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[Christiaan Mantz](#)*

Posted Date: 10 May 2024

doi: 10.20944/preprints202405.0716.v1

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Solving Apollonius' problem through relativity

Christiaan Mantz

Amsterdam, The Netherlands

March 26, 2024

Abstract

We solve the problem of Apollonius by applying the Lorentz transformation within the framework of special relativity to go to a frame of simultaneity, where the solutions are the circumcenter. We transform back to get the solutions in a general frame. We show that the solutions in a general frame are the foci of the ellipse going through the centers of the given 3 circles.

Keywords: Apollonius' problem, conics, Relativity, foci, Lorentz transformation, frame of simultaneity

Apollonius' problem and Lorentz invariance

Apollonius problem can be stated as follows. Given 3 circles, find the circles that touch these three circles. Let $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ be the three given circles, with centers $A = [a_1, a_2], B = [b_1, b_2], C = [c_1, c_2]$ and with corresponding radii $\pm r_A = a_3, \pm r_B = b_3, \pm r_C = c_3$. We can view the problem in 3 dimensions, in particular in (2,1)-dim spacetime. There, the circles are given by the position vectors $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$. In (2,1)-dim the problem we need to solve is, where do the following 3 light cones meet

$$\begin{aligned}(x - a_1)^2 + (y - a_2)^2 &= (t - a_3)^2 \\(x - b_1)^2 + (y - b_2)^2 &= (t - b_3)^2 \\(x - c_1)^2 + (y - c_2)^2 &= (t - c_3)^2\end{aligned}\tag{1}$$

When we subtract these equations in pairs we obtain

$$\begin{aligned}x(b_1 - a_1) + y(b_2 - a_2) - t(b_3 - a_3) &= \frac{b_1^2 + b_2^2 - b_3^2}{2} - \frac{a_1^2 + a_2^2 - a_3^2}{2} \\x(c_1 - b_1) + y(c_2 - b_2) - t(c_3 - b_3) &= \frac{c_1^2 + c_2^2 - c_3^2}{2} - \frac{b_1^2 + b_2^2 - b_3^2}{2} \\x(a_1 - c_1) + y(a_2 - c_2) - t(a_3 - c_3) &= \frac{a_1^2 + a_2^2 - a_3^2}{2} - \frac{c_1^2 + c_2^2 - c_3^2}{2}\end{aligned}\tag{2}$$

Observe that this linear set of equations (2) is Lorentz invariant in the sense that it is formulated in terms of a relativistic inner product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 - u_3 v_3\tag{3}$$

The same is true for the set of equations that defines the problem, Eq.1. With this observation, the solution to Apollonius' problem can be obtained in the following manner. Solve for the plane through the points and intersect it with the line given by the linear set of equations, Eq.2. To get the solutions, we only need to compute the relativistic length from the point of intersection with any point of the given points, using the relativistic inner product, Eq.3. We need to move in the direction of the line given by Eq.2 up and down, starting from the point of intersection, to get the solutions. Let's explain why this is true.

Suppose the circles are of the same size for example, then the solution is the circumcenter.

CLMMantz@gmail.com

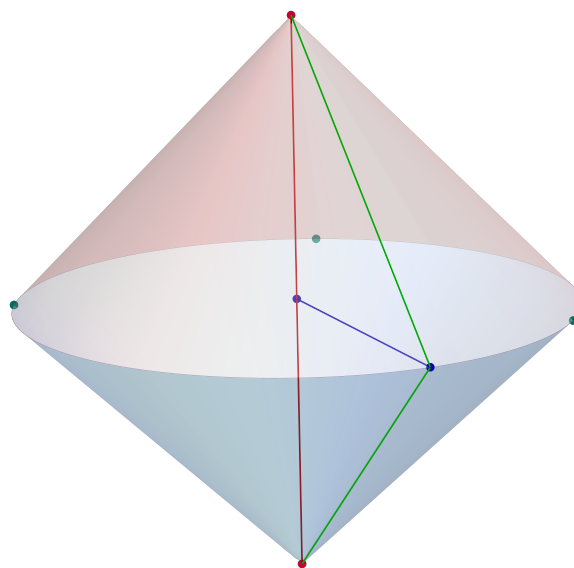


Figure 1: The 3 given circles are being displayed as 3 green points in (2,1)-dim. There is a point in time where the three given circles are shrunk to a point and since the circles are the same size, these points are on a horizontal plane. The circumcenter of the circle going through the given green points is shown in blue. The red points are the solutions in (2,1)-dim spacetime and have, in the case that the green circles are the same size, the same x, y coordinates as the blue circumcenter.

