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Ismail's Ratio Conquers New Horizons the Non-stationary M/G/1 Queue's State Variable Closed Form Expression

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Ismail's Ratio Conquers New Horizons the Non-Stationary $M/G/1$ Queue's State Variable CLOSED form Expression

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Abstract: This paper investigates the search for an exact analytic solution to the state variable of the non-stationary $M/G/1$ queue. Currently, the only known solution to this problem is through simulation. However, this study proposes a constant ratio β (Ismail's ratio) that relates the time-dependent mean arrival and mean service rates, offering an exact analytical solution. The stability dynamics of the time-varying $M/G/1$ queueing system are then examined numerically in relation to time, β, and the queueing parameters.

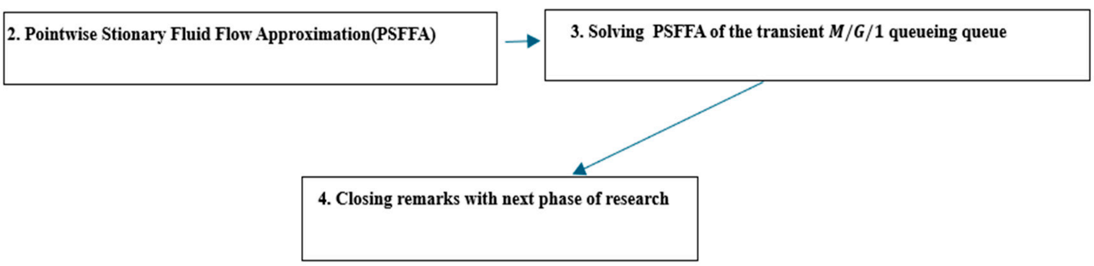
Keywords: Time Varying $M/G/1$ queueing system; number of customers; mean arrival rate; time; The Pointwise Stionary Fluid Flow Approximation(PSFFA); Time Varying queueing systems(TVQSs)

1. Introduction

The field of transient/non-stationary analysis has limited literature, which can be categorized into simulation, transient analysis, analysis, and applications techniques. These categories encompass various approaches to studying systems that change over time, including simulations, analysing transient behaviour, and exploring non-stationary phenomena. In certain cases, mathematical transformations are employed to analysing non-stationary queueing systems. However, evaluating these expressions can be computationally complex. As a result, there has been a focus on numerically determining the transient behaviour of such systems instead of deriving closed form expressions.

The current exposition contributes to solving for first time ever, the longstanding unsolved problem of obtaining the state variable of the time varying $M/G/1$ queueing system.

The following flowchart shows how this paper is organized.



2. PSFFA

Therefore, the time varying $M/G/1$ queueing system's PSFFA model[2–5] reads:

$$x^{\cdot} = -\mu\left(\frac{(x+1)-\sqrt{(x^2+2\xi x+1)}}{1-\xi}\right)+\lambda,\ \xi = C^2 = \text{squared coefficient of variation} \quad (1)$$

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TVQSs' life example [6] is depicted by Figure1.

