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The Language of Spheres in Physics

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Keywords: Numbers; Symmetry; Group Theory; Spheres

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Article

The Language of Spheres in Physics

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Abstract: Physical laws find expression through combinations of mathematical symbols, numbers, functions, geometries, and relations. These amalgamations unfold within a mathematical model designed to represent the "objective reality" of the system under scrutiny. The language used and the associated concepts are in a perpetual state of evolution, mirroring the ongoing expansion of the phenomena accessible to our scientific understanding. In this contribution, we delve into fundamental, at times seemingly elementary, elements of the mathematical language inherent to the physical sciences, guided by the overarching principles of symmetry and group theory. Our focus turns to the captivating realm of spheres, those strikingly symmetric entities that manifest prominently within our geometric landscape. By exploring the interplay between mathematical abstraction and the tangible beauty of symmetry, we seek to deepen our understanding of the underlying structures that govern our interpretation of the physical world.

Keywords: numbers; symmetry; group theory; spheres

MSC: 58D19; 58J70; 76M60; 52B15

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1. Introduction

The precision of mathematical models in physics is inherently limited when viewed from a physicist standpoint. Such models are transient rather than eternal, subject to the passage of (entropic) time and contingent upon the scale at which they operate. To align with experimental observations and predictions, these models often undergo modifications, sometimes requiring radical transformations. In the dynamic interplay between the modeler and the modeled object, a degree of probability emerges, representing the level of epistemic confidence in the model's appropriateness [1].

This recognition of inherent uncertainty gives rise to the imperative of introducing some level of coarse-graining into the initial mathematical model, conceived to portray an ontic entity or fact. Take, for example, the inconceivability of irrational numbers in human perception, yet physical laws, for several centuries now, have been articulated in terms of real numbers, denoted as $\mathbb{R}$, constructed from the abstract concept of limits (such as the limit of Cauchy sequences of rational numbers).

Nevertheless, these profoundly abstract mathematical concepts have evolved to such a degree of familiarity that they now stand as an indispensable foundation for formulating descriptions of our environment. Serving as a robust backdrop, they assist us in imposing structure and establishing a set of physical laws capable of navigating the intricacies of complexity and chaos. In our pursuit of order, we inevitably organize observations through the lens of symmetry.

Beneath the allure of this aesthetic concept lies the intricate framework of mathematics, particularly the concept of a group, with all its potential generalizations, deformations, approximations, and the inherent challenges of incompleteness and partial versions. It is within this mathematical tapestry that we find the tools and principles allowing us to not only conceptualize but also articulate the laws that govern the universe, providing a coherent and profound understanding of our complex reality.

In this contribution, we harness the potency of these tools to reexamine the first elements of the sequence of unit spheres, denoted as $\mathbb{S}^n$, and explore their profound physical implications across various domains of physics. This exploration extends into realms such as atomic, molecular, nuclear, particle, and condensed matter physics, shedding light on the intricate interplay between the

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mathematical abstraction of spheres and their tangible relevance in understanding the fundamental structures and behaviors that underpin the diverse facets of the physical world.

We do not aspire to introduce entirely original content in this paper. Rather, this article is conceived as a fusion of fundamental (somewhat very basic) and advanced discussions on the mathematics of symmetry. The journey begins with a recapitulation of essential concepts pertaining to groups and semi-groups (Section 2), followed by a freshman level reminder of algebraic and topological aspects of numbers (Section 3). We then show the significant role played by simple or semi-simple Lie groups in elucidating symmetries across classical and quantum physics realms (Section 4).

The core of this presentation centers on a reexamination of the realm of spheres (Section 5), and the ensuing exploration of the associated geometry of Hopf fibrations (Section 6). Furthermore, we unravel the mesmerizing simplifications that the spherical symmetry contributes to explaining the Balmer lines, detailing the spectral line emissions of the hydrogen atom—a proto-quantum formalism dating back to 1885 (Section 7). The paper concludes with a succinct summary in Section 8.

For those seeking a more in-depth understanding, Appendices A and B provide additional details on group actions and the Lie algebra formalism.

2. Semi-Groups and Groups

In this section we remind the main definitions concerning semi-groups, groups, and structures built from them. Usually these notions are supposed known, but remind them lies in the self-contained spirit of this contribution.

2.1. Definitions

A semi-group is a set $G = \{g \in G\}$ equipped with an operation that combines any two elements $g_1 \in G$, $g_2 \in G$ to form a third element $g_3 = g_1 g_2 \in G$ while being associative, i.e. $(g_1 g_2) g_3 = g_1 (g_2 g_3) \equiv g_1 g_2 g_3$.

A group is a semi-group having an identity element e, $eg = ge = g$ and inverse elements, $g^{-1} g = g g^{-1} = e$.

A group or semi-group is *abelian* if the operation is commutative: $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$, *e.g.*, translations in the line or to rotations on the circle or in the plane. Rotations in space are non-commutative.

Groups can be finite (*e.g.*, permutations, point groups of symmetry of molecules), discrete (*e.g.*, space group of symmetry of crystals), continuous (*e.g.*, Euclidean displacements), Lie groups (infinitesimal transformations can be considered as well).

Continuous groups can be compact (parameters of transformation are bounded, *e.g.*, rotations on the sphere) or non compact, *e.g.*, translations in the plane.

Two elements g_1, g_2 of a group G are said *conjugate* if there exists $g \in G$ such that $g_2 = g g_1 g^{-1}$. This is an equivalence relation whose equivalence classes are called *conjugacy classes* (one famous example is the Rubik cube where the essential property for winning is precisely the set of conjugate classes of one permutation group, see Figure 1, and [2] for explanation).

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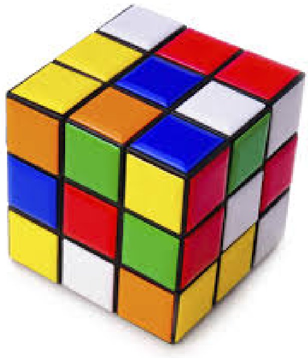

Figure 1. Rubik cube.

2.2. Example: Chirality

Chirality is the simplest example of a group: $G = \{I, P\}$, with $IP = PI = P$, $I^2 = I = P^2$. P is like mirror image. A figure is chiral if it cannot be mapped to its mirror image by rotations and translations alone. Examples are shown in Figures 2 and 3.

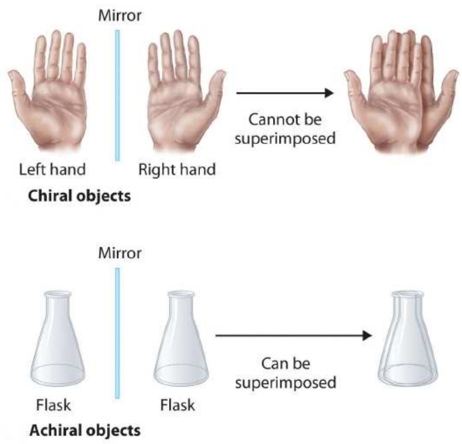

Figure 2. From https://chem.libretexts.org/.

A chiral object and its mirror image are said to be enantiomorphs.

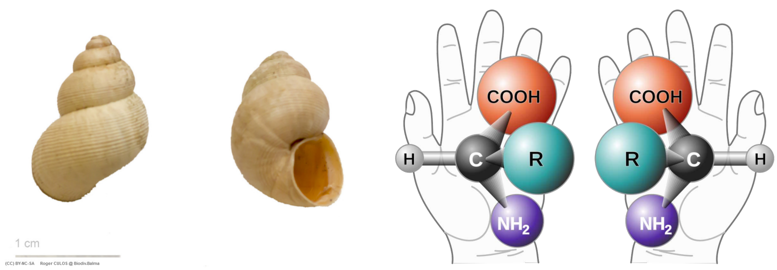

Figure 3. From https://balma.biodiv.fr/spip.php?observation7352.

An interesting question about mirror is shown in Figure 4: Why does a mirror reverse left to right, but not top to bottom?

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Figure 4. Why does a mirror reverse left to right, but not top to bottom? Answer is found at https://www.canberratimes.com.au/story/7661679/why-does-a-mirror-reverse-left-to-right-but-not-top-to-bottom/.

2.3. Building Groups from Subgroups

A subset H of a group G is a *subgroup* of G if it contains e and is stable versus group operations: $HH := \{h_1h_2\,,\, h_1, h_2 \in H\} \subset H$ and $H^{-1} := \{h^{-1}\,,\, h \in H\} \subset H$. *Direct* product of two groups G_1 and G_2 consists in glueing the two groups together, without interaction: $G = G_1 \times G_2$ is simply their Cartesian product, endowed with the group law:

$$(g_1, g_2)(g_1', g_2') = (g_1g_1', g_2g_2')\,, \quad g_1, g_1' \in G_1,\, g_2, g_2' \in G_2\,.$$

Neutral element is (e_1, e_2). With the identifications $g_1 \sim (g_1, e_2)$, $g_2 \sim (e_1, g_2)$, both G_1 and G_2 are invariant subgroups of $G_1 \times G_2$ (think to the plane as the product of two lines, or to the torus as the product of two circles).

For *semi-direct* product of two groups G_1 and G_2, one suppose given a homomorphism α from G_2 into the group $\mathrm{Aut}\,G_1$ of automorphisms of G_1. Then one defines the semidirect product $G = G_1 \rtimes G_2$ as the Cartesian product, endowed with the group law:

$$(g_1, g_2)(g_1', g_2') = (g_1\alpha_{g_2}(g_1'), g_2g_2')$$

Neutral element is (e_1, e_2) and the inverse of (g_1, g_2) is is $(g_1, g_2)^{-1} = \left(\left[\alpha_{g_2^{-1}}(g_1)\right]^{-1}, g_2^{-1}\right)$ (think to the affine group of the line, which combines dilation and translation).

2.4. Group Cosets

Let H be a subgroup of G. One says that $g_1 \equiv g_2 \mod H$ if there exists $h \in H$ such that $g_2 = g_1h$. Denoting $\bar{g} = gH$ the equivalence class of g mod H, the set of equivalence classes is denoted G/H and called the left coset of G by H. Right coset $H\backslash G$ is defined in a similar way as the set of equivalence classes $\{Hg\,,\, g \in G\}$.

2.5. Group Actions

Irreducibility linked to symmetry (and invariance) is the mathematical concept of a group, more precisely group of transformations (see Appendix A for more details). Concretely, elements in a group G, like the group of Euclidean displacements in the plane (semi-direct product of rotations and

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translations), transform into themselves objects in a set X, like the set of triangles, in such a way that we can compose such transformations in an associative way and we can transform back too:

$$g \in G\,,\; x \in X\,,\; g \cdot x = y \in X \Leftrightarrow x = g^{-1} \cdot y\,,\; (gg') \cdot x = g \cdot (g' \cdot x)$$

3. Numbers and Groups

In this section we implement the group material exposed above through the sequence of numbers, from the most elementary ones to some sophisticated objects like octonions and Cayley algebras.

3.1. Sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_p$

In preamble, one should remind that Arithmetic with (cardinal) Hindu-Arabic numerals is easy while it is inextricable with (ordinal) Roman numerals (*e.g.*, Figure 5).

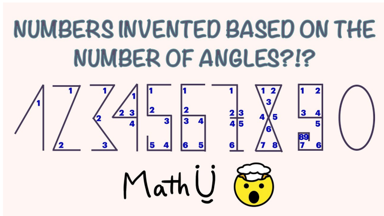

Figure 5. The Hindu-Arabic numeral system and counting angles, *On the Calculation with Hindu Numerals by al-Kwharizmi (about 825 AD)* and *On the Use of Indian Numerals* by Abu Yusuf Yaqub Ibn Ishdaq al-Kindi (830 AD). Eventually the Hindu-Arabic way of writing numbers and computations replaced the Roman Numerals. From [3].

The set of **natural numbers** $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ (note that including 0 is now the common convention among set theorists and logicians) equipped with addition, is not a group (no inverse), just an abelian semi-group. Equipped with multiplication, $\mathbb{N}$ is an abelian semi-group as well.

The set of **rational integers** $\mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = -\mathbb{N}^* \cup \mathbb{N}$, equipped with addition, is an abelian group. Equipped with multiplication, it is a semi-group.

