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MS 460 New Methods for Multivariate Normal Moments

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Abstract: Normal moments are the building blocks of the Hermite polynomials, which in turn are the building blocks of the Edgeworth expansions for the distribution of parameter estimates. Isserlis (1918) gave the bivariate normal moments and 2 special cases of trivariate moments. Beyond that, convenient expressions for multivariate variate normal moments are still not available. We compare 3 methods for obtaining them, the most powerful being the differential method. We give simpler formulas for the bivariate moment than that of Isserlis, and explicit expressions for the general moments of dimensions 3 and 4.

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1. Introduction

Suppose that $X \sim \mathcal{N}_p(\mathbf{0}, V)$, the p-dimensional normal distribution with mean $\mathbf{0} \in N^p$ and covariance V. For $N = \{0, 1, 2, \cdots\}$, $n \in N^p$, and $x \in R^p$, define

$$|n| = \sum_{j=1}^{p} n_j,\ n! = \Pi_{j=1}^{p} n_j!,\ x^n = x_1^{n_1} \cdots x_p^{n_p},\ V_{r_1 \cdots r_p} = E\ X_{r_1} \cdots X_{r_p}, \tag{1}$$

$$\mu_n = \mu_{n_1 \cdots n_p} = E\ X^n = E\ X_1^{n_1} \cdots X_p^{n_p} = V_{1^{n_1} \cdots p^{n_p}} \tag{2}$$

$= q_{1^{n_1} \cdots p^{n_p}}$ in the notation of Isserlis. Since $-X$ has the same distribution as X, $\mu_n = 0$ if its *order*, $|n|$, is odd. Also,

$$p = V = 1 \Rightarrow \mu_{2n} = \nu_{2n} \text{ where}$$
$$\nu_{2n} = 1\,3 \cdots (2n-1) = (2n)!/(2^n n!) = E\ N^n \text{ for } X \sim \mathcal{N}_1(0,1). \tag{3}$$

The p-variate normal distribution and its moments play a central role in the Edgeworth expansions for the distribution of the standardized vector sample mean, and more generally for a wide class of vector estimates based on a sample of size n. See Withers (1984) for $p = 1$, and Withers and Nadarajah (2014) and Withers (2024) for $p > 1$. The first r terms of these Edgeworth expansions need most of the multivariate Hermite polynomials and normal moments of order $3r$. The Edgeworth expansions need the moments and Hermite polynomials of $Y = V^{-1}X \sim \mathcal{N}_p(0, V^{-1})$, that is, with $V = (V_{jk})$ replaced by $V^{-1} = (V^{jk})$.

Isserlis (1918) showed that the general $2r$th moment of order $2r$ is

$$V_{12\cdots,2r} = E\ X_1 X_2 \cdots X_{2r} = \sum^{\nu_{2n}} V_{j_1 j_2} \cdots V_{j_{2r-1} j_{2r}}, \tag{4}$$

where summation is over all ν_{2n} permutations $j_1, \cdots, j_{2r}$ of $1, 2, \cdots, 2r-1, 2r$ giving distinct terms. This is a special case of the formula for a multivariate moment in terms of the cumulants of a *general* random vector $X \in R^p$,

$$E\ X_{i_1} X_{i_2} \cdots X_{i_r} = B^{i_1, i_2, \cdots, i_r}(\kappa) = \sum_{k=1}^{r} B_k^{i_1, i_2, \cdots, i_r}(\kappa)$$

with $i_1, i_2, \cdots, i_r$ replaced by $1, 2, \cdots, 2r$ where $B^{i_1, i_2, \cdots, i_r}(\kappa)$ and $B_k^{i_1, i_2, \cdots, i_r}(\kappa)$ are the multivariate complete and partial exponential Bell polynomials, as given in (3.1) of Withers and Nadarajah (2013).

Here we give several methods for deriving normal moments, and a number of new results. Section 2 uses *the method of successive specialisation* using the step down rule (1). This soon becomes unwieldy without writing software. Section 3 summarises the results of Isserlis and Soper for bivariate moments. Section 4 gives *the method of successive generalisation using quasi-differential operators*. This powerful method has been used to obtain multivariate moments in terms of multivariate cumulants from the univariate formulas: see 3.29 of Stuart and Ord (1987). But it can also be used to obtain multivariate from univariate moments for *parametric* distributions, as we demonstrate here.

Section 5 gives *the multinomial method*, but in detail only for bivariate moments. Sections 6 and 7 give for the 1st time the general moments of dimensions 3 and 4. and illustrate how it can be extended to find moments with $p > 4$. Section 8 shows that these results are easily extended to moments of the multivariate normal with non-zero mean.

Jinadasa and Tracy (1986) and Tracy and Sultan (1993) and Holmquist (2007). gave central moments by matrix differentiation of $E\ e^{t'X} = e^{t'Vt/2}$. However the results are in terms of vec, $\otimes$ and permutation matrices, so make interpretation difficult. Its differentiation with respect to t can be used to give moments in terms of multivariate Bell polynomials, but again these take some effort to understand. Phillips, K. (2010) gives software in R for moments. For applications to quantum mechanics and field theory, see Simon (1974, page 9).

$$\text{Set } [r]_j = r!/(r-j)! = r\cdots(r-j+1),\ 1/k! = 0 \text{ for } k = -1, -2, \ldots \tag{5}$$

2. The Method of Successive Specialisation

From (4) follows the new and more useful *step down* recurrence rule,

$$E\ X_1 \ldots X_{2r} = \mu_{1^{2r}} = V_{12\cdots,2r} = \sum_{k=1}^{2r-1} V_{k,2r} V^{(k)}_{12\cdots,2r-1} \tag{1}$$

where $V^{(k)}_{12\cdots,2r-1}$ is $V_{12\cdots,2r}$ with k and $2r$ removed. This gives the normal moments of order $2r$ in terms of those of order $2r-2$. For example,

$$E\ X_1 \ldots X_4 = \mu_{1111} = V_{1234} = \sum_{k=1}^{3} V_{k4} V^{(k)}_{123} = V_{14}V_{23} + V_{24}V_{13} + V_{34}V_{12},$$

$$E\ X_1 \ldots X_6 = \mu_{1^6} = V_{12\cdots6} = \sum_{k=1}^{5} V_{k6} V^{(k)}_{12\cdots5}$$

$$= V_{16}V_{2345} + V_{26}V_{1345} + V_{36}V_{1245} + V_{46}V_{1235} + V_{56}V_{1234},$$

$$E\ X_1 \ldots X_8 = \mu_{1^8} = V_{12\cdots8} = \sum_{k=1}^{7} V_{k8} V^{(k)}_{12\cdots7} = V_{18}V_{23\cdots7} + \cdots V_{78}V_{12\cdots6}.$$

By (1) with $2r$ replaced by 1, that is, with $X_{2r} = X_1$,

$$E\ X_1^2 X_2 \cdots X_{2r-1} = V_{1^2 23\cdots,2r-1} = \mu_{21^{2r-2}} = \sum_{k=1}^{2r-1} V_{k1} V^{(k)}_{123\cdots,2r-1}. \tag{2}$$

For example taking $r = 3$ then replacing 5 by 1 then 4 by 1 gives

$$E\ X_1^2 X_2 \cdots X_5 = V_{1^2 2345} = \mu_{21111} = V_{11}V_{2345} + \sum_{2345}^{4} V_{12}V_{1345}, \tag{3}$$

$$E\ X_1^3 X_2 X_3 X_4 = \mu_{3111} = V_{111234} = 3V_{11}V_{1234} + 6V_{12}V_{13}V_{14}, \tag{4}$$

$$E\ X_1^4 X_2 X_3 = V_{1^4 23} = \mu_{411} = 3V_{11}(4V_{12}V_{13} + V_{11}V_{23}), \tag{5}$$

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where $\sum_{2345}^{4}$ in (3) sums over the 4 distinct terms obtained by permuting 2345. This shows that there is an error in $q_{1^4 23}$ p139 of Isserlis (1918): his 12 should be 4. His other formulas on p139 pass **the** $V_{ij} \equiv 1$ **test**: under this condition the moments of order $2r$ are ν_{2r}. This provides a useful check on moment formulas.

We now give all $\mu_{r_1 r_2 \cdots}$ of (2) of order $n = r_1 + r_2 + \cdots$ up to $n = 8$. These are obtained from the bottom up. For example $X_7 = X_1$ in (20) gives (19), and $X_7 = X_1$ in (19) gives (18).

Without loss of generality, we assume that

$$V_{jj} \equiv 1. \tag{6}$$

To emphasize this, when $V_{jj} \equiv 1$, we set $\rho_{12\cdots}$ for $V_{12\cdots}$ of (1), and $\rho_{1^{n_1}\cdots p^{n_p}}$ for $V_{1^{n_1}\cdots p^{n_p}}$.

Moments of order 2.

$$E\ X_1^2 = \mu_2 = \rho_{11} = 1,\ E\ X_1 X_2 = \mu_{11} = \rho_{12}.$$

Moments of order 4.

$$E\ X_1^4 = \mu_4 = \nu_4 = 3,\ E\ X_1^3 X_2 = \mu_{31} = 3\rho_{12},\ E\ X_1^2 X_2^2 = \mu_{22} = 1 + 2\rho_{12}^2,$$

$$E\ X_1^2 X_2 X_3 = \mu_{211} = \rho_{23} + 2\rho_{12}\rho_{13}, \tag{7}$$

$$E\ X_1 \cdots X_4 = \mu_{1111} = \rho_{1234} = \rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}. \tag{8}$$

Moments of order 6 in terms of moments of order 2,4.

$$E\ X_1^6 = \mu_6 = \nu_6 = 15,\ E\ X_1^5 X_2 = \mu_{51} = 15\rho_{12},$$

$$E\ X_1^4 X_2^2 = \rho_{1^4 2^2} = \mu_{42} = 3(1 + 4\rho_{12}^2),$$

$$E\ X_1^4 X_2 X_3 = \rho_{1^4 23} = \mu_{411} = 3(\rho_{23} + 4\rho_{12}\rho_{13}) \text{ by (5)}, \tag{9}$$

$$E\ X_1^3 X_2^3 = \rho_{1^3 2^3} = \mu_{33} = 3(3\rho_{12} + 2\rho_{12}^3),$$

$$E\ X_1^3 X_2^2 X_3 = \mu_{321} = 3(1 + 2\rho_{12}^2)\rho_{13} + 6\rho_{12}\rho_{23} \text{ by (11)}, \tag{10}$$

$$E\ X_1^3 X_2 X_3 X_4 = \mu_{3111} = 3(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23} + 2\rho_{12}\rho_{13}\rho_{14}), \tag{11}$$

$$E\ X_1^2 X_2^2 X_3^2 = \mu_{222} = 1 + 2\rho_{12}^2 + 2\rho_{13}^2 + 2\rho_{23}^2 + 8\rho_{12}\rho_{13}\rho_{23}, \tag{12}$$

$$E\ X_1^2 X_2^2 X_3 X_4 = \mu_{2211} = (\mu_{21^4} \text{ at } X_5 = X_2) = \rho_{2^2 34} + 2\rho_{12}\rho_{1234} + \sum_{34}^{2} \rho_{13}\rho_{12^2 4}$$

$$= \rho_{34}(1 + 2\rho_{12}^2) + 2\sum_{12}^{2} \rho_{13}\rho_{14} + 4\rho_{12}\sum_{34}^{2} \rho_{13}\rho_{24}, \tag{13}$$

$$E\ X_1^2 X_2 \cdots X_5 = \mu_{21^4} = \rho_{1^2 2345} = \rho_{2345} + \sum_{2345}^{4} \rho_{12}\rho_{1345} \text{ of (8) by (3)}$$

$$= \sum_{2345}^{3} \rho_{23}\rho_{45} + 2\sum_{2345}^{6} \rho_{12}\rho_{13}\rho_{45}, \tag{14}$$

$$E\ X_1 \cdots X_6 = \mu_{1^6} = \rho_{123456} = \sum^{15} \rho_{12}\rho_{34}\rho_{56}.$$

Of these moments, μ_{3111}, μ_{2211}, μ_{21^4} are new. There is an error in $\mu_{411} = q_{1^4 23}$ of Isserlis but his $q_{1^\lambda 23}$ is correct.