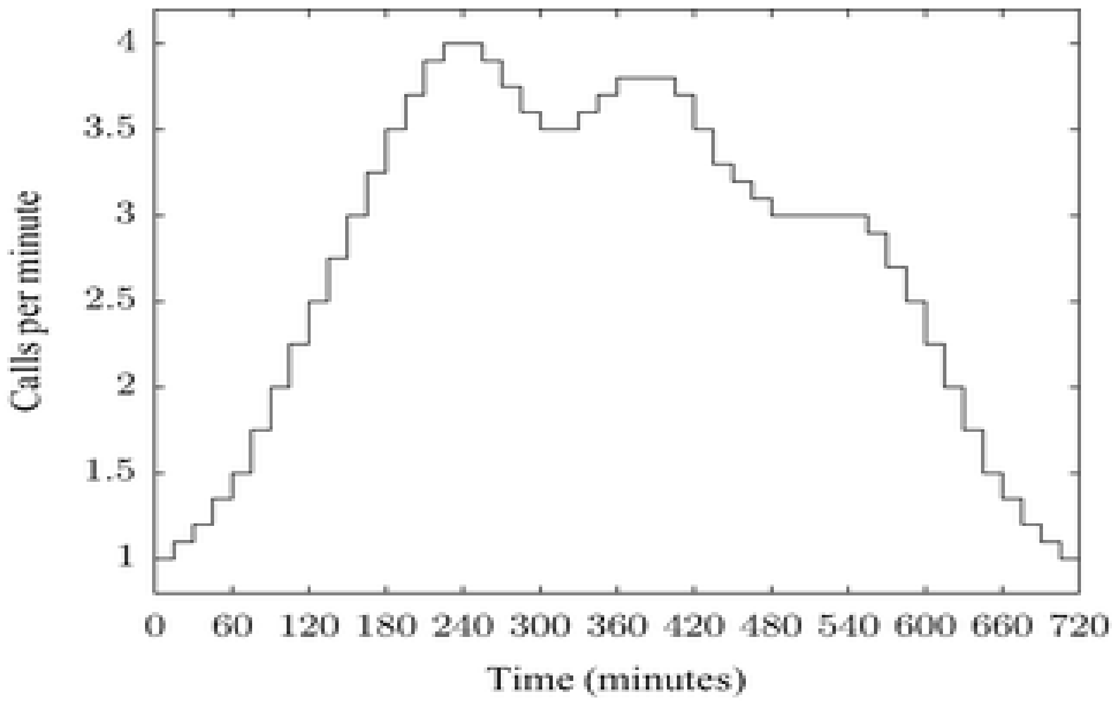

Figure 1.

3. Solving the non-stationary $M/G/1$ queueing system's PSFFA (c,f., (1))

Theorem 3.1. Ismail's ration, β solves (2.1), with a closed form expression to read as:

$$\frac{(|\left(1-\frac{a(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|)^A\left(|\left(1-\frac{b(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|\right)^B(|\left(1-\frac{(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|)^C}{\left(1+\frac{(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)^D}$$

$$=\eta e^{-\frac{1}{(1-\xi)}\int\mu dt} \qquad (2)$$

Proof

We have

$$x^{\cdot}=-\mu\left(\frac{(x+1)-\sqrt{(x^2+2\xi x+1)}}{1-\xi}\right)+\lambda,\ \xi=C^2 \qquad \text{(c.f., (1))}$$

This translates to

$$\frac{[illegible]}{((x+1)-\sqrt{(x^2+2\xi x+1)})-\beta(1-\xi))}=-\frac{\mu dt}{(1-\xi)},\ \beta=\frac{\lambda(t)}{\mu(t)}>0 \qquad (2)$$

Let $x=\sqrt{(1-\xi^2)}\ cschy-\xi$, then $x^{\cdot}=-\sqrt{(1-\xi^2)}y^{\cdot}\ cothy\ cschy$. Thus, we have

$$\frac{-cothy\ cschydy}{((cschy-cothy)-\frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}})}=-\frac{\mu dt}{(1-\xi)}$$

This means

$$\frac{coshydy}{-(1+sinhy-coshy)+\frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}}sinhy)sinhy}=-\frac{\mu dt}{(1-\xi)}$$

Equivalently,

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$$\frac{\frac{\frac{2}{(1-\beta)(1-\xi)}}{\sqrt{1-\xi^2}}(e^{3y}+e^{y})dy}{[e^{2y}-\frac{e^{y}}{\frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}}}+[\frac{2}{\left(\frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}}\right)}-1]]} = -\frac{\mu dt}{(1-\xi)} \quad (3)$$