The set of **rational integers modulo** p, $p = 2, 3, \cdots$: $\mathbb{Z}_p = \{0, 1, 2, \cdot, p-1\,, \mathrm{mod}\, p\}$. Equipped with addition, it is a finite abelian group with p elements. Chirality corresponds to $\mathbb{Z}_2$

3.2. Example: Cubic Relations

Richard Kerner, in a comprehensive series of papers (refer to [4] for a detailed review), posited that cubic (or ternary) relations possess the ability to encapsulate diverse symmetries in relation to the permutation group S_3 or its cyclic subgroup $\mathbb{Z}_3$.

Significantly, these insights find applications in the $\mathbb{Z}_3$-graded generalization of Grassmann algebras, manifesting in their realization within the realm of generalized exterior differential forms. Moreover, they play an important role in the development of gauge theory founded upon this differential calculus and in the ternary generalization of Clifford algebras. These advancements not only unlock new perspectives in mathematical structures but also extend their reach to the realm of physics, providing a framework for describing families of quarks.

3.3. Set $\mathbb{Q}$

The set $\mathbb{Q}$ of **rational numbers** is defined as the set of equivalence classes of pairs of integers (p, q) with $q \neq 0$, $(p_1, q_1) \sim (p_2, q_2) \Leftrightarrow p_1 q_2 = p_2 q_1$. The equivalence class of (p, q) is denoted $\dfrac{p}{q}$.

The set $\mathbb{Q}$, together with the addition and multiplication operations shown below, forms a **field**:

$$(p_1, q_1) + (p_2, q_2) = (p_1 q_2 + p_2 q_1, q_1 q_2)\,, \quad (p_1, q_1) \times (p_2, q_2) = (p_1 p_2, q_1 q_2)\,.$$

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Actually $\mathbb{Q}$ is the prototype of the concept of field: a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational.

3.4. $\mathbb{Q}$ Is Not "Complete"

Let us equip the field $\mathbb{Q}$ with the *metrical topology* associated with the distance $d(x,y) := |x-y|$ between two rationals x and y. This topology gives a sense to the notion of a *Cauchy sequence* $(x_n)_{n\in\mathbb{N}}$ of rationals: one says that $(x_n)_{n\in\mathbb{N}}$ is Cauchy if for all $\epsilon > 0$, there exist $N > 0$ such that, for all $m, n \geq N$, one has

$$|x_m - x_n| < \epsilon\,, \quad i.e.\,, \lim_{m,n\to\infty} |x_m - x_n| = 0\,.$$

If (x_n) is a convergent rational sequence (that is, $\mathbb{Q} \ni x_n \to x \in \mathbb{Q}$), then it is a Cauchy sequence. On the other hand not all Cauchy sequences of rationals are convergent in $\mathbb{Q}$.

Our intuitive sense suggests that when a sequence of terms progressively approaches one another, they must be converging towards a specific value. Take, for instance, the sequence $1, 1.4, 1.41, 1.414, \cdots$, an approximation for the positive solution to the algebraic equation $x^2 = 2$, or the sequence $3, 3.1, 3.14, 3.142, 3.1416, 3.14159, \cdots$, which approximates the non-algebraic (or transcendant) mythic length of the circumference of the unit circle. Both sequences adhere to the Cauchy criterion, signifying their terms draw near each other, yet they don't converge to a rational number.

This observation prompts the existence of certain "irrational" numbers, denoted as "$\sqrt{2}$" and "π", towards which these sequences converge. Consequently, to encompass these irrational values, as well as all other numbers that Cauchy sequences seem to approach, one extends the number system beyond rationals [5].

3.5. Unreal "Real" Numbers Complete Real Rationals!

It is this intuition which motivated Cauchy to use such sequences to define the real numbers: $\mathbb{R}$ is a completion of $\mathbb{Q}$. The real numbers are constructed as *equivalence classes of Cauchy sequences* [6]. Two Cauchy sequences (x_n) and (y_n) of rational numbers are equivalent if $|x_n - y_n| \to 0$, *i.e.*, if the sequence $(x_n - y_n)$ tends to 0. The real numbers $\mathbb{R}$ are the equivalence classes of Cauchy sequences of rational numbers. That is, each such equivalence class is a real number. Through this construction, $\mathbb{R}$ inherits the algebraic field structure of $\mathbb{Q}$, and is complete for the metric topology: all Cauchy sequences of real numbers are convergent in $\mathbb{R}$. In this sense one says that $\mathbb{R}$ is the closure of $\mathbb{Q}$: $\mathbb{R} = \overline{\mathbb{Q}}$, *i.e.*, $\mathbb{R}$ contains all limit points of $\mathbb{Q}$. However there is a big difference: while $\mathbb{Q}$ is *countable*, *i.e.*, there exist a one-to-one map $\mathbb{Z} \mapsto \mathbb{Q}$, the field $\mathbb{R}$ is *uncountable*.

3.6. $\mathbb{C}$ Is "Algebraically Closed" and Complete

The algebraic equation $x^2 + 1 = 0$ has no solution in $\mathbb{R}$. So let us equip the set $\mathbb{R} \times \mathbb{R} := \{z = (x,y)\,,\ x, y \in \mathbb{R}\}$ (Cartesian product) with the commutative addition and multiplication operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\,, \quad (x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\,. \tag{3.1}$$

Both operations afford $\mathbb{R} \times \mathbb{R}$ with a commutative field structure, $0 \equiv (0,0)$ and $1 \equiv (1,0)$ being the neutral elements for addition and multiplication respectively. This is the field of complex numbers and is denoted by $\mathbb{C}$. One introduces the "imaginary" number $\mathrm{i} \equiv (0,1)$, imaginary because it is the square root of -1 [7–9]. Indeed, one checks that $\mathrm{i}^2 = -1$, and so i is solution of $z^2 + 1 = 0$ (like the real $\sqrt{2}$ is solution of $x^2 - 2 = 0$). The complex numbers also form the real vector space of dimension two, with $\{1, \mathrm{i}\}$ as a standard basis. This justifies the notation $z = x1 + y\mathrm{i} \equiv x + \mathrm{i}y$.

Introducing the mirror symmetry with respect to the real axis, $z = x + \mathrm{i}y \mapsto \bar{z} := x - \mathrm{i}y$, one checks that $z\bar{z} = x^2 + y^2 \equiv |z|^2$. Equipped with the topology associated with the distance $|z_1 - z_2|$, $\mathbb{C}$ is complete, and, as a vector space, is isomorphic to the Euclidean plane.

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Any algebraic equation $a_n z^n + a_n z^n + \cdots a_1 z + a_0 = 0$, with complex coefficients, has n roots in $\mathbb{C}$, *i.e.*, $\mathbb{C}$ is algebraically closed (fundamental theorem of the algebra)

3.7. Complex Numbers Emerging from Tensor Product of Two Planes

There exists an interesting interpretation of the multiplication law (3.1) in terms of *tensor product* of two planes. The tensor product stands as a cornerstone, arguably the most distinctive one, in the realm of quantum mechanics. In essence, the Cartesian product, traditionally employed to model two systems independently in classical physics, undergoes a transformative process known as quantization. This metamorphosis results in the emergence of the tensor product of two vector spaces, a fundamental framework indispensable for understanding the intricacies of quantum mechanical systems.

First, we remind that a vector space V over a field $\mathbb{F}$, *e.g.*, $\mathbb{Q}$ or $\mathbb{R}$, is a set whose elements or vectors, may be added together and multiplied ("scaled") by elements ("scalars") in $\mathbb{F}$. Two essential properties must be satisfied: the distributivity of scalar multiplication with respect to the vector addition and the distributivity of scalar multiplication with respect to field addition. Then, the tensor product $V \otimes W$ of two vector spaces V and W (over the same field) is the vector space over $\mathbb{F}$ is defined as the vector space over $\mathbb{F}$ consisting of all bilinear forms from $V \times W$ to $\mathbb{F}$. Note that $\dim(V \otimes W) = \dim V \dim W$ while $\dim(V \times W) = \dim V + \dim W$.

Hence the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$ of the plane with itself, with basis $(\mathbf{e}_1, \mathbf{e}_2)$, is the four-dimensional vector space of bilinear forms

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R}^2 \ni (\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, \mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2) &\mapsto \varphi(\mathbf{x}, \mathbf{y}) \\ &= \sum_{i,j=1,2} x_i y_j \varphi_{ij} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv {}^{\mathrm{t}}\mathbf{x} A_\varphi \mathbf{y}\,, \end{aligned}$$

where $\varphi_{ij} = \varphi\left(\mathbf{e}_i, \mathbf{e}_j\right)$. Any 2×2 real matrix A can decompose as $A = a_0 \mathbb{1}_2 + a_1\sigma_1 + a_2\tau_2 + a_3\sigma_3$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices $\sigma_1, \sigma_2 := -\mathrm{i}\tau_2, \sigma_3$ were introduced by Pauli [10]. Hence, we have for the matrix elements of A_φ:

$$a_0 = \frac{\varphi_{11} + \varphi_{11}}{2}, \quad a_1 = \frac{\varphi_{21} + \varphi_{12}}{2}, \quad a_2 = \frac{\varphi_{21} - \varphi_{12}}{2}, \quad a_3 = \frac{\varphi_{11} - \varphi_{11}}{2}.$$

Due to the algebraic relations

$$\sigma_1\tau_2 = \sigma_3 = -\tau_2\sigma_1\,, \quad \tau_2\sigma_3 = \sigma_1 = -\sigma_3\tau_2\,, \quad \sigma_3\sigma_1 = -\tau_2 = -\sigma_1\sigma_3\,, \tag{3.2}$$

one can give the decomposition of the matrix A_φ the bi-complex form:

$$A_\varphi = a_0\mathbb{1}_2 + a_2\tau_2 + \sigma_1\left(a_1\mathbb{1}_2 + a_2\tau_2\right) = \begin{pmatrix} a_0 + a_3 & a_1 - a_2 \\ a_1 + a_2 & a_0 - a_3 \end{pmatrix} \tag{3.3}$$

$$\equiv \mathfrak{z}_1 + \sigma_1\mathfrak{z}_2 = \mathfrak{z}_1 + {}^{\mathrm{t}}\mathfrak{z}_2\sigma_1\,. \tag{3.4}$$

Due to $\tau_2^2 = -\mathbb{1}_2$, which means that the $\pm\tau_2$ are square roots of $-\mathbb{1}_2$, whilst $\sigma_1^2 = \mathbb{1}_2 = \sigma_3^2$, the expression

$$\mathfrak{z} = x\mathbb{1}_2 + \tau_2 y = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \tag{3.5}$$

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is a matrix representation of the complex number $z = x + \mathrm{i}y$ with the correspondences $1 \mapsto \mathbb{1}$, $\mathrm{i} \mapsto \tau_2$, and $\bar{z} \mapsto {}^t\mathfrak{z}$. The multiplication law defined in (3.1) perfectly aligns with the matrix multiplication operation represented as $\mathfrak{z}_1\mathfrak{z}_2$. Now any matrix of the type (3.5) can be factorised as

$$\begin{aligned} \mathfrak{z} &= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \sqrt{x^2+y^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \sqrt{x^2+y^2}\exp\theta\tau_2\,, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}\,, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}\,. \end{aligned} \tag{3.6}$$

This is the matrix form of the expression of the complex number $z = re^{\mathrm{i}\theta}$ in terms of its polar coordinates, (r,θ), $r \geq 0$, $\theta \in [0, 2\pi)$ mod (2π).

The correspondence $\mathrm{i} \leftrightarrow \tau_2$ is not inherently absolute; an alternative choice could be $\mathrm{i} \leftrightarrow -\tau_2$. This ambiguity arises from the existence of two possible orientations when defining trigonometric functions: clockwise and anticlockwise. It is crucial to recognize that this feature should not be disregarded when transitioning from a formalism based on complex numbers to one expressed in terms of real numbers [11].

