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Keywords: delta-homotopy perturbation method; convergence; partial differential equation; Burger's equation; Bratu's equation

Article

The δ-Homotopy Perturbation Method

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Abstract: Homotopy perturbation and analysis methods have been widely used to obtain both approximate and exact solutions to nonlinear problems. In general, these two methods are based on the Taylor series with respect to an embedding parameter. Many researchers have compared the two methods and raised more concerns on the homotopy perturbation method (HPM) because the homotopy analysis method (HAM) contains a convergence-control parameter $\hbar$. For this reason, in this article, a more general form of HPM is introduced as the δ-homotopy perturbation method (δ-HPM), which contains a control parameter δ. The introduction of parameter δ in this new modification gives a better way to adjust and control the convergence region and the rate of the series solution. We confirm through the given examples in this study that the HPM is a special case of the δ-HPM. The error and convergence analysis of this proposed method are also presented.

Keywords: δ-homotopy perturbation method; convergence; partial differential equation; Burger's equation; Bratu's equation

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1. Introduction

The notion of homotopy, which is based on the introduction of a parameter p that varies from zero to one, has been used by many researchers to solve nonlinear problems in science and engineering. The problem simplifies to a somewhat trivial problem when $p = 0$, which is usually linear and whose solution can be found relatively easily. A family of solutions are obtained when p is incremented to one, which approaches the desired solution as p tends to one.

In 1992, Liao used the basic concept of homotopy to develop a general analytical method called homotopy analysis method (HAM) [43,44]. Liao reiterated the method by inserting an auxiliary parameter $\hbar$, beside another auxiliary function in his problem formulation. This parameter $\hbar$ can be employed in controlling the convergence of solution series obtained as the power series in p. The HAM has been implemented on a wide class of boundary and initial value problems [1–3,10–12,19,27,28, 45–51,58,60]. The further search of expanding the convergence region led to a modification of HAM called q-HAM, proposed by El-Tawil and Huseen [17]. The q-HAM have been successful employed to various problems in science and engineering [6–8,17,18,35–41,55–57]. On the contrary, He [29] later explained that it was not necessary to use the auxiliary parameter $\hbar$ and proposed another analytical method called the homotopy perturbation method (HPM). This method has gained the attention of researchers and have been used to solve linear and nonlinear problems [16,21–25,29–33,59,62]. However some issues of HPM have been published in [1,53,61,63]. To overcome these issues, an improved modification of HPM called the parameterized homotopy perturbation method (PHPM) was proposed in [4,5]. However, the new modification proposed in this study can be seen as a good refinement of existing numerical methods and can be used to study nonlinear models that describe natural phenomena.

In this paper, we propose a more general form of HPM namely, δ-homotopy perturbation method, (δ-HPM) which guarantees a convergent series solution. The introduction of a parameter δ in this modification help in adjusting and controlling the convergence region and the convergence rate of solution series. The examples provided in this paper confirm that when the approximation solutions obtained by using HPM is divergent, we can achieve a convergent series solution simply by selecting an appropriate value of δ from the so-called δ-curve where the horizontal line test is employed to attain the intervals containing the optimal values.

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This paper is organized as follows: In Section 2, we present the fundamental idea of the proposed method, the error analysis and convergence theorem. The numerical examples, numerical comparison, absolute errors in tabular form and δ-curves of each example are presented in Section 3. Finally, Section 4 gives the conclusion.

2. Analysis of the proposed algorithm

2.1. Fundamental idea of the δ-HPM

Here, we present the basic idea of δ-HPM. We refer the readers to He's works [29–34] for a good understanding of HPM where more developments can be found. To describe the proposed method, consider the nonlinear differential equation

$$\mathcal{F}(y(x,t)) - g(x,t) = 0, \quad x,t \in \Omega, \tag{1}$$

subject to some boundary conditions

$$\mathfrak{B}\left(y, \frac{\partial y}{\partial n}\right) = 0, \quad x,t \in \partial\Omega. \tag{2}$$

Here, $y(x,t)$ is an unknown function of x and t, $\mathcal{F}$ represents the differential operator, $\mathfrak{B}$ signifies boundary operator, $g(x,t)$ is a known analytic function, $\partial\Omega$ is the boundary of the domain Ω, and $\frac{\partial y}{\partial n}$ is the differentiation in the direction normal to $\partial\Omega$. The operator $\mathcal{F}$ in Eq. (1) can be written in the form $\mathcal{F} = \mathcal{M} + \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are respectively linear and nonlinear operators. Thus Eq. (1) is reformulated as

$$\mathcal{M}(y(x,t)) + \mathcal{N}(y(x,t)) - g(x,t) = 0. \tag{3}$$

In the view of He's homotopy construction in [31], we formulate the homotopy $\mathcal{G}(w(x,t;p),p)$ as:

$$\mathcal{G}(w(x,t;p),p) = \left(1 - \frac{p}{\delta}\right)\Big[\mathcal{M}(w(x,t;p)) - \mathcal{M}(y_0(x,t))\Big] + p\Big[\mathcal{F}(w(x,t;p)) - g(x,t)\Big] = 0. \tag{4}$$

Equivalently,

$$\begin{aligned}\mathcal{M}(w(x,t;p)) - \mathcal{M}(y_0(x,t)) &= \frac{p}{\delta}\Big[\mathcal{M}(w(x,t;p)) - \mathcal{M}(y_0(x,t))\Big] - p\Big[\mathcal{M}(w(x,t;p)) + \mathcal{N}(w(x,t;p)) - g(x,t)\Big] \\ &= p\Big(\frac{1}{\delta} - 1\Big)\mathcal{M}(w(x,t;p)) - \frac{p}{\delta}\mathcal{M}(y_0(x,t)) - p\big(\mathcal{N}(w(x,t;p)) - g(x,t)\big).\end{aligned} \tag{5}$$

