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A Comprehensive Review on Solving the System of Equations $AX = C$ and $XB = D$

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Article

A Comprehensive Review on Solving the System of Equations $AX = C$ and $XB = D$

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Abstract: This survey provides a review of the theoretical research on the classic system of matrix equations $AX = C$ and $XB = D$, which has wide-ranging applications across fields such as control theory, optimization, image processing, and robotics. The paper discusses various solution methods for the system, focusing on specialized approaches, including generalized inverse methods, matrix decomposition techniques, and solutions in the forms of Hermitian, extreme rank, reflexive, and conjugate solutions. Additionally, specialized solving methods for specific algebraic structures, such as Hilbert spaces, Hilbert C^*-modules, and quaternions, are presented. The paper explores the existence conditions and explicit expressions for these solutions, along with examples of their application in electronic network and color image.

Keywords: system of matrix equations; general solution; special solution; Moore-Penrose inverse; matrix decomposition

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1. Introduction

Systems of equations, particularly

$$\begin{cases} AX = C, \\ XB = D, \end{cases} \tag{1}$$

are essential tools in linear algebra and have widespread applications in diverse scientific and engineering disciplines. These equations often appear in various domains such as control theory, optimization, image processing, system identification, and robotics [1–7]. Specifically, the matrix system $AX = C, XB = D$ can represent the state-space model of a dynamic system, where A and B correspond to system transformations, X represents the system state, and C and D are the output matrices [8]. Solving these equations provides the system's state at a given time. In signal processing, particularly in filter design and signal reconstruction, the filter matrix X transforms input signals A to output C, ensuring the transformed signal interacts correctly with B to produce output D [9]. This concept extends to image processing, where matrices A and B represent operations (e.g., encryption), and C and D are the original and transformed images. Solving for X gives the required transformation. In robotics and computer vision, this matrix system arises in rigid body transformations. Dual quaternions represent 3D transformations, where A and B may represent rotation and translation, and X is the transformation matrix [10]. This system is vital for solving inverse kinematics problems, such as determining joint parameters of a robotic arm.

This paper aims to summarize the theoretical results related to the matrix equation system (1), mainly focusing on the conditions and corresponding expressions for the existence of general solutions, least squares solutions, and minimum norm solutions. The paper also highlights generalized inverse methods and matrix decomposition methods in real and complex fields, as well as special solving methods for certain algebraic structures, such as Hilbert C^*- modules, Hilbert spaces, rings, dual numbers, quaternions, split quaternions, and dual quaternions.

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The most widely used and earliest approach for solving system (1) is based on generalized inverses or inner inverses. For special forms of solutions, such as Hermitian, nonnegative definite, maximal and minimal rank solutions, and generalized (anti-)reflexive solutions, this class of methods provides a rich theoretical framework. On the other hand, matrix decomposition is a powerful tool for solving more complex special forms of solutions. Due to the different forms resulting from various matrix decompositions, these special forms can be used to construct corresponding special solutions. Related research covers symmetric, mirror-symmetric, bi-(skew-)symmetric, and orthogonal solutions over the real numbers, as well as unitary, (semi-)positive definite, generalized reflexive, generalized conjugate, and Hamiltonian solutions over the complex numbers.

For certain special algebraic structures, there are specialized solving methods. For example, in Hilbert C^*-modules, Hilbert spaces, and rings, inner inverses are widely used. For quaternions, dual numbers, and dual quaternions, generalized inverses can also be applied to solve the system (1). However, in the case of split quaternions, matrix representations are the more widely used approach for solving the system. Additionally, some researchers have discussed the use of determinants to express the form of solutions for quaternion systems. This paper also provides examples of applying the system (1) to dual quaternion matrices and dual split quaternion tensors for the encryption and decryption of color images and videos.

The remainder of the paper is organized as follows. Chapter 2 introduces generalized inverse methods for solving the general solution, Hermitian and nonnegative definite solutions, maximal and minimal rank solutions, and generalized reflexive solutions. The study of system (1) in Hilbert C^*-modules, Hilbert spaces, and rings is presented in Chapter 3. Chapter 4 discusses eigenvalue decomposition, singular value decomposition, and generalized singular value decomposition of matrices, along with research conclusions for some special solutions of system (1). Chapters 5 and 6 focus on the studies of dual numbers and quaternions, respectively. Chapter 7 introduces examples of using the system (1) in the encryption and decryption of color images and videos. Finally, Chapter 8 summarizes the content of the paper.