3.8. 2D Lattices

3.8.1. General

Consider two complex numbers ω_1, ω_2 such that ω_2/ω_1 is <u>not</u> real. They define the ("Bravais") lattice in the plane $\Lambda(\omega_1,\omega_2) = \{n_1\omega_1 + n_2\omega_2\,,\, n_1, n_2 \in \mathbb{Z}\} \equiv \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with *lattice basis* (or primitive) $\{\omega_1,\omega_2\}$, and with *fundamental parallelogram* [or *primitive (unit) cell*] determined by $(0,\omega_1,\omega_2)$. Λ is a subgroup of $\mathbb{C}$ for the addition (~ translation). Two pairs (ω_1,ω_2), (α_1,α_2) are *equivalent* if they generate the same lattice, *i.e.*, if there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{Z}$, $ad - bc = \pm 1$, such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

The set of such matrices form the noncommutative *modular group* $\mathrm{PSL}(2,\mathbb{Z})$ (relevant for Weirstrass elliptic functions).

There are 5 possible Bravais lattices in 2-dimensional space: monoclinic (arbitrary ω_1, ω_2), tetragonal or (square $\omega_2/\omega_1 = e^{\pm\mathrm{i}\pi/2}$) , hexagonal ($\omega_2/\omega_1 = e^{\pm 2\mathrm{i}\pi/3}$), orthorhombic (rectangular $\omega_2/\omega_1 = \rho e^{\pm\mathrm{i}\pi/2}$, $1 \neq \rho > 0$), and orthorhombic centered (rectangular centered). They are shown in Figure 6.

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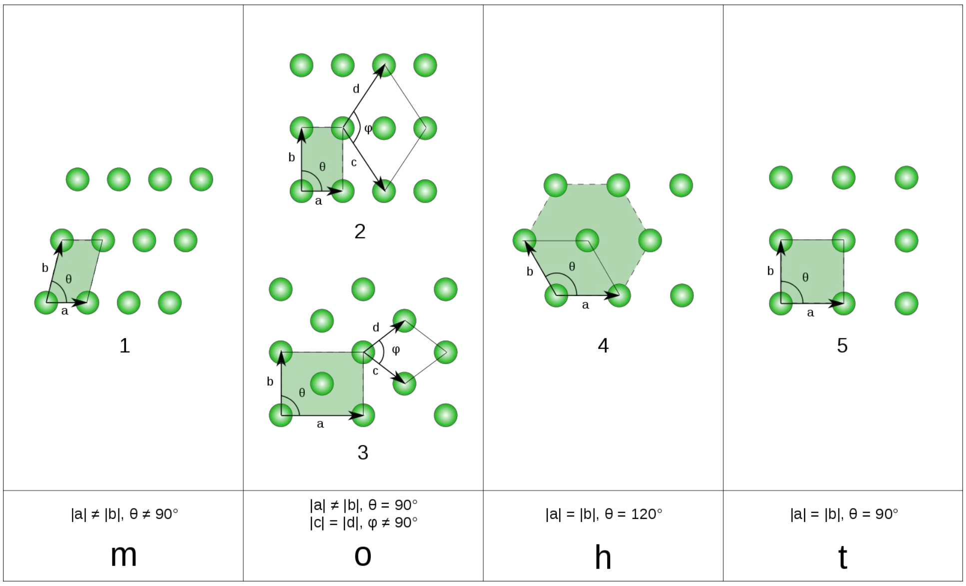

Figure 6. Five 2D lattice types. From https://en.m.wikipedia.org/wiki/File:2d-bravais.svg.

3.8.2. Graphene Example

A (two-dimensional) crystal is a periodic array of atoms. For the graphene (hexagonal lattice or honeycomb), Figure 7, the primitive translation vectors are alternatively defined as:

$$\omega_1 = a\left(\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right), \quad \omega_2 = a\sqrt{3}, \quad a \approx 1.42\,\mathring{A},$$

which means that $\omega_2/\omega_1 = e^{-\mathrm{i}\pi/3}$.

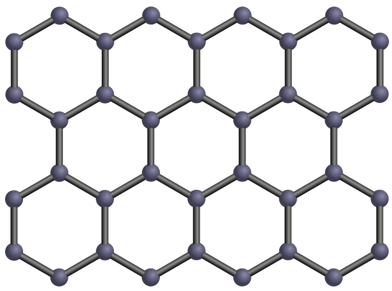

Figure 7. An illustration of atomic organisation of Graphene. From https://pixabay.com/fr/vectors/graphène-graphite-benzène-chimie-161773/.

4. Symmetry and Group(s)

4.1. Symmetry in Physics

D. Gross, in his enlightening 1995 Physics Today article [12] exploring Wigner's legacy, aptly observed that

> *Einstein's great advance in 1905 was to put symmetry first, to regard the symmetry principle as the primary feature of nature that constrains the allowable dynamical laws.*

In contemporary physics, the principles of symmetry hold a paramount status as the most fundamental components of our description of nature. This prominence owes a significant debt to the influential contributions of the Princeton group, comprising notable figures such as Wigner, Bargmann, Weyl, Gödel, von Neumann, ..., from Princeton University and/or the Institute for Advanced Study

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(IAS). Their collective insights and groundbreaking work have played a pivotal role in elevating the significance of symmetry principles, making them a cornerstone in our understanding of the fundamental laws governing the physical universe.

The foundation of our comprehension regarding the existence of elementary particles is rooted in the relativity principle. This principle posits that physical laws remain invariant under fundamental transformations of inertial frames, specifically translation (implying homogeneity in space-time), rotation (indicating isotropy in space-time), and boosts. Together, these transformations constitute a relativity or kinematic group, known as the Galileo or Poincaré group. Notably, both groups exhibit a structure that can be described as a semi-direct product of subgroups. This conceptual framework has paved the way for a profound understanding of the nature and existence of elementary particles in the realm of physics. One should add the importance of

- discrete translational and rotational symmetries in classifying molecules, crystals, quasicrystals,
- continuous rotational symmetries (together with parity and time reversal) in explaining and classifying atomic spectra,
- unitary symmetries for describing and classifying nuclear spectra,
- unitary symmetries for describing strong and weak interactions (gauge fields) and successfully predicting the existence of new elementary particles.

In his 1963 Nobel lecture quoted by D. Gross, Wigner pointed out that progress in physics was partly based on the ability to separate the analysis of a physical phenomenon into two parts:

1. *initial conditions that are arbitrary, complicated, unpredictable, as forming the "phase space" of a physical system in a wide sense, and laws of nature that summarize the regularities which are independent of the initial conditions*.
2. Then, Wigner argued that *symmetry or invariance principles provide a structure and coherence to the laws of nature: they summarize the regularities of the laws that are independent of the specific dynamics. Without regularities in the laws of nature we would be unable to discover the laws themselves*.

One should also mention what Pierre Curie noticed in 1894 [13] (Neuman-Minnigerode-Curie Principle):

- When certain causes produce certain effects, the elements of symmetry of the causes must be found in the effects produced.
- When certain effects reveal a certain asymmetry, this asymmetry must be found in the causes which gave rise to it.

Note that the reciprocal of these two assertions is not true. More precisely, the characteristic symmetry of a phenomenon is that symmetry which is best compatible with the existence of the phenomenon. A phenomenon can exist in surroundings which possess its characteristic symmetry or at least one subgroup of its characteristic symmetry. In other words, certain symmetry elements can exist together with certain phenomena but they are not necessary. But it is necessary that certain symmetry elements do not exist (Neuman-Minnigerode-Curie Principle).

4.2. Lie Groups and Lie Algebras

Building upon the foundational principles of symmetries and natural laws, as eloquently discussed by Levy-Leblond [14], we delve deeper into one of their unescapable mathematical frameworks: the theory of Lie groups and algebras.

To a Lie group G corresponds its Lie algebra $\mathfrak{g} \ni X \mapsto \exp X = g \in G$, where the notation exp takes it susual sense when we deal with group of matrices (see Appendix B for more mathematical details). Simple or semi-simple Lie groups (Lie algebras), like (pseudo-)rotation groups or unitary groups or symplectic groups, are ubiquitous in Physics, see Table 1 for the nomenclature.

Simple Lie algebras have been classified by Cartan through their one-to-one correspondence to possible (irreducible reduced) root systems in $\mathbb{R}^n$. There are four classes of classical (irreducible reduced) root systems in $\mathbb{R}^n$:

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1. A_n associated with unitary groups $\mathrm{SU}(n+1)$ or their complexified $\mathrm{SL}(n+1,\mathbb{C})$ or the real form $\mathrm{SL}(n+1,\mathbb{R})$.
2. B_n associated with orthogonal (rotation) groups $\mathrm{SO}(2n+1)$ or their pseudo- rotation counterparts $\mathrm{SO}(p,q,p+q=2n+1)$.
3. C_n associated with symplectic groups $\mathrm{Sp}(2n)$.
4. D_n associated with orthogonal (rotation) groups $\mathrm{SO}(2n)$ or their pseudo- rotation counterparts $\mathrm{SO}(p,q,p+q=2n)$.

Moreover there are five exceptional cases: G_2, F_4, E_6, E_7, E_8.

Table 1. Nomenclature for the most current Lie groups and algebras. The symbol $\mathrm{M}(n,\mathbb{K})$ is for the algebra of $n\times n$ matrices with coefficients in $\mathbb{K}=\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$, etc. Note that $J=\begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix}$ and that $\mathrm{Sp}(n)$ is frequently written as $\mathrm{Sp}(2n)$. (M,g): Manifold M with metric g, $\mathcal{X}(M)$: vector fields on M, $\mathcal{L}_X$: Lie derivative, (M,ω): Symplectic manifold M with symplectic 2-form ω.

Lie Group	Lie Algebra
General Linear Group: $\mathrm{GL}(n,\mathbb{K})=\{A\in\mathrm{M}(n,\mathbb{K})\,,\,\det A\neq 0\}$	$\mathfrak{gl}(n,\mathbb{K})=\{A\in\mathrm{M}(n,\mathbb{K})\}$
Special Linear Group: $\mathrm{SL}(n,\mathbb{K})=\{A\in\mathrm{GL}(n,\mathbb{K})\,,\,\det A=1\}$	$\mathfrak{sl}(n,\mathbb{K})=\{A\in\mathfrak{gl}(n,\mathbb{K})\,,\,\mathrm{Tr}\,A=0\}$
Special Orthogonal Group: $\mathrm{SO}(n,\mathbb{K})=\{A\in\mathrm{SL}(n,\mathbb{K})\,,\,A\,{}^{\mathrm{t}}\!A=\mathbb{1}_n\}$	$\mathfrak{so}(n,\mathbb{K})=\{A\in\mathfrak{sl}(n,\mathbb{K})\,,\,A+{}^{\mathrm{t}}\!A=0\}$
Special Unitary Group: $\mathrm{SU}(n)=\{A\in\mathrm{SL}(n,\mathbb{C})\,,\,AA^{\dagger}=\mathbb{1}_n\}$	$\mathfrak{su}(n)=\{A\in\mathfrak{sl}(n,\mathbb{C})\,,\,A+A^{\dagger}=0\}$
Symplectic Group: $\mathrm{Sp}(n)=\{A\in\mathrm{GL}(2n,\mathbb{R})\,,\,AJ\,{}^{\mathrm{t}}\!A=J\}$	$\mathfrak{sp}(n)=\{A\in\mathfrak{gl}(2n,\mathbb{R})\,,\,AJ+J\,{}^{\mathrm{t}}\!A=0\}$
Group of Isometries of (M,g): $\mathrm{G}=\{\phi:M\mapsto M\,\vert\,\phi\text{ preserves }g\}$	$\mathfrak{g}=\{X\in\mathcal{X}(M)\,\vert\,\mathcal{L}_X g=0\}$
Group of Symplectomorphisms of (M,ω): $\mathrm{G}=\{\phi:M\mapsto M\,\vert\,\phi^{*}\omega=\omega\}$	$\mathfrak{g}=\{X\in\mathcal{X}(M)\,\vert\,\mathcal{L}_X\omega=0\}$

4.3. Physics and Low-Dimensional Root Systems

There exist isomorphisms between low-dimensional root systems (see Appendix B). Borrowing its terminology to atomic spectroscopy one can enunciate that the lowest levels in the Cartan classification are occupied by the most familiar or well-known symmetries of our physics discipline:

(i) $A_1\simeq B_1\simeq C_1$, ($\mathrm{SO}(3)\simeq\mathrm{SU}(2)/\mathbb{Z}_2$, $\mathrm{SL}(2,\mathbb{R})\simeq\mathrm{SU}(1,1)\simeq\mathrm{Sp}(2,\mathbb{R})\simeq\mathrm{SO}_0(2,1)\times\mathbb{Z}_2$): space isotropy, spherical rotator, geometric optical devices, q-bits, Lorentz invariance, infinite well, isospin, kinematic groups of 1 + 1 de Sitter and Anti de Sitter space-times, ...

