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Osamu Yasuda *

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Analytic Formulae for T Violation in Neutrino Oscillations

Osamu Yasuda

Department of Physics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan; yasuda@phys.se.tmu.ac.jp

Abstract: Recently, a concept known as μTRISTAN, which involves the acceleration of μ^+, has been proposed. This initiative has led to considerations of a new design for a neutrino factory. Additionally, leveraging the polarization of μ^+, measurements of T violation in neutrino oscillations are also being explored. In this paper, we present analytical expressions for T violation in neutrino oscillations within the framework of standard three flavor neutrino oscillations, a scenario involving nonstandard interactions, and a case of unitarity violation. We point out that examining the energy spectrum of T violation may be useful for probing new physics effects.

Keywords: neutrino oscillation; T violation; μTRISTAN

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1. Introduction

Results from various neutrino oscillation experiments have nearly determined the three mixing angles and the absolute values of the mass squared differences in the standard three flavor mixing scenario within the lepton sector [1]. The remaining undetermined parameters, such as the mass ordering, the octant of the atmospheric neutrino oscillation mixing angle, and the CP phase, are expected to be resolved by the high-intensity neutrino long-baseline experiments currently under construction, such as T2HK and DUNE. Once the CP phase is established, the standard three flavor lepton mixing scheme will be solidified, concluding the studies of the Standard Model with three massive neutrinos. To explore physics beyond this framework using neutrino oscillations, experiments in previously unexplored channels will be necessary.

Recently, a concept known as μTRISTAN [2] has been proposed, which involves creating a low-emittance μ^+ beam using ultra-cold muon technology and accelerating it to energies suitable for a μ^+ collider. This proposal has reignited interest [3] in the neutrino factory concept [4,5], which could be developed en route to achieving a muon collider. At such a neutrino factory, the decay of μ^+ in the storage ring would produce $\overline{\nu}_\mu$ and ν_e. Ref. [6] explored the potential to polarize the μ beam to reduce the flux of ν_μ or $\overline{\nu}_\mu$, thereby enabling the measurement of $\nu_e \to \nu_\mu$ transitions. If this can be achieved, it would allow for the measurement of T violation in neutrino oscillations, i.e., the difference between the oscillation probabilities $P(\nu_\mu \to \nu_e)$ and $P(\nu_e \to \nu_\mu)$.

T violation in neutrino oscillations has been discussed by many researchers in the past [6–26]. T violation has attracted significant attention primarily because its structure is simpler than that of CP violation, which compares $P(\nu_\mu \to \nu_e)$ with $P(\overline{\nu}_\mu \to \overline{\nu}_e)$ and involves complications due to the presence of the matter effect. In this paper, we derive the analytical forms of T violation in three scenarios: the standard three flavor scheme, a scenario with nonstandard interactions, and a case of unitarity violation. We also briefly comment on the feasibility of probing new physics effects by examining the energy dependence of T violation.

In Section 2, we review the formalism by Kimura, Takamura, and Yokomakura [29,30] to derive analytical formulas for the oscillation probabilities. In Section 3, we derive the analytic forms for T violation in the three cases: the standard three flavor mixing framework, a scenario involving flavor-dependent nonstandard interactions, and a case with unitarity violation. In Section 4, we summarize our conclusions.

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2. Analytical Formula for Oscillation Probabilities

It has been known [31] (See also earlier works [32–34].) that after eliminating the negative energy states by a Tani-Foldy-Wouthusen-type transformation, the Dirac equation for neutrinos propagating in matter is reduced to the familiar form:

$$i\frac{d\Psi}{dt} = \left(U\mathcal{E}U^{-1} + \mathcal{A}\right)\Psi, \tag{1}$$

where U is the PMNS matrix,

$$\Psi \equiv \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

is the flavor eigenstate,

$$\mathcal{E} \equiv \mathrm{diag}(E_1, E_2, E_3) \tag{2}$$

is the diagonal matrix of the energy eigenvalue $E_j \equiv \sqrt{m_j^2 + \vec{p}^2}$ $(j = 1, 2, 3)$ of each mass eigenstate with momentum $\vec{p}$, and the matrix

$$\mathcal{A} \equiv \sqrt{2}G_F\left\{\mathrm{diag}\left(N_e - \frac{N_n}{2}, -\frac{N_n}{2}, -\frac{N_n}{2}\right)\right\}.$$

stands for the matter effect, which is characterized by the Fermi coupling constant G_F, the electron density N_e and the neutron density N_n. Throughout this paper we assume for simplicity that the density of matter is constant. The 3×3 matrix on the right hand side of Equation (1) is hermitian and can be formally diagonalized by a unitary matrix $\widetilde{U}$ as:

$$U\mathcal{E}U^{-1} + \mathcal{A} = \widetilde{U}\widetilde{\mathcal{E}}\widetilde{U}^{-1}, \tag{3}$$

where

$$\widetilde{\mathcal{E}} \equiv \mathrm{diag}\left(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\right)$$

is a diagonal matrix with the energy eigenvalue $\widetilde{E}_j$ in the presence of the matter effect. Equation (1) can be easily solved, resulting the flavor eigenstate at the distance L:

$$\Psi(L) = \widetilde{U}\exp\left(-i\widetilde{\mathcal{E}}L\right)\widetilde{U}^{-1}\Psi(0). \tag{4}$$

Thus we have the probability amplitude $A(\nu_\beta \to \nu_\alpha)$ of the flavor transition $\nu_\beta \to \nu_\alpha$:

$$A(\nu_\beta \to \nu_\alpha) = \left[\widetilde{U}\exp\left(-i\widetilde{\mathcal{E}}L\right)\widetilde{U}^{-1}\right]_{\alpha\beta}. \tag{5}$$

From Equation (5) we observe that the shift $\mathcal{E} \to \mathcal{E} - \mathbf{1}E_1$, where $\mathbf{1}$ stands for the 3×3 identity matrix, changes only the overall phase of the probability amplitude $A(\nu_\beta \to \nu_\alpha)$, and this phase does not affect the value of the probability $P(\nu_\beta \to \nu_\alpha) = |A(\nu_\beta \to \nu_\alpha)|^2$ of the flavor transition $\nu_\beta \to \nu_\alpha$. In the following discussions, therefore, for simplicity, we define the diagonal energy matrix $\mathcal{E}$ and the potential one $\mathcal{A}$ as follows:

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$$\begin{aligned}
\mathcal{E} &\equiv \mathrm{diag}(E_1, E_2, E_3) - E_1 \mathbf{1} \\
&= \mathrm{diag}(0, \Delta E_{21}, \Delta E_{31})
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathcal{A} &\equiv \sqrt{2} G_F \left\{ \mathrm{diag}\left(N_e - \frac{N_n}{2}, -\frac{N_n}{2}, -\frac{N_n}{2} \right) + \left(\frac{N_n}{2} \right) \mathbf{1} \right\} \\
&= \mathrm{diag}(A, 0, 0)
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
\Delta E_{jk} &\equiv E_j - E_k \simeq \frac{m_j^2 - m_k^2}{2|\vec{p}|} \equiv \frac{\Delta m_{jk}^2}{2|\vec{p}|} \equiv \frac{\Delta m_{jk}^2}{2E} \\
A &\equiv \sqrt{2} G_F N_e.
\end{aligned} \tag{8}$$