For convenience in the narration of this paper, the following notations are used uniformly. Symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^{m\times n}$, $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$ represent the real number field, the complex number field, the set of $m\times n$ matrices over the real numbers, the set of complex vectors with n elements, and the set of $m\times n$ matrices over the complex numbers, respectively. O and I denote appropriately sized zero matrices and identity matrices. For an arbitrary matrix A, $\bar{A}$, A^{T} and A^* represent the conjugate, transpose and conjugate transpose of A, respectively. For an $m\times n$ matrix A over the real numbers, complex numbers, or quaternions, $\mathrm{rank}(A)$ represents the rank of A and $\mathscr{R}(A)$ expresses the range (column space) of A. For a complex square matrix A, it is (semi-)positive definite if and only if for every $v\in\mathbb{C}^n$, we have $v^*Av > (\geq) 0$. For two complex square matrices A and B of the same size, we say that $A > (\geq) B$ in the Löwner partial ordering if $A-B$ is (semi-)positive definite. The symbols $i_+(A)$ and $i_-(A)$ denote the numbers of positive and negative eigenvalues of a Hermitian complex matrix A, counted with multiplicities. Note that for Hermitian nonnegative definite matrix A, denote $A^{\frac{1}{2}}$ is the matrix satisfying $A^{\frac{1}{2}}A^{\frac{1}{2}}=A$. The $||\cdot||$ mentioned below all represents the Frobenius-norm of a matrix.

2. The Generalized Inverse Methods for Solving (1)

Since 1954, Penrose has described a generalization of the inverse of non-singular matrices through the unique solution of a system of four matrix equations [11]. This area has since attracted considerable attention.

For $A\in\mathbb{C}^{m\times n}$, there exists a unique $A^\dagger$ satisfying the following system:

$$AA^\dagger A = A,\quad A^\dagger AA^\dagger = A^\dagger,\quad (AA^\dagger)^* = AA^\dagger,\quad (A^\dagger A)^* = A^\dagger A,$$

where $A^\dagger$ is called the general inverse or the Moore-Penrose inverse of A. In the following discussion, we denote the symbols $\mathcal{L}_A = I - A^\dagger A$ and $\mathcal{R}_A = I - AA^\dagger$.

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Penrose proposed the necessary and sufficient conditions for the matrix system $AX = C, XB = D$, along with an expression for its solutions in terms of the general inverse.

Theorem 1 (General solutions using the Moore-Penrose inverse for (1) over $\mathbb{C}$)**.** *[11] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$. The matrix system (1) is solvable if and only if the equations $AX = C$ and $XB = D$ are consistent, and the condition $AD = CB$ holds, or equivalently,*

$$AA^{\dagger}C = C,\ DB^{\dagger}B = D,\ AD = CB.$$

Under these conditions, the general solution is given by

$$X = A^{\dagger}C + DB^{\dagger} - A^{\dagger}ADB^{\dagger}. \tag{2}$$

Remark 1. *The concept of the general inverse and Theorem 1 can also be extended to von Neumann regular rings, particularly to the quaternion algebra [12].*

Later, the concept of the g-inverse of a complex matrix was introduced by Rao and Mitra [13]. For $A \in \mathbb{C}^{m\times n}$, if the matrix A^{-} satisfies

$$AA^{-}A = A,$$

then A^{-} is defined as the g-inverse of A.

Remark 2. *The g-inverse of a complex matrix is not necessarily unique.*

The g-inverse can also be used to represent the solution of linear matrix systems. Theorem 1 can be restated in terms of the g-inverse as follows.

Theorem 2 (General solutions using the g-inverse for (1) over $\mathbb{C}$)**.** *[16] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$. The matrix system (1) is solvable if and only if*

$$AA^{-}C = C,\ DB^{-}B = D,\ AD = CB.$$

When (1) is consistent, the general solution is expressed as

$$X = A^{-}C + DB^{-} - A^{-}ADB^{-} + (I - A^{-}A)V(I - B^{-}B),$$

where $V \in \mathbb{C}^{m\times n}$ is arbitrary.