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Moments of order 8 in terms of lower moments. Again we use ν_{2n} of (3) and $\mu_{n_1 n_2 \ldots} = \rho_{1^{n_1} 2^{n_2} \ldots} = E\, X_1^{n_1} X_2^{n_2} \ldots$.

$$\mu_8 = \rho_{1^8} = \nu_8 = 105,\ \mu_{71} = \rho_{1^7 2} = 105\rho_{12},\ \mu_{62} = \rho_{1^6 2^2} = 15(1 + 6\rho_{12}^2),$$

$$\mu_{611} = (\mu_{51^3} \text{ at } X_4 = X_2) = \rho_{1^6 23} = 15(\rho_{23} + 6\rho_{12}\rho_{13}), \quad (15)$$

$$\mu_{53} = \rho_{1^5 2^3} = 15(3\rho_{12} + 4\rho_{12}^3),$$

$$\mu_{521} = (\mu_{51^3} \text{ at } X_4 = X_2) = \rho_{1^5 2^2 3} = 15(\rho_{13} + 2\rho_{12}\rho_{23} + 4\rho_{12}^2\rho_{13}), \quad (16)$$

$$\mu_{51^3} = (\mu_{41^4} \text{ at } X_5 = X_1) = \rho_{1^5 234} = 4\rho_{1^3 234} + \sum_{234}^{3} \rho_{12}\rho_{1^4 34} \text{ of (9),(11),(13)},$$

$$\mu_{44} = \rho_{1^4 2^4} = (\mu_{431} \text{ at } X_3 = X_2) = 3(3 + 24\rho_{12}^2 + 8\rho_{12}^4),$$

$$\mu_{431} = \rho_{1^4 2^3 3} = (\mu_{4211} \text{ at } X_4 = X_2)$$

$$= 3\rho_{1^2 2^3 3} + 3\rho_{12}\rho_{1^3 2^2 3} + \rho_{13}\rho_{1^3 2^3} \text{ of (26) and (3)}$$

$$= 9\rho_{12}(1 + 4\rho_{13} + 4\rho_{12}\rho_{23}) + 24\rho_{12}^3\rho_{13} \text{ by (29)},$$

$$\mu_{422} = (\mu_{4211} \text{ at } X_4 = X_2) = 3\rho_{1^2 2^3 3} + 3\rho_{12}\rho_{1^3 2^2 3} + \rho_{13}\rho_{1^3 2^3}. \text{ See (28)}.$$

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$$\mu_{4211} = \rho_{1^4 2^2 34} = (\mu_{41^4} \text{ at } X_5 = X_2) = 3\rho_{1^2 2^2 34} + 2\rho_{12}\rho_{1^3 234}$$
$$+ \sum_{34}^{2} \rho_{13}\rho_{1^3 2^2 4} \text{ of (13), (11) and (3): see (15), (10);}$$
$$\mu_{41^4} = \rho_{1^4 2345} = (\mu_{31^5} \text{ at } X_6 = X_1)$$
$$= 3\rho_{112345} + \sum_{2345}^{4} \rho_{12}\rho_{1^3 345} \text{ of (14) and (11),} \tag{17}$$
$$\mu_{332} = \rho_{1^3 2^3 3^2} = (\mu_{3311} \text{ at } X_4 = X_3) = 2\rho_{1^2 3^3 2} + 3\rho_{12}\rho_{1^2 2^2 3^2} + 2\rho_{13}\rho_{1^2 2^3 3},$$
$$\mu_{3311} = \rho_{1^3 2^3 34} = (\mu_{32111} \text{ at } X_5 = X_2) = 2\rho_{1^2 3^3 4} + 3\rho_{12}\rho_{1^2 2^2 34} + \sum_{34}^{2} \rho_{13}\rho_{1^2 2^3 4}:$$

see (17).

$$\mu_{3221} = \rho_{1^3 2^2 3^2 4} = (\mu_{321^3} \text{ at } X_5 = X_4) = 2\rho_{1^2 2^2 34^2} + 2\rho_{12}\rho_{1^2 234^2} + \rho_{13}\rho_{1^2 2^2 4^2}$$
$$+ 2\rho_{14}\rho_{1^2 2^2 34}: \text{ See (20).}$$
$$\mu_{32111} = \rho_{1^3 2^2 345} = (\mu_{31^5} \text{ at } X_6 = X_2) = 2\rho_{1^2 2^2 345} + 2\rho_{12}\rho_{1^2 2345} + \sum_{345}^{3} \rho_{13}\rho_{1^2 2^2 45},$$
$$\mu_{31^5} = (\mu_{21^6} \text{ at } X_7 = X_1) = \rho_{1^3 2\cdots 6} = 2\rho_{1\cdots 6} + \sum_{2\cdots 6}^{5} \rho_{12}\rho_{113456}, \tag{18}$$
$$\mu_{2^4} = \rho_{1^2 2^2 3^2 4^2} = (\mu_{2^3 1^2} \text{ at } X_5 = X_4) = \rho_{3^2 4^2} + 2\sum_{12}^{2}\sum_{12}^{2} \rho_{1344} + 2\rho_{12}\rho_{123^2 4^2}$$
$$+ 2\rho_{14}(\rho_{24} + 2\rho_{23}\rho_{34}) + 2\rho_{13}\rho_{2344} + 2\rho_{12}\sum_{34}^{2} \rho_{13}\rho_{2344},$$
$$\mu_{2^3 1^2} = \rho_{1^2 2^2 3^2 45} = (\mu_{2^2 1^4} \text{ at } X_6 = X_3)$$
$$= \rho_{3^2 45} + \sum_{12}^{2}(2\rho_{13}\rho_{1345} + \sum_{45}^{2} \rho_{14}\rho_{1335}) + 2\rho_{12}\rho_{123^2 45} + \sum_{45}^{2} \rho_{14}(\rho_{25} + 2\rho_{23}\rho_{35})$$
$$+ 2\rho_{13}\rho_{2345} + \rho_{12}(2\rho_{13}\rho_{2345} + \sum_{45}^{2} \rho_{14}\rho_{2335}),$$
$$\mu_{2^2 1^4} = \rho_{1^2 2^2 3\cdots 6} = (\mu_{21^6} \text{ at } X_7 = X_2) = \rho_{2^2 3\cdots 6} + 2\rho_{12}\rho_{1\cdots 6} + \sum_{3\cdots 6}^{4} \rho_{13}(\rho_{1456}$$
$$+ \sum_{1456}^{4} \rho_{12}\rho_{2456}) = \rho_{3\cdots 6} + \sum_{12}^{2}\sum_{3\cdots 6}^{4} \rho_{13}\rho_{1456} + 2\rho_{12}\rho_{1\cdots 6} + \sum_{3\cdots 6}^{12} \rho_{13}\rho_{24}\rho_{1256} + \rho_{12}\,\alpha_2$$
$$\text{where } \alpha_2 = \sum_{3\cdots 6}^{4} \rho_{13}\rho_{2456} = \sum_{3\cdots 6}^{12} \rho_{13}\rho_{24}\rho_{56},$$

$$\mu_{21^6} = \rho_{1^2 234567} = \rho_{234567} + \sum_{2\cdots 7}^{6} \rho_{12}\rho_{134567} \text{ by (2),} \tag{19}$$

$$\mu_{1^8} = \sum^{105} \rho_{12}\rho_{34}\rho_{56}\rho_{78} = \sum_{k=1}^{7} \rho_{k8}\rho^{k}_{12\cdots 7} = \rho_{18}\rho_{2\cdots 7} + \cdots + \rho_{78}\rho_{12\cdots 6}, \tag{20}$$

where $\rho^k_{12\cdots 7}$ is $\rho_{12\cdots 7}$ with k removed. These iterative expressions for μ_{51^3}, μ_{41^4}, μ_{31^5}, $\mu_{2^2 1^4}$, μ_{21^6} are new. However without software, it is easy to make an error with these substitutions. For μ_{3221}, see (20).

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NOTE 2.1. *This relation giving moments in terms of the covariance is a special case of the following. For any random variable $X \in R$ with rth cumulant κ_r, $r \geq 1$, its rth moment is given by*

$$m_r = B_r(\kappa) = \sum_{k=1}^{r} B_{rk}(\kappa),\ r \geq 1,\ \text{where } \kappa = (\kappa_1, \kappa_2, \cdots)$$

where $B_{rk}(\kappa)$ is the partial exponential Bell polynomial defined by (4) below. For any random vector $X \in R^p$ with rth cumulant κ_r, $r \in N^p$, this becomes

$$m_r = B_r(\kappa) = \sum_{k=1}^{|r|} B_{rk}(\kappa) \text{ where } |r| = \sum_{k=1}^{p} r_k$$

and $B_{rk}(\kappa)$ is the multivariate partial exponential Bell polynomial. It may be written down from the univariate form. See Withers and Nadarajah (2012, 2013, 2014).

