'$ with the decomposition $Y_i = A_i + M_i$, where $M_i \in \mathcal{M}^O_{\mathrm{loc}}$, $A_i \in \mathcal{V}$, $M_{i,0} = 0, i = 1, \ldots, d$. Let $(T_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ be sequences of totally inaccessible stopping times and totally inaccessible wide sense stopping times, respectively. For $(E, \mathcal{E})$, consider the Lusin space where $E = (\mathbb{R}^d \setminus \{0\}) \cup \{\delta^d\} \cup \{\delta^g\}$ such that δ^d, and δ^g are some supplementary points or are the set of processes with finite variation on any segment $[0, t]$, **P**-a.s.; $\mathcal{E} = \mathcal{B}(E)$. On the σ-algebra $\tilde{\mathcal{E}} = \mathcal{B}(E) \times \mathcal{E}$, define integer random measures on $(\mathbb{R}_+ \times E, \tilde{\mathcal{E}})$

$$\mu^r(\Gamma) = \sum_{n\geq 1} \mathbf{1}_\Gamma(T_n, \beta^r_{T_n}), \qquad \mu^g(\Gamma) = \sum_{n\geq 1} \mathbf{1}_\Gamma(U_n, \beta^g_{U_n})$$

where $\mathbf{1}_\Gamma(\cdot)$ is an indicator function of a set $\Gamma \in \tilde{\mathcal{E}}$, $\beta^r_t = \Delta Y_t$ if $\Delta Y_t \neq 0$ and $\beta^r_t = \delta$ if $\Delta Y_t = 0$, $\beta^g_t = \Delta^+ Y_t$ if $\Delta^+ Y_t \neq 0, \beta^g_t = \delta$ if $\Delta^+ Y_t = 0$, $t > 0$. Under the *un*usual conditions on probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, any optional semimartingale Y can be decomposed as follows:

$$Y_t = Y_0 + B_t + C_t + \int_{]0,t]\times E} h(x) d(\mu^r - \nu^r) + \int_{[0,t[\times E} h(x) d(\mu^g - \nu^g)$$
$$+ \int_{]0,t]\times E} (x - h(x)) d\mu^r + \int_{[0,t[\times E} (x - h(x)) d\mu^g$$

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or in short notation

$$Y = Y_0 + B + C + \sum_{j=r,g} \left[h(x) * (\mu^j - \nu^j) + (x - h(x)) * \mu^j \right], \tag{3}$$

Here, Y_0 is $\mathcal{F}_0$-measurable random variable, $B \in \mathcal{V}, B_0 = 0$, and $C \in \mathcal{M}^O_{loc}, C_0 = 0$, and it is continuous. The ν^j represent the compensators of μ^j (refer to chapter 7 of Abdelghani and Melnikov (2020) for a detailed definition of compensators). The function $h(x)$ is a truncation function, meaning it is a bounded function with compact support that behaves as $h(x) = x$ around the origin. As an example, one could choose $h(x) = x1_{\{|x|\leq 1\}}$. Additionally, $W * \mu$ denotes the integral process, and $W * (\mu - \nu)$ denotes the stochastic integral with respect to the compensated random measure $(\mu - \nu)$.

The collection of the processes B, C and the measures ν^r, ν^g is called the (local) characteristics of the semimartingale Y with respect to the probability measure P. Let us call them a quadruplet and denote them by

$$Q(Y|P) = (B, C, \nu^r, \nu^g). \tag{4}$$

In addition, there exists an increasing predictable process A, predictable processes b, c and two transition kernels F^r and F^g from $(\Omega \times [0, T], \mathbf{P})$ into $(E, \mathcal{E})$ such that

$$B = b \cdot A, \qquad B_t = \int_0^t b_s dA_s$$
$$C = c \cdot A, \qquad C_t = \int_0^t b_s dA_s$$
$$\nu^j = F^j \otimes A, \qquad \nu^j([0,t] \times E) = \int_0^t \int_E F^j_s(dx) dA_s, \; j = r, g$$

Every optional semimartingale Y with characteristics $Q(Y|P) = (B, C, \nu^r, \nu^g)$ can be associated with an optional Laplace cumulant process defined by

$$G(u) = \langle u, B\rangle + \frac{1}{2}\langle u, Cu\rangle + (e^{\langle u,x\rangle} - 1 - \langle u, h(x)\rangle) * \nu^r + (e^{\langle u,x\rangle} - 1 - \langle u, h(x)\rangle) * \nu^g \tag{5}$$

Furthermore, expressing the optional Laplace cumulant process can be achieved using the process A, represented as $G(u) = g(u) \cdot A$ with

$$g(u) = \langle u, b\rangle + \frac{1}{2}\langle u, cu\rangle + (e^{\langle u,x\rangle} - 1 - \langle u, h(x)\rangle) * F^r + (e^{\langle u,x\rangle} - 1 - \langle u, h(x)\rangle) * F^g \tag{6}$$

Another significant distinction of the calculus of optional semimartingales on the *un*usual probability space, in comparison to regular semimartingales on the usual probability space, is their integration. To conclude this section and emphasize this difference, let us briefly discuss the results presented in Chapter 7 of Abdelghani and Melnikov (2020) concerning integration with respect to optional semimartingales.

The optional stochastic integral with respect to optional semimartingales X is defined in terms of the stochastic integrals with respect to its components A and m as

$$\begin{aligned} Y_t = \int_0^t h_s \, dX_s = h \circ X_t = \int_0^t h_s \, dA_s + \int_0^t h_s \, dm_s \\ = h \circ A_t + h \circ m_t = h \circ A^r_t + h \circ A^g_t + h \circ m^r_t + h \circ m^g_t, \end{aligned} \tag{7}$$

Where

$$h \circ A^r_t = \int_{0+}^t h_{s-} dA^r_s, \;\; h \circ A^g_t = \int_0^{t-} h_s dA^g_{s+},$$

$$h \circ m^r_t = \int_{0+}^t h_{s-} dm^r_s, \;\; h \circ m^g_t = \int_0^{t-} h_s dm^g_{s+},$$

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Moreover, given that $X = X^r + X^g = X^c + X^d + X^g$, we can express the integral with respect to X as

$$h \circ X = h \circ X^r + h \circ X^g = h \circ X^c + h \circ X^d + h \circ X^g,$$

where $X^r = A^r + m^r = A^c + m^c + A^d + m^d$ and $X^g = A^g + m^g$.

3. Exponentially Optional semimartingale models

In this section, we delve into specific details and lemmas tailored for Exponentially Optional Semimartingale models. The forthcoming discussions will cover various results, including equivalent statements for exponentially special optional semimartingales, a martingale version of the Lévy–Khintchine formula applicable to optional semimartingales, multiplicative decomposition for optional semimartingale, and a corollary addressing the uniqueness of the exponential optional compensator.

To start this section, we present definitions for a special optional semimartingale and an exponential optional compensator. This initial step aims to show differences between the findings in this section and those in the references Jacod and Shiryaev (1987) and Eberlein et al. (2009).

Definition 9. *An optional semimartingale X is called* exponentially special optional semimartingales *if* $\exp(X - X_0)$ *is a special optional semimartingale.*

Definition 10. *Let X be an optional semimartingale. A strongly predictable process $V \in \mathcal{V}$ is called exponential optional compensator of X if* $\exp(X - X_0 - V) \in \mathcal{M}^O_{loc}(P)$.

In the following lemma, our objective is to reformulate the outcome presented in Corollary II.2.42 of Jacod and Shiryaev (1987) for exponentially special optional semimartingales. This restatement serves as an extension of the original result by Jacob and Shiryaev.