Define $\frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}} = \zeta$

$$\left[e^{2y}+\frac{2e^{y}}{\zeta}-1\right] = 0 \Longrightarrow e^{y} = \frac{\left(\frac{1}{\zeta}\pm\sqrt{(\frac{1}{\zeta^2}-4\left(\frac{2}{\zeta}-1\right)}\right)}{2} = \frac{\left(\frac{1}{\zeta}\pm\sqrt{(\frac{1}{\zeta^2}-\frac{8}{\zeta}+4}\right)}{2} = a, b$$

where

$$a = \frac{\left(\frac{1}{\zeta}+\sqrt{(\frac{1}{\zeta^2}-\frac{8}{\zeta}+4}\right)}{2}$$

$$b = \frac{\left(\frac{1}{\zeta}-\sqrt{(\frac{1}{\zeta^2}-\frac{8}{\zeta}+4}\right)}{2}$$

$$\frac{\frac{2}{\zeta}(e^{3y}+e^{y})dy}{[e^{2y}-\frac{e^{y}}{\zeta}+[\zeta-1]][e^{2y}-1]} = -\frac{\mu dt}{(1-\xi)} \quad (4)$$

$$\frac{\frac{2}{\zeta}(e^{3y}+e^{y})}{[e^{2y}-\frac{e^{y}}{\zeta}+[\zeta-1]][e^{2y}-1]} = \left[\frac{A}{(e^{y}-a)}+\frac{B}{(e^{y}-b)}+\frac{C}{(e^{y}-1)}+\frac{D}{(e^{y}+1)}\right]$$

Hence, it is implied that:

$$-bA-aB+\big(1-(a+b)\big)C-(a+b+1)D = 0 \quad (5)$$

$$\therefore B = \frac{(1-a)(C-D)}{\left(\frac{2}{\zeta}+a\right)} \quad (6)$$

which implies

$$A = \frac{2}{\zeta}-\frac{\left(1+\frac{2}{\zeta}\right)C}{\left(\frac{2}{\zeta}+a\right)}-\frac{\left(1+\frac{2}{\zeta}+2a\right)D}{\left(\frac{2}{\zeta}+a\right)} \quad (7)$$

which yields

$$C = \frac{\frac{4}{\zeta}\left(\frac{2}{\zeta}+a\right)-\left[\left(1+\frac{2}{\zeta}+2a\right)-\big(ab-(a+b)\big)\left(\frac{2}{\zeta}+a\right)\right]D}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)} \quad (8)$$

$$bA+aB+abC-abD = 0$$

$$b\left[\frac{2}{\zeta}-\frac{\left(1+\frac{2}{\zeta}\right)C}{\left(\frac{2}{\zeta}+a\right)}-\frac{\left(1+\frac{2}{\zeta}+2a\right)D}{\left(\frac{2}{\zeta}+a\right)}\right]+a\left[\frac{(1-a)(C-D)}{\left(\frac{2}{\zeta}+a\right)}\right]+abC-abD = 0$$

Therefore, we have

$$D = \frac{\frac{2b}{\zeta}+a\left(ab-\frac{b\left(1+\frac{2}{\zeta}\right)}{\left(\frac{2}{\zeta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right]\right)[\frac{\frac{4}{\zeta}\left(\frac{2}{\zeta}+a\right)}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)}]}{(ab+b\left[\frac{\left(1+\frac{2}{\zeta}+2a\right)}{\left(\frac{2}{\zeta}+a\right)}\right]+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right])+a\left(ab-\frac{b\left(1+\frac{2}{\zeta}\right)}{\left(\frac{2}{\zeta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right]\right)[\frac{\left[\left(1+\frac{2}{\zeta}+2a\right)-\big(ab-(a+b)\big)\left(\frac{2}{\zeta}+a\right)\right]}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)}}$$

(9)

$$C=\frac{\frac{4}{\zeta}\left(\frac{2}{\zeta}+a\right)-\left[\left(1+\frac{2}{\zeta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\zeta}+a\right)\right]D}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)} \qquad (10)$$

$$A=\frac{2}{\zeta}-\frac{\left(1+\frac{2}{\zeta}\right)C}{\left(\frac{2}{\zeta}+a\right)}-\frac{\left(1+\frac{2}{\zeta}+2a\right)D}{\left(\frac{2}{\zeta}+a\right)},\ B=\frac{(1-a)(C-D)}{\left(\frac{2}{\zeta}+a\right)} \qquad (11)$$