3. Bivariate Moments of Soper and Isserlis

Soper (1915-16) and Isserlis (1918) gave the general bivariate normal moment with correlation $\rho = \rho_{12}$ when $V_{jj} \equiv 1$ in terms of ν_{2n} of (3), as

$$\mu_{rs} = E\ X_1^r X_2^s = \sum_{0 \leq k \leq s/2} m_{rsk} (1-\rho^2)^k \rho^{s-2k},\ m_{rsk} = \binom{s}{2k} \nu_{2k} \nu_{r+s-2k}. \tag{1}$$

Putting $s = 0, 1, 2, 3, 4, 5$ then expanding $(1-\rho^2)^k$ and simplifying, gives

$$\mu_{r0} = E\ X_1^r = \nu_r,\ \mu_{r1} = E\ X_1^r X_2 = \nu_{r+1}\rho, \tag{2}$$
$$\mu_{r2} = E\ X_1^r X_2^2 = \nu_r(1 + r\rho^2),\ \mu_{r3} = E\ X_1^r X_2^3 = \nu_{r+1}[3\rho + (r-1)\rho^3], \tag{3}$$
$$\mu_{r4} = E\ X_1^r X_2^4 = \nu_r[3 + 6r\rho^2 + r(r-2)\rho^4], \tag{4}$$
$$\mu_{r5} = E\ X_1^r X_2^5 = \nu_{r+1}[15\rho + 10(r-1)\rho^3 + (r-1)(r-3)\rho^5]. \tag{5}$$

Special cases are

$$\mu_{11} = \rho,\ \mu_{31} = 3\rho,\ \mu_{51} = 15\rho,\ \mu_{71} = 105\rho,\ \mu_{91} = 945\rho,\ \mu_{11,1} = 10395\rho.$$
$$\mu_{22} = 1 + 2\rho^2,\ \mu_{42}/3 = 1 + 4\rho^2,\ \mu_{62}/15 = 1 + 6\rho^2, \mu_{82}/105 = 1 + 8\rho^2,$$
$$\mu_{10,2}/945 = 1 + 10\rho^2,\ \mu_{13} = 3\rho,\ \mu_{33}/3 = 3\rho + 2\rho^3,\ \mu_{53}/15 = 3\rho + 4\rho^3,$$
$$\mu_{73}/105 = 3\rho + 6\rho^3,\ \mu_{93}/945 = 3\rho + 8\rho^3.$$
$$\mu_{04} = 3,\ \mu_{24} = 3 + 12\rho^2,\ \mu_{44}/3 = 3 + 24\rho^2 + 8\rho^4,$$
$$\mu_{64}/15 = 3(1 + 12\rho^2 + 8\rho^4),\ \mu_{84}/105 = 3(1 + 16\rho^2 + 16\rho^4).$$

By swapping r and s, this includes all bivariate moments up to order 10. (1) is not a symmetric rule. Setting $Y_1 = X_2$ and $Y_2 = X_1$ we see that (6) $\Rightarrow$ $\mu_{rs} = \mu_{sr}$ giving its *dual*

$$\mu_{rs} = \sum_{0 \leq k \leq r/2} m_{srk} (1-\rho^2)^k \rho^{r-2k}. \tag{6}$$

This has about $r/2$ terms while (1) has about $s/2$ terms if $s > r$. The same is true if we expand $(1-\rho^2)^k$ to write (1) as

$$\mu_{rs} = \sum_{-1 \leq n \leq s/2} M_{rsn} \rho^{s-2n} \text{ where} \tag{7}$$
$$M_{rsn} = \sum_{0 \leq i \leq s/2} [m_{rsk}(-1)^k \binom{k+1}{i+1}]_{k=n+i+1}.$$

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The method becomes increasingly cumbersome as s increases. In §4 we give a new and simpler formula for the bivariate moment μ_{rs}.

4. Successive Generalisation Using Quasi-Differential Operators

An alternative to deriving particular cases from the general moment, is to derive bivariate moments from univariate moments, then trivariate moments from bivariate moments, and so on, by the method of 3.29 of Stuart and Ord (1987), given there to obtain multivariate moments in terms of multivariate cumulants. But equally well it applies to relations among moments. To increase consistency with the notation there, we set

$$\mu(ij) = V_{ij},\ \mu(1^{n_1}\cdots p^{n_p}) = E\ X_1^{n_1}\cdots X_p^{n_p} = \mu_n \text{ of } (2). \tag{1}$$

We do not assume that $V_{jj} \equiv 1$. Define the operator $\partial(1.2)$ by

$$\partial(1.2)\ \mu(1^r 2^s 3^t \ldots) = r\mu(1^{r-1}2^{s+1}3^t \ldots).$$
$$\text{So } \partial(1.2)^k\ \mu(1^r 2^s \ldots) = [r]_k\ \mu(1^{r-k}2^{s+k}\ldots). \tag{2}$$

We show how to obtain $\mu_{n_1\cdots n_{p+1}}$ by applying $\partial(j.p+1)^m$ to $\mu_{n_1\cdots n_p}$ for any $j = 1, \ldots, p$. Applying $\partial(1.2)$ to

$$\mu(1^{2r}) = E\ X_1^{2r} = \nu_{2r}\mu(1^2)^r,$$

gives

$$2r\mu(1^{2r-1}2) = \nu_{2r}\ r\mu(1^2)^{r-1}\ 2\mu(12), \tag{3}$$

or dividing by $2r$,

$$\mu(1^{2r-1}2) = \nu_{2r}\ \mu(1^2)^{r-1}\mu(12),\ \text{that is, } E\ X_1^{2r-1}X_2 = \nu_{2r}V_{11}^{r-1}\ V_{12},$$

proving (2). Applying this operator a 2nd, 3rd and 4th time gives (3)–(5). We can apply the operator k times using Faa di Bruno's rule for differentiating $h(x) = f(g(x))$, [4c] p137 of Comtet (1974): for $k \geq 0$, its kth derivative is

$$h_{.k}(x) = \sum_{j=0}^{k} B_{kj}\ f_{.j}(g(x))$$

where $B_{kj} = B_{kj}(g)$ is the partial exponential Bell polynomial in $g = (g_1, g_2, \cdots)$ and $g_k = g_{.k}(x)$. Comtet tables them on p307–308. $B_{kj}(g)$ is defined by

$$\left(\sum_{k=1}^{\infty} g_k t^k/k!\right)^j/j! = \sum_{k=j}^{\infty} B_{kj}t^k/k!. \text{ So } B_{k0} = I(k=0). \tag{4}$$

We apply this with

$$d/dx = \partial(1.2),\ g(x) = \mu(1^2),\ f(g) = g^r,\ h(x) = \mu(1^2)^r.$$
$$\text{So } f_{.j}(g(x)) = [r]_j\mu(1^2)^{r-j},\ g_1 = 2\mu(12),\ g_2 = 2\mu(2^2),\ g_k = 0 \text{ for } k \geq 3,$$

and $B_{kj}/k!$ is the coefficient of t^k in $[2\mu(12)t + \mu(2^2)t^2]^j/j!$:

$$B_{kj} = [k]_j[2\mu(12)]^{2j-k}\mu(2^2)^{k-j}/(2j-k)!,\ k/2 \leq j \leq k.$$

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So by (2) we obtain for $k \geq 1$,

$$\mu(1^{2r-k}2^k)2^r r!/(2r-k)! = [2r]_k \mu(1^{2r-k}2^k)/\nu_{2r}$$
$$= \partial(1.2)^k \mu(1^{2r})/\nu_{2r} = \partial(1.2)^k \mu(1^2)^r = \sum_{j=1}^{k} B_{kj}[r]_j \mu(1^2)^{r-j} \tag{5}$$
$$= \sum_{k/2\leq j\leq min(k,r)} [r]_j[k]_j \mu(1^2)^{r-j}[2\mu(12)]^{2j-k}\mu(2^2)^{k-j}/(2j-k)! = H^{12}_{rk}, \tag{6}$$

say. Dividing by $r!k!$ gives

Theorem 1. *At $s = 2r-k$, $\mu_{sk} = \mu(1^s 2^k) = E\ X_1^s X_2^k$ is given by*

$$\mu_{sk}\ 2^r/s!k! = \sum_{k/2\leq j\leq min(k,r)} B_{r-j}C_{2j-k}D_{k-j} \text{ at } 2r = s+k \text{ where} \tag{7}$$
$$B_r = \mu(1^2)^r/r!,\ C_r = C^{12}_r = [2\mu(12)]^r/r!,\ D_r = \mu(2^2)^r/r!, \tag{8}$$
$$\mu(j^2) = V_{jj} = \sigma_j^2 \text{ say},\ \mu(12) = V_{12} = \rho_{12}\sigma_1\sigma_2.$$

This is our 1st new formula for the bivariate normal moments.

Corollary 1. *At $s = 2r-k$, if $V_{jj} \equiv 1$, then $\mu_{sk}/\nu_{2r} = H^{12}_{rk}/[2r]_k$*

$$\text{where } H^{12}_{rk} = \sum_{k/2\leq j\leq min(k,r)} [k]_j[r]_j\ (2\rho_{12})^{2j-k}/(2j-k)!. \tag{9}$$
$$\text{So, } \mu_{sk}\ 2^r/s! = \sum_{k/2\leq j\leq min(k,r)} [k]_j(2\rho_{12})^{2j-k}/(2j-k)!(r-j)!. \tag{10}$$

$$\text{So, } H^{12}_{r0} = 1,\ H^{12}_{r1} = 2r\rho_{12},\ H^{12}_{r2} = 2r + 4[r]_2\rho_{12}^2,$$
$$H^{12}_{r3} = 12[r]_2\rho_{12} + 8[r]_3\rho_{12}^3,\ H^{12}_{r4}/4 = 3[r]_2 + 12[r]_3\rho_{12}^2 + 4[r]_4\rho_{12}^4. \tag{11}$$

Swapping s and k in (7) and (10) give equivalent formulas. (10) is simpler than Soper and Isserlis's (1) as that is a polynomial in both ρ_{12} and $1-\rho_{12}^2$.
Putting $\rho_{12} = 1$ in Corollary 4.1 gives the new identity $[H^{12}_{rk}]_{\rho_{12}=1} =$

$$\sum_{k/2\leq j\leq min(k,r)} [k]_j[r]_j\ 2^{2j-k}/(2j-k)! = [2r]_k \text{ for } 0 \leq k \leq 2r. \tag{12}$$

5. The Multinomial Method

We now obtain (2)–(5) by another new method. Given

$$t \in R^p,\ t'X \sim \mathcal{N}_1(0, v_t) \text{ where } v_t = t'Vt, \text{ so } E\ (t'X)^{2r} = \nu_{2r}\ v_t^r. \tag{1}$$

By the multinomial theorem for $|n|$ and $n!$ of (1),

$$(t'X)^{2r} = \sum_{|n|=2r} \binom{2r}{n} t^n X^n \text{ where } \binom{2r}{n} = (2r)!/n!.$$
$$\text{So, } \nu_{2r}\ v_t^r = E\ (t'X)^{2r} = \sum_{|n|=2r} \binom{2r}{n} t^n \mu_n,\ \text{ and } \mu_n = E_n F_n \text{ where}$$
$$E_n = \nu_{2r}/\binom{2r}{n} = n!/2^r r!, \text{ and } F_n = \text{coeff}(t^n) \text{ in } v_t^r,\ |n| = 2r, \tag{2}$$

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F_n can be obtained from a 2nd application of the multinomial theorem. Suppose that $V_{jj} \equiv 1$. So

$$\rho_{jk} = E\ X_j X_k \text{ and } v_t = 2 \sum_{1\leq j<k\leq p} \rho_{jk} t_j t_k + \sum_{j=1}^{p} t_j^2. \tag{3}$$

For $c = (c_{jk})$ a symmetric $p \times p$ matrix of integers in N and $a \in N^p$, set

$$\rho^c = \Pi_{1\leq j<k\leq p}\, \rho_{jk}^{c_{jk}},\ c_j = \sum_{k\neq j} c_{jk}.$$

By (3), for $a, c \in N^p$, the coefficient of t^n in v_t^r is

$$F_n = \sum\{ \binom{r}{ac} 2^{|c|} \rho^c :\ 2a_j + c_j = n_k,\ k = 1, \cdots, p\},\ |n| = 2r \tag{4}$$

where $\binom{r}{ac} = r!/a!c!,\ a! = \Pi_{j=1}^{p} a_j!$ and $c! = \Pi_{1\leq j<k\leq p} c_{jk}!$.