(ii) $A_1\times A_1$ ($\mathrm{SU}(2)\times\mathrm{SU}(2)\simeq\mathrm{SO}(4)$, $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)\simeq\mathrm{SO}_0(2,2)$): orbital degeneracy of the Coulomb potential or Kepler problem [18], two-dimensional square infinite well (quantum dot), ...

(iii) A_2 ($\mathrm{SU}(3)$): spatial harmonic oscillator, standard model color, q-trits, ...

(iv) $B_2\simeq C_2$ ($\mathrm{SO}_0(4,1)\simeq\mathrm{Sp}(2,2)/\mathbb{Z}_2$, $\mathrm{SO}_0(3,2)\simeq\mathrm{Sp}(4)/\mathbb{Z}_2$): kinematic groups of 3+1 de Sitter [19] and Anti de Sitter [20] space-times, dynamical symmetry of the H atom discrete or continuous spectrum [18], ...

(v) $A_3\simeq D_3$ ($\mathrm{SO}(6)\simeq\mathrm{SU}(4)/\mathbb{Z}_2$, $\mathrm{SO}_0(4,2)\simeq\mathrm{SU}(2,2)/\mathbb{Z}_2$, $\mathrm{SO}_0(5,1)$): gauge symmetries, conformal group of 3 + 1 de Sitter, Anti de Sitter and Minkowski space-times respectively, and their roles in the interpretation of the dark universe [15–17], dynamical symmetry of the H atom discrete and continuous spectrum, AdS/DS-CFT, ...

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5. Playing with Spheres

We now explore the world of unit spheres in light of group theory.

5.1. The Circle and Its Complexified Version

The unit circle is defined in the complex plane as $\mathbb{S}^1 = \{e^{\mathrm{i}\theta}\,, \theta_0 \leq \theta < \theta_0 + 2\pi\,, \mod 2\pi\}$. It is a compact multiplicative commutative group, denoted U(1). One should think to the importance of this group in mathematics (*e.g.*, Fourier series and integral) and in physics (*e.g.*, gauge invariance in electrodynamics).

The complexified unit circle $\mathbb{S}^1_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 1\} = \{e^{\mathrm{i}z}\,, z = x + \mathrm{i}y \in \mathbb{C}\}$ is also a group. It is the non-compact multiplicative commutative group

$$\begin{aligned}
&\mathrm{SO}(2,\mathbb{C}) = \\
&\left\{\begin{pmatrix}\cos(x+\mathrm{i}y) & -\sin(x+\mathrm{i}y)\\ \sin(x+\mathrm{i}y) & \cos(x+\mathrm{i}y)\end{pmatrix} = \begin{pmatrix}\cos(x) & -\sin(x)\\ \sin(x) & \cos(x)\end{pmatrix}\begin{pmatrix}\cosh(y) & -\mathrm{i}\sinh(y)\\ \mathrm{i}\sinh(y) & \cosh(\mathrm{i}y)\end{pmatrix}\right\}\\
&= \{\exp \tau_2 x \, \exp \sigma_2 y\,,\ x, y \in \mathbb{R}\} \simeq \mathrm{U}(1) \times \mathbb{R}\,. \qquad (5.1)
\end{aligned}$$

An example is found with the motion of a particle on the unit circle: the corresponding phase space, *i.e.*, the set of pairs (angular position, angular momentum) is the infinite cylinder

$$\begin{aligned}
\mathrm{Cyl} &= \{(\theta\,, J \propto \dot{\theta})\,, \theta \in \mathbb{R} \mod 2\pi\,, J \in \mathbb{R}\}\\
&\equiv \left\{e^{\mathrm{i}\theta}\,(\cosh J + (-1)\sinh J)\,, \theta \in \mathbb{R} \mod 2\pi\,, J \in \mathbb{R}\right\} \cong \mathbb{S}^1_{\mathbb{C}}\,,
\end{aligned}$$

whose the matrix form (5.1) is obtained through the maps $\mathrm{i} \mapsto \tau_2$ for the first factor and $(-1) \mapsto \sigma_2 = \mathrm{i}\tau_2$ for the second factor.

We could as well consider the free motion on the two-dimensional de Sitter space-time [19]. The latter has the topology $\mathbb{S}^1 \times \mathbb{R}$. It may be visualized as the one-sheet hyperboloid $\{(y^0, y^1, y^2) \in \mathbb{R}^3 \mid (y^0)^2 - (y^1)^2 - (y^2)^2 = -R^2\}$.

The phase space for such a motion is also a one-sheet hyperboloid with the same cylinder topology. In this case both the configuration space and the phase space have $\mathrm{SO}_0(1,2) \simeq \mathrm{SU}(1,1)/\mathbb{Z}_2$ as invariance group.

5.2. The Sphere $\mathbb{S}^3$ through Its Multifaceted Aspects

Our familiar sphere $\mathbb{S}^2$ is "poor": no intrinsic group structure, save the fact that it includes mythological structures, like the five platonic solids (see next subsection). Then let us consider the next one in the sequence: the sphere $\mathbb{S}^3$. It may be identified with the compact multiplicative non-commutative group SU(2) corresponding to A_1. Think to the importance of this group in mathematics (*e.g.*, harmonic analysis in space) and in physics (angular momentum, spin, q-bit, ...).

5.2.1. Sphere $\mathbb{S}^3$ and Group SU(2)

The relation of $\mathbb{S}^3$ to SU(2) is given by:

$$\begin{aligned}
&\mathrm{SU}(2) \ni U_\xi = \begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix} \equiv \begin{pmatrix}\xi_0 + \mathrm{i}\xi_3 & -\xi_2 + \mathrm{i}\xi_1\\ \xi_2 + \mathrm{i}\xi_1 & \xi_0 - \mathrm{i}\xi_3\end{pmatrix}\,,\ \xi_0\,, \xi_i \in \mathbb{R}\,, i = 1,2,3\,,\\
&\det(U_\xi) = |\alpha|^2 + |\beta|^2 = \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\,,\\
&\text{i.e.}\,,\ \xi \equiv (\xi_0, \xi_1, \xi_2, \xi_3) \equiv (\xi_0, \boldsymbol{\xi}) \in \mathbb{S}^3\,.
\end{aligned}$$

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5.2.2. Sphere $\mathbb{S}^3$ and Rotations in $\mathbb{R}^3$

Let us explain the relation between $\mathbb{S}^3$ and the proper rotations in space $\mathbb{R}^3$. Any proper rotation in space is determined by a unit vector $\mathbf{n}$ defining the rotation axis and a rotation angle $0 \leq \theta < 2\pi$ about $\mathbf{n}$ in the direct sense.

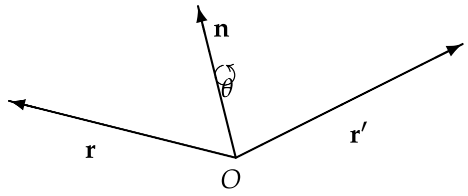

Figure 8. Proper rotation of vector $\mathbf{r}$ to $\mathbf{r}'$ by angle θ about unit vector $\mathbf{n}$ in space $\mathbb{R}^3$.

The action of such a rotation, $\mathcal{R}(\mathbf{n},\theta)$, on a vector $\mathbf{r}$ is given by:

$$\begin{aligned}
\mathbf{r}' &\overset{\text{def}}{=} \mathcal{R}(\mathbf{n},\theta) \bullet \mathbf{r} = \mathbf{r}\cdot\mathbf{n}\,\mathbf{n} + \cos\theta\,\mathbf{n}\times(\mathbf{r}\times\mathbf{n}) + \sin\theta\,(\mathbf{n}\times\mathbf{r}) \\
&= \left[\cos\theta\,\mathbb{1}_3 + (1-\cos\theta)\mathbf{n}\,{}^{t}\mathbf{n} + \sin\theta\,\mathbf{n}\times\right]\mathbf{r}\,, \\
\mathbf{n}\times &:= \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}
\end{aligned}$$

The correspondence between elements rotations $\mathcal{R}(\mathbf{n},\theta)$ and the matrix form $U_\xi \in \mathrm{SU}(2)$ of $\xi = (\xi_0, \boldsymbol{\xi}) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\,\mathbf{n}\right)$ is understood as :

$$\begin{gathered}
\mathbf{r}' = (x_1', x_2', x_3') = \mathcal{R}(\mathbf{n},\theta) \bullet \mathbf{r} \\
\Updownarrow \\
\begin{pmatrix} \mathrm{i}x_3' & -x_2' + \mathrm{i}x_1' \\ x_2' + \mathrm{i}x_1' & -\mathrm{i}x_3' \end{pmatrix} = U_\xi \begin{pmatrix} \mathrm{i}x_3 & -x_2 + \mathrm{i}x_1 \\ x_2 + \mathrm{i}x_1 & -\mathrm{i}x_3 \end{pmatrix} U_\xi^\dagger\,.
\end{gathered} \tag{5.2}$$

5.3. Sphere $\mathbb{S}^3$ and Quaternions (Hamilton, 1843)

Let us equip the set $\mathbb{C}\times\mathbb{C} = \mathbb{C}^2 := \{q = (u,v)\,,\ u,v \in \mathbb{C}\}$ with the commutative addition and non-commutative multiplication operations:

$$(u,v) + (u',v') = (u+u', v+v')\,, \quad (u,v)(u',v') = (uu' - v\overline{v'}, uv' + v\overline{u'})\,. \tag{5.3}$$

Both operations afford $\mathbb{C}^2$ with a noncommutative field structure, $0 \equiv (0,0)$ and $1 \equiv (1,0)$ being the neutral elements for addition and multiplication respectively. This is the field of quaternions [21,22] and is denoted by $\mathbb{H}$. One identifies $(u,0)$ with the complex u and one introduces the "quaternionic imaginary" $\hat{\jmath} \equiv (0,1)$. One checks that $\hat{\jmath}^2 = -1$ and the fundamental commutation rule

$$u\hat{\jmath} = \hat{\jmath}\bar{u} \quad \forall\, u \in \mathbb{C}\,, \quad \text{i.e.} \quad \mathrm{i}\hat{\jmath} = -\hat{\jmath}\mathrm{i}\,.$$

Hence we have the bicomplex notation for quaternions:

$$q = (u,v) = u + v\hat{\jmath} = u + \hat{\jmath}\bar{v}\,. \tag{5.4}$$

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With the notations $\hat{\imath} := (\mathrm{i}, 0)$ and $\hat{k} := (0, \mathrm{i}) = \hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath}$, one has as well $\hat{\imath} = \hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath}$, and $\hat{\jmath} = \hat{k}\hat{\imath} = -\hat{\imath}\hat{k}$, with $\hat{\imath}^2 = -1\hat{\jmath}^2$. Hence, one can decompose any quaternion as

$$q = (u, v) = u + v\hat{\jmath} \quad \text{with} \quad \begin{cases} u &= q_0 + q_1\hat{\imath} \\ v &= q_2 + q_3\hat{\imath} \end{cases} \quad \text{(bicomplex notation)}, \tag{5.5}$$

$$q = (u, v) = q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k} \equiv (q_0, \mathbf{q}) \quad \text{(scalar-vector notation)}. \tag{5.6}$$

Quaternions form the real vector space $\mathbb{R}^4$ of dimension 4, with $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$ as a standard basis. Quaternions with null scalar component, *i.e.*, $q = (0, \mathbf{q}) \equiv \mathbf{q}$, are said pure vector quaternions.

With the bicomplex notation, one easily understands the origin of the multiplication law given in (5.3). With the scalar-vector notations, the quaternionic multiplication reads as

$$qq' = (q_0q_0' - \mathbf{q}\cdot\mathbf{q}', q_0\mathbf{q}' + q_0'\mathbf{q} + \mathbf{q}\times\mathbf{q}').$$

Introducing the mirror symmetry with respect to the 0th axis, $q = (q_0, \mathbf{q}) \mapsto \bar{q} := (q_0, -\mathbf{q})$, one checks that $q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \equiv |q|^2$, $\overline{qq'} = \bar{q}'\bar{q}$ (involutive), $|qq'| = |q||q'|$, and the inverse of $q \neq 0$ is given by $q^{-1} = \bar{q}/|q|$.