In 3 dim the line goes up and down from the circumcenter by a length $\sqrt{-x^2 - y^2 + z^2}$ where here $-x^2 - y^2 = 0$.

In the general case, where the circles are not the same size,

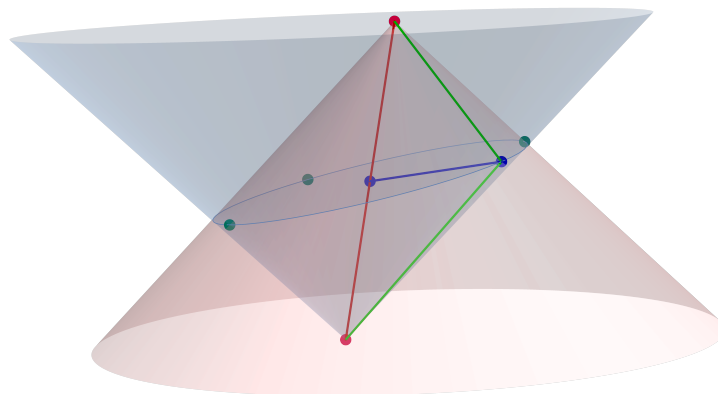


Figure 2: The 3 given circles are shown as 3 green points. Since they are of different sizes they are not on a horizontal plane. The red points are the solutions to the problem. They are the circles that touch the 3 given circles. Also the associated cones of the red points are being displayed. They meet in an ellipsoid, which goes through the 3 green points.

we can just Lorentz transform the problem into a situation where the circles are the same size. This is the frame of simultaneity. We can do this, since the linear set of equations (2) is Lorentz invariant.

We can solve of the (2,1)-circumcenter through intersecting the line given by Eq.2 and the plane through the 3 given points. This is true because, if in one frame, the frame of simultaneity, the intersection is the circumcenter, then in any Lorentz transformed frame the circumcenter becomes just the (2,1)-circumcenter. We can view this as the definition of the (2,1)-circumcenter. The solutions are just the relativistic length up and down analogues to the case when the line is vertical in the case the given circles are all the same size.

Foci of the ellipse

The solutions to Apollonius problem are the foci of the ellipse through the given 3 points.

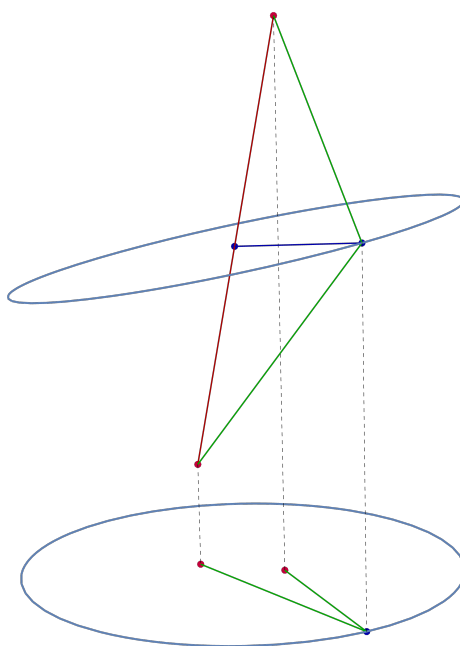


Figure 3: The (2,1)-dim configuration is being projected on a horizontal plane.

The ellipse is just the Lorentz transformed circle going through the centers of the 3 given circles. We show in 2-dim that the ellipse goes through the centers of the three given circles and that the centers of the solution circles are the foci of the ellipse

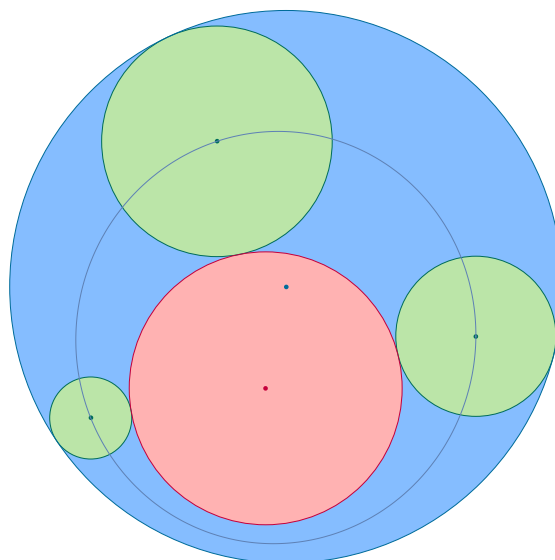


Figure 4: The blue and the red circle are the solutions to the problem, which touch the 3 given green circles. The ellipse goes through the centers of the 3 given circles and the centers of the solutions are the foci of the ellipse.