The embedding parameter p varies from zero to a nonzero parameter δ, $y_0(x,t)$ is the initial approximation of Eq. (1) satisfying the boundary conditions Eq. (2). From Eq. (4) with $p = 0$ and $p = \delta$, we obtain the following:

$$\begin{aligned}\mathcal{M}(w(x,t;0)) - \mathcal{M}(y_0(x,t)) &= 0, \\ \mathcal{F}(w(x,t;\delta)) - g(x,t) &= 0.\end{aligned} \tag{6}$$

The variation of p from zero to δ corresponds to precisely that of $w(x,t;p)$ from $y_0(x,t)$ to the solution $y(x,t)$. This process is called in topology, deformation process. Now, we make the critical assumption that the solution of Eq. (4) can be expressed as a power series in p given as

$$w(x,t;p) = \sum_{r=0}^{\infty} w_r(x,t)p^r. \tag{7}$$

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By setting $p = \delta$, the solution of Eq. (1) can be obtained as

$$y(x,t) = \lim_{p \to \delta} w(x,t;p) = \sum_{r=0}^{\infty} Y_r(x,t;\delta) = \sum_{r=0}^{\infty} w_r(x,t)\delta^r. \tag{8}$$

Remark 1. *It should be noted that the special case with $\delta = 1$ is the standard HPM.*

2.2. Convergence and error analysis

Theorem 1. *Let $Y_m(x,t;\delta)$ and $y(x,t)$ be defined in Banach space $\mathcal{B}$ [42]. Then the series solution defined in Eq. (8) as*

$$\sum_{r=0}^{\infty} Y_r(x,t;\delta) = \sum_{r=0}^{\infty} w_r(x,t)\delta^r, \tag{9}$$

is convergent for a prescribed value of δ, if

$$\|Y_{r+1}\| \leq \frac{\beta}{|\delta|}\|Y_r\|, \quad \forall\ Y_0 \in \mathcal{B}, \tag{10}$$

for some β such that $0 < \beta < |\delta|$.

Proof. We first define a sequence of partial sums $\{P_m\}_{m=0}^{\infty}$ as follows

$$\begin{aligned} P_0 &= w_0, \\ P_1 &= w_0 + w_1\delta, \\ P_2 &= w_0 + w_1\delta + w_2\delta^2, \\ &\vdots \\ P_m &= w_0 + w_1\delta + w_2\delta^2 + \cdots + w_m\delta^m. \end{aligned} \tag{11}$$

Then, we show that $\{P_m\}_{m=0}^{\infty}$ is a Cauchy sequence in the Banach space $\mathcal{B}$. Thus, for a nonzero parameter δ, we have

$$\|P_{m+1} - P_m\| = \|Y_{m+1}\| \leq \frac{\beta}{|\delta|}\|Y_m\| \leq \left(\frac{\beta}{|\delta|}\right)^2 \|Y_{m-1}\| \leq \cdots \leq \left(\frac{\beta}{|\delta|}\right)^{m+1} \|Y_0\|. \tag{12}$$

For every $m, n \in \mathbb{N}$ with $m \geq n$, and applying the triangle inequality, we obtain

$$\begin{aligned} \|P_m - P_n\| &= \|(P_m - P_{m-1}) + (P_{m-1} - P_{m-2}) + \cdots + (P_{n+1} - P_n)\| \\ &\leq \|P_m - P_{m-1}\| + \|P_{m-1} - P_{m-2}\| + \cdots + \|P_{n+1} - P_n\| \\ &\leq \left(\frac{\beta}{|\delta|}\right)^m \|Y_0\| + \left(\frac{\beta}{|\delta|}\right)^{m-1} \|Y_0\| + \cdots + \left(\frac{\beta}{|\delta|}\right)^{n+1} \|Y_0\| \\ &\leq \left(\frac{\beta}{|\delta|}\right)^{n+1} \left(\left(\frac{\beta}{|\delta|}\right)^{m-n-1} + \left(\frac{\beta}{|\delta|}\right)^{m-n-2} + \cdots + \left(\frac{\beta}{|\delta|}\right) + 1 \right) \|Y_0\| \\ &\leq \left(\frac{\beta}{|\delta|}\right)^{n+1} \left(\frac{1 - \left(\frac{\beta}{|\delta|}\right)^{m-n}}{1 - \frac{\beta}{|\delta|}} \right) \|Y_0\|. \end{aligned} \tag{13}$$

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Since $0 < \beta < |\delta|$, we have $1 - \left(\frac{\beta}{|\delta|}\right)^{m-n} < 1$. Hence, Eq. (13) implies

$$\|P_m - P_n\| \leq \frac{\left(\frac{\beta}{|\delta|}\right)^{n+1}}{1 - \frac{\beta}{|\delta|}} \|Y_0\|. \tag{14}$$

Considering that $\|Y_0\| < \infty$, we have

$$\lim_{m\to\infty} \|P_m - P_n\| = 0. \tag{15}$$

Therefore, $\{P_m\}_{m=0}^{\infty}$ is a Cauchy sequence in the Banach space $\mathcal{B}$ and every Cauchy sequence is a convergent sequence. Accordingly, the series solution defined in Eq. (8) is convergent. □

Theorem 2. *Suppose that the truncated series*

$$\sum_{r=0}^{\Lambda} Y_r(x,t;\delta) = \sum_{r=0}^{\Lambda} w_r(x,t)\delta^r, \tag{16}$$

is utilized as an approximate solution of (1). Then the maximum absolute truncation error is projected as

$$\left\| y(x,t) - \sum_{r=0}^{\Lambda} Y_r(x,t;\delta) \right\| \leq \frac{\left(\frac{\beta}{|\delta|}\right)^{\Lambda+1}}{1 - \frac{\beta}{|\delta|}} \|Y_0\|. \tag{17}$$