The multinomial method for bivariate moments.

Set $\rho = \rho_{12},\ c = c_{12}$. So $v_t = 2\rho t_1 t_2 + t_1^2 + t_2^2$ and the coefficient of t^n in v_t^r needed for (2) is

$$\begin{aligned} &\mu_n/E_n = 2^r r! \mu_n/n! = F_n \\ &= \sum\{ \binom{r}{ca_1a_2} (2\rho)^c :\ c + 2a_1 = n_1,\ c + 2a_2 = n_2\} \text{ where} \\ &2r = |n|, \text{ and } \binom{r}{ca_1a_2} = r!/(c!a_1!a_2!). \end{aligned} \tag{5}$$

Taking $n = (2r - i, i)$ for $i \leq 5$ gives (2)–(5).

Example 1. *Take* $n = (2r-5, 5)$. *Then* $(c, a_2) = (1, 2),\ (3, 1)$ *or* $(5, 0)$. *So* F_n *sums over* $(c, a_1, a_2) = (1, r-3, 2),\ (3, r-4, 1),\ (5, r-5, 0)$ *giving*

$$F_n = \sum_{j=3}^{5} [r]_j \alpha_j \text{ where } \alpha_3 = \rho,\ \alpha_4 = 4\rho^3/3,\ \alpha_5 = 4\rho^5/15.$$

So since $E_n = (2r-5)!5!/2^r r!$, *by (5),* $r = 3, 4, 5, 6$ *give*

$$\begin{aligned} &\mu_{15} = 15\rho,\ \mu_{35} = 8(3\rho + 4\rho^3),\ \mu_{55} = 15(15\rho + 40\rho^3 + 8\rho^5), \\ &\mu_{75} = 105(5\rho + 20\rho^3 + 8\rho^5). \end{aligned}$$

Example 2. *Take* $n = (2r-6, 6)$. *Then* $(c, a_2) = (0, 3), (2, 2), (4, 1)$ *or* $(6, 0)$. *So* F_n *sums over* $(c, a_1, a_2) = (c, a_2) = (0, r-3, 3), (2, r-4, 2), (4, r-5, 1), (6, r-6, 0)$*:*

$$F_n = \sum_{j=3}^{6} [r]_j \alpha_j \text{ where } \alpha_3 = 1/6,\ \alpha_4 = \rho^2,\ \alpha_5 = \rho^4/3,\ \alpha_6 = 4\rho^6/45.$$

By (5), $E_n = (2r-6)!6!/2^r r!$, *so* $r = 3, 4, 5, 6, 7$ *give* $\mu_{06}, \mu_{26}, \mu_{46}$ *above, and*

$$\mu_{66} = 45(5 + 90\rho^2 + 120\rho^4 + 16\rho^6),\ \mu_{86} = 5.7.9(5 + 120\rho^2 + 240\rho^4 + 64\rho^6).$$

Moments with $p > 2$ can be dealt with similarly, but the operator method of §6 is easier.

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6. Trivariate Moments

Here we give μ_{rst} for arbitrary r, s, t. Define B_r, C_r, D_r as in (8). Assuming $V_{jj} \equiv 1$, Isserlis (1918) gave (12) and the following 2 trivariate moments.
q_I is his notation.

$$\mu_{2r,11} = E\ X_1^{2r} X_2 X_3 = q_{1^{2r}23} = \nu_{2r}(\rho_{23} + 2r\rho_{12}\rho_{13}), \tag{1}$$
$$\mu_{2r-1,21} = q_{1^{2r-1}2^2 3} = \nu_{2r}[\rho_{13} + 2\rho_{12}\rho_{23} + 2(r-1)\rho_{12}^2\rho_{13}] \tag{2}$$
$$= \rho_{13}a_{2r-1,2} + \rho_{23}b_{2r-1,2} \text{ where } a_{2r-1,2} = \nu_{2r}[1 + 2(r-1)\rho_{12}^2],\ b_{2r-1,2} = 2\nu_{2r}\rho_{12}.$$

So $\mu_{211}, \mu_{411}, \mu_{611}, \mu_{321}, \mu_{521}$ are given by (7), (5), (15), (10), (16).

Trivariate moments using the differential operator method.
We now extend the operator $\partial(1.2)$ of (2) to $\partial(i.j)$. So

$$\partial(1.3)\ \mu(1^r 2^s 3^t 4^u \cdots) = r\mu(1^{r-1} 2^s 3^{t+1} 4^u \cdots).$$

That is, $\partial(1.3)\ \mu_{rstu...} = r\mu_{r-1,s,t+1,u,...}$. By (2), for B_r, C_r, D_r of (8),

$$\partial(1.3)\ B_r = 2B_{r-1}\ \mu(13),\ \partial(1.3)\ C_r = 2C_{r-1}\ \mu(23). \tag{3}$$
$$\text{So, } \partial(1.3)\ \mu(1^2)^r = 2r\mu(1^2)^{r-1}\mu(13),$$
$$\partial(1.3)\ \mu(12)^r = r\mu(12)^{r-1}\mu(23),\ \partial(1.3)^e\ \mu(12)^r = [r]_e\mu(12)^{r-e}\mu(23)^e. \tag{4}$$

Applying $\partial(1.3)$ to (7) gives

Theorem 2. *For $2r = s + k + 1$, $\mu_{sk1} = \mu(1^s 2^k 3)$ is given by*

$$\mu_{sk1}\ 2^{r-1}/s!k! = \sum [B_{r-j-1}C_J\mu(13) + B_{r-j}C_{J-1}\mu(23)]_{J=2j-k}\ D_{k-j} \tag{5}$$

summed over $k/2 \le j \le min(k, r)$.

For example (5) with $k = 0$ and (11) with $k = 1$ reduce to (2) with r changed to $2r - 1$:

$$\mu_{2r-1,1} = \mu(1^{2r-1}2) = \nu_{2r}\mu(1^2)^{r-1}\mu(12).$$

Corollary 2. *When $V_{jj} \equiv 1$, for $2r = s + k + 1$,*

$$\mu_{sk1}\ 2^{r-1}/s! = \sum_j [k]_j 2^J [R\rho_{13}\rho_{12}^J + J\rho_{23}\rho_{12}^{J-1}/2]/J!R! \tag{6}$$

summed over $k/2 \le j \le min(k, r)$ at $J = 2j - k, R = r - j$.

For $k = 1, 2$ (6) gives Isserlis's (1), (2) but otherwise (6) is new.

Corollary 3.

$$\mu_{sk1} = \rho_{13}a_{sk} + \rho_{23}b_{sk} \text{ where} \tag{7}$$
$$a_{2r,3}/\nu_{2r} = 3(1 + 2r\rho^2), b_{2r,3}/\nu_{2r} = 2r[3\rho + 2(r-1)\rho^3]. \tag{8}$$
$$\text{So for } \mu_{211},\ a_{21} = 2\rho, b_{21} = 1;\ \text{ for } \mu_{321},\ a_{32}/3 = 1 + 2\rho^2, b_{32}/3 = 2\rho;$$
$$\text{for } \mu_{431},\ a_{43}/3 = 4(3\rho + 2\rho^3), b_{43}/3 = 1 + 4\rho^2; \tag{9}$$

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$$\begin{aligned}
&for\ \mu_{541},\ a_{54}/15 = 3 + 24\rho^2 + 8\rho^4, b_{54}/15 = 4(3\rho + 4\rho^3);\\
&for\ \mu_{631},\ a_{63}/15 = 3(1 + 6\rho^2),\ b_{63}/15 = 6(3\rho + 4\rho^3);\\
&for\ \mu_{651},\ a_{65} = 90(15\rho + 40\rho^3 + 8\rho^5),\ b_{65} = 225(1 + 12\rho^2 + 8\rho^4);\\
&for\ \mu_{741},\ a_{74}/105 = 3(1 + 12\rho^2 + 8\rho^4),\ b_{74}/105 = 12(\rho + 2\rho^3);\\
&for\ \mu_{831},\ a_{83}/\nu_8 = 3(1 + 8\rho^2),\ b_{83}/\nu_8 = 24(\rho + 2\rho^3).
\end{aligned}$$

By (2),

$$\begin{aligned}
&\partial(2.3)\ \mu(1^r 2^s) = s\mu(1^r 2^{s-1}3),\ \text{ that is, } \partial(2.3)\ \mu_{rs} = s\ \mu_{r,s-1,1},\\
&\text{so } \partial(2.3)\ C_r = 2C_{r-1}\mu(13),\ \partial(2.3) D_r = 2D_{r-1}\mu(23). \qquad (10)
\end{aligned}$$

Applying $\partial(2.3)$ to (7) gives an alternative to Theorem 6.1:

Theorem 3. *For $2r = s + k$, $\mu_{s,k-1,1} = \mu(1^s 2^{k-1}3)$ is given by*

$$\begin{aligned}
&\mu_{s,k-1,1}2^{r-1}/s!(k-1)!\\
&= \sum_{k/2 \le j \le min(k,r)} B_{r-j}[C_{J-1}D_K\mu(13) + C_J D_{K-1}\mu(23)] \qquad (11)\\
&at\ J = 2j - k, K = k - j. \qquad (12)
\end{aligned}$$

Corollary 4. *When $V_{jj} \equiv 1$, for $2r = s + k$, and J, K of (12),*

$$\begin{aligned}
&\mu_{s,k-1,1}\ 2^{r-1}/s!(k-1)!\\
&= \sum_{k/2 \le j \le min(k,r)} [J\rho_{13}(2\rho_{12})^{J-1} + K\rho_{23}(2\rho_{12})^J]/J!K!(r-j)!.
\end{aligned}$$