5.4. *Quaternions and Space Rotations*

Expressed in "polar" coordinates $q = |q|\xi$, $\xi \in \mathbb{S}^3 \sim \mathrm{SU}(2)$, one obtains the matrix realisation of quaternions:

$$q = (q_0, \mathbf{q}) \leftrightarrow A_q = \begin{pmatrix} q_0 + \mathrm{i}q_3 & \mathrm{i}q_1 - q_2 \\ \mathrm{i}q_1 + q_2 & q_0 - \mathrm{i}q_3 \end{pmatrix}. \tag{5.7}$$

Hence, the correspondence (5.2) between elements $\xi = (\xi_0, \boldsymbol{\xi}) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{n}\right) \in \mathbb{S}^3$ and rotations $\mathcal{R}(\mathbf{n}, \theta)$ in space is now understood in terms of quaternions:

$$\mathbf{r}' = \mathcal{R}(\mathbf{n}, \theta) \bullet \mathbf{r} \;\Leftrightarrow\; (0, \mathbf{r}') = \xi(0, \mathbf{r})\bar{\xi}. \tag{5.8}$$

5.5. *Coquaternions (Cockle, 1849) versus Quaternions*

It is interesting to compare the matrix realisation of quaternions with the decomposition (3.3) of the (Lie) algebra $\mathrm{M}(2, \mathbb{R})$ composed of 2×2 real matrices. Forgetting this matrix realisation and considering the matrix basis $(\mathbb{1}_2 \equiv 1, \sigma_1, \tau_2, \sigma_3)$ as pure numbers

$$(\mathbb{1}_2 \leftrightarrow 1, \sigma_1 \leftrightarrow \boldsymbol{\ell}_1, \tau_2 \leftrightarrow \boldsymbol{\ell}_2, \sigma_3 \leftrightarrow \boldsymbol{\ell}_3),$$

obeying the same algebraic relations as those given in (3.2), one obtains the noncommutative ring $\mathbb{H}_{\mathrm{co}}$ of $q = (a_0, \mathbf{a})$, $a_\mu \in \mathbb{R}$, $\mu = 0, 1, 2, 3$, named *coquaternions* [23,24] or *split-quaternions*. We remind that rings are algebraic structures that generalize fields in the sense that multiplicative inverses need not exist. With this scalar-vector form, the multiplication law for coquaternions reads as

$$qq' = \left((a_0a_0' + a_1a_1' - a_2a_2' + a_3a_3', a_0\mathbf{a}' + a_0'\mathbf{a} + \mathbf{a}\times_{\mathrm{co}}\mathbf{a}'\right), \quad \mathbf{a}\times_{\mathrm{co}}\mathbf{a}' := \begin{cases} a_2a_3' - a_2a_3' \\ a_1a_3' - a_3a_1' \\ a_1a_2' - a_2a_1' \end{cases}$$

If instead we start from the bicomplex form (3.4) and adopt the one-to-one correspondence matrix $\leftrightarrow$ coquaternion

$$\mathrm{M}(2, \mathbb{R}) \ni A = \mathfrak{z}_1 + \sigma_1\mathfrak{z}_2 \leftrightarrow q = z_1 + \boldsymbol{\ell}_1 z_2 \equiv (z_1, z_2), \quad z_1 = a_0 + a_2\boldsymbol{\ell}_2, \quad z_2 = a_1 + a_3\boldsymbol{\ell}_2,$$

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then the multiplication law reads

$$qq' = (z_1, z_2)(z_1', z_2') = \left(z_1 z_1' + \bar{z}_2 z_2', \bar{z}_1 z_2' + z_2 z_1'\right),$$

a formula to be compared with (5.3).

Ring $\mathbb{H}_{co}$ is equipped with the isotropic quadratic form

$$N(q) := a_0^2 - a_1^2 + a_2^2 - a_3^2 = |z_1|^2 - |z_2|^2 = qq^*, \tag{5.9}$$

where $q^* = (a_0, -\mathbf{a})$ is the *conjugate* of q. Each coquaternion q with $N(q) \neq 0$ has inverse $q^{-1} = q^*/N(q)$. One checks that $N(qq') = N(q)N(q')$.

The multiplicative group of coquaternions with $N(q) = 1$ form the unit pseudo-sphere in $\mathbb{R}^4$:

$$\mathtt{H}^3 := \left\{\varsigma \in \mathbb{H}_{co}, qq^* = a_0^2 - a_1^2 + a_2^2 - a_3^2 = 1\right\}.$$

The topology of $\mathtt{H}^3$ is that of the Cartesian product $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \sim \mathbb{T}^2 \times \mathbb{R}$. This is inferred from the parametrisation with (θ, ψ, t):

$$a_0 = \cosh t \cos\theta, \quad a_2 = \cosh t \sin\theta, \quad a_1 = \sinh t \cos\psi, \quad a_3 = \sinh t \sin\psi.$$

Like the matrix realisation of $\mathbb{S}^3$ is $\mathrm{SU}(2)$, the matrix realisation of $\mathtt{H}^3$ is the group $\mathrm{SL}(2,\mathbb{R})$. The symmetry group of the quadratic form (5.9) on $\mathbb{R}^4$ is $\mathrm{O}(2,2)$.

5.6. Complex Quaternions

The ring of complex quaternions is the tensor product $\mathbb{H} \otimes \mathbb{C} \equiv \mathbb{H}_{\mathbb{C}}$. It is just obtained from $\mathbb{H}$ by extending real scalars to complex ones in (5.6):

$$\mathbb{H}_{\mathbb{C}} = \left\{z = z_0 + z_1\hat{\imath} + z_2\hat{\jmath} + z_3\hat{k} \equiv (z_0, \mathbf{z}), z_a \in \mathbb{C}, a = 0,1,2,3\right\}.$$

Note the biquaternion form of a complex quaternion: $z = p + \mathrm{i}q$, $p, q \in \mathbb{H}$ (the correct writing should be $z = p \otimes 1 + q \otimes \mathrm{i}$), which corresponds to the isomorphism $\mathbb{H}_{\mathbb{C}} \cong \mathbb{H} \oplus \mathbb{H}$. The matrix representation of $\mathbb{H}_{\mathbb{C}}$ is analogous to (5.7):

$$\mathbb{H}_{\mathbb{C}} \ni z = (z_0, \mathbf{z}) \leftrightarrow A_z = \begin{pmatrix} z_0 + \mathrm{i}z_3 & \mathrm{i}z_1 - z_2 \\ \mathrm{i}z_1 + z_2 & z_0 - \mathrm{i}z_3 \end{pmatrix}.$$

This establishes the ring isomorphism $\mathbb{H}_{\mathbb{C}} \cong M_2(\mathbb{C})$. For more in-depth information, refer to [20] where these objects are efficiently utilized to describe the two-fold covering $\mathrm{Sp}(4,\mathbb{R})$ of the Anti-de Sitter group $\mathrm{SO}_0(2,3)$.

5.7. The Complexified Sphere $\mathbb{S}^3_{\mathbb{C}}$

The complexified sphere $\mathbb{S}^3_{\mathbb{C}} = \{z = (z_0, \mathbf{z}) \in \mathbb{C}^4 \mid z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1\}$ is also a group. It is the (universal) covering $\mathrm{SL}(2,\mathbb{C})$ of the Lorentz group $\mathrm{SO}_0(3,1) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. Let us consider the motion of a particle on $\mathbb{S}^3$. The phase space is the cotangent bundle

$$T^*(\mathbb{S}^3) = \{z = (q,p) \in \mathbb{R}^4 \times \mathbb{R}^4 \simeq \mathbb{H} \times \mathbb{H} \mid |q|^2 = 1, q \cdot p = 0\}.$$

This phase space can be identified with the complex sphere $\mathbb{S}^3_{\mathbb{C}}$ through the following parametrization of the latter:

$$z = q \cosh|p| + \mathrm{i}\frac{p}{|p|}\sinh|p| \in \mathbb{H}_{\mathbb{C}}, \quad |q| = 1, \quad q \cdot p = 0.$$

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We could as well consider the free motion on the 3 + 1-dimensional de Sitter spacetime. The latter may be viewed as a one-sheeted hyperboloid embedded in a five-dimensional Minkowski space with metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,-1)$:

$$M_{dS} \equiv \left\{ x \in \mathbb{R}^5;\; x^2 = \eta_{\alpha\beta}\, x^\alpha x^\beta = -\frac{3}{\Lambda_{\mathrm{dS}}} \right\}, \quad \alpha,\beta = 0,1,2,3,4\,.$$

For the past years it has becoming a realistic model for our spacetime because of the observed non zero value of the cosmological constant Λ, a model accompanied of concepts like "dark energy" or "quintessence"). The phase space for the motion of a test particle on the de Sitter space-time has also the topology $\mathbb{S}^3 \times \mathbb{R}^3$ or equivalently the complexified $\mathbb{S}^3_{\mathbb{C}} \simeq \mathrm{SL}(2,\mathbb{C})$. In this case both configuration space and phase space have $\mathrm{SO}_0(1,4) \simeq \mathrm{Sp}(2,2)/\mathbb{Z}_2$ as invariance group. More details are found in the volume [19].

5.8. The Sphere $\mathbb{S}^3$, Its Icosians and Its Icosian Ring

The group SU(2) has a finite subgroup denoted here by $\mathcal{I}$ with 120 elements, named icosians by Hamilton. There exists an icosian game invented by Hamilton [25,26]. It is a kind of Eulerian path through the 20 vertices of the dodecahedron, the dual polyhedron of the icosahedron (see Figure 9, and the nice illustration in [27]): a path such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point. Winning strategies use the icosian group $\mathcal{I}$.

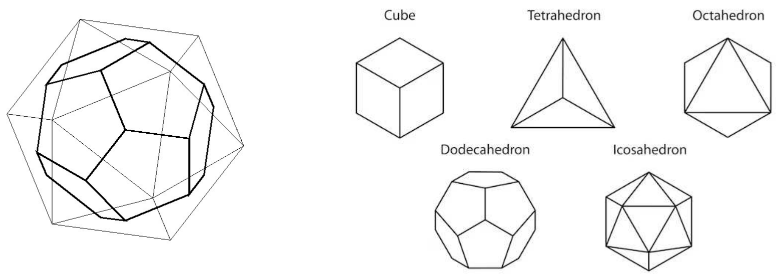

Figure 9. Dodecahedron and its dual, the icosahedron (left). With its twenty triangular faces, twelve vertices and thirty edges, the latter is one of the five Platonic Solids (right). Among the Platonic solids (Theaetetus of Athens, 417- 369 BC; Plato, 428/427–348/347 BC; Euclid, fl. c. 300 BC; Johannes Kepler, 1571-1630), either the dodecahedron or the icosahedron may be seen as the best approximation to the sphere. Remind that a platonic solid is a convex polyhedron with vertices in a sphere $\mathbb{S}^2_r$ of radius r. The faces of a platonic solid are congruent regular polygons, with the same number of faces meeting at each vertex. All edges are congruent, as are its vertices and angles.

One has the isomorphism $\mathcal{I}/\mathbb{Z}_2 \simeq Y$, where Y is the symmetry group (60 proper rotations) of the icosahedron (or of its dual, the dodecahedron).

Let us consider, together with the quaternionic unity 1, three unit-norm quaternions describing through (5.8) 72° rotations around three distinct five-fold axes of the icosahedron,

$$\alpha = \frac{1}{2}(\tau,1,1/\tau,0)\,, \quad \beta = \frac{1}{2}(\tau,0,1,1/\tau)\,, \quad \gamma = \frac{1}{2}(\tau,1/\tau,0,1)\,.$$

Here, $\tau \overset{\mathrm{def}}{=} \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$ is the golden mean, with $-1/\tau = \frac{1-\sqrt{5}}{2} = \tau - 1$, both being algebraic integers, solutions of the algebraic equation with integer coefficients

$$x^2 = x + 1\,.$$

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The $\mathbb{Z}$-span of $\mathcal{I}$ is the icosian ring $\mathbb{I}$ [28] which fills densely $\mathbb{R}^4 \simeq \mathbb{H}$. One shows that

$$\mathbb{I} = \mathbb{Z}[\tau] + \mathbb{Z}[\tau]\alpha + \mathbb{Z}[\tau]\beta + \mathbb{Z}[\tau]\gamma\,,$$

where $\mathbb{Z}[\tau] = \{m + n\tau \mid m, n \in \mathbb{Z}\}$ is the extension ring of the golden mean.