Proof. This follows from inequality (13) in Theorem 1. In particular, for $m \geq \Lambda$, we have

$$\|P_m - P_\Lambda\| \leq \left(\frac{\beta}{|\delta|}\right)^{\Lambda+1} \left(\frac{1 - \left(\frac{\beta}{|\delta|}\right)^{m-\Lambda}}{1 - \frac{\beta}{|\delta|}} \right) \|Y_0\|. \tag{18}$$

As $m \to \infty$, for a prescribed value of δ, $P_m \to y(x,t)$ and $1 - \left(\frac{\beta}{|\delta|}\right)^{m-\Lambda} < 1$. Hence,

$$\left\| y(x,t) - \sum_{r=0}^{\Lambda} Y_r(x,t;\delta) \right\| \leq \left(\frac{\beta}{|\delta|}\right)^{\Lambda+1} \frac{|\delta|}{|\delta| - \beta} \|Y_0\|, \tag{19}$$

where $Y_0 = w_0 = y_0$. This completes the proof. □

3. Numerical examples

Example 1. *Consider the partial differential equation [24,53]*

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = 2\frac{\partial^3 y}{\partial x^2 \partial t}, \quad x \in \mathbb{R},\ t > 0, \tag{20}$$

with the initial condition

$$y(x,0) = e^{-x}. \tag{21}$$

The exact solution is

$$y(x,t) = e^{-x-t}. \tag{22}$$

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In order to solve Eq. (20) with initial condition Eq. (21), we construct the following homotopy:

$$\left(1-\frac{p}{\delta}\right)\left[\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}\right]+p\left[\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}-2\frac{\partial^3 w}{\partial x^2\partial t}\right]=0. \tag{23}$$

Equivalently,

$$\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}=\frac{p}{\delta}\left[\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}\right]-p\left[\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}-2\frac{\partial^3 w}{\partial x^2\partial t}\right]. \tag{24}$$

Substituting Eq. (7) into Eq. (24) and setting the identical powers of p equal to each other, we obtain

$$\begin{aligned}
p^{(0)} &: \frac{\partial w_0}{\partial t}-\frac{\partial y_0}{\partial t}=0,\\
p^{(1)} &: \frac{\partial w_1}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_0}{\partial t}-\frac{1}{\delta}\frac{\partial y_0}{\partial t}-\frac{\partial w_0}{\partial x}+2\frac{\partial^3 w_0}{\partial x^2\partial t}, \quad w_1(x,0)=0,\\
p^{(2)} &: \frac{\partial w_2}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_1}{\partial t}-\frac{\partial w_1}{\partial x}+2\frac{\partial^3 w_1}{\partial x^2\partial t}, \quad w_2(x,0)=0,\\
&\vdots\\
p^{(r)} &: \frac{\partial w_r}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_{r-1}}{\partial t}-\frac{\partial w_{r-1}}{\partial x}+2\frac{\partial^3 w_{r-1}}{\partial x^2\partial t}, \quad w_r(x,0)=0, \quad r=2,3,4,\cdots.
\end{aligned} \tag{25}$$

For simplicity we take $w_0(x,t)=y_0(x,t)=e^{-x}$ and we obtain the following recurrence relations:

$$\begin{aligned}
w_1(x,t) &= \int_0^t\left[-\frac{\partial w_0(x,\xi)}{\partial x}+2\frac{\partial^3 w_0(x,\xi)}{\partial x^2\partial\xi}\right]d\xi,\\
w_r(x,t) &= \int_0^t\left[\left(\frac{1}{\delta}-1\right)\frac{\partial w_{r-1}(x,\xi)}{\partial\xi}-\frac{\partial w_{r-1}(x,\xi)}{\partial x}+2\frac{\partial^3 w_{r-1}(x,\xi)}{\partial x^2\partial\xi}\right]d\xi, \quad r=2,3,4,\cdots.
\end{aligned} \tag{26}$$

The components of the δ-HPM solution are obtained from Eq. (26) as follows:

$$\begin{aligned}
w_0(x,t) &= e^{-x},\\
w_1(x,t) &= te^{-x},\\
w_2(x,t) &= \frac{te^{-x}}{2\delta}\Big(\delta t+2\delta+2\Big),\\
w_3(x,t) &= \frac{te^{-x}}{6\delta^2}\Big(\delta^2t^2+6\delta^2t+6\delta^2+6\delta t+12\delta+6\Big),\\
w_4(x,t) &= \frac{te^{-x}}{24\delta^3}\Big(\delta^3t^3+12\delta^3t^2+36\delta^3t+24\delta^3\\
&\quad +12\delta^2t^2+72\delta^2t+72\delta^2+36\delta t+72\delta+24\Big),\\
w_5(x,t) &= \frac{te^{-x}}{120\delta^4}\Big(\delta^4t^4+20\delta^4t^3+120\delta^4t^2+240\delta^4t+120\delta^4\\
&\quad +20\delta^3t^3+240\delta^3t^2+720\delta^3t+480\delta^3+120\delta^2t^2\\
&\quad +720\delta^2t+720\delta^2+240\delta t+480\delta+120\Big),\\
w_6(x,t) &= \frac{te^{-x}}{720\delta^5}\Big(\delta^5t^5+30\delta^5t^4+300\delta^5t^3+1200\delta^5t^2+1800\delta^5t\\
&\quad +720\delta^5+30\delta^4t^4+600\delta^4t^3+3600\delta^4t^3+7200\delta^4t\\
&\quad +3600\delta^4+300\delta^3t^3+3600\delta^3t^2+10800\delta^3t+7200\delta^3\\
&\quad +1200\delta^2t^2+7200\delta^2t+7200\delta^2+1800\delta t+3600\delta+720\Big),
\end{aligned} \tag{27}$$