By (10), applying $\partial(2.3)$ to (11) gives

Theorem 4. *For $2r = s + k$, and J, K of (12),*

$$\begin{aligned}
&\mu_{s,k-2,2}\ 2^{r-1}/s!(k-2)! = \sum_{k/2 \le j \le min(k,r)} B_{r-j}\ [2C_{J-2}D_K\mu(13)^2\\
&+ C_{J-1}D_{K-1}4\mu(13)\mu(23) + C_J D_{K-2}2\mu(23)^2 + C_J D_{K-1}\mu(33)]. \qquad (13)
\end{aligned}$$

Corollary 5. *When $V_{jj} \equiv 1$, for $2r = s + k$ and J, K of (12),*

$$\begin{aligned}
&\mu_{s,k-2,2}\ 2^{r-1}/s!(k-2)! = \sum_{k/2 \le j \le k} [2(2\rho_{12})^{J-2}\rho_{13}^2/(J-2)!K!\\
&+ 4\rho_{13}\rho_{23}(2\rho_{12})^{J-1}/(J-1)!(K-1)!\\
&+ \{1/(K-1)! + 2\rho_{23}^2/(K-2)!\}\ (2\rho_{12})^J/J!]/(r-j)!. \qquad (14)
\end{aligned}$$

Here are some examples. $k = 2$ gives the 1st equation in (3) with r changed to $2r - 2$ and X_2 replaced by X_3. (13) with $k = 3$ gives (2) with λ changed to $2r - 3$ and X_2 replaced by X_3. (14) with $k = 4$ gives for $2r = s + 4$,

$$\begin{aligned}
&\mu_{2r-4,22} = \nu_{2r-4}A_r \text{ where } A_r = a + (2r-4)b + 4(2r-4)(2r-6)\ c, \qquad (15)\\
&a = 1 + 2\rho_{23}^2,\ b = \rho_{12}^2 + \rho_{13}^2 + 4\rho_{12}\rho_{13}\rho_{23},\ c = \rho_{12}^2\rho_{13}^2.\\
&\text{So, } \mu_{222} = a + 2b \text{ as in (12)},\ \mu_{422} = 3(a + 4b + 8c),\ \mu_{622} = 15(a + 6b + 24c),\\
&\mu_{822} = 105(a + 8b + 48c), \mu_{10,22} = 945(a + 10b + 80c),\ \mu_{12,22} = \nu_{12}(a + 12b + 120c).
\end{aligned}$$

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(14) with $k=5$ gives for $2r=s+5$,

$$\mu_{s32}\,2^{r-1}/s!3! = \sum_{j=3}^{5} a_j/(r-j)! \text{ where } a_3 = 4\rho_{13}\rho_{23} + 2(1+2\rho_{23}^2)\rho_{12}, \tag{16}$$

$$a_4 = 4\rho_{12}\rho_{13}^2 + 8\rho_{13}\rho_{23}\rho_{12}^2,\ a_5 = 8\rho_{12}^2\rho_{13}^2.$$

$$\text{So, } 2\mu_{332}/9 = a_3 + a_4,\ \mu_{532}/45 = a_3/2 + a_4 + a_5.$$

(14) with $k=6$ gives $2r=s+6$,

$$\mu_{2r-6,42}\,2^{r-1}/(2r-6)!4! = \sum_{j=3}^{6} a_j/(r-j)! \text{ where} \tag{17}$$

$$a_3 = 1/2 + 2\rho_{23}^2,\ a_4 = \rho_{13}^2 + 8\rho_{12}\rho_{13}\rho_{23} + 2\rho_{12}^2 + 2\rho_{23}^2,$$

$$a_5 = 4\rho_{12}^2\rho_{13}^2 + 16\rho_{12}^3\rho_{13}\rho_{23}/3 + 1 + 2\rho_{12}^4/3,\ a_6 = 4\rho_{12}^4\rho_{13}^2/3.$$

$$\text{So, } \mu_{442}/36 = a_3/2 + a_4 + a_5,\ \mu_{642}/540 = a_3/6 + a_4/2 + a_5 + a_6.$$

(14) with $k=6$ gives $2r=s+7$,

$$\mu_{2r-7,52}\,2^{r-1}/(2r-7)!5! = \sum_{j=4}^{7} a_j/(r-j)! \text{ where} \tag{18}$$

$$a_4 = 2\rho_{13}\rho_{23} + (1+4\rho_{23}^2)\rho_{12},$$

$$a_5 = 2\rho_{12}\rho_{13}^2 + 8\rho_{12}^2\rho_{13}\rho_{23} + 4\rho_{12}^3(1+2\rho_{23}^2)/3,$$

$$a_6 = 8\rho_{12}^3\rho_{13}(\rho_{13}+2\rho_{23})/3 + 4\rho_{12}^5/15,\ a_7 = 8\rho_{12}^5\rho_{13}^2/15.$$

$$\text{So, } 8\mu_{552}/15 = \sum_{j=4}^{6} a_j/(6-j)!,\ 8\mu_{752}/315 = \sum_{j=4}^{7} a_j/(7-j)!.$$

Now apply $\partial(1.3)^t$ to (7) with $u=s-t$ using Faa di Bruno's rule of (5) with 2 changed to 3, and Leibniz' rule for the tth derivative of a product. So,

Theorem 5. *For B_r, C_r, D_r of (8) and $2r = u+k+t$,*

$$\mu_{ukt}\,2^r/u!k! = \sum_{k/2\le j\le min(k,r)} D_{k-j}\partial(1.3)^t[B_{r-j}C_{2j-k}], \tag{19}$$

$$\textit{where } \partial(1.3)^t[B_hC_J] = \sum_{d+e=t}\binom{t}{d} B_{h.3d}C_{J.3e} \tag{20}$$

$$B_{h.3d} = \partial(1.3)^d B_h = H_{hd}^{13}/h!,\ C_{J.3e} = \partial(1.3)^e C_J = C_{J-e}[2\mu(23)]^e, \tag{21}$$

and H_{hd}^{13} is H_{rk}^{12} of (6) with $2, r, k$ replaced by $3, h, d$. That is,

$$\mu_{ukt}\,2^r/u! = \sum_{k/2\le j\le min(k,r)} [k]_j\mu(2^2)^{k-j} I_{r-j,2j-k,t}^{123}\,2^{2j-k}/(r-j)!(2j-k)!,$$

$$I_{hJt}^{123} = \partial(1.3)^t\,\mu(1^2)^h\mu(12)^J = \sum_{d+e=t}\binom{t}{d}[J]_e\,H_{hd}^{13}\,\mu(12)^{J-e}\mu(23)^e. \tag{22}$$

(21) follows from (4). (22) follows from

$$\partial(1.3)^e\mu(12)^J = [J]_e\mu(12)^{J-e}.$$

This is our 1st formula for the general trivariate moment. Example 7.1 will need

$$I^{123}_{h0t} = H^{13}_{ht},\ I^{123}_{h1t} = H^{13}_{ht}\mu(12) + tH^{13}_{h,t-1}\mu(23),$$
$$I^{123}_{h2t} = H^{13}_{ht}\mu(12)^2 + 2tH^{13}_{h,t-1}\mu(12)\mu(23) + [t]_2 H^{13}_{h,t-2}\mu(23)^2. \tag{23}$$
$$\text{Set } L_{hJt} = h!\ \text{RHS}(20) = \sum_{d+e=t} \binom{t}{d} H^{13}_{hd}(2\rho_{12})^g(2\rho_{23})^e/g! \text{ at } g = J-e.$$

Corollary 6. *When $V_{jj} \equiv 1$, for $2r = u+k+t$ and H^{12}_{rk} of (9),*

$$\mu_{ukt}\ 2^r/u! = \sum_{k/2\le j\le min(k,r)} [k]_j\ L_{r-j,2j-k,t}/(r-j)! \tag{24}$$
$$\text{where } L_{hJt} = 2^J \sum_{e=0}^{J} [t]_e \rho^g_{12}\rho^e_{23} H^{13}_{h,t-e}/g!e! \text{ at } g = J-e. \tag{25}$$

$$\text{So } L_{h0t} = H^{13}_{ht},\ L_{h1t} = 2\rho_{12}H^{13}_{ht} + 2t\rho_{23}H^{13}_{h,t-1},$$
$$L_{h2t} = 2\rho^2_{12}H^{13}_{ht} + 4t\rho_{12}\rho_{23}H^{13}_{h,t-1} + 2[t]_2\rho^2_{23}H^{13}_{h,t-2},$$
$$L_{h3t} = 4\rho^3_{12}H^{13}_{ht}/3 + 4t\rho^2_{12}\rho_{23}H^{13}_{h,t-1} + 4[t]_2\rho_{12}\rho^2_{23}H^{13}_{h,t-2} + 4[t]_3\rho^3_{23}H^{13}_{h,t-3}/3,$$
$$L_{hJ1} = 2^J[2h\rho^J_{12}\rho_{13}/J! + \rho^{J-1}_{12}\rho_{23}/(J-1)!.$$

Putting $\tilde{L}_{hJt} = L_{hJt}$ when $\rho_{ij} \equiv 1$, gives the new identity

$$[2r]_{k+t} = \sum_{k/2\le j\le min(k,r)} [k]_j[r]_j\ \tilde{L}_{r-j,2j-k,t}/(r-j)!$$
$$\text{where } \tilde{L}_{hJt} = 2^J \sum_{e=0}^{min(J,t)} [t]_e[2h]_{t-e}/(J-e)!e!.$$

Some examples: $k = 3$ gives for $2r-3 = u+t$,

$$\mu_{u3t}\ (r-2)!2^r/u!3! = L_{r-2,1t} + (r-2)L_{r-3,3t}.$$
$$t = 3, u = 2r-6, h = r-2 \Rightarrow \mu_{u33}\ (r-2)!2^r/u!3! = L_{h13} + hL_{h-1,33}.$$

Set $a = \rho_{12}\rho_{13}$, and $b = \rho_{12}^2 + \rho_{13}^2$.