5.9. The Root System E_8 and Quasicrystals

Let us now consider the root system E_8. It is made of 240 vectors in $\mathbb{R}^8$. The corresponding root lattice $\mathbb{E}_8$ is the $\mathbb{Z}$-span of E_8. Because of the crystallographic properties of E_8, we have:

$$\mathbb{E}_8 = \sum_{i=1}^{8} \mathbb{Z}\alpha_i\,,$$

where the α_i's are the basis roots of the corresponding Dynkin diagram shown in (8).

Subsequently, by selecting a particular four-dimensional plane within $\mathbb{R}^8$ possessing a distinct "golden mean" orientation and executing the orthogonal projection of the root lattice $\mathbb{E}_8$ onto this hyperplane, the result is the formation of the icosian ring $\mathbb{I}$.

Similar manipulations on the D_6 root lattice projected onto $\mathbb{R}^3$, on the A_4 root lattice projected onto the plane $\mathbb{R}^2$, and on the A_2 root lattice projected onto the real line $\mathbb{R}$, accompanied by "cuts" based on the choice of appropriate "windows", produce standard models for three-, two- and one-dimensional quasicrystals (*e.g.*, icosahedral quasicrystals, Penrose tilings, Fibonacci chain ...), as is shown in Figure 10, [29].

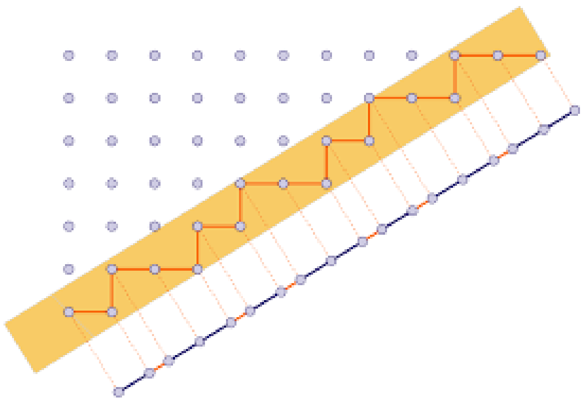

Figure 10. Cut and Project trick is a method to get a quasiperiodic set in $\mathbb{R}$ from a lattice in $\mathbb{R}^2$ (From https://thespectrumofriemannium.wordpress.com/tag/classical-groups/)

In Figure 11 is shown an example of observed quasicrystalline surface.

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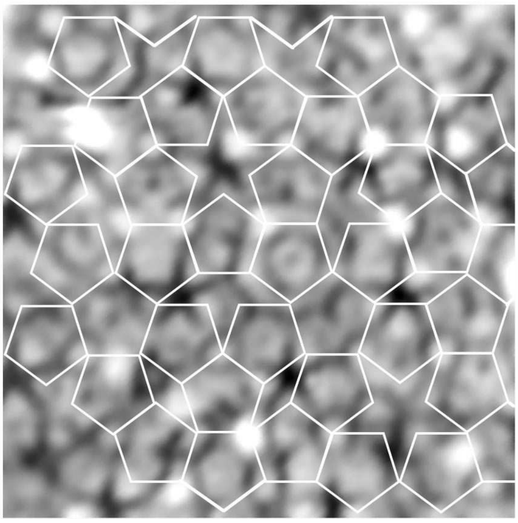

Figure 11. Quasicrystalline five-fold surface of $\mathbf{Al}_{70}$ $\mathbf{Pd}_{21}$ $\mathbf{Mn}_9$, from [30].

5.10. *Algebraic Modelling of Quasicrystals*

Observed self-similarity factors β in quasicrystals are quadratic Pisot-Vijayaraghavan (PV) numbers, *i.e.*, roots > 1 of some quadratic polynomial, whose leading coefficient is 1, whose coefficients are integers, and the other root is < 1 in absolute value. In the case above, the factor is the golden mean $\beta = \tau = \frac{1+\sqrt{5}}{2}$. The other ones are

$$\beta = \tau^2 = \frac{3+\sqrt{5}}{2}, \ \beta = \delta = 1+\sqrt{2}, \ \beta = \theta = 2+\sqrt{3}.$$

Each such β determines a discrete set of the line, $\mathbb{Z}_\beta$, the set of "beta-integers" [31], aimed to play the role of integers in the realm of such quasicrystalline studies. Un example is shown in Figure 12 with the first tau-integers around the origin.

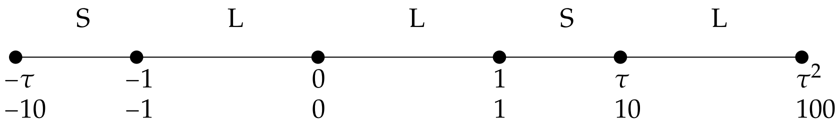

Figure 12. First elements of $\mathbb{Z}_\tau$ (tau-integers) around the origin and associated tiling.

Beta-lattices Γ are precisely aimed to replace lattices in the context of quasicrystals. They are based on beta-integers, like lattices are based on integers:

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i \,,$$

with $(\mathbf{e}_i)$ a base of $\mathbb{R}^d$. Examples are shown in Figures 13 and 14.

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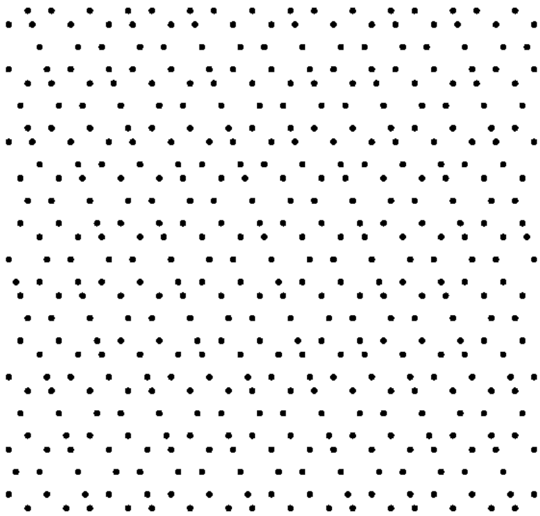

Figure 13. τ-lattice $\Gamma_1(\tau) = \mathbb{Z}_\tau + e^{\frac{i\pi}{5}}\mathbb{Z}_\tau$ in $\mathbb{R}^2$.

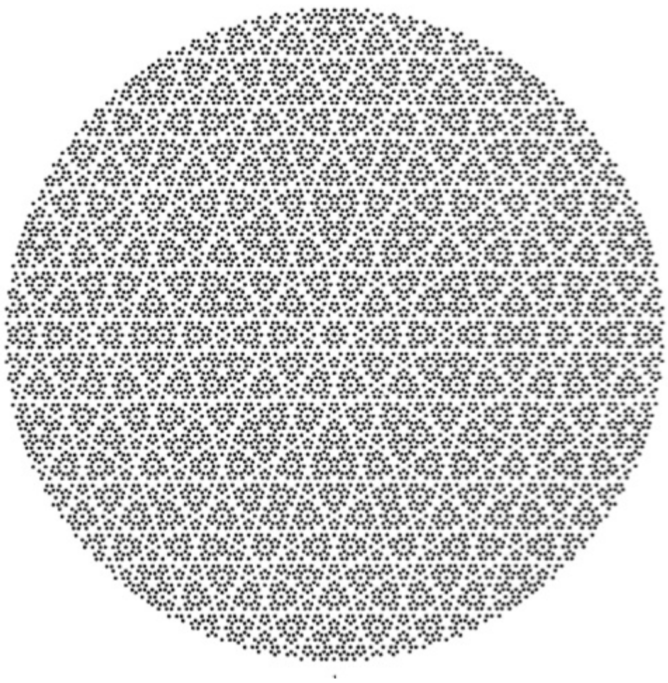

Figure 14. A mathematical five-fold quasicrystal (from [32]).

5.11. Sphere $\mathbb{S}^7$ and Octonions (Graves, 1843, Cayley, 1845)

Like $\mathbb{Q}, \mathbb{C}, \mathbb{H}$, the algebra $\mathbb{O}$ of octonions [33,34] is built by providing the Cartesian product $\mathbb{H}\times\mathbb{H} = \{(p,q)\,,\, p,q \in \mathbb{H}\}$ with the commutative addition and the non-commutative and non-associative product:

$$(p,q) + (p',q') = (p+p',q+q')\,, \quad (p,q)(p',q') = (pp' - \overline{q'}q, q'p + q\overline{p'})\,.$$

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Let us introduce the 8 octonionic basis numbers: $1 \equiv (1,0)$ (unity) and

$$\hat{\imath} \equiv (\hat{\imath},0)\,,\ \hat{\jmath} \equiv (\hat{\jmath},0)\,,\ \hat{k} \equiv (\hat{k},0)\,, \tag{5.10}$$

$$\hat{l} \equiv (0,1)\,,\ \hat{\imath}\hat{l} = (0,\hat{\imath}) \equiv \underline{\hat{\imath}}\,,\ \hat{\jmath}\hat{l} = (0,\hat{\jmath}) \equiv \underline{\hat{\jmath}}\,,\ \hat{k}\hat{l} = (0,\hat{k}) \equiv \underline{\hat{k}}\,, \tag{5.11}$$

with the commutation rules

$$\hat{\imath}\hat{l} = (0,\hat{\imath}) = -\hat{l}\hat{\imath}\,, \quad \hat{\jmath}\hat{l} = (0,\hat{\jmath}) = -\hat{l}\hat{\jmath}\,, \quad \hat{k}\hat{l} = (0,\hat{k}) = -\hat{l}\hat{k}\,. \tag{5.12}$$

We then obtain the biquaternion notation for octonions:

$$\mathbb{O} \ni x = p + q\hat{l} = p_0 + p_1\hat{\imath} + p_2\hat{\jmath} + p_3\hat{k} + q_0\hat{l} + q_1\underline{\hat{\imath}} + q_2\underline{\hat{\jmath}} + q_3\underline{\hat{k}} = p + \ell\bar{q}\,. \tag{5.13}$$

In octonionic calculations we should be aware of the non-associativity of the product, *e.g.*, $(\hat{\jmath}\hat{k})\hat{l} = \hat{\imath}\hat{l}$ whilst $\hat{\jmath}(\hat{k}\hat{l}) = -\hat{\imath}\hat{l}$.

The octonionic conjugate of the octonion x is defined as

$$x = (p,q) \mapsto \bar{x} := (\bar{p},-q)\,.$$

This operation allows to define the Euclidean norm of $x \in \mathbb{O}$ and its inverse if $x \neq 0$.

$$x\bar{x} = \bar{x}x = |p|^2 + |q|^2 \equiv |x|^2\,, \quad x^{-1} = \frac{\bar{x}}{|x|^2}\,.$$

Conjugation is involutive: $\overline{xx'} = \overline{x'}\bar{x}$, and one checks that $|xx'| = |x||x'|$.

Like the bicomplex notation (5.4)-(5.5) for quaternions, we also have the quadricomplex notation for octonions:

$$x = (p,q) = (t,u) + \hat{\jmath}(v,w) \quad \text{where} \quad \begin{cases} t = p_0 + p_3\hat{k} & , \quad u = q_0 + q_3\hat{k} \\ v = p_2 + p_1\hat{k} & , \quad w = q_2 - q_1\hat{k} \end{cases}. \tag{5.14}$$

Finally, the Hurwitz theorem states that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the only normed division algebras over the real numbers (up to isomorphism), normed in the sense that $|xx'| = |x|\,|x'|$. These four algebras also form the only alternative, finite-dimensional division algebras over the real numbers .

A general rotation in $\mathbb{R}^8$, *i.e.*, an element among the 28 elements of SO(8), can be written through successive right multiplications:

$$\mathbb{O} \ni x \mapsto ((((((x\,\eta_1)\,\eta_2)\,\eta_3)\,\eta_4)\,\eta_5)\,\eta_6)\,\eta_7\,, \quad \eta_i \in \mathbb{O}\,,\ |\eta_i| = 1\,.$$

For a recent review on the role of octonions in Nuclear Physics, see [35].