Lemma 1. *Suppose X is an exponentially special optional semimartingale. The following statements are equivalent:*

a) X is an optional semimartingale, and it admits the local quadruplet of characteristics $Q(X|P) = (B, C, \nu^r, \nu^g)$.

b) For each $z \in \mathbb{R}$, the process $e^{zX} - (e^{zX_-}) \circ G(z) \in \mathcal{M}^O_{loc}$.

Proof. $(a) \Rightarrow (b)$: We consider the following decomposition of X :

$$X = M + B + \sum_{s \le t} [\Delta X_s - h(\Delta X_s)] + \sum_{s<t} \left[\Delta^+ X_s - h(\Delta^+ X_s)\right].$$

Applying the change of variables formula to $f(x) = e^{zx}$ and X, we get

$$\begin{aligned} e^{zX} - e^{zX_0} =& u e^{zX_-} \circ B + z e^{zX_-} \circ M \\ & - \frac{1}{2} z^2 e^{zX_-} \cdot C \\ & + e^{zX_-} \left(e^{zx} - 1 - zh(x)\right) * \mu^r \\ & + e^{zX} \left(e^{zx} - 1 - zh(x)\right) * \mu^g. \end{aligned} \tag{8}$$

Let us consider the right-hand side of (8); the second term is an optional local martingale; other terms are optional processes with a finite variation. However, since e^{zX} is a special optional semimartingale, $e^{zX_-} \left(e^{zx} - 1 - zh(x)\right) * \mu^r + e^{zX} \left(e^{zx} - 1 - zh(x)\right) * \mu^g$ is actually a process of locally integrable variation. Thus,

$$e^{zX_-} \left(e^{zx} - 1 - zh(x)\right) * (\mu^r - \nu^r) + e^{zX} \left(e^{zx} - 1 - zh(x)\right) * (\mu^g - \nu^g) \in \mathcal{M}^O_{loc},$$

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and the result follows.

$(b) \Rightarrow (a)$: By hypothesis, e^{zX} is an optional semimartingale for each $z \in \mathbb{R}$. Then, since $\log(x) \in C^2$, X is an optional semimartingale.

Let $(B^*, C^*, \nu^{r*}, \nu^{g*})$ be a good version of the characteristics of X. For each $z \in \mathbb{R}$ we associate to $(B^*, C^*, \nu^{r*}, \nu^{g*})$ a process $G^*(z)$. We have proved the implication $(a) \Rightarrow (b)$, thus $e^{zX} - e^{zX} \circ G^*(z) \in \mathcal{M}^O_{loc}$. Then the hypothesis and the uniqueness of the canonical decomposition of the special semimartingale e^{zX} show that $e^{zX} \circ G^*(z) = e^{zX} \circ G(z)$ up to an evanescent set. Using the orthogonality of G^r and G^g and integrating the processes e^{-zX_-} and e^{-zX}, we obtain that $G(z)$ and $G^*(z)$ are indistinguishable. Therefore, the set N of all ω for which there exists $z \in \mathbb{Q}$ and $t \in \mathbb{Q}_+$ with $G(z)_{t(\omega)} \neq G^*(z)_{t(\omega)}$ is P-null.

Now, we observe that the optional cumulant function is continuous and, as a result, completely characterized by its values on $\mathbb{Q}$. Consequently, outside of the set N, we have $B^*_t := B^{r*}_t + B^{g*}_t, B_t := B^r + B^g$, $C^*_t = C_t$ and $\nu^{j*}([0,t] \times \cdot) = \nu^j([0,t] \times \cdot), j = r, g$, for all $t \in \mathbb{Q}_+$ (see Gnedenko and Kolmogorov (1954)). This relationship also holds for all $t \in \mathbb{R}+$ due to the right-continuity of $B^{r*}, B^r, C^*, C, \nu^{r*}, \nu^r$, and the left-continuity of $B^{g*}, B^g, \nu^{g*}, \nu^g$. Therefore, (B, C, ν^r, ν^g) also serves as a version of the local characteristics of X. □

A significant finding in Jacod and Shiryaev (1987) (II.2.48 Corollary) and Eberlein et al. (2009) presents a martingale version of the Lévy–Khintchine formula specifically tailored for regular semimartingales. In this context, our goal is to broaden this result to encompass optional semimartingales.

Corollary 1. *Suppose X is an exponentially special optional semimartingale and $\mathcal{E}(G(z)) \neq 0$ for all $z \in \mathbb{R}$. If for each $z \in \mathbb{R}$, $\frac{e^{zX}}{\mathcal{E}(G(z))} \in \mathcal{M}^O_{loc}$. Then X is an optional semimartingale with characteristics (B, C, ν^r, ν^g).*

Proof. Denote $M(z) = \frac{e^{zX}}{\mathcal{E}(G(z))} \in \mathcal{M}^O_{loc}$. Then by Lemma 5.2.3 in Pak (2021) we get

$$
\begin{aligned}
e^{zX} =&\mathcal{E}(G(z))M(z) \\
=&e^{zX_0} + \mathcal{E}(G(z)) \cdot M^r_t(z) + M_-(z) \cdot \mathcal{E}^r_t(G(z)) \\
&+ \mathcal{E}_+(G(z)) \odot M^g_{t+}(z) + M(z) \odot \mathcal{E}^g_{t+}(G(z)) \\
=&e^{zX_0} + \mathcal{E}(G(z)) \cdot M^r_t(z) + M_-(z)\mathcal{E}^r_-(G(z)) \cdot G^r_t(z) \\
&+ \mathcal{E}_+(G(z)) \odot M^g_{t+}(z) + M(z)\mathcal{E}^r(G(z)) \odot G^g_{t+}(z) \\
=&e^{zX_0} + \mathcal{E}(G(z)) \cdot M^r_t(z) + \mathcal{E}_+(G(z)) \odot M^g_{t+}(z) \\
&+ e^{zX} \circ G_t(z)
\end{aligned}
$$

Note that we have used the definition of $\mathcal{E}(G(z))$ above and the notation $\odot$ for the optional integral with respect to the optional martingale process. It follows from the above that $e^{zX} - e^{zX_-} \circ G(z) \in \mathcal{M}^O_{loc}$. The result follows from $(b) \Rightarrow (a)$ of Lemma 1. □

In addition to the canonical decomposition for an optional semimartingale, we want to establish that every optional semimartingale also allows for a multiplicative decomposition, as demonstrated in the following theorem.

Theorem 1. *Let X be an optional semimartingale with $X_0 = 1$, such that X, X_- and X_+ take their values in $(0, \infty)$. Then X admits a multiplicative decomposition $X = LD$, where $L \in \mathcal{M}^O_{loc}$, $L > 0$ and $D \in \mathcal{V}$ is a positive strongly predictable process and $L_0 = D_0 = 1$, if and only if X is a special semimartingale.*

In this case, the multiplicative decomposition is unique (up to evanescence), and is given as follows, where $X = 1 + M + A$ is the canonical (additive) decomposition of X, and $H^r := \frac{1}{X_- + \Delta A}$ and $H^g := \frac{1}{X + \Delta^+ A}$ are necessarily locally bounded and positive:

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$$L = \mathcal{E}\left(H \circ M\right), \quad D = \frac{1}{\mathcal{E}\left(-H \circ A\right)} \tag{9}$$

Proof. If a multiplicative decomposition $X = LD$ exists, by Lemma 5.2.3 in Pak (2021) we have

$$X = LD = 1 + \int_{]0,t]} D_s dL^r_s + \int_{]0,t]} L_{s-} dD^r_s + \int_{[0,t[} D_{s+} dL^g_{s+} + \int_{[0,t[} L_s dD^g_{s+}. \tag{10}$$

But $\int_{]0,t]} D_s dL^r_s + \int_{[0,t[} L_s dD^g_{s+}$ is an optional local martingale and $\int_{]0,t]} L_{s-} dD^r_s + \int_{[0,t[} L_s dD^g_{s+}$ is a strongly predictable process with finite variation, thus X is an optional special semimartingale.