Subsequently, researchers have investigated a series of special solutions to system (1) using generalized inverses, including Hermitian solutions, nonnegative solutions, maximal and minimal rank solutions, general (anti-)reflexive solutions, real nonnegative or real positive solutions, and others.

The earliest research on the Hermitian and nonnegative definite solutions of system (1) was conducted by Mitra et al. [14], followed by their subsequent work on the possible minimal rank of the solutions [15]. Mitra's focus on matrix equation systems continued, and in 1990, he extended the study to a more general form of the system $A_1XB_1 = C_1, A_2XB_2 = C_2$ [16]. After 2000, research on system (1) became more in-depth: Peng and other scholars investigated the (anti-)reflexive solutions of the system [17–23], while Liu et al. focused on the least squares solutions and the rank of the solutions, exploring the ranks of matrix blocks and the corresponding conditions using block matrix formulations [24,25]. Due to the unique properties of Hermitian matrices, Wang et al. examined the existence conditions and expressions for Hermitian solutions to system (1) that satisfy various inequality constraints, as well as the ranks and inertia indices of these solutions [26–28]. Additionally, some scholars have focused on bi-(skew-)symmetric solutions and reducible solutions [29,30].

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2.1. Hermitian, Nonnegative Solutions

Hermitian and nonnegative matrices are among the most widely applied special types of matrices, and their associated properties have been thoroughly studied. A complex square matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian if $A = A^*$.

In 1976, Khatri and Mitra considered the necessary and sufficient conditions for the existence of Hermitian and nonnegative definite solutions to system (1), and provided expressions for the solutions when they exist. The main results are stated in Theorem 3.

Theorem 3 (Hermitian and nonnegative solutions for (1) over $\mathbb{C}$)**.** *[14] Let $A, C \in \mathbb{C}^{p\times n}$ and $B, D \in \mathbb{C}^{n\times q}$ such that the system (1) is solvable. Define*

$$M = \begin{bmatrix} CA^* & CB \\ D^*A^* & D^*B \end{bmatrix}.$$

(a) The system (1) has Hermitian solutions if and only if M is Hermitian. Under this condition, a general Hermitian solution is given by

$$X = \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} C \\ D^* \end{bmatrix} + [C^*, D] \begin{bmatrix} A \\ B^* \end{bmatrix}^{-*} - \begin{bmatrix} A \\ B^* \end{bmatrix}^- M \begin{bmatrix} A \\ B^* \end{bmatrix}^{-*} + \left[I - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix} \right] U \left[I - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix} \right]^*,$$

where $U \in \mathbb{C}^{n\times n}$ is an arbitrary Hermitian matrix.

(b) The system (1) has nonnegative definite solutions if and only if M is nonnegative definite and $\operatorname{rank}(M) = \operatorname{rank}[C^*, D]$. *Under this condition, general nonnegative definite solutions have the form of*

$$X = [C^*, D]M^- \begin{bmatrix} C \\ D^* \end{bmatrix} + \left[I - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix} \right] U \left[I - \begin{bmatrix} A \\ B^* \end{bmatrix}^- \begin{bmatrix} A \\ B^* \end{bmatrix} \right]^*,$$

where $U \in \mathbb{C}^{n\times n}$ is an arbitrary nonnegative definite matrix.

Based on Theorem 3, the following theorem considers the solvability conditions and explicit expressions for the Hermitian solutions to the system (1) with inequality constraints:

$$AX = C,\ XB = D \text{ subject to } MXM^* \geq N \tag{3}$$

and

$$AX = C,\ XB = D \quad \text{subject to } MXM^* \geq N \geq O, \tag{4}$$

for given $M \in \mathbb{C}^{m\times n}$ and Hermitian $N \in \mathbb{C}^{m\times m}$.

Theorem 4 (Hermitian solutions for (3) and (4) over $\mathbb{C}$)**.** *[28] Given $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$, $M \in \mathbb{C}^{m\times n}$ and Hermitian $N \in \mathbb{C}^{m\times m}$. Assume*

$$P = \begin{bmatrix} A \\ B^* \end{bmatrix},\ Q = \begin{bmatrix} C \\ D^* \end{bmatrix},\ T = \begin{bmatrix} CA^* & CB \\ D^*A^* & D^*B \end{bmatrix},\ \widehat{M} = M\mathcal{L}_P,\ \widehat{P} = \mathcal{L}_P\widehat{M}^\dagger,$$
$$\widehat{N} = N - M\left[P^\dagger Q + Q^*(P^\dagger)^* - P^\dagger T(P^\dagger)^*\right]M^*,\ \widetilde{N} = N - MQ^*T^\dagger QM^*.$$