6. Higher Spheres and Hopf Maps and Fibrations

What about $\mathbb{S}^n, n \geq 4$? There is nothing really special from a strictly group point of view when we deal with higher dimensional spheres. Naturally many interesting topological features exist, like the Hopf fibrations [36,37]:

6.1. Hopf Fibrations

They are the following relations involving spheres the $\mathbb{S}^n$, $n = 1,2,3,4,5,7,8,15$:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \xrightarrow{p} \mathbb{S}^2\,, \quad \mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \xrightarrow{p} \mathbb{S}^4\,, \quad \mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \xrightarrow{p} \mathbb{S}^8\,,$$

where "$\hookrightarrow$" stands for embedding, and p is the Hopf map. The first one is displayed in Figure 15.

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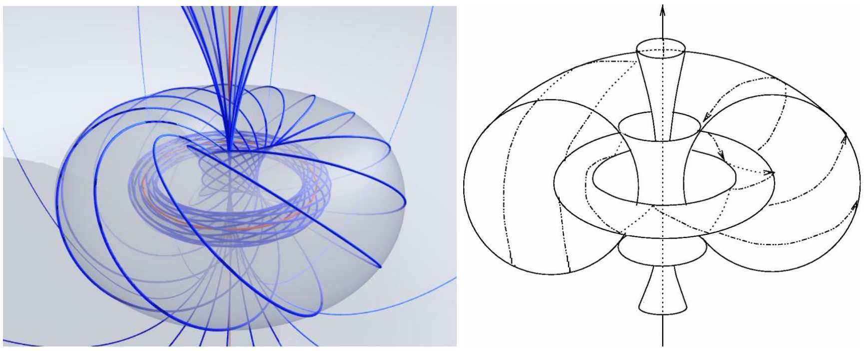

Figure 15. Hopf fibration (from http://www-fourier.ujf-grenoble.fr/b̃kloeckn/images.html). This fiber bundle structure, denoted $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \stackrel{p}{\to} \mathbb{S}^2$, consists of the fiber space $\mathbb{S}^1$, embedded in the total space $\mathbb{S}^3$, and the projection p (Hopf's map) of $\mathbb{S}^3$ onto the base space $\mathbb{S}^2$. Stereographic projection of the Hopf fibration induces a remarkable structure on $\mathbb{R}^3$, in which space is filled with nested tori made of linking Villarceau circles: each fiber projects to a circle in space (one of which is a "circle through infinity", i.e. a line). Each torus is the stereographic projection of the inverse image of $\mathbb{S}^2$ latitude circle. (Topologically, a torus is the product of two circles.) One of these tori is illustrated by the image of linking keyrings on the right.

More generally, the Hopf construction gives circle bundles: $\mathbb{S}^{2n+1} \stackrel{p}{\to} \mathbb{C}\mathbf{P}^n$ over complex projective space. This is actually the restriction of the topological line bundle over $\mathbb{C}\mathbf{P}^n$ to the unit sphere in $\mathbb{C}^{n+1}$.

One may also regard $\mathbb{S}^1$ as lying in $\mathbb{R}^2$ and factor out by unit real multiplication to obtain $\mathbb{R}\mathbf{P}^1 = \mathbb{S}^1$ and a fiber bundle $\mathbb{S}^1 \stackrel{p}{\to} \mathbb{S}^1$ with fiber $\mathbb{S}^0$.

Note that one might regard $\mathbb{S}^{4n-1}$ as lying in $\mathbb{H}^n$ (quaternionic n-space) and factor out by unit quaternion ($\simeq \mathbb{S}^3$) multiplication to get $\mathbb{H}\mathbf{P}^n$. In particular, since $\mathbb{S}^4 = \mathbb{H}\mathbf{P}^1$, there is a bundle $\mathbb{S}^7 \stackrel{p}{\to} \mathbb{S}^4$ with fiber $\mathbb{S}^3$.

A similar construction with the octonions yields a bundle $\mathbb{S}^{15} \stackrel{p}{\to} \mathbb{S}^8$ with fiber $\mathbb{S}^7$.

These bundles are sometimes also called Hopf bundles (real, quaternionic, and octonionic Hopf bundles). As a consequence of Adams' theorem, these are the only fiber bundles with spheres as total space, base space, and fiber.

Hopf fibration surfaces show up in the description of quantum entanglement [38,39]. In particle physics they underly the mathematics of the Dirac monopole. It also appears in general relativity, for instance in the Robinson congruence (see for instance [40]).

6.2. *Group Interpretation of the First Hopf Map:* $\mathbb{S}^3 \stackrel{p}{\to} \mathbb{S}^2$

Let us consider the element $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$ of $\mathbb{S}^3 \cong \mathrm{SU}(2)$ which brings the north pole of $\mathbb{S}^2$ to the direction $(x, y, z) \in \mathbb{S}^2$:

$$\xi(0,0,0,1)\bar{\xi} = (0, x, y, z)\,, \quad x^2 + y^2 + z^2 = 1\,.$$

This gives the quadratic relations which exemplify the projection p (Hopf's map) of $\mathbb{S}^3$ onto the base space $\mathbb{S}^2$:

$$\mathbb{S}^3 \ni \left\{ \begin{array}{ll} x = & 2(\xi_0\xi_2 + \xi_1\xi_3) \\ y = & 2(\xi_2\xi_3 - \xi_0\xi_1) \\ z = & \xi_0^2 + \xi_3^2 - \xi_1^2 - \xi_2^2 \end{array} \right. .$$

One easily checks that the Hopf inverse of a point $\in \mathbb{S}^2$ is a circle $\mathbb{S}^1 \subset \mathbb{S}^3$ and that the Hopf inverse of a circle $\mathbb{S}^1 \subset \mathbb{S}^2$ is a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

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6.3. Group Interpretation of the Second Hopf Map: $\mathbb{S}^7 \overset{p}{\to} \mathbb{S}^4$

We here consider the special group isomorphism proper to $B_2 \cong C_2$ (see Figure A1):

$$\mathrm{Spin}(5) \equiv \mathrm{SO}(5) \times \mathbb{Z}_2 \simeq G = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a,b,c,d \in \mathbb{H}, \; g\,{}^t\overline{g} = \mathbb{1}_2 \right\}.$$

It is understood through the action of the group G on specific 2×2-quaternionic matrices

$$g \begin{pmatrix} y_4 & p \\ \bar{p} & -y_4 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} y_4' & p' \\ \bar{p}' & -y_4' \end{pmatrix}, \quad p \in \mathbb{H}, \, y_4 \in \mathbb{R}, \quad y_4^2 + \|p\|^2 = 1.$$

Just consider the group element $g = \begin{pmatrix} a & b \\ -\overline{b_\alpha} & \bar{a} \end{pmatrix} \in G$, $b_\alpha = \bar{a}ba/\|a\|^2$, which brings the "north pole" of $\mathbb{S}^4$, $(y_4 = 1\,,\, p = 0)$, to the point $(y_4, p) \in \mathbb{S}^4$. We have $\|a\|^2 + \|b\|^2 = 1$, *i.e.*, $(a,b) \in \mathbb{S}^7$. This gives the quadratic relations:

$$p = -2ba\,, \quad y_4 = \|a\|^2 - \|b\|^2\,.$$

The Hopf inverse of a point $\in \mathbb{S}^4$ is the sphere $\mathbb{S}^3 \subset \mathbb{S}^7$ and the Hopf inverse of a sphere $\mathbb{S}^3 \subset \mathbb{S}^4$ is $\mathbb{S}^3 \times \mathbb{S}^3 \simeq \mathrm{SO}(4)$.

6.4. Group Interpretation of the Third Hopf Map: $\mathbb{S}^{15} \overset{p}{\to} \mathbb{S}^8$

The octonionic Hopf fibration finds its elegant description through the lens of the automorphism group G2 of the normed algebra of octonions. Actually, one can grasp its essence by considering the group Spin(9) (where Spin(n) serves as the two-fold covering of SO(n)), as explained in [41].

7. Kepler Problem and Spheres

7.1. 1-Dimensional Kepler Problem

Consider the energy of a particle of mass m trapped in a well determined on the positive line $q \geq 0$ by the attractive Kepler-Coulomb potential:

$$V(q) = -\frac{k}{q}, \quad k > 0\,,$$

and $V(q) = \infty$ for $q < 0$. Its energy is given by

$$E = \frac{p^2}{2m} - \frac{k}{q}\,.$$

Regularizing the Kepler problem consists in considering the alternative equation:

$$\Theta \overset{\mathrm{def}}{=} q(p_0^2 + p^2) - 2mk = 0\,, \quad q > 0\,.$$

where $p_0 = \sqrt{-2mE}$. Let us see how the Balmer laws [42] come to light from the symmetry A_1. Let us onsider the three observables defined on the upper half-plane viewed as the phase space $\{(q,p) \mid q > 0\,,\, p \in \mathbb{R}\}$:

$$d = qp\,, \quad \gamma_0 = \frac{1}{2}q(p^2 + p_0^2)\,, \quad \gamma_1 = \frac{1}{2}q(p^2 - p_0^2)\,.$$

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Their Poisson brackets $\left(\{f,g\} \stackrel{\text{def}}{=} \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)$ obey the commutation relations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$:

$$\{d,\gamma_0\} = \gamma_1\,, \quad \{d,\gamma_1\} = \gamma_0\,, \quad \{\gamma_0,\gamma_1\} = p_0^2\, d\,.$$

After quantization of these classical observables, say $A \mapsto \hat{A}$, Poisson brackets become commutators, quantum counterparts become (essentially) self-adjoint operators acting in some Hilbert space of quantum states, and a specific unitary irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ is involved. Precisely, that one for which the generator $\widehat{\gamma_0}$ has non zero positive integers multiples of p_0 as spectral values. This is due to the fact the inverse operator $\widehat{\gamma_0}^{-1}$ is compact. We remind that an operator A on a infinite-dimensional Hilbert space $\mathcal{H}$ is said to be compact if it can be written in the form $A = \sum_{n=1}^{\infty} \lambda_n \langle \mathbf{f}_n, \cdot \rangle \mathbf{e}_n$ where $\{\mathbf{e}_n\}$ and $\{\mathbf{f}_n\}$ are orthonormal sets (not necessarily complete) and the λ_n's form a sequence of positive numbers with limit zero (they can accumulate only at zero). Since the classical equation $\Theta = 2\gamma_0 - 2mk = 0$ becomes $\widehat{\Theta} = 2(\widehat{\gamma_0} - 2mk)\psi = 0$, the spectral condition leads directly to the Balmer-like quantization of the energy:

$$n\, p_0 - mk = 0 \;\Leftrightarrow\; E = -\frac{mk^2}{2n^2}\,.$$

7.2. Phase Space of the Regularized 1d-Kepler Problem: The Complexified Circle with Zero Radius

Let us show that the phase space of the regularized $1d$-Kepler problem is the complexified circle with zero radius.

Following the Fock method [43,44] for displaying the symmetry of the H-atom, let us introduce the circle variable

$$\xi = \left(\frac{p_0^2 - p^2}{p_0^2 + p^2}, \frac{2p_0 p}{p_0^2 + p^2}\right) \in \mathbb{S}^1\,.$$

(Inverse stereographic projection transforming the compactified momentum real line into the unit circle). By imposing that this change of variable be part of a canonical transformation of the phase space, the conjugate variable π_ξ to ξ reads:

$$\pi_\xi = \left(-qp\,,\, q\frac{p_0^2 - p^2}{2p_0}\right) = \left(-d\,,\, \frac{\gamma_1}{p_0}\right).$$

One easily checks the two constraints:

$$\left|\pi_\xi\right| = \frac{mk}{p_0} \text{ on the energy shell and } \xi \cdot \pi_\xi = 0\,.$$

Therefore, the pair $\left(\xi, \frac{p_0}{mk}\pi_\xi\right)$ parametrizes the complex circle in $\mathbb{C}^2$ with null radius:

$$\mathbb{S}^1_{\mathbb{C}}(0) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 0\}\,.$$

This is a geometric model for the phase space of the one-dimensional regularized Kepler problem, which can be identified to $T_0^*\mathbb{S}^1$ (cotangent bundle on $\mathbb{S}^1$ with the zero-section deleted, named Moser manifold or Kepler manifold).