Suppose that we have two multiplicative decompositions $X = LD = \hat{L}\hat{D}$, for which we can write (10). The uniqueness of the canonical decomposition yields

$$\int_{]0,t]} L_{s-} dD^r_s + \int_{[0,t[} L_s dD^g_{s+} = \int_{]0,t]} \hat{L}_{s-} d\hat{D}^r_s + \int_{[0,t[} \hat{L}_s d\hat{D}^g_{s+} \ \ (a.s.).$$

Since D^r and D^g are orthogonal, we have

$$\int_{]0,t]} L_{s-} dD^r_s = \int_{]0,t]} \hat{L}_{s-} d\hat{D}^r_s, \quad \int_{[0,t[} L_s dD^g_{s+} = \int_{[0,t[} \hat{L}_s d\hat{D}^g_{s+} \ \ (a.s.).$$

Since $L, \hat{L}, D, \hat{D}$ are positive and $L/\hat{L} = \hat{D}/D$, we deduce that

$$\int_{]0,t]} 1/D_{s-} dD^r_s = \int_{]0,t]} 1/\hat{D}_{s-} d\hat{D}^r_s$$

and

$$\int_{[0,t[} 1/D_s dD^g_{s+} = \int_{[0,t[} 1/\hat{D}_s d\hat{D}^g_{s+}$$

then if

$$U = \int_{]0,t]} 1/D_{s-} dD^r_s + \int_{[0,t[} 1/D_s dD^g_{s+},$$

we also have

$$\int_{]0,t]} 1/\hat{D}_{s-} d\hat{D}^r_s + \int_{[0,t[} 1/\hat{D}_s d\hat{D}^g_{s+} = U,$$

which means, we have $D = \hat{D} = \mathcal{E}(U)$, and this proves the uniqueness.

It remains to prove the following: if $X = 1 + M + A$ is the canonical decomposition of the optional special semimartingale X (with $X > 0, X_- > 0$ and $X_+ > 0$), and if $N = H \circ M$ and $B \circ A$, then the strongly predictable process H is both locally bounded and positive. Moreover, $X = LD$, where $L = \mathcal{E}(N)$ and $D = 1/\mathcal{E}(-B)$ (as defined in (9)). Note that N, and L are optional local martingale, while B, and D are strongly predictable with finite variation. First, we observe that

$${}^pX = {}^pA + {}^pM = A_- + \Delta A + M_- = X_- + \Delta A,$$

where the notation ^{p} denotes a predictable projection of the process. Then for any predictable stopping time T, and since $X > 0$, we get $X_{T-} + \Delta A_T = \mathbf{E}(X_T|\mathcal{F}_T) > 0$ a.s. on $\{T < \infty\}$. Then Theorem 2.4.52 in Abdelghani and Melnikov (2020) yields that $X_- + \Delta A > 0$ outside P-null set, and we deduce $0 < H^r < \infty$. Next, notice that

$${}^oX_+ = {}^oA_+ + {}^oM_+ = A + \Delta^+ A + M = X + \Delta^+ A,$$

where ^{o} denotes an optional projection of the process. Then for any stopping time T, and since $X > 0$, we get $X_T + \Delta^+ A_T = \mathbf{E}(X_{T+}|\mathcal{F}_T) > 0$ a.s. on $\{T < \infty\}$. Then Theorem 2.4.52 in Abdelghani and Melnikov (2020) yields that $X + \Delta^+ A > 0$ outside P-null set, and we have we deduce $0 < H^g < \infty$. Furthermore, Lemma H^r and H^g are locally bounded because they both are strongly predictable. Since

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we know that H is locally bounded we can define L and D as above, and it remains to prove that $X = LD = \mathcal{E}(N)/\mathcal{E}(-B)$.

Observe that $\Delta B = \frac{\Delta A}{X_- + \Delta A} < 1$ and $\Delta^+ B = \frac{\Delta^+ A}{X + \Delta^+ A} < 1$ identically, hence $\mathcal{E}(-B) > 0$ and $\mathcal{E}(-B)_- > 0, \mathcal{E}(-B)_+ > 0$. Then we apply the change of variables formula to the function $f(x,y) = x/y$, or rather to a C^2 function coinciding with f for y outside an arbitrary small neighborhood of 0, to obtain with $D^* = 1/D$ and $\hat{X} = LD$:

$$
\begin{aligned}
\hat{X}_t =& 1 + \hat{X}_- \cdot N^r_t + \hat{X}_- \cdot B^r_t + \sum_{s \le t} \left(\frac{1 + \Delta N_s}{1 - \Delta B_s} - 1 - \Delta N_s - \Delta B_s \right) \hat{X}_{s-} \\
& + \hat{X} \odot N^g_{t+} + \hat{X} \odot B^g_{t+} + \sum_{s < t} \left(\frac{1 + \Delta^+ N_s}{1 - \Delta^+ B_s} - 1 - \Delta^+ N_s - \Delta^+ B_s \right) \hat{X}_s \\
=& 1 + \hat{X}_- \cdot N^r_t + \hat{X}_- \cdot B^r_t + \sum_{s \le t} \hat{X}_{s-} \frac{\Delta N_s + \Delta B_s}{1 - \Delta B_s} \Delta B_s \\
& + \hat{X} \odot N^g_{t+} + \hat{X} \odot B^g_{t+} + \sum_{s < t} \hat{X}_{s-} \frac{\Delta^+ N_s + \Delta^+ B_s}{1 - \Delta^+ B_s} \Delta^+ B_s \\
=& 1 + \hat{X}_- \left(1 + \frac{\Delta B_s}{1 - \Delta B_s} \right) \cdot N^r_t + \hat{X}_- \left(1 + \frac{\Delta B_s}{1 - \Delta B_s} \right) \cdot B^r_t \\
& + \hat{X} \left(1 + \frac{\Delta^+ B_s}{1 - \Delta^+ B_s} \right) \odot N^g_{t+} + \hat{X} \left(1 + \frac{\Delta^+ B_s}{1 - \Delta^+ B_s} \right) \odot B^g_{t+} \\
=& 1 + \frac{\hat{X}_- H^r}{1 - \Delta B} \cdot X^r_t + \frac{\hat{X} H^g}{1 - \Delta^+ B} \odot X^g_{t+} \\
=& 1 + \frac{\hat{X}_-}{X_-} \cdot X^r_t + \frac{\hat{X}}{X} \odot X^g_{t+}.
\end{aligned}
$$

Note that $\Delta B = H^r \Delta A, \Delta^+ B = H^g \Delta^+ A$ and $1 - H^r \Delta A = H^r X_-, 1 - H^g \Delta^+ A = H^g X$, hence $\frac{H^r}{1 - \Delta B} = \frac{1}{X_-}$ and $\frac{H^g}{1 - \Delta^+ B} = \frac{1}{X}$. Then

$$
\frac{1}{\hat{X}_-} \cdot \hat{X}^r + \frac{1}{\hat{X}} \odot \hat{X}^g = \frac{1}{X_-} \cdot X^r + \frac{1}{X} \odot X^g
$$

Thus, similarly to how we proved in the beginning, we deduce that $\hat{X} = X$. □

To conclude this section, we present a corollary addressing the uniqueness of the exponential optional compensator.