Thus the appearance oscillation probability $P(\nu_\beta \to \nu_\alpha)$ $(\alpha \neq \beta)$ is given by

$$\begin{aligned}
& P(\nu_\beta \to \nu_\alpha) \\
&= \left| \left[\tilde{U} \exp\left(-i \tilde{\mathcal{E}} L \right) \tilde{U}^{-1} \right]_{\alpha\beta} \right|^2 \\
&= \left| \sum_{j=1}^{3} \tilde{X}_j^{\alpha\beta} e^{-i \tilde{E}_j L} \right|^2 \\
&= \left| e^{-i \tilde{E}_1 L} \sum_{j=1}^{3} \tilde{X}_j^{\alpha\beta} e^{-i \Delta \tilde{E}_{j1} L} \right|^2 \\
&= \left| \sum_{j=1}^{3} \tilde{X}_j^{\alpha\beta} \left(e^{-i \Delta \tilde{E}_{j1} L} - 1 \right) \right|^2
\end{aligned} \tag{9}$$

$$\begin{aligned}
&= \left| (-2i) \sum_{j=2}^{3} e^{-i \Delta \tilde{E}_{j1} L/2} \tilde{X}_j^{\alpha\beta} \sin\left(\frac{\Delta \tilde{E}_{j1} L}{2} \right) \right|^2 \\
&= 4 \left| e^{-i \Delta \tilde{E}_{31} L/2} \tilde{X}_3^{\alpha\beta} \sin\left(\frac{\Delta \tilde{E}_{31} L}{2} \right) + e^{-i \Delta \tilde{E}_{21} L/2} \tilde{X}_2^{\alpha\beta} \sin\left(\frac{\Delta \tilde{E}_{21} L}{2} \right) \right|^2 \\
&= 4 \left| \tilde{X}_3^{\alpha\beta} \sin\left(\frac{\Delta \tilde{E}_{31} L}{2} \right) + e^{i \Delta \tilde{E}_{32} L/2} \tilde{X}_2^{\alpha\beta} \sin\left(\frac{\Delta \tilde{E}_{21} L}{2} \right) \right|^2
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\tilde{X}_j^{\alpha\beta} &\equiv \tilde{U}_{\alpha j} \tilde{U}_{\beta j}^* \\
\Delta \tilde{E}_{jk} &\equiv \tilde{E}_j - \tilde{E}_k
\end{aligned}$$

have been defined,

$$\sum_{j=1}^{3} \tilde{X}_j^{\alpha\beta} = \delta_{\alpha\beta} = 0 \ \text{ for } \alpha \neq \beta \tag{11}$$

was subtracted in Equation (9) and throughout this paper the indices $\alpha, \beta = (e, \mu, \tau)$ and $j, k = (1, 2, 3)$ stand for those of the flavor and mass eigenstates, respectively. Once we know the eigenvalues $\tilde{E}_j$ and the quantity $\tilde{X}_j^{\alpha\beta}$, the oscillation probability can be expressed analytically.[1] So the only non-trivial

[1] In the case of three neutrino flavors in matter, the energy eigenvalues, $\tilde{E}_j$, can in principle be analytically determined using the cubic equation root formula [35]. However, the analytic expression for $\tilde{E}_j$ involving the inverse cosine function

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problem in the standard case is to obtain the expression for $\widetilde{X}_j^{\alpha\beta}$, and this was done by Kimura, Takamura and Yokomakura [29,30]. Their arguments are based on the trivial identities. From the unitarity condition of the matrix $\widetilde{U}$, we have

$$\delta_{\alpha\beta} = \left[\widetilde{U}\widetilde{U}^{-1}\right]_{\alpha\beta} = \sum_j \widetilde{U}_{\alpha j}\widetilde{U}^*_{\beta j} = \sum_j \widetilde{X}_j^{\alpha\beta}. \tag{12}$$

Next we take the (α, β) component of the both hand sides in Equation (3):

$$\left[U\mathcal{E}U^{-1} + \mathcal{A}\right]_{\alpha\beta} = \left[\widetilde{U}\widetilde{\mathcal{E}}\widetilde{U}^{-1}\right]_{\alpha\beta} = \sum_j \widetilde{U}_{\alpha j}\widetilde{E}_j\widetilde{U}^*_{\beta j} = \sum_j \widetilde{E}_j\widetilde{X}_j^{\alpha\beta} \tag{13}$$

Furthermore, we take the (α, β) component of the square of Equation (3):

$$\left[\left(U\mathcal{E}U^{-1} + \mathcal{A}\right)^2\right]_{\alpha\beta} = \left[\widetilde{U}\widetilde{\mathcal{E}}^2\widetilde{U}^{-1}\right]_{\alpha\beta} = \sum_j \widetilde{U}_{\alpha j}\widetilde{E}_j^2\widetilde{U}^*_{\beta j} = \sum_j \widetilde{E}_j^2\widetilde{X}_j^{\alpha\beta} \tag{14}$$

Putting Equations (12)–(14) together, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ \widetilde{E}_1 & \widetilde{E}_2 & \widetilde{E}_3 \\ \widetilde{E}_1^2 & \widetilde{E}_2^2 & \widetilde{E}_3^2 \end{pmatrix} \begin{pmatrix} \widetilde{X}_1^{\alpha\beta} \\ \widetilde{X}_2^{\alpha\beta} \\ \widetilde{X}_3^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} Y_1^{\alpha\beta} \\ Y_2^{\alpha\beta} \\ Y_3^{\alpha\beta} \end{pmatrix} \tag{15}$$

with

$$Y_j^{\alpha\beta} \equiv \left[\left(U\mathcal{E}U^{-1} + \mathcal{A}\right)^{j-1}\right]_{\alpha\beta} \quad \text{for } j = 1, 2, 3,$$

which can be easily solved by inverting the Vandermonde matrix:

$$\begin{pmatrix} \widetilde{X}_1^{\alpha\beta} \\ \widetilde{X}_2^{\alpha\beta} \\ \widetilde{X}_3^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}}(\widetilde{E}_2\widetilde{E}_3, & -(\widetilde{E}_2 + \widetilde{E}_3), & 1) \\ \dfrac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}(\widetilde{E}_3\widetilde{E}_1, & -(\widetilde{E}_3 + \widetilde{E}_1), & 1) \\ \dfrac{1}{\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}(\widetilde{E}_1\widetilde{E}_2, & -(\widetilde{E}_1 + \widetilde{E}_2), & 1) \end{pmatrix} \begin{pmatrix} Y_1^{\alpha\beta} \\ Y_2^{\alpha\beta} \\ Y_3^{\alpha\beta} \end{pmatrix}. \tag{16}$$

3. Analytic Form of T Violation

In this section, we derive the analytic form of T violation in the cases with and without unitarity, using the formalism described in Section 2.

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is not practically useful. Therefore, below we will calculate $\widetilde{E}_j$ using perturbation theory with small parameters, such as $\Delta E_{21}/\Delta E_{31} = \Delta m^2_{21}/\Delta m^2_{31} \approx 1/30$ and those relevant to nonstandard scenarios.