(a) The system (3) has Hermitian solutions if and only if $PP^\dagger Q = Q$, T is Hermitian, and

$$\mathcal{R}_{\widehat{M}}\widehat{N}\mathcal{R}_{\widehat{M}} \leq O,\ \operatorname{rank}(\mathcal{R}_{\widehat{M}}\widehat{N}\mathcal{R}_{\widehat{M}}) = \operatorname{rank}(\mathcal{R}_{\widehat{M}}\widehat{N}).$$

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At this point, the Hermitian solution $X \in \mathbb{C}^{n\times n}$ can be expressed as

$$X = X_0 + \widehat{P}W_1(\widehat{P})^* + \mathcal{L}_P\mathcal{L}_{\widehat{M}}V_1\mathcal{L}_P + \mathcal{L}_P V_1^*\mathcal{L}_{\widehat{M}}\mathcal{L}_P,$$

where

$$X_0 = P^\dagger Q + Q^* P^{\dagger *} - P^\dagger T P^{\dagger *} + \widehat{P}\widehat{N}\left[I - \mathcal{R}_{\widehat{M}}\widehat{N}^\dagger\mathcal{R}_{\widehat{M}}\widehat{N}\right]\widehat{P}^*,$$

$W_1 \in \mathbb{C}^{n\times n}$ is an arbitrary nonnegative definite Hermitian matrix, $V_1 \in \mathbb{C}^{n\times n}$ is arbitrary.

(b) The system (4) has Hermitian nonnegative definite solutions if and only if T is a nonnegative Hermitian matrix, and

$$\operatorname{rank}(T) = \operatorname{rank}(Q),\ \widehat{M}\widehat{M}^\dagger\widetilde{N} = \widetilde{N}.$$

At this point, the Hermitian nonnegative definite solution $X \in \mathbb{C}^{m\times n}$ can be described as

$$X = Q^*T^\dagger Q + \mathcal{L}_P\left[\dot{M}\dot{N}\dot{M}^* + \mathcal{L}_{\widehat{M}}U\mathcal{L}_{\widehat{M}}^*\right]\mathcal{L}_P^*,$$

where

$$\dot{N} = \widehat{N} + \widehat{M}\widehat{M}^\dagger W_1\widehat{M}\widehat{M}^\dagger,\ \dot{M} = \widehat{M}^\dagger + \mathcal{L}_{\widehat{M}}W\dot{N}^{\frac{1}{2}\dagger},$$

with arbitrary U, W, and nonnegative definite Hermitian $W_1 \in \mathbb{C}^{n\times n}$.

Remark 3. *In Theorem 4, selecting $M = I$ and $N = O$ can yield the Hermitian nonnegative solutions to (1).*

In [28], the authors consider the maximal rank and inertia of the Hermitian solution to (3) using matrix decomposition methods, which will be introduced in the next section.

Additionally, Ke and Ma [31] have supplemented the results for the symmetric solutions to system (1) over $\mathbb{R}$.

Theorem 5 (Symmetric solutions for (1) over $\mathbb{R}$)**.** *[31] Given $A, C \in \mathbb{R}^{p\times n}$ and $B, D \in \mathbb{R}^{n\times q}$. Denote $K = C^{\mathrm{T}}\mathcal{L}_A$, $N = \mathcal{R}_C A^{\mathrm{T}}$, $Q_1 = D^{\mathrm{T}} - C^{\mathrm{T}}A^\dagger B - KDC^\dagger$ and $Q = B^{\mathrm{T}} - A^\dagger BA^{\mathrm{T}} - \mathcal{L}_A DC^\dagger A^{\mathrm{T}} - \mathcal{L}_A K^\dagger Q_1 N$. The system (1) has a symmetric solution $X \in \mathbb{R}^{n\times n}$ if and only if the system of matrix equations*

$$AY = B,\ YC = D,\ YA^{\mathrm{T}} = B^{\mathrm{T}},\ C^{\mathrm{T}}Y = D^{\mathrm{T}}$$

has a solution $Y \in \mathbb{R}^{n\times n}$. In this case, the symmetric solution to (1) is given by