For $h = r-2,\ \mu_{2r-6,33}\ h!2^r/(2r-6)!3! = \sum_{j=0}^{2} c_j\ \rho_{23}^j$ where (26)

$c_0/16 = 3[h]_2\ a/2 + [h]_3\ ab + 2[h]_4\ a^3/3,$

$c_1/12 = h + 2[h]_2\ b + 4[h]_3\ a^2,\ c_2 = 48[h]_2\ a,\ c_3 = 8h.$

So, $r = 5, u = 4, h = 3$ gives $4\mu_{433}/3 = \sum_{j=0}^{2} c_j\rho_{23}^j$ where

$c_0 = 48(3a + 2ab),\ c_1 = 36(1 + 4b + 8a^2),\ c_2 = 288a,\ c_3 = 24.$

$r = u = 6, h = 4$ gives $16\mu_{633}/45 = \sum_{j=0}^{2} c_j\rho_{23}^j$ where

$c_0/32 = 9a + 12ab + 8a^3,\ c_1/48 = 1 + 6b + 24a^2,\ c_2 = 576a,\ c_3 = 32.$

$r = 7, u = 8, h = 5$ gives $4\mu_{833}/63 = \sum_{j=0}^{2} c_j\rho_{23}^j$ where

$c_0 = 160(3a + 6ab + 8a^3),\ c_1 = 60(1 + 8b + 48a^2),\ c_2 = 960a,\ c_3 = 40.$

By Corollary 6.5 with $k = 4$,

$$\text{for } 2r-4 = u+t,\ \mu_{u4t}2^r/u!4! = \sum_{j=2}^{4} L_{r-j,2j-4,t}/(r-j)!(4-j)!. \tag{27}$$

$$\text{So, } \mu_{242}/3 = 1 + 2\rho_{13}^2 + 4\rho_{12}^2 + 4\rho_{23}^2 + 16\rho_{12}\rho_{13}\rho_{23} + 8\rho_{12}^2\rho_{23}^2, \tag{28}$$

$$\mu_{341}/9 = \rho_{13} + 4\rho_{12}(\rho_{23} + \rho_{12}\rho_{13}) + 8\rho_{12}^3\rho_{23}/3, \tag{29}$$

$$L_{143} = 16(\rho_{12}\rho_{23}^3 + \rho_{12}^3\rho_{23} + 3\rho_{12}^3\rho_{23}^2\rho_{13},$$

$$\mu_{343}/9 = 3\rho_{13} + 2\rho_{13}^3 + 12[\rho_{12}^2\rho_{13} + \rho_{12}\rho_{23}(1 + 2\rho_{13}^2) + \rho_{13}\rho_{23}^2]$$
$$+ 8[\rho_{12}\rho_{23}^3 + \rho_{12}^3\rho_{23} + 3\rho_{12}^2\rho_{23}^2\rho_{13}],$$
$$\mu_{444}/9 = 3 + 24b + 72a + 8c + 96d + 128ab + 192a^2 \text{ where}$$
$$a = \Pi_{123}^3\ \rho_{12},\ b = \sum_{123}^{3}\rho_{12}^2,\ c = \sum_{123}^{3}\rho_{12}^4,\ d = \sum_{123}^{3}\rho_{12}^2\rho_{13}^2 = (b^2 - c)/2.$$

An alternative to Theorem 6.4 is given by applying $\partial(2.3)^t$ to (7) with $u = k - t$.

Trivariate moments by the multinomial method.

We now apply (4) with $p = 3$. Set $c = (c_{12}, c_{13}, c_{23})$. Then

$$\rho^c = \rho_{12}^{c_{12}}\rho_{13}^{c_{13}}\rho_{23}^{c_{23}},\ c_1 = c_{12} + c_{13},\ c_2 = c_{12} + c_{23},\ c_3 = c_{13} + c_{23}.$$

Example 3. *Take $n = (2r-2, 1, 1)$. Then $a_2 = a_3 = 0$ and either $c = (0,0,1)$ or $(1,1,0)$, so that C_n sums over either $a = (r-1, 0, 0)$, $c = (0,0,1)$ or $a = (r-2, 0, 0)$, $c = (1,1,0)$ giving*

$$C_n = 2r\rho_{23} + 4[r]_2\rho_{12}\rho_{13},\ B_n = \nu_{2r}/[2r]_2 = \nu_{2r-2}/(2r),$$
$$\mu_n/\nu_{2r-2} = \rho_{23} + 2(r-1)\alpha \text{ where } \alpha = \rho_{12}\rho_{13}:$$
$$\mu_{211} = \rho_{23} + 2\alpha,\ \mu_{411}/3 = \rho_{23} + 4\alpha,\ \mu_{611}/15 = \rho_{23} + 6\alpha,$$
$$\mu_{811}/105 = \rho_{23} + 8\alpha,\ \mu_{10,11}/945 = \rho_{23} + 10\alpha.$$

This was given by Isserlis with $\lambda = 2r - 2$.

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Example 4. *Take* $n = (2r-3, 2, 1)$.
Then C_n *sums over either* $c = (0,1,0), a = (r-2,1,0)$ *or*
$c = (1,0,1), a = (r-2,0,0)$ *or* $c = (2,1,0), a = (r-3,0,0)$ *giving*

$$C_n/2[r]_2 = \alpha_0 + (r-2)\alpha_1 \text{ where } \alpha_0 = \rho_{13} + 2\rho_{12}\rho_{23},\ \alpha_1 = 2\rho_{12}^2\rho_{23},$$
$$B_n = \nu_{2r}/D \text{ where } D = \binom{2r}{2r-3,2,1} = [2r]_3/2 = 2(2r-1)[r]_2 \Rightarrow$$
$$B_n = \nu_{2r-2}/2[r]_2,\ \mu_n/\nu_{2r-2} = \alpha_0 + (r-2)\alpha_1 :$$
$$\mu_{121} = \alpha_0 \text{ as in (7)},\ \mu_{321}/3 = \alpha_0 + \alpha_1 \text{ as in (10)},$$
$$\mu_{521} = \alpha_0 + 2\alpha_1 \text{ as in (16)},\ \mu_{721} = \alpha_0 + 3\alpha_1,\ \mu_{921} = \alpha_0 + 4\alpha_1.$$

This agrees with (1) with $\lambda = 2r-3$.

Example 5. *Take* $n = (2r-4, 3, 1)$. $2a_3 + c_{13} + c_{23} = 1$ *so* $a_3 = 0$ *and* $(c_{13}, c_{23}) = (01)$ *(Case 1) or* (10) *(Case 2).*
Case 1: $2a_2 + c_{12} + c_{23} = 3$ *so* $a_2 = 0$ *and* $(c_{12}, c_{23}) = (21)$ *(I) say, or* (30)*, (II) say. If (I) then* $\binom{r}{ac} = \binom{r}{r-3,00201} = [r]_3/2,\ (2\rho)^c = 2^3\rho_{12}^2\rho_{23}$.
If (II) then $\binom{r}{ac} = \binom{r}{r-2,10001} = [r]_2,\ (2\rho)^c = 2\rho_{23}$.
Case 2: $2a_2 + c_{12} + c_{23} = 3$ *so* $a_2 = 0$ *and* $(c_{12}, c_{23}) = (30)$ *(I) say, or* (10)*, (II) say. If (I) then* $\binom{r}{ac} = \binom{r}{r-4,00310} = [r]_4/6,\ (2\rho)^c = 2^4\rho_{12}^3\rho_{13}$.
If (II) then $\binom{r}{ac} = \binom{r}{r-3,10110} = [r]_3,\ (2\rho)^c = 4\rho_{12}\rho_{13}$.
Also $B_n = \nu_{2r}/\binom{2r}{2r-4,31} = 3\nu_{2r-4}/2[r]_2$*. So finally,*

$$\mu_n/\nu_{2r-4} = \sum_{j=0}^{2}[r-2]_j\alpha_j \text{ for } \alpha_0 = 3\rho_{23},\ \alpha_1 = 6\rho_{12}(\rho_{13} + \rho_{12}\rho_{23}),\ \alpha_2 = 4\rho_{12}^3\rho_{13}.$$

For example $r = 2, \cdots, 6$ *give (2), (10), (9),*

$$\mu_{631}/15 = \alpha_0 + 3\alpha_1 + 6\alpha_2,\ \mu_{831}/105 = \alpha_0 + 4\alpha_1 + 12\alpha_2.$$

Example 6. *Take* $n = (2r-4, 2, 2)$.
Then C_n *sums over* $a = (r-2,0,0),\ c = (0,0,2),\ a = (r-3,0,0),\ c = (1,1,1),\ a = (r-4,0,0),\ c = (2,2,0),\ a = (r-3,0,1),\ c = (2,0,0)$,
$a = (r-3,1,0),\ c = (0,2,0),\ a = (r-2,1,1),\ c = (0,0,0)$*, giving*

$$C_n/[r]_2 = \sum_{j=0}^{2}[r-2]_j\alpha_j \text{ where}$$
$$\alpha_0 = 1 + 2\rho_{23}^2,\ \alpha_1 = 2\sum_{j=1}^{2}\rho_{1j}^2 + 8\rho_{12}\rho_{13}\rho_{23},\ \alpha_2 = 4\rho_{12}^2\rho_{13},$$
$$B_n = \nu_{2r-4}/[r]_2 \Rightarrow \mu_n/\nu_{2r-4} = \sum_{j=0}^{2}[r-2]_j\alpha_j.$$
$$\text{So, } \mu_{022} = \alpha_0,\ \mu_{222} = \alpha_0 + \alpha_1,\ \mu_{422}/3 = \alpha_0 + 2\alpha_1 + 4\alpha_2,$$
$$\mu_{622}/15 = \alpha_0 + 3\alpha_1 + 6\alpha_2,\ \mu_{822}/105 = \alpha_0 + 4\alpha_1 + 12\alpha_2.$$

Moments with $p > 3$ can be obtained by either of the last 2 methods.

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7. Moments of Dimension 4

We use H^{12}_{rk} of (6) and (9). Recall that by (21), for $C_J = [2\mu(12)]^J/J!$, $\partial(1.3)^e C_J = C_{J-e}[2\mu(23)]^e$. That is,

$$\partial(1.3)^e \mu(12)^J = [J]_e \mu(12)^{J-e}\mu(23)^e.$$

By (2), applying $\partial(1.4)^v$ to (19) and setting $w = u - v$ gives

Theorem 6. *For $B_r, C^{12}_r = C_r, D_r$ of (8) and I^{123}_{hJt} of (22),*

$$\mu_{wktv}\, 2^r/w!k! = \sum_{k/2\le j\le min(k,r)} D_{k-j}E_{r-j,2j-k,tv}/(r-j)! \text{ at } 2r = w+k+t+v,$$

$$E_{hJtv} = h!\partial(1.4)^v RHS(20) = \sum_{0\le e_1\le min(J,t)} \binom{t}{e_1} F_{hJ,t-e_1,e_1 v}, \quad (1)$$

$$F_{hJd_1e_1v} = h!\partial(1.4)^v B_{h.3d_1}C_{J.3e_1} = \sum_{0\le e_2\le min(J-e_1,v)} \binom{v}{e_2} S_{hd_1,v-e_2}T_{Je_1e_2}, \quad (2)$$

$$S_{hd_1d_2} = h!\,\partial(1.4)^{d_2}B_{h.3d_1} = \partial(1.4)^{d_2}H^{13}_{hd_1}$$
$$= \sum_{d_1/2\le j\le min(d_1,h)} [h]_j[d_1]_j\, I^{134}_{h-j,2j-d_1,d_2}\mu(3^2)^{d_1-j}2^{2j-d_1}/(2j-d_1)!,$$

$$T_{Je_1e_2} = \partial(1.4)^{e_2}C_{J.3e_1} = T'\ [2\mu(23)]^{e_1}, T' = \partial(1.4)^{e_2}C_{J-e_1} = [2\mu(24)]^{e_2}C_{J-e_1-e_2}.$$