Corollary 2. *A real-valued optional semimartingale X has a unique (up to indistinguishably) exponential optional compensator V if X is exponentially special optional semimartingales.*

Proof. Suppose that X is exponentially special optional semimartingales. By Theorem 1, there exists a unique positive process $D \in \mathcal{V}$ such that $D_0 = 1$ and $L := \frac{\exp(X - X_0)}{D} \in \mathcal{M}^O_{loc}$. Since $\exp(X - X_0) > 0$ and $\frac{\exp(X_- - X_0)}{D_-} = L_- < \infty$, we have $D_- > 0$. Therefore, $V := \log(D) \in \mathcal{V}$. □

4. Multidimensional optional dual measures

In this section, our objective is to extend the findings presented in Eberlein et al. (2009) to the case of optional semimartingales. Eberlein, Papapantoleon, and Shiryaev, in their study, provided a characterization for one-dimensional semimartingales. These semimartingales were defined as scalar products of the driving semimartingale and d-dimensional vectors, under a suitable equivalent probability measure. Our goal is to generalize this result for *optional* semimartingales H, emphasizing its relevance to interest rate modeling through the optional stochastic integral in the Esscher transform.

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Let $L(H)$ represent the space of predictable integrable processes concerning the semimartingale H. As previously defined in Section 2, $v' \circ H$ signifies the optional stochastic integral of v' with respect to the optional semimartingale H. Additionally, $\tilde{G}(v)$ is used to denote the logarithmic transform of the cumulant process $G(v)$, expressed as $\mathcal{E}(G(v)) = \exp(\tilde{G}(v))$.

Theorem 2. *Let H be an $\mathbb{R}^d$-valued optional semimartingale with characteristics $Q(H|P) = (B, C, \nu^r, \nu^g)$. Let u be a vector in $\mathbb{R}^d$. Consider an $\mathbb{R}^d$-valued strongly predictable process v, such that $v \in L(H)$ and $v' \circ H$ is exponentially special optional semimartingale. Define the measure P_v via the Radon-Nikodym derivative*

$$\frac{dP_v}{dP} = \exp\left(v' \circ H_T - \tilde{G}(v)_T\right),$$

assuming that $e^{v' \circ H - \tilde{G}(v)} \in \mathcal{M}^O(P)$. Then the process H^u with $H^u = \langle u, H \rangle = u'H$ is a 1-dimensional optional semimartingale with characteristics $Q(H^u|P_v) = (B^u, C^u, \nu^{ur}, \nu^{ug})$ of the form

$$\begin{aligned}
B^u &= u'B + u'dCv + \sum_{j=r,g} \left(h(u'x) \frac{e^{v'x}}{1 + W^j(v)} - u'h(x) \right) * \nu^j, \\
C^u &= u'Cu, \\
\nu^{ur}(\omega, t, E) &= 1_E(u'x) \frac{e^{v'x}}{1 + W^r(v)} * \nu^r, \\
\nu^{ug}(\omega, t, E) &= 1_E(u'x) \frac{e^{v'x}}{1 + W^g(v)} * \nu^g,
\end{aligned} \tag{11}$$

where $E \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, and $W^j(v)_t = \int (e^{v'w} - 1)\nu^j(\{t\} \times dw)$, $j = r, g$.

Proof. Using Corollary 1 in section 3, it is sufficient to show that

$$\frac{e^{zH^u}}{\mathcal{E}(G^u(z))} \in \mathcal{M}^O_{loc}(P_v) \tag{12}$$

for all $z \in \mathcal{Z}$, where $\mathcal{Z} \subseteq \mathbb{R}$ open, such that

$$1_{\{|y|>1\}} e^{zy} * \nu^{ur} \in \mathcal{V}, \quad 1_{\{|y|>1\}} e^{zy} * \nu^{ug} \in \mathcal{V}. \tag{13}$$

Here G^u denotes the Laplace cumulant process associated with the characteristics

$$(B^u, C^u, \nu^{ur}, \nu^{ug})$$

Using Lemma 3.1 in Gasparyan (1988), (12) is equivalent to

$$Z \frac{e^{zH^u}}{\mathcal{E}(G^u(z))} = \frac{e^{v' \circ H}}{\mathcal{E}(G(v))} \frac{e^{zH^u}}{\mathcal{E}(G^u(z))} \in \mathcal{M}^O_{loc}(P) \tag{14}$$

Furthermore, using (11), condition (13) transforms to

$$1_{\{|u'x|>1\}} e^{(zu+v)'x} * \nu^r \in \mathcal{V}, \quad 1_{\{|u'x|>1\}} e^{(zu+v)'x} * \nu^g \in \mathcal{V}, \tag{15}$$

because $W^r(v)$ and $W^g(v)$ do not depend on x. Now, the exponent of the numerator in (14) is

$$v' \circ H + zH^u = v' \circ H + zu'H = (v + zu)' \circ H,$$

which has the unique exponential optional compensator by Corollary 2 in section 3

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$$G(v+zu)$$

under the measure P, for the cumulant defined in (5); that is,

$$\frac{e^{(v+zu)'\circ H}}{\mathcal{E}(G(v+zu))} \in \mathcal{M}^O_{loc}(P). \tag{16}$$

Therefore, to complete the proof, it suffices to show

$$\mathcal{E}(G(v))\mathcal{E}(G^u(z)) = \mathcal{E}(G(v+zu)). \tag{17}$$

Now, using the multiplication rule for optional stochastic exponential (see Lemma 8.1.2, Abdelghani and Melnikov (2020)) we get

$$\mathcal{E}(G(v))\mathcal{E}(G^u(z)) = \mathcal{E}(G(v) + G^u(z) + [G(v), G^u(z)]).$$

Hence, (17) reduces to showing that

$$G(v) + G^u(z) + [G(v), G^u(z)] = G(v+zu). \tag{18}$$

On the right-hand side of (18), we have

$$\begin{aligned} G(v+zu) =&(v+zu)' \cdot B + \frac{1}{2}\int (v+zu)' dC(v+zu) \\ &+ \sum_{j=r,g} \left(e^{(v+zu)'x} - 1 - (v+zu)'h(x)\right) * \nu^j \end{aligned} \tag{19}$$

Where operator $\cdot$ is the lebesgue integral. On the left-hand side, we have similarly

$$G(v) = v' \cdot B + \frac{1}{2}\int v' dCv + \sum_{j=r,g}\left(e^{v'x} - 1 - v'h(x)\right) * \nu^j. \tag{20}$$

The second term on the left-hand side of (18), using (11), is

$$\begin{aligned} G^u(z) =& zB^u + \frac{1}{2}z^2C^u + \sum_{j=r,g}(e^{zy} - 1 - zh(y)) * \nu^{uj} \\ =& zu'B + \int zv'dCu + \sum_{j=r,g} z\left(h(u'x)\frac{e^{v'x}}{1+W^j(v)} - u'h(x)\right) * \nu^j \\ &+ \frac{1}{2}z^2u'Cu + \sum_{j=r,g}\left(e^{zu'x} - 1 - zh(u'x)\right)\frac{e^{v'x}}{1+W^j(v)} * \nu^j \\ =& zu'B + \int zv'dCu + \frac{1}{2}z^2u'Cu \\ &+ \sum_{j=r,g}\left(\frac{e^{(zu+v)'x} - e^{v'x}}{1+W^j(v)} - zu'h(x)\right) * \nu^j. \end{aligned} \tag{21}$$

The last term on the left-hand side of (18) is

$$[G(v), G^u(z)] = \sum_{t\leq \cdot}\Delta G(v)_t \Delta G^u(z)_t + \sum_{t<\cdot}\Delta^+ G(v)_t \Delta^+ G^u(z)_t \tag{22}$$

since G, G^u are optional processes of finite variation. Now, we proceed as follows. First, we show that the drift terms on the left-hand side and right-hand side of (18) are equal. Then we show that the diffusive terms are equal. Finally, we prove that the jump terms on the left-hand side and right-hand

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side are equal. The drift term and the diffusive term are rather easy to handle; indeed from (20) and (21), we have that the drift term of the left-hand side of (18) is

$$v' \cdot B + zu'B = (v + zu'B) \cdot B \tag{23}$$

Similarly, the diffusive term of the left-hand side of (18) is

$$\frac{1}{2}\left(\int v'dCv + \int zv'dCu + \int zu'dCv + z^2u'Cu\right) = \frac{1}{2}\int (v+zu)'dC(v+zu), \tag{24}$$

since the matrix C is symmetric. Hence, both terms agree with the right-hand side.