7.3. Phase Space of the Three-Dimensional Regularized Kepler Problem

The above material is easily generalisable to three dimensions. Indeed, the phase space of the three-dimensional regularized Kepler problem is the complexification of $\mathbb{S}^3$ with radius 0.

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Following Fock again in dealing with the more realistic 3-dimensional Coulomb-Kepler problem, let us introduce the $\mathbb{S}^3$ variable

$$\xi = \left(\frac{p_0^2 - \|\mathbf{p}\|^2}{p_0^2 + \|\mathbf{p}\|^2}, \frac{2p_0\,\mathbf{p}}{p_0^2 + \|\mathbf{p}\|^2}\right) \in \mathbb{S}^3 .$$

(Inverse stereographic projection transforming the $3d$-compactified momentum space into 3-sphere).

By imposing that this change of variable be part of a canonical transformation of the phase space, the conjugate variable π_ξ to ξ together with ξ parametrizes the complexification of $\mathbb{S}^3$ in $\mathbb{C}^4$ with null radius. This is the phase space of the three dimensional regularized Kepler problem, which can be identified with $T_0^*\mathbb{S}^3$ (cotangent bundle on $\mathbb{S}^3$ with the zero-section deleted, Moser manifold or Kepler manifold, as named by Souriau [45]).

The three-dimensional aspect of this structure explains the accidental degeneracy of the discrete spectrum of the hydrogen atom with its SO(4) symmetry.

8. Conclusion

This contribution is an invitation to embark on an exploration of the realm governed by numbers, symmetries, and intricate symmetry groups, and, as a appealing illustration, to delve into the profound significance encapsulated within the fundamental aspects of the circle and higher-dimensional spheres. Its aim is to share our appreciation for their value in shaping our mathematical comprehension of the natural world and establishing the foundational principles that underpin the laws of physics.

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Abbreviations

The following abbreviations are used in this manuscript:

UIR   Unitary irreducible representation

Appendix A. Group Actions

A *transformation* of a set S is a one-to-one mapping of S onto itself. A group G is realized as a transformation group of a set S if to each $g \in G$, there is associated a transformation $s \mapsto g \cdot s$ of S where for any two elements g_1 and g_2 of G and $s \in S$, we have $(g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$.

The set S is then called a G-space. A transformation group is *transitive* on S if, for each s_1 and s_2 in S, there is a $g \in G$ such that $s_2 = g \cdot s_1$. In that case, the set S is called a *homogeneous* G-space.

A (linear) *representation* of a group G is a continuous function $g \mapsto T(g)$ which takes values in the group of nonsingular continuous linear transformations of a vector space $\mathcal{V}$, and which satisfies the functional equation $T(g_1 g_2) = T(g_1)T(g_2)$ and $T(e) = I_d$, the identity operator in $\mathcal{V}$, where e is the identity element of G. It follows that $T(g^{-1}) = (T(g))^{-1}$. That is, $T(g)$ is a homomorphism of G into the group of nonsingular continuous linear transformations of $\mathcal{V}$.

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A representation is *unitary* if the linear operators $T(g)$ are unitary with respect to the inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{V}$. That is, $\langle T(g)\,v_1|T(g)\,v2\rangle = \langle v_1|v_2\rangle$ for all vectors v_1, v_2 in $\mathcal{V}$. A representation is *irreducible* if there is no non-trivial subspace $\mathcal{V}_0 \subset \mathcal{V}$ such that for all vectors $v_o \in \mathcal{V}_0$, $T(g)\,v_o$ is in $\mathcal{V}_0$ for all $g \in G$. That is, there is no non trivial subspace $\mathcal{V}_0$ of $\mathcal{V}$ which is invariant under the operators $T(g)$.

Appendix B. Lie Algebra Material

Appendix B.1. General

Let $\mathfrak{g}$ be a complex Lie algebra, that is, a complex vector space with an antisymmetric bracket $[\cdot,\cdot]$ that satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\,,\ \forall\, X,Y,Z \in \mathfrak{g}\,. \tag{A1}$$

For $X, Y \in \mathfrak{g}$, the relation $(\mathrm{ad}_X)(Y) = [X,Y]$ gives a linear map ad: $\mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ (endomorphisms of $\mathfrak{g}$), called the *adjoint representation* of $\mathfrak{g}$.

Next, if $\dim\mathfrak{g} < \infty$, it makes sense to define

$$B(X,Y) = \mathrm{Tr}[(\mathrm{ad}_X)(\mathrm{ad}_Y)],\ X,Y \in \mathfrak{g}\,. \tag{A2}$$

B is a symmetric bilinear form on $\mathfrak{g}$, called the *Killing form* of $\mathfrak{g}$.

Alternatively, one may choose a basis $\{X_j\,,\, j = 1,\dots,n\}$ in $\mathfrak{g}$, in terms of which the commutation relations read

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k,\ i,j = 1,\dots,n\,, \tag{A3}$$

where c_{ij}^k are called the structure constants and $n = \dim\mathfrak{g}$.

Then it is easy to see that $g_{ij} = \sum_{k,m=1}^{n} c_{ik}^m\, c_{jm}^k = B(X_i, X_j)$ defines a metric on $\mathfrak{g}$, called the Cartan-Killing metric.

Appendix B.2. Roots and Cartan Classification

The Lie algebra $\mathfrak{g}$ is said to be *simple*, resp. *semisimple*, if it contains no nontrivial ideal, i.e., an additive subgroup $\mathcal{I}$ of $\mathfrak{g}$ that "absorbs" multiplication from the left by elements of $\mathfrak{g}$, i;e., $[\mathcal{I},\mathfrak{g}] \subset \mathcal{I}$, resp. Abelian ideal. A semisimple Lie algebra may be decomposed into a direct sum of simple ones. Furthermore, $\mathfrak{g}$ is semisimple iff the Killing form is nondegenerate (Cartan's criterion).

Let $\mathfrak{g}$ be semisimple. Choose in $\mathfrak{g}$ a Cartan subalgebra $\mathfrak{h}$, i.e., a maximal nilpotent subalgebra (it is in fact maximal Abelian and unique up to conjugation). The dimension ℓ of $\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$. A *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a linear form on $\mathfrak{h}$, $\alpha \in \mathfrak{h}^*$, for which there exists $X \neq 0$ in $\mathfrak{g}$ such that $(\mathrm{ad}_H)X = \alpha(H)X,\ \forall\, H \in \mathfrak{h}$.

Then one can find (Cartan, Chevalley) a basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$, with the following properties. $\{H_j, j = 1,\dots,\ell\}$ is a basis of $\mathfrak{h}$ and each generator E_α is indexed by a nonzero root α, in such a way that the commutation relations (A3) may be written in the following form:

$$[H_i, H_j] = 0\,,\quad i,j = 1,\dots,\ell\,, \tag{A4}$$

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha\,,\quad i = 1,\dots,\ell\,, \tag{A5}$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha \equiv \sum_{i=1}^{\ell} \alpha^i H_i \in \mathfrak{h}\,, \tag{A6}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta}\,, \tag{A7}$$

where $N_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root.

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Let Δ denote the set of roots of $\mathfrak{g}$. Note that the nonzero roots come in pairs: $\alpha \in \Delta \Leftrightarrow -\alpha \in \Delta$, and no other nonzero multiple of a root is a root.

Accordingly, the set of nonzero roots may be split into a subset Δ_+ of positive roots and the corresponding subset of negative roots $\Delta_- = \{-\alpha, \alpha \in \Delta_+\}$.

The set Δ_+ is contained in a simplex (convex pyramid) in $\mathfrak{h}^*$, the edges of which are the so-called simple positive roots, i.e., positive roots which cannot be decomposed as the sum of two other positive roots. Of course, the same holds for Δ_-.

The consideration of root systems is the basis of the Cartan classification of simple Lie algebras into four infinite series $A_\ell, B_\ell, C_\ell, D_\ell$ and five exceptional algebras G_2, F_4, E_6, E_7, E_8.

Appendix B.3. Dynkin Diagrams: General

The roots of a complex Lie algebra form a lattice of rank n in the dual of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where n is the Lie algebra rank of $\mathfrak{g}$. Hence, the root lattice can be considered as a lattice in $\mathbb{R}^n$. A vertex, or node, in the Dynkin diagram is drawn for each Lie algebra simple root, which corresponds to a generator of the root lattice. Between two nodes α and β, an edge is drawn if the simple roots are not perpendicular. One undirected single edge is drawn if the angle between them is $2\pi/3$, a directed double edge if the angle is $3\pi/4$, and a directed triple edge is drawn if the angle is $5\pi/6$. There are no other possible angles between Lie algebra simple roots. The term "directed edge" means that double and triple edges point toward the shorter (black circle) vector. Dynkin diagrams are shown in Figures A1, A2, A3, A4.

Appendix B.4. From Top to Bottom, Dynkin Diagrams for A_n, B_n, C_n, D_n

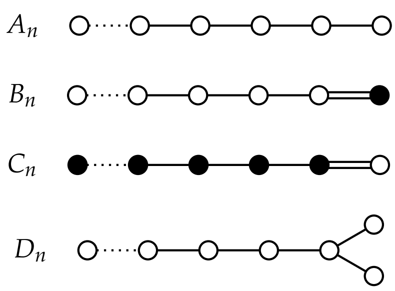

Figure A1. Dynkin diagrams A_n, B_n, C_n, D_n

Appendix B.5. Dynkin Diagrams G_2, F_4, E_6, E_7, E_8

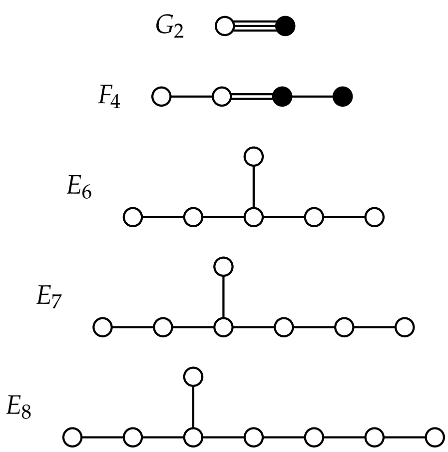

Figure A2. From top to bottom: Dynkin diagrams for G_2, F_4, E_6, E_7, E_8

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Appendix B.6. Examples: Roots $A_1 \times A_1$, A_2, B_2

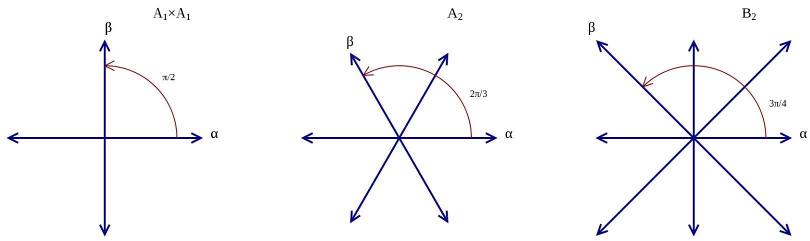

Figure A3. rank-2 root systems $A_1 \times A_1$, A_2, B_2 (from http://en.wikipedia.org/wiki/Root_system)

Appendix B.7. Examples: Roots C_2, D_2, G_2

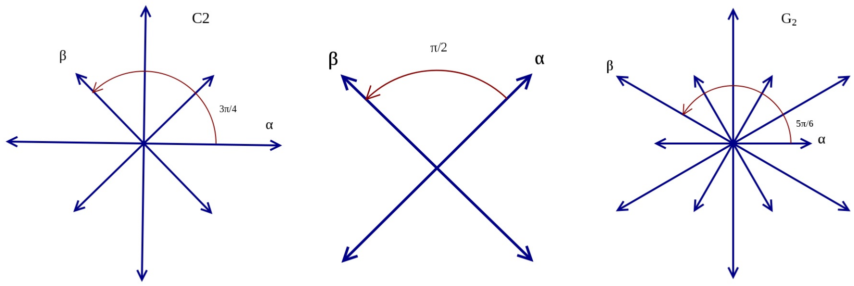

Figure A4. rank-2 root systems C_2, D_2, G_2 (from http://en.wikipedia.org/wiki/Root_system)

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