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$$\vdots$$

The $(\Lambda+1)$ term approximate solution for problem (20)-(21) is

$$Y_\Lambda(x,t;\delta)=\sum_{r=0}^{\Lambda}w_r(x,t)\delta^r=w_0+w_1\delta+w_2\delta^2+w_3\delta^3+\cdots+w_\Lambda\delta^\Lambda. \tag{28}$$

Setting $\delta=1$ in Eq. (28) with reference to Eq. (27), we have

$$Y_6(x,t;1)=\frac{e^{-x}}{720}\left(t^6+66t^5+1470t^4+13320t^3+46440t^2+45360t+720\right), \tag{29}$$

which is precisely same solution obtained by Ganji et al. [24] using HPM. As a consequence, the HPM solution is undoubtedly a special case of the δ-HPM solution with $\delta=1$. Liang et al. [53] shows that for a given $x\geq 0$, the HPM solution Eq. (29) increases monotonously to infinity as t increases and very quickly the relative error increases monotonously. Indeed, one can easily verify that the HPM solution series is divergent for every t and x except at $t=0$, nevertheless corresponds to $y(x,0)=e^{-x}$, the given initial condition. In a nutshell, the radius of convergence of the series solution Eq. (29) by HPM is zero. The use of HPM in this example might produce a divergent approximations. But it is important to obtain a convergent series solution. For this reason, we introduce the parameter δ to ensure the convergence of the solution series by δ-HPM. In this example we demonstrated that, if the approximations given by the standard HPM is divergent, one can still achieve a convergent solution series simply by choosing a suitable δ value from the δ-curve. In other words, to select an appropriate δ that guarantees a convergent series solution, we first plot the δ-curve of 16^{th} order approximate solution, as shown in Figure 1, and use the line segment nearly parallel to the horizontal axis as a valid region of δ. Table 1 shows that the 16^{th} order approximate solution series obtained by the standard HPM is divergent for every t and x except at $t=0$ and the absolute error monotonously increases very quickly. However, the 16^{th} order approximate solution series obtained by δ-HPM with $\delta=-1$ is in good agreement with the exact solution. Furthermore, it is easily seen from Eq. (27) that the δ-HPM solutions with $\delta=-1$ yields

$$\begin{aligned}
Y_1(x,t;-1) &= e^{-x}(1-t),\\
Y_2(x,t;-1) &= e^{-x}\left(1-t+\frac{t^2}{2!}\right),\\
Y_3(x,t;-1) &= e^{-x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}\right),\\
Y_4(x,t;-1) &= e^{-x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}\right),\\
Y_5(x,t;-1) &= e^{-x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}\right),\\
Y_6(x,t;-1) &= e^{-x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}\right),\\
&\vdots\\
Y_\Lambda(x,t;-1) &= e^{-x}\sum_{r=0}^{\Lambda}\frac{(-t)^r}{r!}.
\end{aligned} \tag{30}$$

Thus,

$$y(x,t) = e^{-x}\lim_{\Lambda\to\infty}\sum_{r=0}^{\Lambda}\frac{(-t)^r}{r!}$$

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$$= \quad e^{-x-t}, \tag{31}$$

which is obviously the exact solution. Therefore, the convergence-control parameter δ in this new modification equips us with a convenient way to guarantee the convergence of series solution.

Table 1. The comparative study of Y_{16} of δ-HPM, HPM and the exact solution for Example 1

			δ-HPM ($\delta = -1$)		HPM ($\delta = 1$)	
x	t	Exact	Approx.	Absolute error	Approx.	Absolute error
	0	0.1353352832	0.1353352832	0	0.1353352832	0
	0.1	0.1224564283	0.1224564283	1.221245×10^{-15}	$1.2332686540 \times 10^{03}$	1.233146×10^{03}
2	0.2	0.1108031584	0.1108031584	7.063794×10^{-15}	$3.3171025221 \times 10^{03}$	3.316992×10^{03}
	0.3	0.1002588437	0.1002588437	9.157952×10^{-14}	$6.5260426500 \times 10^{03}$	6.525942×10^{03}
	0.4	0.0907179533	0.0907179533	6.301903×10^{-14}	$1.1189446548 \times 10^{04}$	1.118936×10^{04}
	0.5	0.0820849986	0.0820849986	2.581269×10^{-15}	$1.7699263871 \times 10^{04}$	1.769918×10^{04}
	0	0.0024787522	0.0024787522	0	0.0024787522	0
	0.1	0.0022428677	0.0022428677	2.081668×10^{-17}	$2.2588103319 \times 10^{01}$	2.258586×10^{01}
6	0.2	0.0020294306	0.0020294306	1.301043×10^{-16}	$6.0754851951 \times 10^{01}$	6.075282×10^{01}
	0.3	0.0018363048	0.0018363048	1.677261×10^{-15}	$1.1952864055 \times 10^{02}$	1.195268×10^{02}
	0.4	0.0016615573	0.0016615573	1.155760×10^{-15}	$2.0494186234 \times 10^{02}$	2.049402×10^{02}
	0.5	0.0015034392	0.0015034392	4.683753×10^{-17}	$3.2417332565 \times 10^{02}$	3.241718×10^{02}