The jump terms are more difficult to manipulate due to the presence of the fixed times of discontinuity for the optional semimartingale H, which entails that the Laplace cumulant process is discontinuous. Regarding the fixed times of discontinuity, we have

$$\begin{aligned} \Delta G(v)_t &= \int (e^{v'x} - 1)\nu^r(\{t\} \times dx) = W^r(v)_t, \\ \Delta^+ G(v)_t &= \int (e^{v'x} - 1)\nu^g(\{t\} \times dx) = W^g(v)_t, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \Delta G^u(z)_t &= \int (e^{zy} - 1)\nu^{ur}(\{t\} \times dy) \\ &= \int (e^{zu'x} - 1)\frac{e^{v'x}}{1 + W^r(v)_t}\nu^r(\{t\} \times dx) \\ &= \int (e^{(v+zu)'x} - e^{v'x})\frac{1}{1 + W^r(v)_t}\nu^r(\{t\} \times dx). \end{aligned} \tag{26}$$

Similarly,

$$\Delta^+ G^u(z)_t = \int (e^{(v+zu)'x} - e^{v'x})\frac{1}{1 + W^g(v)_t}\nu^g(\{t\} \times dx). \tag{27}$$

An important observation here is that $W(v)$ does not depend on the integrating variable x. Hence, we can pull it out of the integration, and get

$$\begin{aligned} \Delta G^u(z)_t =& \frac{1}{1 + W^r(v)_t}\int (e^{(v+zu)'x} - e^{v'x})\nu^r(\{t\} \times dx), \\ \Delta^+ G^u(z)_t =& \frac{1}{1 + W^g(v)_t}\int (e^{(v+zu)'x} - e^{v'x})\nu^g(\{t\} \times dx). \end{aligned} \tag{28}$$

Hence, for the last term in (18), we can calculate further using (28)

$$\begin{aligned} [G(v), G^u(z)] =& \sum_{t\leq \cdot}\Delta G(v)_t \Delta G^u(z)_t + \sum_{t<\cdot}\Delta^+ G(v)_t \Delta^+ G^u(z)_t \\ =& \sum_{t\leq \cdot}\frac{W^r(v)_t}{1 + W^r(v)_t}\int (e^{(v+zu)'x} - e^{v'x})\nu^r(\{t\} \times dx) \\ &+ \sum_{t<\cdot}\frac{W^g(v)_t}{1 + W^g(v)_t}\int (e^{(v+zu)'x} - e^{v'x})\nu^g(\{t\} \times dx) \\ =& \sum_{t\leq \cdot}\frac{1 + W^r(v)_t - 1}{1 + W^r(v)_t}\int (e^{(v+zu)'x} - e^{v'x})\nu^r(\{t\} \times dx) \end{aligned} \tag{29}$$

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$$
\begin{aligned}
&+\sum_{t<\cdot}\frac{1+W^g(v)_t-1}{1+W^g(v)_t}\int(e^{(v+zu)'x}-e^{v'x})\nu^g(\{t\}\times dx)\\
=&\sum_{t\leq\cdot}\bigg(\int(e^{(v+zu)'x}-e^{v'x})\nu^r(\{t\}\times dx)\\
&-\frac{1}{1+W^r(v)_t}\int(e^{(v+zu)'x}-e^{v'x})\nu^r(\{t\}\times dx)\bigg)\\
&+\sum_{t<\cdot}\bigg(\int(e^{(v+zu)'x}-e^{v'x})\nu^g(\{t\}\times dx)\\
&-\frac{1}{1+W^g(v)_t}\int(e^{(v+zu)'x}-e^{v'x})\nu^g(\{t\}\times dx)\bigg).
\end{aligned}
$$

Now, using the above observations, we can show that the jump terms on the left-hand side and right-hand side of (18) are equal. Indeed, we can express the integrals with respect to the compensators ν^r and ν^g as sums in (20) and (21). Then we have that the jump term on the left-hand side of (18) is

$$
\begin{aligned}
&\sum_{t\leq\cdot}\int\bigg(e^{v'x}-1-v'h(x)+\frac{e^{(zu+v)'x}-e^{v'x}}{1+W^r(v)_t}\\
&-zu'h(x)+e^{(v+zu)'x}-e^{v'x}\\
&-\frac{1}{1+W^r(v)_t}\int(e^{(v+zu)'x}-e^{v'x})\bigg)\nu^r(\{t\}\times dx)\\
&+\sum_{t\leq\cdot}\int\bigg(e^{v'x}-1-v'h(x)+\frac{e^{(zu+v)'x}-e^{v'x}}{1+W^g(v)_t}\\
&-zu'h(x)+e^{(v+zu)'x}-e^{v'x}\\
&-\frac{1}{1+W^g(v)_t}\int(e^{(v+zu)'x}-e^{v'x})\bigg)\nu^g(\{t\}\times dx)\\
=&\sum_{t\leq\cdot}\int\left(e^{(v+zu)'x}-1-(v+zu)'h(x)\right)\nu^r(\{t\}\times dx)\\
&+\sum_{t<\cdot}\int\left(e^{(v+zu)'x}-1-(v+zu)'h(x)\right)\nu^g(\{t\}\times dx)\\
=&\sum_{t\leq\cdot}\left(e^{(v+zu)'x}-1-(v+zu)'h(x)\right)*\nu^r\\
&+\left(e^{(v+zu)'x}-1-(v+zu)'h(x)\right)*\nu^g
\end{aligned}
$$

which equals the corresponding quantity on the right-hand side of (18). The proof is finished. □

5. Duality relations

In this section, we establish an equivalent relation, namely the duality relation, between the prices of Margrabe options, and quanto options with European call and put options in a market characterized by an unusual probability space. This analysis takes into account a portfolio composed of optional processes.

To initiate, we aim to establish the duality relationship between the value of a swap option, specifically a Margrabe option or Spread option, with a payoff of $M_T = (S^1_T - S^2_T)^+$, and the payoff of European call and put option. It is important to note that we represent the payoff of a European call option at maturity T by $C_T = (S_T - K)^+$, where K is the strike price, and for the put option, it is denoted by $P_T = (K - S_T)^+$.

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Theorem 3. *Assume that the asset price processes S^1 and S^2 are exponential special optional semimartingales and $e^{H^i} \in \mathcal{M}^O(P)$, $i = 1, 2$. Then we can relate the value of a swap option and a European option via the following equality:*

$$M_T = P_T^v(1, S_T^u) = C_T^\theta(S_T^\zeta, 1), \tag{30}$$

Or

$$\mathbf{E}(S_T^1 - S_T^2)^+ = \mathbf{E}_v(1 - S_T^u)^+ = \mathbf{E}_\theta(S_T^\zeta - 1)^+, \tag{31}$$

where the characteristics $(C^u, \nu^{ur}, \nu^{ug})$ and $(C^\zeta, \nu^{\zeta r}, \nu^{\zeta g})$ of $H^u = \log S^u$ and $H^\zeta = \log S^\zeta$, respectively, are given by Theorem 2 for $v = (1,0)'$, $u = (-1,1)'$, and $\theta = (0,1)', \zeta = (1,-1)'$.