Corollary 7. *For $2r = w+k+t+v$, $V_{jj}\equiv 1$, E_{hJtv} of (1) and (2), and I^{123}_{hJt} of (22),*

$$\mu_{wktv}\, 2^r/w! = \sum_{k/2\le j\le min(k,r)} [k]_j\, E_{r-j,2j-k,tv}/(r-j)! \text{ where}$$

$$S_{hd_1d_2} = \sum_{d_1/2\le j\le min(d_1,h)} [h]_j[d_1]_j\, I^{134}_{h-j,2j-d_1,d_2}2^{2j-d_1}/(2j-d_1)!, \quad (3)$$

$$T_{Je_1e_2} = (2\rho_{23})^{e_1}(2\rho_{24})^{e_2}C_{J-e_1-e_2} = 2^J\rho_{23}^{e_1}\rho_{24}^{e_2}\rho_{12}^{J-e_1-e_2}/(J-e_1-e_2)!. \quad (4)$$

For Examples 7.1 and 7.2, and H^{12}_{rk} of (6), we need

$$S_{h0d_2} = I^{134}_{h0d_2} = H^{14}_{hd_2},\ S_{h1d_2}/2h = I^{134}_{h-1,1d_2} = H^{14}_{h-1,d_2}\rho_{13} + d_2H^{14}_{h-1,d_2-1}\rho_{34},$$
$$S_{h2d_2}/2h = H^{14}_{h-1,d_2} + 2(h-1)I^{134}_{h-2,2,d_2}. \quad (5)$$

So $S_{h00} = 1,\ S_{h01} = 2h\rho_{14},\ S_{h02} = H^{14}_{h2} = 2h + 4[h]_2\rho^2_{14},\ S_{h10} = 2h\rho_{13},$
$S_{h11} = 2h\rho_{34} + 4[h]_2\rho_{13}\rho_{14},\ S_{h12} = 4[h]_2\,(\rho_{13} + 2\rho_{14}\rho_{34}) + 8[h]_3\rho_{13}\rho^2_{14}.$

$$S_{h20} = 2h + 4[h]_2\rho^2_{13},\ S_{h21} = 4[h]_2\rho^{(2)}_{14.3} + 8[h]_3\rho^2_{13}\rho_{14}, \quad (6)$$

$$S_{h22}/4 = [h]_2(1+2\rho^2_{34}) + 2[h]_3(\rho^2_{13} + \rho^2_{14} + 4\rho_{13}\rho_{14}\rho_{34}) + 4[h]_4\rho^2_{13}\rho^2_{14}, \quad (7)$$

$$S_{h23}/8 = 3[h]_3(\rho_{14} + 2\rho_{13}\rho_{34} + 2\rho_{14}\rho^2_{34}) + 2[h]_4(\rho^3_{14} + 3\rho^2_{13}\rho_{14} + 6\rho_{13}\rho^2_{14}\rho_{34})$$
$$+ 4[h]_5\rho^2_{13}\rho^3_{14},$$
$$T_{Je_10} = (2\rho_{23})^{e_1}C_{J-e_1},\ T_{Je_11} = (2\rho_{23})^{e_1}(2\rho_{24})C_{J-e_1-1},$$
$$T_{J00} = C_J,\ T_{J01} = 2\rho_{24}C_{J-1},\ T_{J10} = 2\rho_{23}C_{J-1},\ T_{J11} = 4\rho_{23}\rho_{24}C_{J-2},$$
$$T_{J20} = 2^J\rho^2_{23}\rho^{J-2}_{12}/(J-2)!,\ T_{J21} = 2^J\rho^2_{23}\rho_{24}\rho^{J-3}_{12}/(J-3)!.$$
$$I^{134}_{h23} = H^{14}_{h3}\rho^2_{13} + 6H^{14}_{h2}\rho_{13}\rho_{34} + 6H^{14}_{h1}\rho^2_{34}.$$

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Example 7. *Take $v = 1, 2r - 1 = w + k + t$, $S_{hd_1d_2}$ of (3), and $T_{Je_1e_2}$ of (4). If $V_{jj} \equiv 1$, then*

$$\mu_{wkt1}\, 2^r / w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,t1} / (r-j)! \text{ where} \tag{8}$$

$$E_{hJt1} = \sum_{0 \le e_1 \le min(J,t)} \binom{t}{e_1} F_{hJ,t-e_1,e_11},\ F_{hJd_1e_11} = S_{hd_10} T_{Je_11} + S_{hd_11} T_{Je_10}.$$

Now take $t = 1, 2r - 2 = w + k$. Then by (11) and (5),

$$\mu_{wk11}\, 2^r / w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,11} / (r-j)! \text{ where} \tag{9}$$

$$E_{hJ11} = F_{hJ011} + F_{hJ101},$$

$$F_{hJ011} = S_{h00} T_{J11} + S_{h01} T_{J10},\ F_{hJ101} = S_{h10} T_{J01} + S_{h11} T_{J00},$$

$$\text{So } E_{hJ11} = 4\rho_{23}\rho_{24} C_{J-2} + 4hc\, C_{J-1} + (2h\rho_{34} + 4[h]_2 \rho_{13}\rho_{14}) C_J$$

$$\text{where } c = \rho_{13}\rho_{24} + \rho_{14}\rho_{23}. \tag{10}$$

$$\text{So for } k = 1,\ w = 2r - 3,\ \mu_{w111}\, 2^r / w! = E_{r-1,111} / (r-1)! \text{ where} \tag{11}$$

$$E_{h111} = 4hb + 8[h]_2 a,\ a = \rho_{12}\rho_{13}\rho_{14},\ b = \rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23} : \tag{12}$$

$$\mu_{1111} = b \Rightarrow (8),\ \mu_{3111} = 3(b + 2a) \Rightarrow (10),\ \mu_{5111} = 15(b + 4a), \tag{13}$$

$$\mu_{7111} = 105(b + 6a),\ \mu_{2r+1,111} = \nu_{2r+2}\,(b + 2ra).$$

$$\text{For } k = 2,\ w = 2r - 4,\ \mu_{w211}\, 2^{r-1} / w! = \sum_{j=1}^{2} E_{r-j,2j-2,11} / (r-j)!, \tag{14}$$

$$\text{where } E_{h011} = 2h\rho_{34} + 4[h]_2 \rho_{13}\rho_{24},$$

$$E_{h211} = 4\rho_{23}\rho_{24} + 8h\rho_{12}c + 4\rho_{12}^2 (h\rho_{34} + 2[h]_2 \rho_{13}\rho_{14}) :$$

$$\mu_{0211} = \rho_{34.2}^{(2)} \Rightarrow (7),$$

$$\mu_{2211} = \rho_{34} + 2\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 4\rho_{12}c + 2\rho_{12}^2\rho_{34} \Rightarrow (13),$$

$$\mu_{4211}/3 = \rho_{34} + 4\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 8\rho_{12}c + 4\rho_{12}^2 \rho_{34.1}^{(2)}, \tag{15}$$

$$\mu_{6211}/15 = \rho_{34} + 6\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 12\rho_{12}c + 6\rho_{12}^2 \rho_{34.1}^{(4)},$$

$$\text{where } \rho_{ij.k}^{(r)} = \rho_{ij} + r\rho_{ik}\rho_{jk}.$$

$$\text{For } k = 3, w = 2r - 5, \mu_{w311}\, 2^{r-1} / 3w! = \sum_{j=2}^{3} E_{r-j,2j-3,11} / (r-j)! \tag{16}$$

$$\text{where } E_{h311} = 8\rho_{12}\rho_{23}\rho_{24} + 8h\rho_{12}^2 c + 8\rho_{12}^3 (h\rho_{34} + 2[h]_2 \rho_{13}\rho_{14})/3 :$$

$$r = 3 \Rightarrow \mu_{1311} = 3b + 6\rho_{21}\rho_{23}\rho_{24} \Rightarrow (11),$$

$$r = 4 \Rightarrow \mu_{3311}/3 = 3b + 6\rho_{12}(\rho_{13}\rho_{14} + \rho_{23}\rho_{24}) + 6\rho_{12}^2 c + 2\rho_{12}^3 \rho_{34}, \tag{17}$$

$$r = 5 \Rightarrow \mu_{5311}/15 = 3b + 12a + 6\rho_{12}\rho_{23}\rho_{24} + 12\rho_{12}^2 c + 4\rho_{12}^3 \rho_{34.1}^{(2)},$$

$$r = 6 \Rightarrow \mu_{7311}/315 = b + 6a + 2\rho_{21}\rho_{23}\rho_{24} + 6\rho_{12}^2 c + 2\rho_{12}^3 \rho_{34.1}^{(4)}.$$

$$r = 7 \Rightarrow \mu_{9311}/945 = 3(b + 6a) + 6\rho_{12}\rho_{23}\rho_{24} + 12\rho_{12}^2 c + 8\rho_{12}^3 \rho_{34.1}^{(6)}.$$

$$r = 8 \Rightarrow 4\mu_{11,311}/10395 = 12(b + 6a) + 24\rho_{12}\rho_{23}\rho_{24} + 120\rho_{12}^2 c + 120\rho_{12}^3 \rho_{34.1}^{(8)}.$$