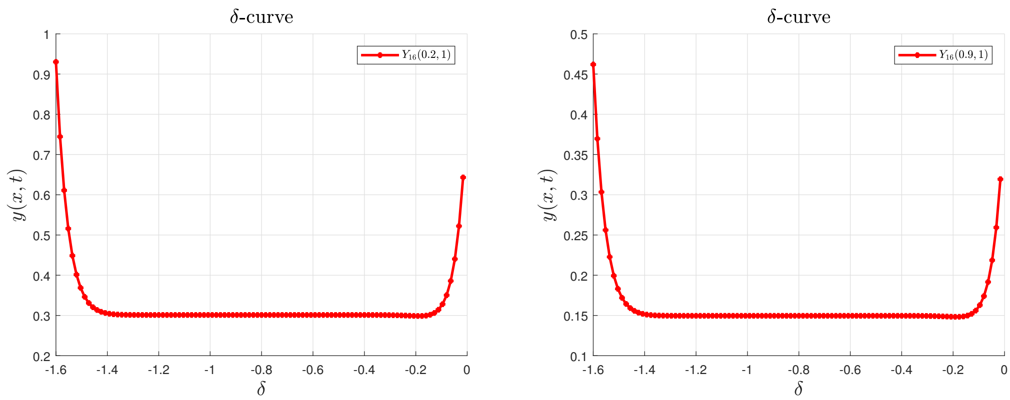

Figure 1. The δ-curve of $Y_{16}(x,t)$ solution with different x at $t = 1$ for Example 1

Example 2. *Consider the nonlinear partial differential Burger's equation [26]*

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = \frac{\partial^2 y}{\partial x^2}, \quad (x,t) \in \mathbb{R} \times \left[0, \frac{1}{2}\right), \tag{32}$$

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with the initial condition

$$y(x,0) = 2x. \tag{33}$$

This equation characterizes various phenomena, for instance, a mathematical model of turbulence and the approximation theory of the flow through a shock wave traveling in a viscous fluid [13]. The exact solution is

$$y(x,t) = \frac{2x}{2t+1}. \tag{34}$$

In order to solve Eq. (32) with initial condition Eq. (33), we construct the following homotopy:

$$\left(1-\frac{p}{\delta}\right)\left[\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}\right]+p\left[\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\right]=0. \tag{35}$$

Equivalently,

$$\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}=\frac{p}{\delta}\left[\frac{\partial w}{\partial t}-\frac{\partial y_0}{\partial t}\right]-p\left[\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\right]. \tag{36}$$

Substituting Eq. (7) into Eq. (36) and setting the identical powers of p equal to each other, we obtain

$$\begin{aligned}
p^{(0)} &: \frac{\partial w_0}{\partial t}-\frac{\partial y_0}{\partial t}=0,\\
p^{(1)} &: \frac{\partial w_1}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_0}{\partial t}-\frac{1}{\delta}\frac{\partial y_0}{\partial t}-w_0\frac{\partial w_0}{\partial x}+\frac{\partial^2 w_0}{\partial x^2}, \quad w_1(x,0)=0,\\
p^{(2)} &: \frac{\partial w_2}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_1}{\partial t}-w_0\frac{\partial w_1}{\partial x}-w_1\frac{\partial w_0}{\partial x}+\frac{\partial^2 w_1}{\partial x^2}, \quad w_2(x,0)=0,\\
&\vdots\\
p^{(r)} &: \frac{\partial w_r}{\partial t}=\left(\frac{1}{\delta}-1\right)\frac{\partial w_{r-1}}{\partial t}-\sum_{m=0}^{r-1}\left(w_m\frac{\partial w_{r-m-1}}{\partial x}\right)-\frac{\partial^2 w_{r-1}}{\partial x^2}, \quad w_r(x,0)=0, \quad r=2,3,4,\cdots.
\end{aligned} \tag{37}$$

Case 1. *For simplicity we take $w_0(x,t) = y_0(x,t) = 2x(1+t)$ and we obtained the following recurrent relation*

$$\begin{aligned}
w_1(x,t) &= \int_0^t \left[-\frac{\partial w_0(x,\xi)}{\partial \xi}-w_0(x,\xi)\frac{\partial w_0(x,\xi)}{\partial x}+\frac{\partial^2 w_0(x,\xi)}{\partial x^2}\right]d\xi,\\
w_r(x,t) &= \int_0^t \left[\left(\frac{1}{\delta}-1\right)\frac{\partial w_{r-1}(x,\xi)}{\partial \xi}+\sum_{m=0}^{r-1}\left(w_m\frac{\partial w_{r-m-1}(x,\xi)}{\partial x}\right)-\frac{\partial^2 w_{r-1}(x,\xi)}{\partial x^2}\right]d\xi, \quad r=2,3,4,\cdot
\end{aligned} \tag{38}$$

The components of the δ-HPM solution are obtain from Eq. (38) as follows:

$$\begin{aligned}
w_0(x,t) &= 2x(1+t),\\
w_1(x,t) &= -6xt-4xt^2-\frac{4}{3}xt^3,\\
w_2(x,t) &= \frac{6(\delta-1)}{\delta}xt+\frac{4(4\delta-1)}{\delta}xt^2+\frac{4(11\delta-1)}{3\delta}xt^3+\frac{16}{3}xt^4+\frac{16}{15}xt^5,\\
w_3(x,t) &= -\frac{6(\delta-1)^2}{\delta^2}xt-\frac{4(7\delta^2-8\delta+1)}{\delta^2}xt^2-\frac{4(42\delta^2-22\delta+1)}{3\delta^2}xt^3\\
&\quad -\frac{16(9\delta-2)}{3\delta}xt^4-\frac{32(11\delta-1)}{15\delta}xt^5-\frac{272}{45}xt^6-\frac{272}{315}xt^7,\\
&\vdots
\end{aligned} \tag{39}$$

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The $(\Lambda+1)$ *term approximate solution for problem (32)-(33) is*

$$Y_\Lambda(x,t;\delta)=\sum_{r=0}^{\Lambda} w_r(x,t)\delta^r = w_0+w_1\delta+w_2\delta^2+w_3\delta^3+\cdots+w_\Lambda\delta^\Lambda. \tag{40}$$

Setting $\delta=1$ *in Eq. (40) with reference to Eq. (39), we have*

$$Y_3(x,t;1)=2x-4xt+8xt^2-16xt^3-32xt^4-\frac{304}{15}xt^5-\frac{272}{45}xt^6-\frac{272}{315}xt^7. \tag{41}$$