Proof. We will use asset S^1 as the numeraire asset; if we use asset S^2 instead, then we get the duality relationship with a call option. The value of the swap option is

$$\begin{aligned} M_T = \mathbf{E}(S_T^1 - S_T^2)^+ =& \mathbf{E}\left[e^{H_T^1}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right] \\ =& \mathbf{E}\left[e^{v'H_T}\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right] \end{aligned} \tag{32}$$

where $v = (1,0)'$. Moreover, $e^{v'H} \in \mathcal{M}^O(P)$ by assumption. Define a new measure P_v via the Radon-Nikodym derivative

$$\frac{dP_v}{dP} = e^{v'H_T},$$

then the pricing problem (32) becomes

$$M_T = \mathbf{E}(S_T^1 - S_T^2)^+ = \mathbf{E}_v\left[\left(1 - \frac{S_T^2}{S_T^1}\right)^+\right]$$

where we define the process $S^u = (S_t^u)_{0 \le t \le T}$ via

$$S_t^u = \frac{S_t^2}{S_t^1} = \frac{e^{H_t^2}}{e^{H_t^1}} = e^{u'H_t} = e^{H_t^u}, \quad 0 \le t \le T,$$

for $u = (-1,1)'$. The triplet of local characteristics of the semimartingale H^u is given by theorem 2 for $v = (1,0)'$ and $u = (-1,1)'$. Now, applying Lemma 3.1 in Gasparyan (1988), we obtain that

$$e^{u'H} \in \mathcal{M}^O(P_v)$$

since

$$e^{u'H}e^{v'H} = e^{H^2} \in \mathcal{M}^O(P)$$

Therefore, we conclude that

$$M_T = \mathbf{E}(S_T^1 - S_T^2)^+ = \mathbf{E}_v(1 - S_T^u)^+ = P_T^v(1, S_T^u).$$

Using the same methodology, we derive the proof of equivalency for the Margrabe option and the call option as well. □

Additionally, we demonstrate another duality, this time between a quanto call option, with the payoff $Q_T^C = S_T^1(S_T^2 - K)^+$, and a European call option with payoff as defined before.

Theorem 4. *Assume that the asset price processes S^1 and S^2 are exponential special optional semimartingales and $e^{H^i} \in \mathcal{M}^O(P)$, $i = 1, 2$. Then we can relate the value of a quanto call option and a European call option via the following duality:*

$$Q_T^C = C_T^v(S_T^u, K) \tag{33}$$

Or

$$\mathbf{E}[S_T^1(S_T^2 - K)^+] = \mathbf{E}_v(S_T^u - K)^+, \tag{34}$$

where the characteristics $(C^u, \nu^{ur}, \nu^{ug})$ of $H^u = \log S^u$ are given by Theorem 2 for $v = (1,0)'$, $u = (0,1)'$. An analogous duality result relates the quanto put option and the European put option.

Proof. The value of the quanto call option is

$$\begin{aligned} Q_T^C = \mathbf{E}[S_T^1(S_T^2 - K)^+] =& \mathbf{E}[e^{H_T^1}(S_T^2 - K)^+] \\ =& \mathbf{E}[e^{v'H}(e^{H_T^2} - K)^+] \\ =& \mathbf{E}_v[(e^{H_T^u} - K)^+], \end{aligned}$$

where $\frac{dP_v}{dP} = e^{v'H_T}$ for $v = (1,0)'$ and $H^u = u'H$ for $u = (0,1)'$. Hence, the statement follows. □

Additionally, we delve into the *Call-Put duality* within this market, exploring two relations for these two contracts. The proof for one of them (a) requires utilizing theorem (2) to establish this duality, while the proof for the other relation (b) involves portfolio value and the Black-Scholes model.

Theorem 5. *Assume that the asset price process S is exponential special optional semimartingales and $e^H \in \mathcal{M}^O(P)$. Then we can relate the value of a European call option and a European put option via the following equivalencies:*

$$a)\ P_T(K, S_T) = KC_T^v(S_T^u, 1/K) \tag{35}$$

where the characteristics $(C^u, \nu^{ur}, \nu^{ug})$ of $H^u = \log S^u$ is given by Theorem 2 for $v = 1$, $u = -1$. Furthermore, we can relate the value of a standard put option with volatility σ to a standard call option via the following equality with volatility $-\sigma$, as follows:

$$b)\ P_T(K, S_T) = C_T(-S_T, -k),\ Or\ P_T(K, S_T, \sigma) = C_T(-S_T, -k, -\sigma) \tag{36}$$

a. The value of a put option is

$$\begin{aligned} P_T(K, S_T) = \mathbf{E}[(K - S_T)^+] &= \mathbf{E}[e^{H_T}(Ke^{-H_T} - 1)^+] \\ &= K\mathbf{E}[e^{H_T}(e^{-H_T} - 1/K)^+] \\ &= K\mathbf{E}[e^{vH_T}(e^{-H_T} - 1/K)^+] \\ &= K\mathbf{E}_v[(e^{-H_T} - 1/K)^+] \end{aligned}$$

where $\frac{dp_v}{dp} = e^{vH_T}$ for $v = 1$ and $H^u = uH$ for $u = -1$. Thus, the statement (a) follows. □

b. To prove this duality, we employ a different approach than the previous, specifically utilizing the Black-Scholes model. In the following, we provide detailed proof of the pricing of a European call option and subsequently investigate the pricing of a European put option. We then establish the relationship between these two.

Assume that the market consists of two types of securities B and S and a portfolio $\pi = (\eta, \xi)$ which is composed of the optional processes η and $\eta.\xi$ is the volume of the reference asset B, while ξ is the volume of the security S. Suppose $B_t > 0$ and $S_t \geq 0$ for all $t \geq 0$ and write the ratio process $R_t = S_t/B_t$. Then, the discounted value of the portfolio is $Y_t = \eta_t + \xi_t R_t$ which is a real-valued optional semimartingale that had right and left limits. Furthermore, we restrict the portfolio, π, to

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be self-financing that is $dY_t = \xi dR_t$ where the interest rate is zero. Now let's consider the augmented Balck-Scholes model with left and right jumps as described in Abdelghani and Melnikov (2020),

$$B_t = B_0 + \int_{]0,t]} rB_s ds \tag{37}$$

$$S_t = S_0 + \int_{]0,t]} S_{s-}(\mu ds + \sigma dW_s + adL^r_s) + \int_{[0,t[} bS_s dL^g_{s+} \tag{38}$$

where $L^r_t = L_t - \lambda t$, $L^g_t = -\bar{L}_{t-} + \gamma t$, and r, μ, σ, a, and b are constants. W is diffusion term and L and $\bar{L}$ are independent Poisson with constant intensity λ and γ, respectively. Let the initial wealth account for B_0 and the initial price be S_0. We can write S as $S_t = S_0\mathcal{E}(H)$, where $H_t = \mu t + \sigma W_t + a(L_t - \lambda t) + b(\gamma t - \bar{L}_{t-})$, with $H_0 = 0$, and $B_t = B_0 \exp(h_t)$ where $h_t = rt$. Furthermore, we are gonna use the stochastic exponential form mentioned in Abdelghani and Melnikov (2020) as follows:

$$\mathcal{E}(H)_t = \exp[H_t - \frac{1}{2}\langle H^c, H^c\rangle] \prod_{0<s\le t}(1+\Delta S_s)e^{-\Delta S_s} \prod_{0\le s<t}(1+\Delta^+ S_s)e^{-\Delta^+ S_s} \tag{39}$$