This finally solves the complicated mathematical computations to

$$\left[\frac{A}{(e^y-a)}+\frac{B}{(e^y-b)}+\frac{C}{(e^y-1)}+\frac{D}{(e^y+1)}\right]dy=-\frac{\mu dt}{(1-\xi)}$$

Integrating both sides

$$[\boldsymbol{A}ln\,|(1-ae^{-y})|+\boldsymbol{B}ln\,|(1-be^{-y})|+Cln|(1-e^{-y})|-Dln(1+e^{-y})]=-\frac{1}{(1-\xi)}\int\mu dt+ln\eta$$

for some non − negative real constant parameter η

Or

$$\frac{|(1-ae^{-y})|^{A}|(1-be^{-y})|^{\boldsymbol{B}}|(1-e^{-y})|^{C}}{(1+e^{-y})^{D}}=\eta e^{-\int\mu dt}$$

This transforms to the final required closed form solution:

$$\frac{|\left(1-ae^{-csch^{-1}\left(\frac{x+\xi}{\sqrt{(1-\xi^2)}}\right)}\right)|^{A}|\left(1-be^{-csch^{-1}\left(\frac{x+\xi}{\sqrt{(1-\xi^2)}}\right)}\right)|^{\boldsymbol{B}}|\left(1-e^{-csch^{-1}\left(\frac{x+\xi}{\sqrt{(1-\xi^2)}}\right)}\right)|^{C}}{\left(1+e^{-csch^{-1}\left(\frac{x+\xi}{\sqrt{(1-\xi^2)}}\right)}\right)^{D}}=\eta e^{-\frac{1}{(1-\xi)}\int\mu dt}$$

By mathematical analysis, it is well known that

$$csch^{-1}\left(\frac{x+\xi}{\sqrt{(1-\xi^2)}}\right)=ln\left(\frac{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}{x+\xi}\right)\ ,\ x\neq-\xi(\text{we are allowed to assign } \xi\in[0,\infty))$$

Thus, one gets

$$\frac{(|\left(1-\frac{a(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|)^{A}\left(|\left(1-\frac{b(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|\right)^{B}(|\left(1-\frac{(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)|)^{C}}{\left(1+\frac{(x+\xi)}{\sqrt{(1-\xi^2)}+\sqrt{((1-\xi^2)+[x+\xi]^2)}}\right)^{D}}$$

$$=\eta e^{-\frac{1}{(1-\xi)}\int\mu dt}$$

where

$$a=\frac{\left(\frac{1}{\zeta}+\sqrt{\left(\frac{1}{\zeta^2}-\frac{8}{\zeta}+4\right)}\right)}{2},b=\frac{\left(\frac{1}{\zeta}-\sqrt{\left(\frac{1}{\zeta^2}-\frac{8}{\zeta}+4\right)}\right)}{2},\ \frac{(1-\beta)(1-\xi)}{\sqrt{1-\xi^2}}=\zeta$$

$$A=\frac{2}{\zeta}-\frac{\left(1+\frac{2}{\zeta}\right)C}{\left(\frac{2}{\zeta}+a\right)}-\frac{\left(1+\frac{2}{\zeta}+2a\right)D}{\left(\frac{2}{\zeta}+a\right)},\ B=\frac{(1-a)(C-D)}{\left(\frac{2}{\zeta}+a\right)}$$

$$C=\frac{\frac{4}{\zeta}\left(\frac{2}{\zeta}+a\right)-\left[\left(1+\frac{2}{\zeta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\zeta}+a\right)\right]D}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)}$$

$$D=\frac{\frac{2b}{\zeta}+a\left(ab-\frac{b\left(1+\frac{2}{\zeta}\right)}{\left(\frac{2}{\zeta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right]\right)\left[\frac{\frac{4}{\zeta}\left(\frac{2}{\zeta}+a\right)}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)}\right]}{(ab+b\left[\frac{\left(1+\frac{2}{\zeta}+2a\right)}{\left(\frac{2}{\zeta}+a\right)}\right]+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right])+a\left(ab-\frac{b\left(1+\frac{2}{\zeta}\right)}{\left(\frac{2}{\zeta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\zeta}+a\right)}\right]\right)\left[\frac{\left[\left(1+\frac{2}{\zeta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\zeta}+a\right)\right]}{\left(\frac{2}{\zeta}\right)+a+[ab-(a+b)]\left(\frac{2}{\zeta}+a\right)}\right]}$$

As required.