Case 2. *Alternatively, by taking* $w_0(x,t)=y_0(x,t)=2x$*, then from Eq. (38), we obtain the following*

$$\begin{aligned}
w_0(x,t) &= 2x,\\
w_1(x,t) &= -4xt,\\
w_2(x,t) &= \frac{4xt(\delta+2\delta t-1)}{\delta},\\
w_3(x,t) &= \frac{-4xt(\delta+2\delta t-1)^2}{\delta^2},\\
&\vdots\\
w_r(x,t) &= \frac{(-1)^r 4xt(\delta+2\delta t-1)^{r-1}}{\delta^{r-1}}, \quad r=1,2,3,\cdots.
\end{aligned} \tag{42}$$

The $(\Lambda+1)$ *term approximate solution for problem (32)-(33) is*

$$\begin{aligned}
Y_\Lambda(x,t;\delta) &= \sum_{r=0}^{\Lambda} w_r(x,t)\delta^r\\
&= 2x+\sum_{r=1}^{\Lambda}(-1)^r 4xt\delta(\delta+2\delta t-1)^{r-1}.
\end{aligned} \tag{43}$$

Setting $\delta=1$ *in Eq. (43) with reference to Eq. (42), we have*

$$Y_\Lambda(x,t;1)=2x\sum_{r=0}^{\Lambda}(-1)^r(2t)^r. \tag{44}$$

which is in agreement with the solution obtained using HPTM [14]. Furthermore, from Eq. (44), we have the exact solution as

$$\begin{aligned}
y(x,t) &= 2x\lim_{\Lambda\to\infty}\sum_{r=0}^{\Lambda}(-1)^r(2t)^r\\
&= \frac{2x}{2t+1}.
\end{aligned} \tag{45}$$

Remark 2. *It should be noted in case 2 that*

$$\lim_{m\to\infty}\left|\frac{w_{m+1}}{w_m}\right| = |\delta+2\delta t-1|<1. \tag{46}$$

Thus, in agreement with Theorem 1 and holds for the values of $-\frac{1}{2}<t<\frac{2-\delta}{2\delta}$*. For example,* $\delta=1$*, holds for the values of* $-\frac{1}{2}<t<\frac{1}{2}$*.*

Remark 3. *From Table 2, we can observe that it is better to seek other values for* δ *besides 1. For example,* $\delta=0.80$ *gives a better approximation when compared with* $\delta=1$*. Thus,* δ*-HPM is more general and reliable than HPM.*

Table 2. The comparative study of Y_5 of δ-HPM, HPM and the exact solution for Example 2 case 1

x	t	Exact	δ-HPM ($\delta = 0.80$) Approx.	δ-HPM Absolute error	HPM ($\delta = 1$) Approx.	HPM Absolute error
0.01	0.08	0.0172413793	0.0172416459	2.665546×10^{-7}	0.0172404940	8.852999×10^{-7}
	0.10	0.0166666667	0.0166670364	3.697086×10^{-7}	0.0166632809	3.385753×10^{-6}
	0.12	0.0161290323	0.0161295280	4.957343×10^{-7}	0.0161188868	1.014542×10^{-5}
	0.14	0.0156250000	0.0156256389	6.389195×10^{-7}	0.0155993038	2.569623×10^{-5}
	0.16	0.0151515152	0.0151522465	7.313646×10^{-7}	0.0150939574	5.755779×10^{-5}
	0.18	0.0147058824	0.0147063985	5.161891×10^{-7}	0.0145884896	1.173927×10^{-4}
	0.20	0.0142857143	0.0142850695	6.447537×10^{-7}	0.0140633199	2.223944×10^{-4}
0.05	0.08	0.0862068966	0.0862082293	1.332773×10^{-6}	0.0862024701	4.426500×10^{-6}
	0.10	0.0833333333	0.0833351819	1.848543×10^{-6}	0.0833164046	1.692877×10^{-5}
	0.12	0.0806451613	0.0806476400	2.478671×10^{-6}	0.0805944342	5.072708×10^{-5}
	0.14	0.0781250000	0.0781281946	3.194597×10^{-6}	0.0779965188	1.284812×10^{-4}
	0.16	0.0757575758	0.0757612326	3.656823×10^{-6}	0.0754697868	2.877889×10^{-4}
	0.18	0.0735294118	0.0735319927	2.580946×10^{-6}	0.0729424482	5.869635×10^{-4}
	0.20	0.0714285714	0.0714253477	3.223769×10^{-6}	0.0703165993	1.111972×10^{-3}

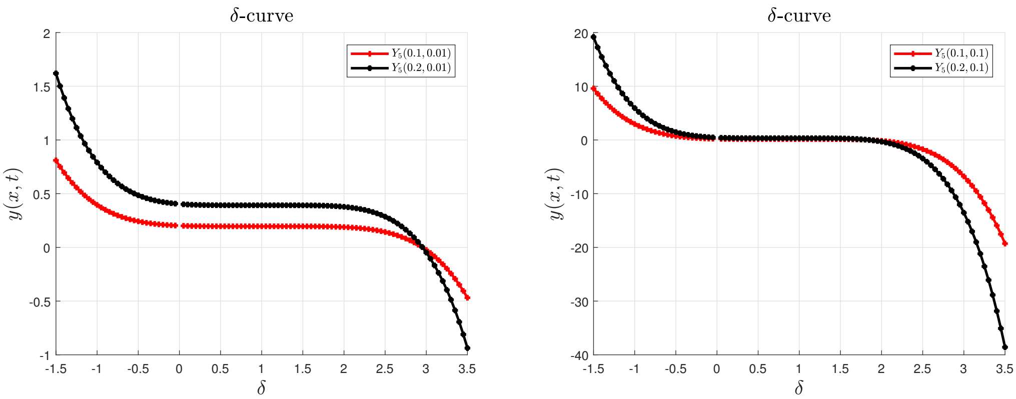

Figure 2. The δ-curve of $Y_5(x,t)$ solution with different x and t for Example 2 case 1