In this case, the Ratio process is:

$$R_t = S_0 \exp\left\{\left(\mu - r - \frac{1}{2}(\sigma^2 - \lambda a^2 - \gamma b^2)\right)t + \sigma W_t\right\} \times \prod_{0<s\le t}\left[(1+a\Delta L_t)e^{-a\Delta L_t}\right] \prod_{0\le s<t}\left[(1-b\Delta^+ \bar{L}_{s-})e^{-b\Delta^+ \bar{L}_{s-}}\right]$$

and is not a local optional martingale. So we need to find a suitable derivative to transfer this ratio under a martingale measure Q such that Y_t is a local optional martingale with respect to Q (see Melnikov and Shiryaev (1996) and for further explanation, you can refer to Abdelghani and Melnikov (2020) chapter 9). In the following, we repeat some of the key concepts. So we want the derivative $Z = \mathcal{E}(N)$ for which we have to find a local martingale N such that $\Psi(h, H, N)$,

$$\begin{aligned}\Psi(h,H,N) =& N_t + H_t - h_t + \langle h^c - N^c, h^c - H^c\rangle_t \\ &+ \sum_{0<s\le t} \frac{(\Delta h_s - \Delta N_s)(\Delta h_s - \Delta H_s)}{1+\Delta h_s} \\ &+ \sum_{0<s\le t} \frac{(\Delta^+ h_s - \Delta^+ N_s)(\Delta^+ h_s - \Delta^+ H_s)}{1+\Delta^+ h_s}\end{aligned}$$

is a local martingale. Thus, for the case of European options, chapter 9 of Abdelghani and Melnikov (2020) takes $N_t = \varsigma W_t + c(L_t - \lambda t) + d(\gamma t - \bar{L}_{t-})$ which is an optional local martingale that will render Z as an optional scaling factor. By this choice of N we get the following:

$$\begin{aligned}\Psi(h,H,N) =& (\varsigma+\sigma)W_t + (a+c+ac)(L_t - \lambda t) \\ &+ (b+d-bd)(\gamma t - \bar{L}_{t-}) \\ &+ (\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma)t\end{aligned} \tag{40}$$

is local martingale if

$$\mu - r + \varsigma\sigma + 2ac\lambda + 2bd\gamma = 0 \tag{41}$$

Thus we have to find (ς, σ, d) such that (41) is satisfied. We find the martingale measure Q by solving this equation which has infinitely many solutions that mean the market of Black-Scholes with left and right jumps is incomplete.

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Now let's turn our attention to the problem at hand, pricing a European call option. As outlined in chapter 9, section 6 of Abdelghani and Melnikov (2020) with the following choice of parameters for equation (41):

$$\left\{c = \frac{-a}{1+a}, d = \frac{-b}{1-b}, \varsigma = \frac{r - \mu + 2\lambda\frac{a^2}{1+a} + 2\gamma\frac{b^2}{1+b}}{\sigma}\right\} \tag{42}$$

leads to the normalized price of R under Q that is:

$$\begin{aligned} R_t =& R_0\mathcal{E}(\Psi(h, H, N)) \\ =& R_0\mathcal{E}((\varsigma + \sigma)W_t) \\ =& S_0 \exp\{(\varsigma + \sigma)W_t - \frac{1}{2}(\varsigma + \sigma)^2 t\}, \\ B_0 =& 1 \end{aligned} \tag{43}$$

It is just a function of the Wiener process and, all the left jumps are absorbed. Thus, R_t is $Q-$martingale. Thus,

$$E_Q[R_T|F_t] = R_t = e^{-rt}B_t \tag{44}$$

As calculated in Abdelghani and Melnikov (2020), with this choice of parameters, the price of a call option at arbitrary time $0 \le t < T$ is:

$$C_t = e^{rt}R_t\Phi\big(\tilde{K}_t + (\varsigma + \sigma)\sqrt{T-t}\big) + e^{-r(T-t)}K\Phi(\tilde{K}_t) \tag{45}$$

where:

$$\tilde{K} = \frac{1}{(\varsigma + \sigma)\sqrt{T}}\Big[\ln\big(\frac{S_0}{K}\big) + (r - \frac{1}{2}(\varsigma + \sigma)^2)T\Big] \tag{46}$$

Now, we proceed with the pricing of a European Put Option for optional processes. The payoff function of a European put option can be written as:

$$P_T = (K - S_T)^+ = (S_T - K)^+ + K - S_T \tag{47}$$

Using the payoff of the call option, we can write:

$$\begin{aligned} P_t =& E_Q[P_T|F_t] \\ =& E_Q[(S_T - K)^+|F_t] + E_Q[k - S_T|F_t] \\ =& C_t + E_Q[k - S_T|F_t] \end{aligned} \tag{48}$$

Where C_t is the price of the call option. Now if we denote:

$$\tilde{P}_T = \frac{P_T}{B_T} \tag{49}$$

We have:

$$\begin{aligned} \tilde{P}_t =& E_Q[\frac{P_T}{B_T}|F_t] \\ =& E_Q[\frac{C_T}{B_T}|F_t] + E_Q[\frac{K - S_T}{B_T}|F_t] \\ =& \tilde{C}_t + Ke^{-rT} - E_Q[R_T|F_t] \end{aligned} \tag{50}$$

To compute $E_Q[R_T|F_t]$, we use the same martingale measure as it is used for pricing the European call option. Thus, by (44) We can rewrite (50) using (49) as follows:

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$$\begin{aligned}\tilde{P}_t =& e^{-rt} P_t \\ =& e^{-rt} C_t + K e^{-rT} - e^{-rt} S_t\end{aligned} \tag{51}$$

By simplifying we get the following relation which is called *Call-Put Parity*.

$$P_t = C_t + K e^{-r(T-t)} - S_t \tag{52}$$

By using this Call-Put Parity relation, we can calculate the fair price of the put option and also establish the duality relation. Using the fact that $\Phi(-x) = 1 - \Phi(x)$, we have:

$$\begin{aligned}P_0 =& C_0 + K e^{-rT} - S_0 \\ =& S_0 \Phi(\tilde{K} + (\varsigma + \sigma)\sqrt{T}) - e^{-rT} K \Phi(\tilde{K}) + K e^{-rT} - S_0 \\ =& S_0 \Big[\Phi(\tilde{K} + (\varsigma + \sigma)\sqrt{T}) - 1\Big] - e^{-rT}(-K)(\Phi(\tilde{K}) - 1) \\ =& (-S_0)\Phi(-\tilde{K} - (\varsigma + \sigma)\sqrt{T}) - e^{-rT}(-K)\Phi(-\tilde{K})\end{aligned}$$

Therefore, using (46) definition, we have:

$$-\tilde{K} = \frac{1}{-(\varsigma + \sigma)\sqrt{T}} \Big[\ln\big(\frac{-S_0}{-K}\big) + (r - \frac{1}{2}(-\varsigma - \sigma)^2)T\Big] \tag{53}$$

Now given the definition of ς in (42), we can see that:

$$-\varsigma = \frac{r - \mu + 2\lambda\frac{a^2}{1+a} + 2\gamma\frac{b^2}{1+b}}{-\sigma}$$

is achieved simply by using $-\sigma$ thus we established the following relation which is called *Call-Put Dulaity*.

$$P_0(S_0, K, \sigma) = C_0(-S_0, -K, -\sigma) \tag{54}$$

And for arbitrary time $0 \le t < T$, we have:

$$\begin{aligned}P_t =& C_t + K e^{-r(T-t)} - S_t \\ =& r^{rt} R_t \Phi(\tilde{K}_t + (\varsigma + \sigma)\sqrt{T-t}) - e^{-r(T-t)} K \Phi(\tilde{K}_t) + K e^{-r(T-t)} - S_t \\ =& (-S_t)\Phi(-\tilde{K}_t - (\varsigma + \sigma)\sqrt{T-t}) - e^{-r(T-t)}(-K)\Phi(-\tilde{K}_t)\end{aligned}$$

where:

$$\tilde{K}_t = \frac{1}{(\varsigma + \sigma)\sqrt{T-t}} \Big[\ln\big(\frac{S_t}{K}\big) + (r - \frac{1}{2}(\varsigma + \sigma)^2)(T-t)\Big] \tag{55}$$

Thus, the *Call-Put Duality* for arbitrary time $0 \le t < T$ is:

$$P_t(S_t, K, \sigma) = C_t(-S_t, -K, -\sigma) \tag{56}$$

Upon examining the duality relation in chapter 4 of the Melnikov (2011) and drawing a parallel, it becomes apparent that the (56) relation bears similarity to the duality relation involving cadlag processes. □

6. Applications and Examples

In this section, we proceed with concrete examples and explicit calculations for the previously obtained results in section 5. The first example delves into an optional version of the well-known Merton's jump-diffusion model and calculate the price of a quanto call option.