If $\xi=0$, the derived solution reduces to the closed form solution of Time Varying $M/G/1$ queueing system.

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Numerical experiment

$$\frac{\left(\left|\left(1-\frac{a(x+\xi)}{\sqrt{1-\xi^2}+\sqrt{(1-\xi^2)+[x+\xi]^2}}\right)\right|\right)^A\left(\left|\left(1-\frac{b(x+\xi)}{\sqrt{1-\xi^2}+\sqrt{(1-\xi^2)+[x+\xi]^2}}\right)\right|\right)^B\left(\left|\left(1-\frac{(x+\xi)}{\sqrt{1-\xi^2}+\sqrt{(1-\xi^2)+[x+\xi]^2}}\right)\right|\right)^C}{\left(1+\frac{(x+\xi)}{\sqrt{1-\xi^2}+\sqrt{(1-\xi^2)+[x+\xi]^2}}\right)^D}$$

$$=\eta e^{-\frac{1}{(1-\xi)}\int \mu dt} \qquad \text{(c.f., Theorem(3.1))}$$

Let $\beta = 2, \xi = 0.5$(instability phase for stable M/G/1 queueing system), $\eta = 1, \mu(t) = \frac{1}{t}$. Hence, $\zeta =$ 0.2886751346, a = 2.834830673 , b = -6.298932288

$A = 7.270191782, B = -31.89121954, C = 107.0377732, D = -62.65366597$

$$\frac{(\left|\left(1-\frac{2.834830673(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|)^{7.270191782}(\left|\left(1-\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|)^{107.0377732}}{\left(1+\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)^{-62.65366597}\left(\left|\left(1+\frac{6.298932288(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|\right)^{31.89121954}}=\frac{1}{t^2}$$

Hence,

$$t=\sqrt{\left[\frac{\left(\left|\left(1+\frac{6.298932288(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|\right)^{31.89121954}\left(1+\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)^{-62.65366597}}{(\left|\left(1-\frac{2.834830673(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|)^{7.270191782}(\left|\left(1-\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|)^{107.0377732}}\right]}$$

It is observed analytically that the assigned time increasing mean service rate, $\mu(t) = \frac{1}{t}$ and for sufficiently large number in the time varying $M/G/1$ queueing system, time will be increasing ultimately to approach infinity. This can be seen by Figures 2–4.