$$X = \frac{1}{2}(Y + Y^{\mathrm{T}}).$$

Or in an equivalent way, equations

$$KK^\dagger Q_1\mathcal{R}_C = Q_1,\ Q\mathcal{L}_N = O,\ \mathcal{R}_{\mathcal{L}_A\mathcal{L}_K}Q = O,\ AD = BC,\ AA^\dagger B = B,\ DC^\dagger C = D$$

hold. At this point, the symmetric solution of (1) can be expressed as

$$\begin{aligned} X = &\frac{1}{2}(A^\dagger B + \mathcal{L}_A DC^\dagger + \mathcal{L}_A K^\dagger Q_1\mathcal{R}_C + QN^\dagger\mathcal{R}_C + \mathcal{L}_A\mathcal{L}_K Z\mathcal{R}_N\mathcal{R}_C) \\ &+ \frac{1}{2}(B^{\mathrm{T}}A^{\mathrm{T}\dagger} + C^{\mathrm{T}\dagger}D^{\mathrm{T}}\mathcal{L}_A + \mathcal{R}_C Q_1^{\mathrm{T}}K^{\mathrm{T}\dagger}\mathcal{L}_A + \mathcal{R}_C N^{\mathrm{T}\dagger} + \mathcal{R}_C\mathcal{R}_N Z^{\mathrm{T}}\mathcal{L}_K\mathcal{L}_A), \end{aligned}$$

where $Z \in \mathbb{R}^{n\times n}$ is an arbitrary matrix.

2.2. Maximal and Minimal Rank Solutions with Inequality Constrain

Through the expression of the solution to the system (1) given by generalized inverses, the case of the rank of the solution can be further studied.

In 1984, Mitra obtained the minimal possible rank solutions to the system (1).

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Theorem 6 (Minimal possible rank solutions for (1) over $\mathbb{C}$)**.** *[15] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$ such that (1) is consistent. Assume without loss of generality that $\operatorname{rank}(C) \leq \operatorname{rank}(D)$. Let X be a solution of the matrix system (1). Then,*

$$\operatorname{rank}(X) \geq \max\{\operatorname{rank}(C), \operatorname{rank}\,(D)\} = \operatorname{rank}\, D.$$

Additionally, $\operatorname{rank}(X) = \operatorname{rank}(D)$ if and only if $\operatorname{rank}(CB) = \operatorname{rank}(C)$.

Decades later, Liu extended Theorem 6, considering the maximal and minimal ranks of the general solutions and the least squares solutions for the system (1).

Theorem 7 (Maximal and minimal rank solutions for (1) over $\mathbb{C}$)**.** *[24] For $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$, the system (1) is consistent with a general solution $X \in \mathbb{C}^{n\times n}$. Then, the maximal and minimal ranks of X are given by*

$$\begin{aligned} &\max \operatorname{rank}(X) = \min\{m + \operatorname{rank}(C) - \operatorname{rank}(A),\ n + \operatorname{rank}(D) - \operatorname{rank}(B)\}, \\ &\min \operatorname{rank}(X) = \operatorname{rank}(C) + \operatorname{rank}(D) - \operatorname{rank}(CB). \end{aligned}$$

The least squares solution for the system (1) can be expressed below.

Theorem 8 (Least squares solutions for (1) over $\mathbb{C}$)**.** *[24] Assume that $A \in \mathbb{C}^{p\times m}, C \in \mathbb{C}^{p\times n}, B \in \mathbb{C}^{n\times q}$, $D \in \mathbb{C}^{m\times q}$.*

(a) The necessary and sufficient condition for (1) to have a least squares solution is

$$A^*ADB^* = A^*CBB^*. \tag{5}$$

(b) If (5) holds, then the general least squares solution of (1) is expressed as

$$X = (A^*A)^\dagger A^*C + DB^*(BB^*)^\dagger - (A^*A)^\dagger (A^*A)DB^*(BB^*)^\dagger + \mathcal{L}_{A^*A} W \mathcal{R}_{BB^*},$$

where $W \in \mathbb{C}^{m\times n}$ is arbitrary.

(c) The least squares solution of (1) is unique if and only if

$$\operatorname{rank}(A) = m \text{ or } \operatorname{rank}(B) = n.$$

In this case, the unique least squares solution is

$$X = (A^*A)^\dagger A^*C \text{ or } X = DB^*(BB^*)^\dagger.$$

Additionally, the maximal and minimal ranks of the least squares solutions for the system (1) are considered.