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3.1. The Three Flavor Case with Unitarity

First let us discuss the case where time evolution is unitary. From Equation (10), we have

$$\begin{aligned}
&P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu) \\
&= 4\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right) \\
&\times\left[e^{i\Delta\widetilde{E}_{32}L/2}\widetilde{X}_3^{e\mu*}\widetilde{X}_2^{e\mu} + e^{-i\Delta\widetilde{E}_{32}L/2}\widetilde{X}_3^{e\mu}\widetilde{X}_2^{e\mu*}\right. \\
&\left.-e^{i\Delta\widetilde{E}_{32}L/2}\widetilde{X}_3^{e\mu}\widetilde{X}_2^{e\mu*} - e^{-i\Delta\widetilde{E}_{32}L/2}\widetilde{X}_3^{e\mu*}\widetilde{X}_2^{e\mu}\right] \\
&= 16\mathrm{Im}\left[\widetilde{X}_2^{e\mu}\widetilde{X}_3^{e\mu*}\right]\sin\left(\frac{\Delta\widetilde{E}_{32}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right),
\end{aligned} \tag{17}$$

Furthermore, from Equation (16), the factor $\mathrm{Im}\left[\widetilde{X}_2^{e\mu}\widetilde{X}_3^{e\mu*}\right]$ in Equation (17) can be rewritten as

$$\begin{aligned}
\mathrm{Im}\left[\widetilde{X}_2^{e\mu}\widetilde{X}_3^{e\mu*}\right] &= \frac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}\frac{1}{\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}} \\
&\times\mathrm{Im}\left[\{Y_3^{e\mu} - (\widetilde{E}_3 + \widetilde{E}_1)Y_2^{e\mu}\}\{Y_3^{e\mu*} - (\widetilde{E}_1 + \widetilde{E}_2)Y_2^{e\mu*}\}\right] \\
&= \frac{1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[Y_2^{e\mu}Y_3^{e\mu*}\right]
\end{aligned} \tag{18}$$

Equations (17) and (18) are applicable for a generic case, as long as unitarity relation (11) holds.

3.1.1. The Standard Three Flavor Case

In the standard three flavor case, $Y_{j+1}^{\alpha\beta} \equiv [(U\mathcal{E}U^{-1} + \mathcal{A})^j]_{\alpha\beta}$ $(j = 1, 2)$ can be expressed as follows:

$$\begin{aligned}
Y_2^{\alpha\beta} &\equiv \left[U\mathcal{E}U^{-1} + \mathcal{A}\right]_{\alpha\beta} \\
&= \sum_{j=2}^{3}\Delta E_{j1}X_j^{\alpha\beta} + A\delta_{\alpha e}\delta_{\beta e}
\end{aligned} \tag{19}$$

$$\begin{aligned}
Y_3^{\alpha\beta} &\equiv \left[\left(U\mathcal{E}U^{-1} + \mathcal{A}\right)^2\right]_{\alpha\beta} \\
&= \sum_{j=2}^{3}(\Delta E_{j1})^2X_j^{\alpha\beta} + A\sum_{j=2}^{3}\Delta E_{j1}\left(\delta_{\alpha e}X_j^{e\beta} + \delta_{\beta e}X_j^{\alpha e}\right) + A^2\delta_{\alpha e}\delta_{\beta e},
\end{aligned} \tag{20}$$

where we have also defined the quantity in vacuum:

$$X_j^{\alpha\beta} \equiv U_{\alpha j}U_{\beta j}^*. \tag{21}$$

From Equations (19) and (20), the factor $\mathrm{Im}\left[Y_2^{e\mu}Y_3^{e\mu*}\right]$ in Equation (18) can be rewritten as

$$\begin{aligned}
&\mathrm{Im}\left[Y_2^{e\mu}Y_3^{e\mu*}\right] \\
&= \mathrm{Im}\left[\left(\Delta E_{21}X_2^{e\mu} + \Delta E_{31}X_3^{e\mu}\right)\right. \\
&\qquad\left.\times\left\{\Delta E_{21}(\Delta E_{21} + A)X_2^{e\mu*} + \Delta E_{31}(\Delta E_{31} + A)X_3^{e\mu*}\right\}\right] \\
&= \mathrm{Im}\left[X_2^{e\mu}X_3^{e\mu*}\right]\Delta E_{21}\Delta E_{31}\Delta E_{32}
\end{aligned} \tag{22}$$

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$\mathrm{Im}\left[X_2^{e\mu}X_3^{e\mu*}\right]$ in Equation (22) is the Jarlskog factor [36] for the lepton sector, and is given in the standard parametrization [1] with the three mixing angles θ_{jk} $(j,k=1,2,3)$ and the Dirac CP phase δ by

$$\begin{aligned}J &\equiv \mathrm{Im}\left[X_2^{e\mu}X_3^{e\mu*}\right]\\ &= -\frac{1}{8}\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&P(\nu_\mu\to\nu_e)-P(\nu_e\to\nu_\mu)\\ &=16J\frac{\Delta E_{21}\Delta E_{31}\Delta E_{32}}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\sin\left(\frac{\Delta\widetilde{E}_{32}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\end{aligned} \tag{23}$$

Equation (23) is a well known formula [10] for the standard three flavor case. It is remarkable that in the standard three flavor case, T violation in matter is proportional to the Jarlskog factor in vacuum. This implies that the only source of T violation is the CP phase δ in the standard three flavor case, and it is the reason why T violation is simpler than CP violation in neutrino oscillations.

3.1.2. The Case with Non-Standard Interactions

As long as unitarity in the three flavor framework is maintained, Equation (18) holds. In this subsection, let us consider the scenario with flavor-dependent nonstandard interactions [37,38] during neutrino propagation. This scenario has garnered significant attention due to its potential implications for phenomenology. In this case the mass matrix is given by

$$U\mathcal{E}U^{-1}+\mathcal{A}+\mathcal{A}_{NP} \tag{24}$$

with

$$\mathcal{A}_{NP}\equiv A\begin{pmatrix}\epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau}\\ \epsilon_{e\mu}^* & \epsilon_{\mu\mu} & \epsilon_{\mu\tau}\\ \epsilon_{e\tau}^* & \epsilon_{\mu\tau}^* & \epsilon_{\tau\tau}\end{pmatrix},$$

where $\mathcal{A}$ and A are given by Equations (7) and (8), respectively. The dimensionless quantities $\epsilon_{\alpha\beta}$ stand for the ratio of the nonstandard Fermi coupling constant interaction to the standard one. Since the matrix (24) is hermitian, time evolution is unitary and all the arguments up to Equation (18) hold also in this case. The oscillation probability is given by Equations (10) and (16), where the standard potential matrix $\mathcal{A}$ must be replaced by $\mathcal{A}+\mathcal{A}_{NP}$.