Two given circles inside another given circle

The problem of Apollonius can be stated in many different situations and these can limit the amount of solutions to the point there aren't any. The given circles can be inside each other for instance. We consider the scenario in which two given circles are inside the third given circle

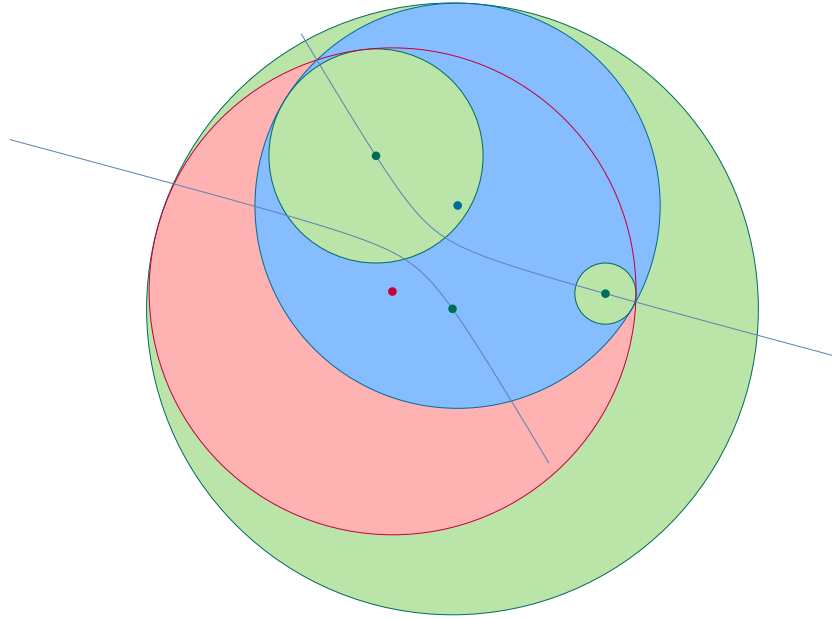


Figure 5: The 3 given circles are shown in green and the solution circles in red and blue. When 2 of the three given circles are inside the third given green circle, the conic section that goes through the centers of the given 3 circles is an hyperbola. The centers of the solution circles are the foci of the hyperbola.

In this case the conic section is a hyperbola instead of an ellipse. The hyperbola goes through the centers of the given 3 circles, and the centers of the blue and the red circle are the foci of the hyperbola. In general there are maximally 8 solutions, so 4 pairs, since the radii of the given circles correspond to 2 points in (2,1)-dim spacetime.

The Lorentz transformation

We can construct the Lorentz transformation in (2,1)-dim by combining three (2,1)-orthonormal basis vectors $e_1 = \frac{u}{\sigma(u)}$, $e_2 = \frac{u \times (u \times v)}{\sigma(u \times (u \times v))}$, $e_3 = \frac{u \times v}{\tau(u \times v)}$ into a matrix Λ , such that the basis vectors form the columns of the matrix

$$\Lambda = e_1 e_2 e_3 = \begin{pmatrix} (e_1)_1 & (e_2)_1 & (e_3)_1 \\ (e_1)_2 & (e_2)_2 & (e_3)_2 \\ (e_1)_3 & (e_2)_3 & (e_3)_3 \end{pmatrix}$$

Using the above notation, the Lorentz transformation is then represented by the matrix Λ

$$\Lambda = \frac{u}{\sigma(u)} \frac{u \times (u \times v)}{\sigma(u \times (u \times v))} \frac{u \times v}{\tau(u \times v)}$$

where $u = \overrightarrow{AB}$ and $v = \overrightarrow{AC}$, $\sigma(u) = \sqrt{u_1^2 + u_2^2 - u_3^2}$ and $\tau(u) = \sqrt{-u_1^2 - u_2^2 + u_3^2}$, and where $v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, -(v_1 w_2 - v_2 w_1))$ is a (2,1)-dim cross product. Note that the (2,1)-dim cross product ensures that the basis vectors are (2,1)-dim perpendicular and the τ and σ functions normalise the vectors. Usually the Gram-Schmidt process is used for creating perpendicular vectors [32]. The obtained Lorentz transformation allows us to go to the frame of simultaneity, where the 3 given circles are the same size.

Conclusion

We have proposed a way of calculating the solutions of Apollonius' problem by solving two matrix equations and computing a relativistic length. The solutions are the foci of the corresponding conic going through the centers of the 3 given circles in 2-dim.

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