Theorem 9 (Maximal and minimal least squares solutions for (1) over $\mathbb{C}$)**.** *[24] For $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$. If the system (1) has a general least squares solution $X \in \mathbb{C}^{n\times n}$, then the maximal and minimal ranks of X are given by*

$$\begin{aligned} &\max \operatorname{rank}(X) = \min\{m + \operatorname{rank}(A^*C) - \operatorname{rank}(A),\ n + \operatorname{rank}(D^*B) - \operatorname{rank}(B)\}, \\ &\min \operatorname{rank}(X) = \operatorname{rank}(A^*C) + \operatorname{rank}(D^*) - \operatorname{rank}(A^*CB). \end{aligned}$$

Liu also presented a set of formulas for the maximal and minimal ranks of the submatrices in a general solution X to the system (1) in [25].

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Suppose that X is a general solution of the system (1), and let X be partitioned into a 2×2 block form:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

In this case, the system (1) can be rewritten as

$$[A_1, A_2]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = [C_1, C_2], \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \tag{6}$$

where $A_1 \in \mathbb{C}^{p\times m_1}, A_2 \in \mathbb{C}^{p\times m_2}, C_1 \in \mathbb{C}^{p\times n_1}, C_2 \in \mathbb{C}^{p\times n_2}, B_1 \in \mathbb{C}^{n_1\times q}, B_2 \in \mathbb{C}^{n_2\times q}, D_1 \in \mathbb{C}^{m_1\times q}, D_2 \in \mathbb{C}^{m_2\times q}$ and $X_1 \in \mathbb{C}^{m_1\times n_1}, X_2 \in \mathbb{C}^{m_1\times n_2}, X_3 \in \mathbb{C}^{m_2\times n_1}, X_4 \in \mathbb{C}^{m_2\times n_2}$, with $m_1 + m_2 = m, n_1 + n_2 = n$. Adopt the following notations for the collections of submatrices X_1, X_2, X_3, and X_4 as

$$S_i = \left\{ X_i \,\middle|\, [A_1, A_2]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = [C_1, C_2], \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \right\}, \ i = 1, 2, 3, 4. \tag{7}$$

The submatrices X_i can be rewritten as the form of

$$X_1 = [I_{n_1}, O]X\begin{bmatrix} I_{p_1} \\ O \end{bmatrix} = P_1XQ_1, \ X_2 = [I_{n_1}, O]X\begin{bmatrix} O \\ I_{p_2} \end{bmatrix} = P_1XQ_2,$$
$$X_3 = [O, I_{n_2}]X\begin{bmatrix} I_{p_1} \\ O \end{bmatrix} = P_2XQ_1, \ X_4 = [O, I_{n_2}]X\begin{bmatrix} O \\ I_{p_2} \end{bmatrix} = P_2XQ_2.$$

Substituting the general solution (2) gives the general expressions for X_i, as follows:

$$X_1 = P_1X_0Q_1 + P_1F_AVE_BQ_1, \ X_2 = P_1X_0Q_2 + P_1F_AVE_BQ_2,$$
$$X_3 = P_2X_0Q_1 + P_2F_AVE_BQ_1, \ X_4 = P_2X_0Q_2 + P_2F_AVE_BQ_2,$$

where $X_0 = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger$.

Theorem 10 (Maximal and minimal rank solutions using block matrices for (1) over $\mathbb{C}$). *[25] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$, such that the matrix system (1) has a general solution. Then,*

(a)
$$\max_{X_1\in S_1} \operatorname{rank}(X_1) = \min\left\{ m_1 + \operatorname{rank}[C_1, A_2] - \operatorname{rank}(A), n_1 + \operatorname{rank}\begin{bmatrix} D_1 \\ B_2 \end{bmatrix} - \operatorname{rank}(B) \right\},$$
$$\min_{X_1\in S_1} \operatorname{rank}(X_1) = \operatorname{rank}[C_1, A_2] + \operatorname{rank}\begin{bmatrix} D_1 \\ B_2 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} C_1B_2 & A_2 \\ B_2 & 0 \end{bmatrix}.$$