The extra complication compared to the standard case is calculations of the eigenvalues $\widetilde{E}_j$ and the elements $[(U\mathcal{E}U^{-1}+\mathcal{A}+\mathcal{A}_{NP})^m]_{\alpha\beta}$ $(m=1,2)$. Here we work with perturbation theory with respect to the small parameters $\Delta E_{21}/\Delta E_{31}=\Delta m^2_{21}/\Delta m^2_{31}\simeq 1/30$ and $\epsilon_{\alpha\beta}$, which we assume to be as small as $\Delta E_{21}/\Delta E_{31}$. Namely, throughout this paper we assume

$$|\Delta E_{31}|\sim A\gg|\Delta E_{21}|\gtrsim A|\epsilon_{\alpha\beta}|.$$

and take into consideration to first order in these small parameters. Then, to first order in them, we get

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$$
\begin{aligned}
Y_2^{e\mu} &\equiv \left[U\mathcal{E}U^{-1} + \mathcal{A} + \mathcal{A}_{NP}\right]_{e\mu} \\
&= \Delta E_{31} X_3^{e\mu} + \Delta E_{21} X_2^{e\mu} + A\epsilon_{e\mu} \qquad (25) \\
Y_3^{e\mu} &\equiv \left[\left(U\mathcal{E}U^{-1} + \mathcal{A} + \mathcal{A}_{NP}\right)^2\right]_{e\mu} \\
&\simeq \{(\Delta E_{31})^2 + A\Delta E_{31}\} X_3^{e\mu} + A\Delta E_{21} X_2^{e\mu} \\
&\quad + A^2\epsilon_{e\mu} + \Delta E_{31}\{X_3, \mathcal{A}_{NP}\}_{e\mu} \\
&= (\Delta E_{31} + A) Y_2^{e\mu} - \Delta E_{31}\Delta E_{21} X_2^{e\mu} - A\Delta E_{31}\epsilon_{e\mu} + \Delta E_{31}\{X_3, \mathcal{A}_{NP}\}_{e\mu}, \qquad (26)
\end{aligned}
$$

where the curly bracket stands for an anticommutator of matrices P and Q: $\{P, Q\} \equiv PQ + QP$, and X_3 is a 3×3 matrix defined by

$$
\begin{aligned}
X_3 &\equiv U\mathrm{diag}(0, 0, 1)U^{-1} \qquad (27) \\
(X_3)_{\alpha\beta} &= U_{\alpha 3}U_{\beta 3}^* = X_3^{\alpha\beta}.
\end{aligned}
$$

From this, the first term on the right hand side of Equation (26) drops in the factor $\mathrm{Im}\left[Y_2^{e\mu} Y_3^{e\mu *}\right]$, and we obtain:

$$
\begin{aligned}
&\mathrm{Im}\left[Y_2^{e\mu} Y_3^{e\mu *}\right] = -\mathrm{Im}\left[Y_2^{e\mu *} Y_3^{e\mu}\right] \\
&\simeq \Delta E_{31}\mathrm{Im}\left[Y_2^{e\mu *}\left(\Delta E_{21} X_2^{e\mu} + A\epsilon_{e\mu} - \{X_3, \mathcal{A}_{NP}\}_{e\mu}\right)\right] \\
&\simeq (\Delta E_{31})^2\mathrm{Im}\left[X_3^{e\mu *}\left(\Delta E_{21} X_2^{e\mu} + A\epsilon_{e\mu} - \{X_3, \mathcal{A}_{NP}\}_{e\mu}\right)\right] \\
&= (\Delta E_{31})^2\mathrm{Im}\left[X_3^{e\mu *}\left\{\Delta E_{21} X_2^{e\mu} + A\left(X_3^{\tau\tau}\epsilon_{e\mu} - X_3^{e\tau}\epsilon_{\tau\mu} - X_3^{\tau\mu}\epsilon_{e\tau}\right)\right\}\right],
\end{aligned}
$$

where we have ignored terms of order $O((\Delta E_{21})^2)$, $O(A^2(\epsilon_{\alpha\beta})^2)$ and $O(A\Delta E_{21}\epsilon_{\alpha\beta})$. Thus we finally get the form for T violation:

$$
\begin{aligned}
&P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu) \\
&\simeq \frac{16(\Delta E_{31})^2}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}} \sin\left(\frac{\Delta\widetilde{E}_{32}L}{2}\right) \sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right) \sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right) \\
&\quad \times\mathrm{Im}\left[X_3^{e\mu *}\left\{\Delta E_{21} X_2^{e\mu} + A\left(X_3^{\tau\tau}\epsilon_{e\mu} - X_3^{e\tau}\epsilon_{\tau\mu} - X_3^{\tau\mu}\epsilon_{e\tau}\right)\right\}\right] \qquad (28)
\end{aligned}
$$

Note that the form of the standard contribution $(\Delta E_{31})^2\Delta E_{21}\mathrm{, Im}\left[X_2^{e\mu} X_3^{e\mu *}\right]$ in Equation (28) differs slightly from that in Equation (22) because we are neglecting terms of order $O((\Delta E_{21})^2)$. The terms proportional to A in the parenthesis in Equation (28) represent the additional contributions to T violation due to nonstandard interactions. These additional contributions are constant with respect to the neutrino energy E, and they exhibit a different energy dependence from that of the standard one, $\Delta E_{21} X_2^{e\mu} = (\Delta m_{21}^2/2E)U_{e2}U_{\mu 2}^*$. Therefore, if the magnitude of the additional contributions from nonstandard interactions is significant enough,[2] then their effects are expected to be observable in the energy spectrum of T violation.

[2] Constraints on the parameters $\epsilon_{\alpha\beta}$ have been provided in Refs. [39–41]. Depending on the sensitivity of each experiment, it may or may not be possible to detect the signal or to improve the existing bounds on $\epsilon_{\alpha\beta}$. The aim of this paper is to derive the analytic form of T violation; estimating experimental sensitivity is beyond its scope.

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3.2. The Three Flavor Case with Unitarity Violation

The discussions in Section 3.1 are based on the assumption that time evolution is unitary. In Ref. [42], the possibility to have a non-unitary leptonic mixing matrix was pointed out. In that case, the relation between the mass eigenstate ν_j and the flavor eigenstate ν_α is given by a nonunitary matrix N:

$$\nu_\alpha = N_{\alpha j}\nu_j$$
$$N \equiv (\mathbf{1}+\eta)U$$

with

$$\eta^\dagger = \eta. \tag{29}$$

In the so-called minimal unitarity violation, which was discussed in Ref. [42], the constraint on the deviation matrix η turned out to be strong. Here we take phenomenologically the form of the nonunitary matrix N and assume that the elements of the deviation matrix η is of order $\Delta E_{21}/\Delta E_{31}$ or smaller, as in Section 3.1.2, namely,

$$|\Delta E_{31}| \sim A \gg |\Delta E_{21}| \gtrsim A|\eta_{\alpha\beta}|.$$

It was argued in Ref. [43] that time evolution in the case of nonunitary mixing matrix can be discussed in terms of the mass eigenstate

$$\Psi_m \equiv \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

and its time evolution is described by

$$i\frac{d\Psi_m}{dt} = \left\{\mathcal{E} + N^T\mathcal{A}N^* - A_n\left(N^TN^* - \mathbf{1}\right)\right\}\Psi_m, \tag{30}$$

where $\mathcal{E}$ and $\mathcal{A}$ are defined by Equations (6) and (7),

$$A_n \equiv \frac{1}{\sqrt{2}}G_F N_n$$

stands for the absolute value of the contribution to the matter effect from the neutral current interaction, and the term $A_n\mathbf{1}$ was added to simplify the calculations without changing the absolute value of the probability amplitude. The 3×3 matrix on the right hand side of Equation (30) can be diagonalized with a unitary matrix W:

$$\mathcal{E} + N^T\mathcal{A}N^* - A_n\left(N^TN^* - \mathbf{1}\right) = W\widetilde{\mathcal{E}}W^{-1}, \tag{31}$$

where

$$\widetilde{\mathcal{E}} \equiv \mathrm{diag}\left(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\right)$$

is the energy eigenvalue matrix in matter with unitarity violation. The mass eigenstate at distance L can be solved as

$$\Psi_m(L) = W\exp(-i\widetilde{\mathcal{E}}L)W^{-1}\Psi_m(0). \tag{32}$$

In cases involving unitarity violation, due to the modified form of the charged current interaction [42], after computing the probability amplitude from Equation (32), we must multiply the probability am-