Example 3. *Consider Bratu's first boundary value problem [9,20,64]*

$$\frac{d^2y}{dx^2} + \lambda e^y = 0, \quad 0 \le x \le 1, \tag{47}$$

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with the boundary conditions

$$y(0) = 0, \quad y(0) = 1. \tag{48}$$

This problem comes originally from a simplification of the solid fuel ignition model in thermal combustion theory. The exact solution of Eq. (47) and Eq. (48) is

$$y(x) = -2\ln\left(\frac{\cosh\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right), \tag{49}$$

where θ satisfies $\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$.

In order to solve Eq. (47) with the boundary condition Eq. (48), we expand e^y into a Taylor series as

$$e^y = \sum_{r=0}^{\infty}\frac{y^r}{r!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots, \tag{50}$$

Now, we construct the following homotopy:

$$\left(1-\frac{p}{\delta}\right)\left[\frac{d^2w}{dx^2} - \frac{d^2y_0}{dx^2}\right] + p\left[\frac{d^2w}{dx^2} + \lambda + \lambda w + \frac{\lambda}{2}w^2 + \frac{\lambda}{6}w^3\right] = 0. \tag{51}$$

Equivalently,

$$\frac{d^2w}{dx^2} - \frac{d^2y_0}{dx^2} = \frac{p}{\delta}\left[\frac{d^2w}{dx^2} - \frac{d^2y_0}{dx^2}\right] - p\left[\frac{d^2w}{dx^2} + \lambda + \lambda w + \frac{\lambda}{2}w^2 + \frac{\lambda}{6}w^3\right]. \tag{52}$$

Substituting Eq. (7) into Eq. (52) and setting the identical powers of p equal to each other, we obtain

$$\begin{aligned}
p^{(0)} &: \quad \frac{d^2w_0}{dx^2} - \frac{d^2y_0}{dx^2} = 0, \quad w_0(0) = 0, \ w_0(1) = 0,\\
p^{(1)} &: \quad \frac{d^2w_1}{dx^2} = \left(\frac{1}{\delta}-1\right)\frac{d^2w_0}{\partial x^2} - \frac{1}{\delta}\frac{d^2y_0}{dx^2} - \lambda - \lambda w_0 - \frac{\lambda}{2}w_0^2 - \frac{\lambda}{6}w_0^3, \ w_1(0) = 0, \ w_1(1) = 0,\\
p^{(2)} &: \quad \frac{d^2w_2}{dx^2} = \left(\frac{1}{\delta}-1\right)\frac{d^2w_1}{dx^2} - \lambda w_1 - \lambda w_0 w_1 - \frac{\lambda}{2}w_0^2 w_1, \ w_2(0) = 0, \ w_2(1) = 0,\\
&\ \vdots\\
p^{(r)} &: \quad \frac{d^2w_r}{dx^2} = \left(\frac{1}{\delta}-1\right)\frac{d^2w_{r-1}}{dx^2} - \lambda w_{r-1} - \frac{\lambda}{2}\sum_{m=0}^{r-1} w_m w_{r-m-1} - \frac{\lambda}{6}\sum_{m=0}^{r-1}\sum_{k=0}^{m} w_k w_{m-k} w_{r-m-1},\\
&\quad\ w_r(0) = 0, \ w_r(1) = 0, \quad r = 2, 3, 4, \cdots.
\end{aligned} \tag{53}$$

Considering $\frac{d^2w_0}{dx^2} - \frac{d^2y_0}{dx^2} = 0$ with boundary conditions $w_0(0) = 0$ and $w_0(1) = 0$, we have $w_0(x) = y_0(x) = 0$. From Eq. (53), we obtain the following recurrence relations

$$\begin{aligned}
w_1(x) &= \int_0^x\int_0^x -\lambda \, d\xi d\xi + C_1 x + C_1', \ w_1(0) = 0, \ w_1(1) = 0,\\
w_r(x) &= \int_0^x\int_0^x \left\{\left(\frac{1}{\delta}-1\right)\frac{d^2w_{r-1}(\xi)}{d\xi^2} - \lambda w_{r-1}(\xi) - \frac{\lambda}{2}\sum_{m=0}^{r-1} w_m(\xi) w_{r-m-1}(\xi)\right.\\
&\quad \left. - \frac{\lambda}{6}\sum_{m=0}^{r-1}\sum_{k=0}^{m} w_k(\xi) w_{m-k}(\xi) w_{r-m-1}(\xi)\right\} d\xi d\xi + C_r x + C_r', \ w_r(0) = 0, \ w_r(1) = 0, \ r = 2, 3, \cdots,
\end{aligned} \tag{54}$$

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where C_1, C_1', C_r, and C_r' are the integration constants to be determined. The components of the δ-HPM solution are obtained from Eq. (54) as follows:

$$
\begin{aligned}
w_0(x) &= 0,\\
w_1(x) &= \frac{\lambda}{2}(x-x^2),\\
w_2(x) &= \frac{\lambda x^2}{24\delta}\Big(\lambda\delta x^2 - 2\lambda\delta x + 12\delta - 12\Big) + \frac{\lambda x}{24\delta}\Big(\lambda\delta - 12\delta + 12\Big),\\
w_3(x) &= \frac{\lambda x^2}{1440\delta^2}\Big(-8\lambda^2\delta^2x^4 + 24\lambda^2\delta^2x^3 - 15\lambda^2\delta^2x^2 - 10\lambda^2\delta^2x - 120\lambda\delta^2x^2\\
&\quad + 240\lambda\delta^2x + 120\lambda\delta x^2 - 240\lambda\delta x - 720\delta^2 + 1440\delta - 720\Big)\\
&\quad + \frac{\lambda x}{1440\delta^2}\Big(9\lambda^2\delta^2 - 120\lambda\delta^2 + 120\lambda\delta + 720\delta^2 - 1440\delta + 720\Big) \qquad (55)\\
w_4(x) &= \frac{\lambda x}{20160\delta^3}\Big(23\delta^3\lambda^3 - 378\delta^3\lambda^2 + 2520\delta^3\lambda - 10080\delta^3\\
&\quad + 378\delta^2\lambda^2 - 5040\delta^2\lambda + 30240\delta^2 + 2520\delta\lambda - 30240\delta + 10080\Big)\\
&\quad + \frac{\lambda x^2}{20160\delta^3}\Big(2520\delta(\lambda x^2 - 2\lambda x + 12) - 42\delta^2(8\lambda^2x^4 - 24\lambda^2x^3\\
&\quad + 15\lambda(\lambda+8)x^2 + 10(\lambda-24)\lambda x + 720) - 10080)\\
&\quad + \delta^3(17\lambda^3x^6 - 68\lambda^3x^5 + 7\lambda^2(11\lambda+48)x^4 + 7(\lambda-144)\lambda^2x^3\\
&\quad - 35\lambda\left(\lambda^2 - 18\lambda - 72\right)x^2 - 21\lambda(\lambda^2 - 20\lambda + 240)x + 10080)\Big)\\
&\ \vdots
\end{aligned}
$$

The $(\Lambda+1)$ term approximate solution for the boundary value problem (47)-(48) is

$$
Y_\Lambda(x;\delta) = \sum_{r=0}^{\Lambda} w_r(x)\delta^r = w_0 + w_1\delta + w_2\delta^2 + w_3\delta^3 + \cdots + w_\Lambda\delta^\lambda. \qquad (56)
$$

Setting $\delta = 1$ in Eq. (56) with reference to Eq. (55), we have

$$
\begin{aligned}
Y_4(x;1) &= \frac{\lambda}{2}(x-x^2) + \frac{\lambda^2}{24}(x^4 - 2x^3 + x) + \frac{\lambda^3}{1440}(-8x^6 + 24x^5 - 15x^4 - 10x^3 + 9x)\\
&\quad + \frac{\lambda^4}{20160}(17x^8 - 68x^7 + 77x^6 + 7x^5 - 35x^4 - 21x^3 + 23x), \qquad (57)
\end{aligned}
$$

which is precisely the same solution obtained by Feng et al. [20] using HPM.

Remark 4. *From Table 3, we can observe that the result with the present modification called δ-HPM is more accurate than the result of HPM (δ-HPM with $\delta = 1$) and PIA [9]. PIA represents perturbation–iteration algorithms.*

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Table 3. The comparative study of Y_3, Y_5 of δ-HPM, HPM, PIA [9] and the exact solution with $\lambda = 1$ for Example 3

x	Exact	Approx.		Approx.		Approx.	
		$Y_5(\delta = 1.15)$	$Y_5(\delta = 1)$	$Y_3(\delta = 1.125)$	$Y_3(\delta = 1)$	PIA(1,1)[9]	PIA(1,2)[9]
0.1	0.04985	0.04985	0.04984	0.04985	0.04970	0.04949	0.04983
0.2	0.08919	0.08919	0.08918	0.08918	0.08892	0.08851	0.08915
0.3	0.11761	0.11761	0.11759	0.11759	0.11723	0.11665	0.11756
0.4	0.13479	0.13479	0.13477	0.13476	0.13434	0.13365	0.13473
0.5	0.14054	0.14054	0.14052	0.14051	0.14006	0.13934	0.14048
0.6	0.13479	0.13479	0.13477	0.13476	0.13434	0.13365	0.13473
0.7	0.11761	0.11761	0.11759	0.11759	0.11723	0.11665	0.11756
0.8	0.08919	0.08919	0.08918	0.08918	0.08892	0.08851	0.08915
0.9	0.04985	0.04985	0.04984	0.04985	0.04970	0.04949	0.04983

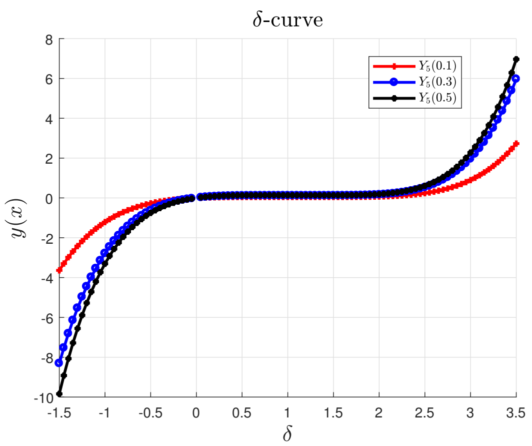

Figure 3. The δ-curve of $Y_5(x)$ solution with different x for Example 3

4. Concluding remarks

In this present paper, we introduced a generalization of the homotopy perturbation method, (HPM) proposed by He [29]. We call the generalization the δ-homotopy perturbation method (δ-HPM). We compared the δ-HPM and the HPM through some models arising from engineering and physics. It is verified in these examples that HPM is actually a special case of δ-HPM for $\delta = 1$. As stated by Liao [52], "the homotopy perturbation method had to use a good enough initial guess," but this is not the case for δ-HPM because of the use of the δ-curve that provides us some range for δ that guarantee a convergent series solution. The parameter δ provides us the flexibility to adjust and control the convergence region including the rate of series solution. Hence, by using δ-HPM, one can overcome a divergent result produced by HPM. A high level of accuracy reveals the complete efficiency and reliability of the proposed method.

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