(b)
$$\max_{X_2\in S_2} \operatorname{rank}(X_2) = \min\left\{ m_1 + \operatorname{rank}[C_2, A_2] - \operatorname{rank}(A), n_2 + \operatorname{rank}\begin{bmatrix} D_1 \\ B_1 \end{bmatrix} - \operatorname{rank}(B) \right\},$$
$$\min_{X_2\in S_2} \operatorname{rank}(X_2) = \operatorname{rank}[C_2, A_2] + \operatorname{rank}\begin{bmatrix} D_1 \\ B_1 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} C_2B_2 & A_2 \\ B_1 & O \end{bmatrix}.$$

(c)
$$\max_{X_3\in S_3} \operatorname{rank}(X_3) = \min\left\{ m_2 + \operatorname{rank}[C_1, A_1] - \operatorname{rank}(A), n_1 + \operatorname{rank}\begin{bmatrix} D_2 \\ B_2 \end{bmatrix} - \operatorname{rank}(B) \right\}$$
$$\min_{X_3\in S_3} \operatorname{rank}(X_3) = \operatorname{rank}[C_1, A_1] + \operatorname{rank}\begin{bmatrix} D_2 \\ B_2 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} C_1B_1 & A_1 \\ B_2 & 0 \end{bmatrix}.$$

(d)
$$\max_{X_4\in S_4} \operatorname{rank}(X_4) = \min\left\{ m_2 + \operatorname{rank}[C_2, A_1] - \operatorname{rank}(A), n_2 + \operatorname{rank}\begin{bmatrix} D_2 \\ B_1 \end{bmatrix} - \operatorname{rank}(B) \right\}$$
$$\min_{X_4\in S_4} \operatorname{rank}(X_4) = \operatorname{rank}[C_2, A_1] + \operatorname{rank}\begin{bmatrix} D_2 \\ B_1 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} C_2B_2 & A_1 \\ B_1 & O \end{bmatrix}.$$

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In addition, by using the ranks of matrix blocks, the necessary and sufficient conditions for the uniqueness of the solution to system (1) are given in block matrix form.

Theorem 11 (Unique conditions of general solutions for (1) over $\mathbb{C}$)**.** *[25] Suppose that matrix system (6) has a solution. Then, the following statements hold.*

(a) The block X_1 in a general solution to (6) is unique if and only if

$$\operatorname{rank}(A_1) = n_1,\ \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{O\} \text{ or } \operatorname{rank}(B_1) = p_1,\ \mathscr{R}(B_1^*) \cap \mathscr{R}(B_2^*) = \{O\}.$$

(b) The block X_2 in a general solution to (6) is unique if and only if

$$\operatorname{rank}(A_1) = n_1,\ \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{O\} \text{ or } \operatorname{rank}(B_2) = p_2,\ \mathscr{R}(B_1^*) \cap \mathscr{R}(B_2^*) = \{O\}.$$

(c) The block X_3 in a general solution to (6) is unique if and only if

$$\operatorname{rank}(A_2) = n_2,\ \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{O\} \text{ or } \operatorname{rank}(B_1) = p_1,\ \mathscr{R}(B_1^*) \cap \mathscr{R}(B_2^*) = \{O\}.$$

(d) The block X_4 in a general solution to (6) is unique if and only if

$$\operatorname{rank}(A_2) = n_2,\ \mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{O\} \text{ or } \operatorname{rank}(B_2) = p_2,\ \mathscr{R}(B_1^*) \cap \mathscr{R}(B_2^*) = \{O\}.$$

Theorem 12 (General solutions using block matrices for (1) over $\mathbb{C}$)**.** *[25] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$, such that the system (6) has a solution.*

(a) Consider $S_1, \cdots, S_4$ in (7) as four independent matrix sets. Then,

$$\begin{aligned}
&\max_{X_i \in S_i}\left([C_1, C_2] - [A_1, A_2]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\right) \\
&= \min\{\operatorname{rank}(A_1) + \operatorname{rank}(A_2) - \operatorname{rank}(A), p_1 + p_2 + \operatorname{rank}(B_1) + \operatorname{rank}(B_2) - 2\operatorname{rank}(B)\}, \\
&\max_{X_i \in S_i}\left(\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} - \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right) \\
&= \min\{\operatorname{rank}(B_1) + \operatorname{rank}(B_2) - \operatorname{rank}(B), n_1 + n_2 + \operatorname{rank}(A_1) + \operatorname{rank}(A_2) - 2\operatorname{rank}(A)\}.
\end{aligned}$$