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plitude by an additional factor of $(NN^\dagger)_{\beta\beta}^{-1/2}$ for the production process and $(NN^\dagger)_{\alpha\alpha}^{-1/2}$ for detection. Defining the modified amplitude

$$\begin{aligned}\hat{A}(\nu_\beta \to \nu_\alpha) &\equiv A(\nu_\beta \to \nu_\alpha)(NN^\dagger)_{\alpha\alpha}^{1/2}(NN^\dagger)_{\beta\beta}^{1/2} \\ &= [N^* W \exp(-i\widetilde{\mathcal{E}}L)W^{-1}N^T]_{\alpha\beta},\end{aligned}$$

the modified probability

$$\hat{P}(\nu_\alpha \to \nu_\beta) \equiv |\hat{A}(\nu_\alpha \to \nu_\beta)|^2,$$

and the quantity

$$\widetilde{\mathcal{X}}_j^{\alpha\beta} \equiv (N^* W)_{\alpha j}(NW^*)_{\beta j},$$

we have the following expression for the appearance oscillation probability:

$$\begin{aligned}
&\hat{P}(\nu_\beta \to \nu_\alpha) \\
&= \left| \left[N^* W \exp\left(-i\widetilde{\mathcal{E}}L\right) W^{-1} N^T \right]_{\alpha\beta} \right|^2 \\
&= \left| \sum_{j=1}^{3} \widetilde{\mathcal{X}}_j^{\alpha\beta} e^{-i\widetilde{E}_j L} \right|^2 \\
&= \left| e^{-i\widetilde{E}_1 L} \sum_{j=1}^{3} \widetilde{\mathcal{X}}_j^{\alpha\beta} e^{-i\Delta\widetilde{E}_{j1} L} \right|^2 \\
&= \left| \sum_{j=1}^{3} \widetilde{\mathcal{X}}_j^{\alpha\beta} \left\{ 1 - \left(1 - e^{-i\Delta\widetilde{E}_{j1} L} \right) \right\} \right|^2 \\
&= \left| [N^* N^T]_{\alpha\beta} - 2i \sum_{j=2}^{3} e^{-i\Delta\widetilde{E}_{j1} L/2} \widetilde{\mathcal{X}}_j^{\alpha\beta} \sin\left(\frac{\Delta\widetilde{E}_{j1} L}{2} \right) \right|^2 \\
&= \left| [\{(\mathbf{1}+\eta)^2\}^T]_{\alpha\beta} + 2e^{-i\Delta\widetilde{E}_{31} L/2 - i\pi/2} \widetilde{\mathcal{X}}_3^{\alpha\beta} \sin\left(\frac{\Delta\widetilde{E}_{31} L}{2} \right) \right. \\
&\qquad \left. + 2e^{-i\Delta\widetilde{E}_{21} L/2 - i\pi/2} \widetilde{\mathcal{X}}_2^{\alpha\beta} \sin\left(\frac{\Delta\widetilde{E}_{21} L}{2} \right) \right|^2 \\
&\simeq 4 \left| \eta_{\beta\alpha} + e^{-i\Delta\widetilde{E}_{31} L/2 - i\pi/2} \widetilde{\mathcal{X}}_3^{\alpha\beta} \sin\left(\frac{\Delta\widetilde{E}_{31} L}{2} \right) \right. \\
&\qquad \left. + e^{-i\Delta\widetilde{E}_{21} L/2 - i\pi/2} \widetilde{\mathcal{X}}_2^{\alpha\beta} \sin\left(\frac{\Delta\widetilde{E}_{21} L}{2} \right) \right|^2 .
\end{aligned} \tag{33}$$

T violation $P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu)$ is a small quantity, and the difference between the probability $P(\nu_\mu \to \nu_e)$ and the modified one $\hat{P}(\nu_\mu \to \nu_e)$ comes from the factor $(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta} = [(\mathbf{1}+\eta)^2]_{\alpha\alpha}[(\mathbf{1}+\eta)^2]_{\beta\beta} \simeq 1 + 2\eta_{\alpha\alpha} + 2\eta_{\beta\beta}$, which has a small deviation from 1. Therefore, T violation of the probability $P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu)$ can be approximated by that of the modified probability $\hat{P}(\nu_\mu \to \nu_e) - \hat{P}(\nu_e \to \nu_\mu)$. Hence T violation is given by

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$$
\begin{aligned}
& P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\mu) \\
\simeq\ & \hat{P}(\nu_\mu \to \nu_e) - \hat{P}(\nu_e \to \nu_\mu) \\
=\ & 4\left|\eta_{\mu e} + e^{-i\Delta\widetilde{E}_{31}L/2 - i\pi/2}\widetilde{\mathcal{X}}_3^{e\mu}\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right) + e^{-i\Delta\widetilde{E}_{21}L/2 - i\pi/2}\widetilde{\mathcal{X}}_2^{e\mu}\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\right|^2 \\
& -4\left|\eta_{e\mu} + e^{-i\Delta\widetilde{E}_{31}L/2 - i\pi/2}\widetilde{\mathcal{X}}_3^{\mu e}\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right) + e^{-i\Delta\widetilde{E}_{21}L/2 - i\pi/2}\widetilde{\mathcal{X}}_2^{\mu e}\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\right|^2 \\
=\ & 4\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\left(\eta_{\mu e}\widetilde{\mathcal{X}}_3^{e\mu *} - \eta_{\mu e}^*\widetilde{\mathcal{X}}_3^{e\mu}\right)\left(e^{i\Delta\widetilde{E}_{31}L/2 + i\pi/2} - e^{-i\Delta\widetilde{E}_{31}L/2 - i\pi/2}\right) \\
& +4\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\left(\eta_{\mu e}\widetilde{\mathcal{X}}_2^{e\mu *} - \eta_{\mu e}^*\widetilde{\mathcal{X}}_2^{e\mu}\right)\left(e^{i\Delta\widetilde{E}_{21}L/2 + i\pi/2} - e^{-i\Delta\widetilde{E}_{21}L/2 - i\pi/2}\right) \\
& -4\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\left(\widetilde{\mathcal{X}}_3^{e\mu}\widetilde{\mathcal{X}}_2^{e\mu *} - \widetilde{\mathcal{X}}_3^{e\mu *}\widetilde{\mathcal{X}}_2^{e\mu}\right) \\
& \times\left(e^{i\Delta\widetilde{E}_{31}L/2 + i\pi/2 - i\Delta\widetilde{E}_{21}L/2 - i\pi/2} - e^{-i\Delta\widetilde{E}_{31}L/2 - i\pi/2 + i\Delta\widetilde{E}_{21}L/2 + i\pi/2}\right) \\
=\ & -16\,\mathrm{Im}\left[\widetilde{\mathcal{X}}_2^{e\mu}\widetilde{\mathcal{X}}_3^{e\mu *}\right]\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{32}L}{2}\right) \\
& +8\,\mathrm{Im}\left[\eta_{\mu e}\widetilde{\mathcal{X}}_3^{e\mu *}\right]\sin\left(\Delta\widetilde{E}_{31}L\right) + 8\,\mathrm{Im}\left[\eta_{\mu e}\widetilde{\mathcal{X}}_2^{e\mu *}\right]\sin\left(\Delta\widetilde{E}_{21}L\right)
\end{aligned} \tag{34}
$$