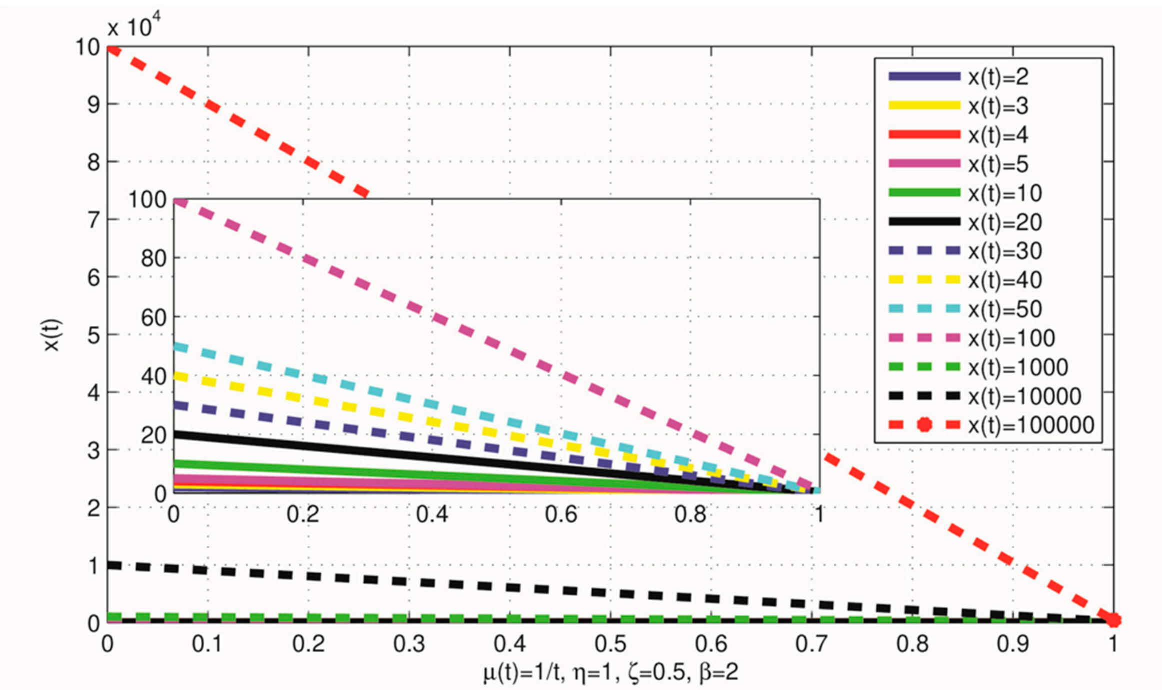

Figure 2.

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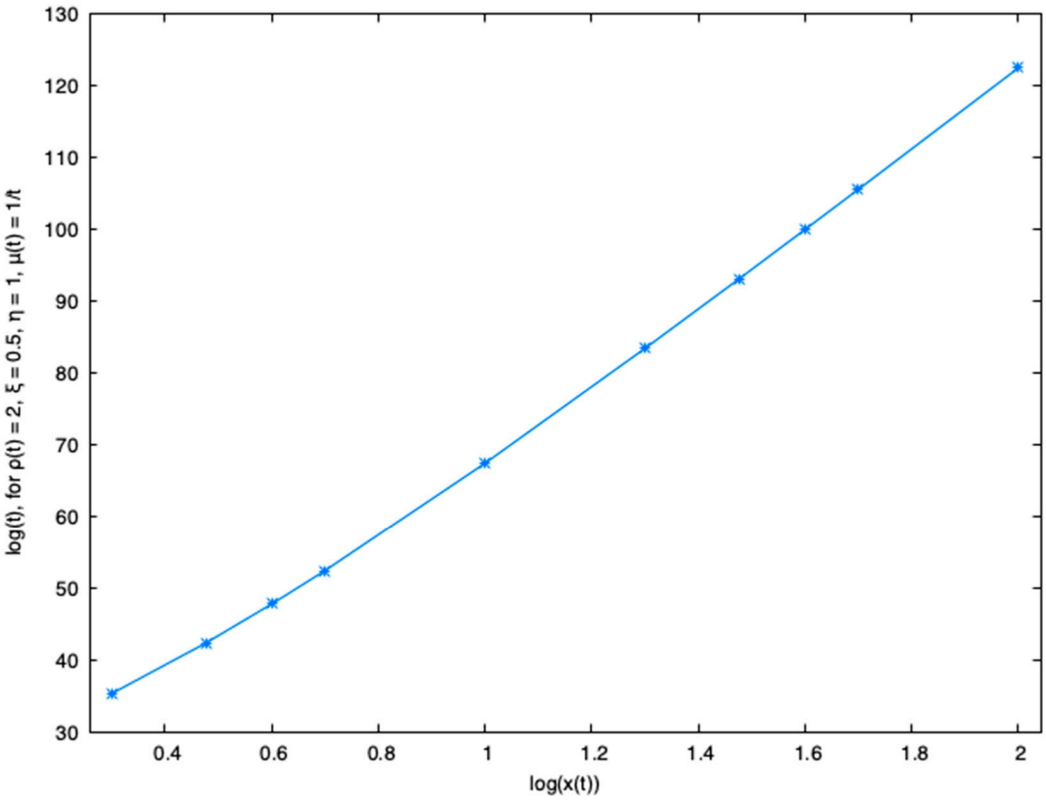

Figure 3.