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Example 1. *Let's consider an optional version of the well-known Merton's jump-diffusion model. The two assets S^1 and S^2 are modeled as optional exponential jump-diffusion processes:*

$$S_t^i = \exp\left(b_i t + W_t^i + \sum_{k=0}^{N_t^{i,r}} U_k^i + \sum_{k=0}^{N_t^{i,g}} Z_k^i\right), \quad i = 1,2, \tag{57}$$

where the drift terms b^i are determined by martingale condition; a Wiener process W_t^i have a variance $\sigma_i \geq 0$ and a correlation coefficient $-1 \leq \rho \leq 1$, i.e. $\langle W^1, W^2 \rangle = \rho$ and

$$c = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix};$$

the process $N_t^{i,r}, i = 1,2$ is a right-continuous Poisson process with intensity λ_i^r while the process $N_t^{i,r}, i = 1,2$ is a left-continuous version of a Poisson process with intensity λ_i^g, the random variables $U_k^i \sim Normal(s_i^r, 0)$ and $Z_k^i \sim Normal(s_i^g, 0)$. The Poisson processes $N_t^{i,r}, i = 1,2$ and $N_t^{i,g}, i = 1,2$ are independent. Thus, the local characteristics of $H = (H^1, H^2)$ are

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; \quad c = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}; \tag{58}$$

and F^r and F^g have the Lebesgue densities f^r and f^g where

$$f^j(x_1, x_2) = \frac{1}{s_1^j s_2^j 2\pi} \exp\left(-\frac{x_1^2}{2(s_1^j)^2} - \frac{x_2^2}{2(s_2^j)^2}\right) \lambda_1^j \lambda_2^j, \quad j = r, g. \tag{59}$$

Now, the price of a quanto call option with strike price K, according to Theorem 4, is equal to the price of a call option with the same strike K, on an asset S^u with characteristics $(B^u, C^u, \nu^{ur}, \nu^{ug})$ provided by Theorem 2 for $v = (1,0)'$ and $u = (0,1)'$. Hence, we can calculate

$$b^u = b_2 + \rho\sigma_1\sigma_2, \tag{60}$$
$$c^u = \sigma_2^2; \tag{61}$$

for the Levy measures, using the independence of the normal variables and completing the square, we have for $y \in \mathbb{R}$, $E \in \mathcal{B}(\mathbb{R} \setminus 0)$:

$$\begin{aligned} 1_E(y) * F^{uj} &= 1_E(x_2) e^{x_1} * F \\ &= \int_{\mathbb{R}^2} 1_E(x_2) e^{x_1} \frac{1}{s_1^j s_2^j 2\pi} \exp\left(-\frac{x_1^2}{2(s_1^j)^2} - \frac{x_2^2}{2(s_2^j)^2}\right) \lambda_1^j \lambda_2^j dx_1 dx_2 \\ &= \lambda_2^j \int_E \frac{1}{s_2^j \sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2(s_2^j)^2}\right) dx_2 \\ &\quad \times \lambda_1^j e^{\left(s_1^j\right)^2/2} \int_{\mathbb{R}} \frac{1}{s_1^j \sqrt{2\pi}} \exp\left(-\frac{(x_1 - \left(s_1^j\right)^2)^2}{2(s_1^j)^2}\right) dx_1 \\ &= \lambda_2^j \lambda_1^j e^{\left(s_1^j\right)^2/2} \int_E \frac{1}{s_2^j \sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2(s_2^j)^2}\right) dx_2, \quad j = r, g. \end{aligned} \tag{62}$$

Therefore, pricing a quanto option in case the two assets follow the optional version of Merton's jump-diffusion model, is equivalent to pricing a call option in a univariate jump-diffusion model of the same class, with

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parameters given by (60)-(62). In particular, left jumps occur according to a compound Poisson process with intensity $\lambda^{ur} = \lambda_2^r \lambda_1^r e^{(s_1^r)^2/2}$, and jump heights are normally distributed with jump variance s_2^r and zero mean, while right jumps occur according to a left-continuous compound Poisson process with intensity $\lambda^{ug} = \lambda_2^g \lambda_1^g e^{(s_1^g)^2/2}$, and jump heights are normally distributed with jump variance s_2^g and zero mean.

Moreover, the second example extends the previous model by incorporating exponential jumps and proceeds to calculate the price of a quanto call option.

Example 2. *Let us now consider the previous example in a case when the random variables U_k^i and Z_k^i are mutually independent and exponentially distributed with parameters α^i and β^i respectively. Thus, the density functions f^r and f^g have the following forms*

$$f^r(x_1, x_2) = \alpha^1 \alpha^2 e^{-\alpha^1 x_1 - \alpha^2 x_2},$$
$$f^g(x_1, x_2) = \beta^1 \beta^2 e^{-\beta^1 x_1 - \beta^2 x_2}.$$

Now, the price of a quanto call option with strike K, according to Theorem 4, is equal to the price of a call option with the same strike K, on an asset S^u with characteristics $(B^u, C^u, \nu^{ur}, \nu^{ug})$ provided by Theorem 2 for $v = (1,0)'$ and $u = (0,1)'$. Consequently, in this case b^u and c^u are the same as in Example 1, and we obtain for $y \in \mathbb{R}$, $E \in \mathcal{B}(\mathbb{R} \setminus \{0\})$:

$$\begin{aligned}
1_E(y) * F^{ur} =& \lambda_1^r \lambda_2^r \int_{\mathbb{R}^2} 1_E(x_2) e^{x_1} \alpha_1 \alpha_2 e^{-\alpha_1 x_1 - \alpha_2 x_2 dx_1 dx_2} \\
=& \lambda_1^r \lambda_2^r \alpha_1 \alpha_2 \int_{\mathbb{R}} e^{x_1(1-\alpha_1)} dx_1 \int_E e^{-\alpha_2 x_2} dx_2 \\
=& \lambda_1^r \lambda_2^r \frac{\alpha_1 \alpha_2}{\alpha_1 - 1} \int_E e^{-\alpha_2 x_2} dx_2.
\end{aligned}$$

The levy measure for left jumps is calculated in a similar way. As a result, pricing a quanto option in case of two assets are modeled as optional jump-diffusion's with exponential jumps, is equivalent to pricing a call option in a univariate optional jump-diffusion model of the same class. In particular, left jumps occur according to a compound Poisson process with intensity $\lambda^{ur} = \lambda_2^r \lambda_1^r \frac{\alpha_1}{\alpha_1 - 1}$, and jump heights are exponentially distributed with a parameter α_2, while right jumps occur according to a left-continuous compound Poisson process with intensity $\lambda^{ug} = \lambda_2^g \lambda_1^g \frac{\beta_1}{\beta_1 - 1}$, and jump heights are exponentially distributed with a parameter β_2.

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