We observe that the energy dependence of T violation in this case is different from that with unitarity, since we have extra contributions which are proportional to $\sin\left(\Delta\widetilde{E}_{31}L\right)$ or $\sin\left(\Delta\widetilde{E}_{21}L\right)$. As in the case with unitarity, $\widetilde{\mathcal{X}}_j^{\alpha\beta}$ can be expressed in terms of the quantity $X_j^{\alpha\beta} \equiv U_{\alpha j}U_{\beta j}^*$ in vacuum, $\widetilde{E}_j$, and $\eta_{\alpha\beta}$. First of all, we note the following relations:

$$
\begin{aligned}
& \sum_j (\widetilde{E}_j)^m \widetilde{\mathcal{X}}_j^{\alpha\beta} \\
=\ & \sum_j (N^*W)_{\alpha j}(\widetilde{E}_j)^m (NW^*)_{\beta j} \\
=\ & \left[N^*\left\{\mathcal{E} + N^T\mathcal{A}N^* - A_n\left(N^TN^* - \mathbf{1}\right)\right\}^m N^T\right]_{\alpha\beta} \\
=\ & \left[N\left\{\mathcal{E} + N^\dagger\mathcal{A}N - A_n\left(N^\dagger N - \mathbf{1}\right)\right\}^m N^\dagger\right]_{\beta\alpha} \\
\equiv\ & \mathcal{Y}_{m+1}^{\alpha\beta} \qquad \text{for } m = 0, 1, 2.
\end{aligned} \tag{35}
$$

Then we rewrite Equations (35) as

$$
\sum_{m=1}^{3} V_{jm}\widetilde{\mathcal{X}}_m^{\alpha\beta} = \mathcal{Y}_j^{\alpha\beta} \qquad \text{for } j = 1, 2, 3, \tag{36}
$$

where $V_{jm} \equiv (\widetilde{E}_m)^{j-1}$ is the element of the Vandermonde matrix V as in the case with unitarity (See Equation (15)). The simultaneous Equation (36) can be solved by inverting V and we obtain

$$
\widetilde{\mathcal{X}}_j^{\alpha\beta} = \sum_{m=1}^{3}(V^{-1})_{jm}\mathcal{Y}_m^{\alpha\beta}. \tag{37}
$$

The factor $\mathrm{Im}\left[\widetilde{\mathcal{X}}_2^{e\mu}\widetilde{\mathcal{X}}_3^{e\mu *}\right]$ can be expressed in terms of $\widetilde{E}_j$ and $\mathcal{Y}_j^{e\mu}$:

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$$
\begin{aligned}
&\mathrm{Im}\left[\widetilde{\mathcal{X}}_2^{e\mu}\widetilde{\mathcal{X}}_3^{e\mu*}\right]\\
&\quad=\frac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}\frac{1}{\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\\
&\quad\times\mathrm{Im}\left[\{\mathcal{Y}_3^{e\mu}-(\widetilde{E}_3+\widetilde{E}_1)\mathcal{Y}_2^{e\mu}+\widetilde{E}_3\widetilde{E}_1\mathcal{Y}_1^{e\mu}\}\{\mathcal{Y}_3^{e\mu*}-(\widetilde{E}_1+\widetilde{E}_2)\mathcal{Y}_2^{e\mu*}+\widetilde{E}_1\widetilde{E}_2\mathcal{Y}_1^{e\mu*}\}\right]\\
&\quad=\frac{1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\left(\widetilde{E}_1^2\mathrm{Im}\left[\mathcal{Y}_1^{e\mu}\mathcal{Y}_2^{e\mu*}\right]-\widetilde{E}_1\mathrm{Im}\left[\mathcal{Y}_1^{e\mu}\mathcal{Y}_3^{e\mu*}\right]+\mathrm{Im}\left[\mathcal{Y}_2^{e\mu}\mathcal{Y}_3^{e\mu*}\right]\right).
\end{aligned}
\tag{38}
$$

The quantities $\mathcal{Y}_j^{e\mu}$ $(j=1,2,3)$ are calculated as follows:

$$
\mathcal{Y}_1^{e\mu}=[NN^\dagger]_{\mu e}=[(\mathbf{1}+\eta)^2]_{\mu e}\simeq 2\eta_{\mu e},
$$

$$
\begin{aligned}
\mathcal{Y}_2^{e\mu}&=\left[N\left\{\mathcal{E}+N^\dagger\mathcal{A}N-A_n\left(N^\dagger N-\mathbf{1}\right)\right\}N^\dagger\right]_{\mu e}\\
&=\left[(\mathbf{1}+\eta)\left\{U\mathcal{E}U^{-1}+(\mathbf{1}+\eta)\mathcal{A}(\mathbf{1}+\eta)-A_n\left((\mathbf{1}+\eta)^2-\mathbf{1}\right)\right\}(\mathbf{1}+\eta)\right]\Big|_{\mu e}\\
&\simeq\left[U\mathcal{E}U^{-1}+\mathcal{A}+\{\eta,U\mathcal{E}U^{-1}\}+2\{\eta,\mathcal{A}\}-2A_n\eta\right]_{\mu e}\\
&\simeq\Delta E_{31}X_3^{\mu e}+\Delta E_{21}X_2^{\mu e}+\Delta E_{31}\{\eta,X_3\}_{\mu e}+2(A-A_n)\eta_{\mu e},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Y}_3^{e\mu}&=\left[N\left\{\mathcal{E}+N^\dagger\mathcal{A}N-A_n\left(N^\dagger N-\mathbf{1}\right)\right\}^2N^\dagger\right]_{\mu e}\\
&=\left[(\mathbf{1}+\eta)\left\{U\mathcal{E}U^{-1}+(\mathbf{1}+\eta)\mathcal{A}(\mathbf{1}+\eta)-A_n\left((\mathbf{1}+\eta)^2-\mathbf{1}\right)\right\}^2(\mathbf{1}+\eta)\right]\Big|_{\mu e}\\
&\simeq\left[U\mathcal{E}^2U^{-1}+\mathcal{A}^2+\{U\mathcal{E}U^{-1},\mathcal{A}\}\right]_{\mu e}+\left[\{U\mathcal{E}U^{-1},\{\eta,\mathcal{A}\}\}+\{\mathcal{A},\{\eta,\mathcal{A}\}\}\right]_{\mu e}\\
&\quad-\left[2A_n\{U\mathcal{E}U^{-1},\eta\}+2A_n\{\mathcal{A},\eta\}\right]_{\mu e}\\
&\quad+\left[\{\eta,U\mathcal{E}^2U^{-1}\}+\{\eta,\mathcal{A}^2\}+\{\eta,\{U\mathcal{E}U^{-1},\mathcal{A}\}\}\right]_{\mu e}\\
&\simeq\Delta E_{31}(\Delta E_{31}+A)X_3^{\mu e}+A\Delta E_{21}X_2^{\mu e}\\
&\qquad+\Delta E_{31}\{X_3,\{\eta,\mathcal{A}\}\}_{\mu e}+\{\mathcal{A},\{\eta,\mathcal{A}\}\}_{\mu e}\\
&\qquad-2A_n\Delta E_{31}\{X_3,\eta\}_{\mu e}-2A_n\{\mathcal{A},\eta\}_{\mu e}\\
&\qquad+\Delta E_{31}^2\{\eta,X_3\}_{\mu e}+\{\eta,\mathcal{A}^2\}_{\mu e}+\Delta E_{31}\{\eta,\{X_3,\mathcal{A}\}\}_{\mu e}\\
&=(\Delta E_{31}+A)\mathcal{Y}_2^{e\mu}-\Delta E_{31}\Delta E_{21}X_2^{\mu e}\\
&\qquad-2(A-A_n)\Delta E_{31}\eta_{\mu e}-(A+2A_n)\Delta E_{31}\{X_3,\eta\}_{\mu e}\\
&\qquad+\Delta E_{31}\{X_3,\{\eta,\mathcal{A}\}\}_{\mu e}+\Delta E_{31}\{\eta,\{X_3,\mathcal{A}\}\}_{\mu e}.
\end{aligned}
$$