Now take $t = 2, 2r - 3 = w + k$. Then by (11) and (5),

$$\mu_{wk21}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,21}/(r-j)! \text{ where} \tag{18}$$

$$E_{hJ21} = F_{hJ201} + 2F_{hJ111} + F_{hJ021},\ F_{hJ201} = S_{h20}T_{J01} + S_{h21}T_{J00},$$
$$F_{hJ111} = S_{h10}T_{J11} + S_{h11}T_{J10},\ F_{hJ021} = S_{h00}T_{J21} + S_{h01}T_{J20}.$$
$$k = 2,\ w = 2r - 5 \Rightarrow \mu_{w221}\, 2^{r-1}/w! = E_{r-1,021}/(r-1)! + E_{r-2,221}/(r-2)!$$
$$\text{where } E_{h021} = S_{h21} \text{ of (6)},\ E_{h221} = F_{h2201} + 2F_{h2111} + F_{h2021},$$
$$F_{h2201} = S_{h20}T_{201} + S_{h21}T_{200},\ F_{h2111} = S_{h10}T_{211} + S_{h11}T_{210},\ F_{h2021} = S_{h01}T_{220}.$$
$$r = 3 \Rightarrow \mu_{1221} = \rho_{14}(1 + 2\rho_{23}^2) + 2\rho_{12}\rho_{24} + 2\rho_{13}\rho_{14}$$
$$+ 4\rho_{23}(\rho_{12}\rho_{34} + 2\rho_{13}\rho_{24}) \Rightarrow (13). \tag{19}$$
$$r = 4 \Rightarrow \mu_{3221} = \sum_{i=1}^{4} a_i \text{ where } a_1 = 3\rho_{14},\ a_2 = 6\rho_{12}\rho_{24} + 6\rho_{13}\rho_{34}, \tag{20}$$
$$a_3 = 6\rho_{14}(\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2) + 12\rho_{23}(\rho_{13}\rho_{24} + \rho_{12}\rho_{34}),$$
$$a_4 = 12\rho_{12}^2\rho_{13}\rho_{34} + 12\rho_{12}\rho_{13}^2\rho_{24} + 24\rho_{12}\rho_{13}\rho_{14}\rho_{23}.$$
$$r = 5 \Rightarrow 2\mu_{5221}/5 = E_{4021}/8 + E_{3221}/2,\ E_{4021} = S_{421} = 48(\rho_{14.3}^{(2)} + 4\rho_{13}^2\rho_{14}),$$
$$\text{by (6)},\ E_{3221} = S_{320}T_{201} + S_{321}T_{200} + 2(S_{310}T_{211} + S_{311}T_{210}) + S_{301}T_{220},$$

$$\Rightarrow \mu_{5221}/15 = \sum_{i=1}^{5} a_i \text{ where } a_1 = \rho_{14},\ a_2 = 2\rho_{12}\rho_{24} + 2\rho_{13}\rho_{34},,$$
$$a_3 = 4\rho_{12}\rho_{23}\rho_{34} + 4(\rho_{12}^2 + \rho_{13}^2)\rho_{14} + 4\rho_{13}\rho_{23}\rho_{24} + 2\rho_{14}\rho_{23}^2,$$
$$a_4 = 16\rho_{12}\rho_{13}\rho_{14}\rho_{23} + 8\sum_{23}^{2} \rho_{12}\rho_{13}^2\rho_{24},\ a_5 = 8\rho_{12}^2\rho_{13}^2\rho_{14}.$$

Now take $t = 3, 2r - 7 = w + k$. By (8),

$$\mu_{wk31}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,31}/(r-j)!,$$
$$w = 2r - 8 \Rightarrow \mu_{w131}\, 2^r/w! = E_{r-1,131}/(r-1)!,$$
$$w = 2r - 9 \Rightarrow \mu_{w231}\, 2^{r-1}/w! = E_{r-1,031}/(r-1)! + E_{r-2,231}/(r-2)!,$$
$$w = 2r - 10 \Rightarrow \mu_{w331}\, 2^{r-1}/3w! = E_{r-2,131}/(r-2)! + E_{r-3,331}/(r-3)!,$$
$$w = 2r - 11 \Rightarrow \mu_{w431}\, 2^{r-3}/3w! = E_{r-2,031}/2(r-2)! + E_{r-3,231}/(r-3)!$$
$$+ E_{r-4,431}/(r-4)!.$$

The reader can now easily work out special cases.

Example 8. *Take $v = 2$, $2r - 2 = w + k + t$. If $V_{jj} \equiv 1$, then*

$$\mu_{wkt2}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,t2}/(r-j)! \text{ where} \tag{21}$$
$$E_{hJt2} = \sum_{0 \le e_1 \le min(J,t)} \binom{t}{e_1} F_{hJ,t-e_1,e_1 2},$$
$$F_{hJd_1e_1 2} = S_{hd_1 0}T_{Je_1 2} + 2S_{hd_1 1}T_{Je_1 1} + S_{hd_1 2}T_{Je_1 0}.$$

Now take $t = 2, 2r-4 = w+k$. *Then*

$$\mu_{wk22}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,22}/(r-j)! \tag{22}$$

where $E_{hJ22} = F_{hJ022} + 2F_{hJ112} + F_{hJ202}$,
$F_{hJ022} = S_{h00}T_{J22} + 2S_{h01}T_{J21} + S_{h02}T_{J20},\ F_{hJ112} = S_{h10}T_{J12} + 2S_{h11}T_{J11}$
$+ S_{h12}T_{J10},\ F_{hJ202} = S_{h20}T_{J02} + 2S_{h21}T_{J01} + S_{h22}T_{J00}$.
So for $k = 2, w = 2r-6 \Rightarrow$

$$\mu_{w222}\, 2^{r-1}/w! = E_{r-1,022}/(r-1)! + E_{r-2,222}/(r-2)! \text{ where} \tag{23}$$

$E_{h022} = S_{h22}$ *of* (7), $E_{h222} = 8\sum_{j=1}^{4}[h]_j\, a_j$,

$a_1 = a_{12} + a_{13},\ a_{12} = \rho_{23}^2 + \rho_{24}^2,\ a_{13} = 4\rho_{23}\rho_{24}\rho_{34}$,
$a_2 = \rho_{12}^2 + \sum_{i=3}^{4} a_{2i},\ a_{23} = 4\rho_{12}\rho_{13}\rho_{23} + 4\rho_{12}\rho_{14}\rho_{24}$,
$a_{24} = 2\rho_{12}^2\rho_{34}^2 + 8\rho_{12}\rho_{13}\rho_{14}\rho_{34} + 8\rho_{12}\rho_{32}\rho_{24}\rho_{34} + 8\rho_{13}\rho_{14}\rho_{23}\rho_{24}$,
$a_3 = \sum_{i=4}^{5} a_{3i},\ a_{34} = 2\rho_{12}^2(\rho_{13}^2 + \rho_{14}^2)$,

$$a_{35} = 8\rho_{12}\rho_{13}\rho_{14}(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}), \tag{24}$$

$a_4 = 4\rho_{12}^2\rho_{13}^2\rho_{14}^2$.

$$r = 3 \Rightarrow \mu_{0222} = 1 + 2\rho_{14}^2 + 2a_1 \Rightarrow (12), \tag{25}$$

$$r = 4 \Rightarrow \mu_{2^4} = 1 + 2\sum_{i=2}^{4} c_i \text{ where } c_2 = \sum(\rho_{ij}^2 : 1 \le i < j \le 4), \tag{26}$$

$c_3/4 = \rho_{12}\rho_{13}\rho_{23} + \rho_{12}\rho_{14}\rho_{24} + \rho_{13}\rho_{14}\rho_{34} + \rho_{23}\rho_{24}\rho_{34}$,
$c_4 = 2\sum_{1234}^{3}(\rho_{12}^2\rho_{34}^2 + 4\rho_{12}\rho_{13}\rho_{23}\rho_{34})$.

$r = 5,\ w = 4 \Rightarrow \mu_{4222} = S_{422}/16 + E_{3222}/4 = 3 + 2\sum_{i=2}^{5} d_5$,
$d_2 = 2\rho_{12}^2 + 2\rho_{13}^2 + 2\rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2$,
$d_3/4 = 2\rho_{12}(\rho_{13}\rho_{23} + \rho_{14}\rho_{24}) + (2\rho_{13}\rho_{14} + \rho_{23}\rho_{24})\rho_{34}$,
$d_4/4 = \rho_{12}^2\rho_{13}^2 + a_{24}/2 + a_{34}/2 = \rho_{12}^2(\rho_{13}^2 + \rho_{14}^2 + \rho_{34}^2) + \rho_{13}^2\rho_{14}^2 + \rho_{14}^2\rho_{23}^2$
$+ 4\rho_{12}(\rho_{14}\rho_{23} + \rho_{13}\rho_{24})\rho_{34} + 4\rho_{13}\rho_{14}\rho_{23}\rho_{24},\ d_5 = 2a_{35}$ *of* (24).

$r = w = 6 \Rightarrow \mu_{6222}/15 = 1 + 2\sum_{i=2}^{5} e_i$ *where*
$e_2 = 3\rho_{12}^2 + 3\rho_{13}^2 + 3\rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2$,
$e_3/4 = 3\rho_{12}\rho_{13}\rho_{23} + 3\rho_{12}\rho_{14}\rho_{24} + 3\rho_{13}\rho_{14}\rho_{34} + \rho_{23}\rho_{24}\rho_{34}$,
$e_4 = \rho_{12}^2(\rho_{13}^2 + \rho_{14}^2 + \rho_{34}^2) + a_{24}/2,\ e_5 = 6a_{35}$.

$$\text{For } k = 3,\ w = 2r-7, \mu_{w322}\, 2^{r-1}/3w! = \sum_{j=2}^{3} E_{r-j,2j-3,22}/(r-j)! \tag{27}$$

where $E_{h122} = 2S_{h12}T_{110} + 2S_{h21}T_{101} + S_{h23}T_{100}$,
$E_{h322} = F_{h3022} + 2F_{h3112} + F_{h3202},\ F_{h3022} = 2S_{h01}T_{321} + S_{h02}T_{320}$,
$F_{h3112} = S_{h10}T_{312} + 2S_{h11}T_{311} + S_{h12}T_{310}$,

$F_{h3202} = S_{h20}T_{302} + 2S_{h21}T_{301} + S_{h23}T_{300}.$
For example, $w = 3, r = 5 \Rightarrow \mu_{3322}/3 = E_{3122}/16 + E_{4322}/64.$

For μ_{wkt3}, put $v = 3$ in Corollary 7.1. Of course the method can be continued for higher moments. For example for moments of dimension 5, one route is to apply $\partial^e_{1.5}$ to Theorem 7.1.

8. Moments of $Z \sim \mathcal{N}_p(\mu, V)$

So far we have given the moments of $X \sim \mathcal{N}_p(0, V)$. It is worth while adding some examples of the non-central moments of $Z = X + \mu$. Consider the case $p = 2$. Then

$$m_{rs} = E\, Z_1^r Z_2^s = \sum_{j=0}^{r} \binom{r}{j} \mu_1^{r-j} \sum_{k=0}^{s} \binom{s}{k} \mu_2^{s-k}\, \mu_{jk}$$

for μ_{jk} of (1). For example

$$\begin{aligned}
m_{r2} &= \sum_{j\ even} \binom{r}{j} \mu_1^{r-j} \nu_j(\mu_2^2 + 1 + j\rho^2) + 2\rho\mu_2 \sum_{j\ odd} \binom{r}{j} \mu_1^{r-j} \nu_{j+1} : \\
m_{22} &= \mu_1^2\mu_2^2 + 4\rho\mu_1\mu_2 + \mu_2^2 + 1 + 2\rho^2, \\
m_{32} &= \mu_1^3(\mu_2^2 + 1) + 3\mu_1(\mu_2^2 + 1 + 2\rho^2) + 6\rho(\mu_1^2 + 1)\mu_2, \\
m_{42} &= \mu_1^4(\mu_2^2 + 1) + 6\mu_1^2(\mu_2^2 + 1 + 2\rho^2) + 3(\mu_2^2 + 1 + 4\rho^2) + 8\rho(\mu_1^3 + \mu_1)\mu_2.
\end{aligned}$$

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