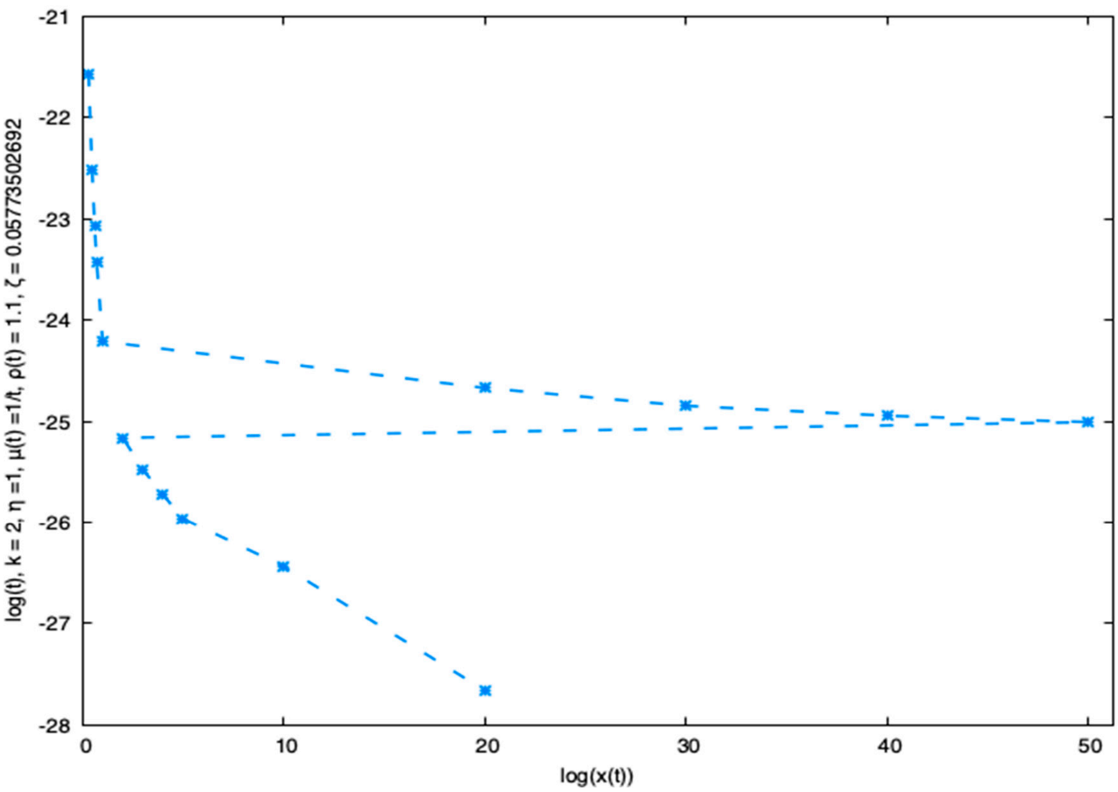

Figure 4.

Mathematically speaking , we have

$$lim_{x(t)\to\infty}\sqrt{\left[\frac{\left(\left|\left(1+\frac{6.298932288(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|\right)^{31.89121954}\left(1+\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)^{-62.65366597}}{\left(\left|\left(1-\frac{2.834830673(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|\right)^{7.270191782}\left(\left|\left(1-\frac{(x+0.5)}{\sqrt{(0.75)}+\sqrt{(0.75+[x+0.5]^2)}}\right)\right|\right)^{107.0377732}}\right]}$$

$$=\sqrt{\left[\frac{(7.298932288)^{31.89121954}(2)^{-62.65366597}}{(1.834830673)^{7.270191782}(0)^{107.0377732}}\right]}=\infty$$

4. Closing remarks with next phase of research

In this work, a challenging topic in queueing theory is examined; more precisely, the underlying queue's state variable is determined. The article offers a solution to this issue by utilizing the non-stationary $M/G/1$ queue's PSFFA analytic solution. Future work will concentrate on resolving open research issues and investigating applications of non-stationary queues in other scientific areas. The study also examines the effects of time and queueing parameters on the underlying queue's performance.

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