In the current scenario involving unitarity violation, we observe a nonvanishing contribution from $\mathcal{Y}_1^{\alpha\beta}$, necessitating knowledge of the explicit form of the energy eigenvalue $\widetilde{E}_1$. Given that $\mathcal{Y}_1^{e\mu}$ is of order $O(\eta_{\alpha\beta})$, to evaluate Equation (38) accurately to first order in both $\Delta E_{21}/\Delta E_{31}$ and $\eta_{\alpha\beta}$, we must calculate $\widetilde{E}_1$ solely to zeroth order in these parameters, i.e., assuming $\Delta E_{21}\to 0$ and $\eta_{\alpha\beta}\to 0$. Under these conditions, the characteristic equation of the 3×3 matrix (31) is defined by

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$$
\begin{aligned}
0 &= \det\Big[\mathbf{1}t - \Big\{\mathcal{E} + N^T \mathcal{A} N^* - A_n\Big(N^T N^* - \mathbf{1}\Big)\Big\}\Big] \\
&\simeq \det\Big[\mathbf{1}t - \mathrm{diag}(0,0,\Delta E_{31}) - U^{-1}\mathrm{diag}(A,0,0)U\Big] \\
&= t\Big\{t^2 - (\Delta E_{31} + A)t + A\Delta E_{31}\cos^2\theta_{13}\Big\} \\
&= t(t-\lambda_+)(t-\lambda_-),
\end{aligned}
$$

where $\lambda_\pm$ are the roots of the quadratic equation and are given by

$$
\begin{aligned}
\lambda_\pm &\equiv \frac{\Delta E_{31} + A \pm \Delta\widetilde{E}_{31}}{2} \\
\Delta\widetilde{E}_{31} &\equiv \sqrt{(\Delta E_{31}\cos 2\theta_{13} - A)^2 + (\Delta E_{31}\sin 2\theta_{13})^2}.
\end{aligned}
$$

From this, we obtain the energy eigenvalues $\widetilde{E}_j$ $(j = 1,2,3)$ to the leading order in $\Delta E_{21}/\Delta E_{31}$ and $\eta_{\alpha\beta}$:

$$
\begin{pmatrix} \widetilde{E}_1 \\ \widetilde{E}_2 \\ \widetilde{E}_3 \end{pmatrix} \simeq \begin{pmatrix} \lambda_- \\ 0 \\ \lambda_+ \end{pmatrix}
$$

The roots $\lambda_- = \widetilde{E}_1$ and $\lambda_+ = \widetilde{E}_3$ satisfy of the quadratic equation

$$
\lambda_\pm^2 - (\Delta E_{31} + A)\lambda_\pm + A\Delta E_{31}\cos^2\theta_{13} = 0.
$$

Hence the first two term on the right hand side of Equation (38) can be rewritten as

$$
\begin{aligned}
&\widetilde{E}_1^2 \mathrm{Im}\Big[\mathcal{Y}_1^{e\mu}\mathcal{Y}_2^{e\mu*}\Big] - \widetilde{E}_1 \mathrm{Im}\Big[\mathcal{Y}_1^{e\mu}\mathcal{Y}_3^{e\mu*}\Big] \\
&= \mathrm{Im}\Big[\mathcal{Y}_1^{e\mu}\Big\{\widetilde{E}_1^2\mathcal{Y}_2^{e\mu*} - \widetilde{E}_1\mathcal{Y}_3^{e\mu*}\Big\}\Big] \\
&\simeq \mathrm{Im}\Big[\mathcal{Y}_1^{e\mu}\Big\{\Delta E_{31} X_3^{\mu e*}\widetilde{E}_1^2 - \Delta E_{31}(\Delta E_{31} + A)X_3^{\mu e*}\widetilde{E}_1\Big\}\Big] \\
&= \mathrm{Im}\Big[\mathcal{Y}_1^{e\mu} A(\Delta E_{31})^2 X_3^{\mu e*}\cos^2\theta_{13}\Big] \\
&= 2A(\Delta E_{31})^2\cos^2\theta_{13}\mathrm{Im}\Big[\eta_{\mu e}X_3^{\mu e*}\Big],
\end{aligned}
$$

whereas the third term on the right hand side of Equation (38) can be written as

$$
\begin{aligned}
&\mathrm{Im}\Big[\mathcal{Y}_2^{e\mu}\mathcal{Y}_3^{e\mu*}\Big] \\
&= -\mathrm{Im}\Big[\mathcal{Y}_2^{e\mu*}\Big\{(\Delta E_{31} + A)\mathcal{Y}_2^{e\mu} - \Delta E_{31}\Delta E_{21}X_2^{\mu e} \\
&\qquad -2(A - A_n)\Delta E_{31}\eta_{\mu e} - (A + 2A_n)\Delta E_{31}\{X_3, \eta\}_{\mu e} \\
&\qquad +\Delta E_{31}\{X_3, \{\eta, \mathcal{A}\}\}_{\mu e} + \Delta E_{31}\{\eta, \{X_3, \mathcal{A}\}\}_{\mu e}\}\Big] \\
&= \Delta E_{31}\mathrm{Im}\Big[X_3^{\mu e*}\Big\{\Delta E_{31}\Delta E_{21}X_2^{\mu e} \\
&\qquad\qquad +2(A - A_n)\Delta E_{31}\eta_{\mu e} + (A + 2A_n)\Delta E_{31}\{X_3, \eta\}_{\mu e} \\
&\qquad\qquad -\Delta E_{31}\{X_3, \{\eta, \mathcal{A}\}\}_{\mu e} - \Delta E_{31}\{\eta, \{X_3, \mathcal{A}\}\}_{\mu e}\}\Big].
\end{aligned}
$$

Thus we obtain the expression for the factor $\mathrm{Im}\Big[\widetilde{\mathcal{X}}_2^{e\mu}\widetilde{\mathcal{X}}_3^{e\mu*}\Big]$:

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$$
\begin{aligned}
&\mathrm{Im}\left[\widetilde{\mathcal{X}}_2^{e\mu}\widetilde{\mathcal{X}}_3^{e\mu*}\right]\\
&=\frac{1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\left(\widetilde{E}_1^2\mathrm{Im}\left[\mathcal{Y}_1^{e\mu}\mathcal{Y}_2^{e\mu*}\right]-\widetilde{E}_1\mathrm{Im}\left[\mathcal{Y}_1^{e\mu}\mathcal{Y}_3^{e\mu*}\right]+\mathrm{Im}\left[\mathcal{Y}_2^{e\mu}\mathcal{Y}_3^{e\mu*}\right]\right)\\
&\simeq\frac{\Delta E_{31}}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[X_3^{\mu e*}\left\{\Delta E_{21}X_2^{\mu e}+4A\eta_{\mu e}+2A_n(\{\eta,X_3\}_{\mu e}-\eta_{\mu e})\right\}\right]
\end{aligned}
\tag{39}
$$

To complete the calculation of Equation (34), we need to estimate the two quantities:

$$
\begin{aligned}
&\mathrm{Im}\left[\eta_{\mu e}\widetilde{\mathcal{X}}_3^{e\mu*}\right]\\
&=\frac{1}{\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[\eta_{\mu e}\left\{\widetilde{E}_1\widetilde{E}_2\mathcal{Y}_1^{e\mu*}-(\widetilde{E}_1+\widetilde{E}_2)\mathcal{Y}_2^{e\mu*}+\mathcal{Y}_3^{e\mu*}\right\}\right]\\
&\simeq\frac{1}{\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[\eta_{\mu e}\left(-\widetilde{E}_1\mathcal{Y}_2^{e\mu*}+\mathcal{Y}_3^{e\mu*}\right)\right]\\
&\simeq\frac{1}{\Delta\widetilde{E}_{31}\lambda_+}\mathrm{Im}\left[\eta_{\mu e}\left(-\lambda_-\Delta E_{31}X_3^{\mu e*}+\Delta E_{31}(\Delta E_{31}+A)X_3^{\mu e*}\right)\right]\\
&=\frac{1}{\Delta\widetilde{E}_{31}\lambda_+}\lambda_+\Delta E_{31}\mathrm{Im}\left[\eta_{\mu e}X_3^{\mu e*}\right]\\
&=\frac{\Delta E_{31}}{\Delta\widetilde{E}_{31}}\mathrm{Im}\left[\eta_{\mu e}X_3^{\mu e*}\right],
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&\mathrm{Im}\left[\eta_{\mu e}\widetilde{\mathcal{X}}_2^{e\mu*}\right]\\
&=\frac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[\eta_{\mu e}\left\{\widetilde{E}_3\widetilde{E}_1\mathcal{Y}_1^{e\mu*}-(\widetilde{E}_3+\widetilde{E}_1)\mathcal{Y}_2^{e\mu*}+\mathcal{Y}_3^{e\mu*}\right\}\right]\\
&\simeq\frac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[-(\widetilde{E}_3+\widetilde{E}_1)\mathcal{Y}_2^{e\mu*}+\mathcal{Y}_3^{e\mu*}\right]\\
&\simeq\frac{-1}{\Delta\widetilde{E}_{21}\Delta\widetilde{E}_{32}}\mathrm{Im}\left[-(\widetilde{E}_3+\widetilde{E}_1)\Delta E_{31}X_3^{\mu e*}+\Delta E_{31}(\Delta E_{31}+A)X_3^{\mu e*}\right]\\
&\simeq 0,
\end{aligned}
\tag{41}
$$

where terms of order $O((\Delta E_{21}/\Delta E_{31})^2)$, $O((\epsilon_{\alpha\beta})^2)$ and $O(\epsilon_{\alpha\beta}\Delta E_{21}/\Delta E_{31})$ have been neglected in Equations (39)–(41). Putting Equations (39)–(41) together, the final expression for T violation is given by

$$
\begin{aligned}
&P(\nu_\mu\to\nu_e)-P(\nu_e\to\nu_\mu)\\
&\simeq 16\frac{\Delta E_{31}}{A\Delta\widetilde{E}_{31}\cos^2\theta_{13}}\sin\left(\frac{\Delta\widetilde{E}_{31}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{21}L}{2}\right)\sin\left(\frac{\Delta\widetilde{E}_{32}L}{2}\right)\\
&\quad\times\mathrm{Im}\left[X_3^{\mu e*}\left\{\Delta E_{21}X_2^{\mu e}+4A\eta_{\mu e}+2A_n(\{\eta,X_3\}_{\mu e}-\eta_{\mu e})\right\}\right]\\
&\quad+8\frac{\Delta E_{31}}{\Delta\widetilde{E}_{31}}\sin\left(\Delta\widetilde{E}_{31}L\right)\mathrm{Im}\left[\eta_{\mu e}X_3^{\mu e*}\right]
\end{aligned}
\tag{42}
$$

Due to the additional contribution proportional to $\sin(\Delta\widetilde{E}_{31}L)$, the energy dependence of Equation (42) in the scenario with unitarity violation differs from that in the scenarios with unitarity, such as the standard case (23) and the nonstandard interaction case (42). Therefore, if the contribution from unitarity violation is significant enough and the experimental sensitivity is sufficiently high, it may be possible to distinguish the unitarity violation scenario from both the standard and nonstandard interaction scenarios by examining the energy spectrum in T violation.

4. Conclusions

In this paper, we have derived the analytical expression for T violation in neutrino oscillations under three different scenarios: the standard three flavor mixing framework, a scenario involving flavor-dependent nonstandard interactions, and a case with unitarity violation. In scenarios preserving unitarity, the T-violating component of the oscillation probability is proportional to $\sin(\frac{\Delta\widetilde{E}_{31}L}{2})\sin(\frac{\Delta\widetilde{E}_{21}L}{2})\sin(\frac{\Delta\widetilde{E}_{32}L}{2})$. However, in the case with unitarity violation, there is an additional contribution proportional to $\sin(\Delta\widetilde{E}_{31}L)$. Should future long-baseline experiments achieve high sensitivity to T violation across a broad energy spectrum, it may become feasible to specifically probe unitarity in the $\nu_\mu \leftrightarrow \nu_e$ channel.

Moreover, we demonstrated that the coefficient of the term $\sin(\frac{\Delta\widetilde{E}_{31}L}{2})\sin(\frac{\Delta\widetilde{E}_{21}L}{2})\sin(\frac{\Delta\widetilde{E}_{32}L}{2})(\Delta E_{31})^2(\Delta\widetilde{E}_{31}\Delta\widetilde{E}_{32}\Delta\widetilde{E}_{21})^{-1}$ varies depending on whether neutrino propagation follows the standard scheme or involves nonstandard interactions. In the standard scenario, this coefficient is proportional to $\Delta E_{21} = \Delta m^2_{21}/2E$. However, in the case with nonstandard interactions, there is an additional contribution that is energy-independent. Thus, it may be possible to observe the effects of nonstandard interactions by examining the energy dependence of T violation.

The purpose of this paper is to derive the analytical expression of T violation, and we did not quantitatively discuss the sensitivity of future experiments. The potential for T violation in neutrino oscillations deserves further study.

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