(b) The four submatrices $X_1, \cdots, X_4$ in (7) are independent. Specifically, for any choice of $X_i \in S_i (i = 1, \cdots, 4)$, the corresponding matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ is a solution of (7) if and only if

$$\mathscr{R}(A_1) \cap \mathscr{R}(A_2) = \{O\},\ \mathscr{R}(B_1^*) \cap \mathscr{R}(B_2^*) = \{O\}.$$

Furthermore, Wang et al. first considered the extremal inertias and ranks of $XX^* - P$ and $X^*X - Q$, where P and Q are Hermitian, and X is a solution of (1). They also derived the necessary and sufficient conditions for special cases such as unitary solvability, contraction solvability, and the left and right minimal solutions to the system (1) [26].

For $A \in \mathbb{C}^{n\times n}$, A is called a unitary matrix if and only if $AA^* = A^*A = I$. Let H be a given set consisting of some matrices in $\mathbb{C}^{n\times n}$, and we say that $A \in H$ is minimal (maximal) if $A \leq W$ (or $A \geq W$) for every $W \in H$. Denote

$$\mathcal{L} = \{XX^* \mid AX = C, XB = D\},\ \mathcal{R} = \{X^*X \mid AX = C, XB = D\}.$$

A solution X is called left (right) minimal or maximal if XX^* (X^*X) is the minimal or maximal matrix of the set $\mathcal{L}$ ($\mathcal{R}$). When $XX^* \leq I$, X is called a contraction matrix. Furthermore, if $XX^* < I$, X is called a strict contraction matrix.

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Theorem 13 (Extreme rank and inertia of $XX^* - P$ for X satisfying (1) over $\mathbb{C}$)**.** *[26] Let $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times q}$ and $P \in \mathbb{C}^{m\times n}$. Suppose that (1) has a solution. Denote the set of all solutions to (1) by S. Then,*

(a) $\max_{X\in S} \operatorname{rank}(XX^* - P) = \min\{r_1, r_2, r_3\}$, *where*

$$r_1 = m + \operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}(B) - \operatorname{rank}(A),$$

$$r_2 = 2m + \operatorname{rank}(CC^* - APA^*) - \operatorname{rank}(A),\ r_3 = \operatorname{rank}\begin{bmatrix} D^* & B^*B \\ P & D \end{bmatrix} + n - 2\operatorname{rank}(B).$$

(b) $\min_{X\in S} \operatorname{rank}(XX^* - P) = \max\{t_1, t_2, t_3, t_4\}$, *where*

$$\begin{aligned} t_1 =& 2\operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - 2\operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} + \operatorname{rank}(CC^* - APA^*), \\ t_2 =& 2\operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} + \operatorname{rank}\begin{bmatrix} D^* & B^*B \\ P & D \end{bmatrix} - 2\operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix} - n, \\ t_3 =& 2\operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix} + i_+\begin{bmatrix} B^*B & D^* \\ D & P \end{bmatrix} \\ &- \operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} + i_+(CC^* - APA^*) - n, \\ t_4 =& 2\operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix} + i_-\begin{bmatrix} B^*B & D^* \\ D & P \end{bmatrix} \\ &- \operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} + i_-(CC^* - APA^*). \end{aligned}$$

(c)

$$\max_{X\in S} i_+(XX^* - P) = \min\{i_1, i_2\},$$
$$\max_{X\in S} i_-(XX^* - P) = \min\{i_3, i_4\},$$

where

$$i_1 = m + i_+(CC^* - APA^*) - \operatorname{rank}(A),\ i_2 = i_-\begin{bmatrix} B^*B & D^* \\ D & P \end{bmatrix} + n - \operatorname{rank}(B),$$

$$i_3 = m + i_-(CC^* - APA^*) - \operatorname{rank}(A),\ i_4 = i_+\begin{bmatrix} B^*B & D^* \\ D & P \end{bmatrix} - \operatorname{rank}(B).$$

(d)

$$\min_{X\in S} i_+(XX^* - P) = \max\{p_1, p_2\},$$
$$\min_{X\in S} i_-(XX^* - P) = \max\{p_3, p_4\},$$

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where

$$p_1 = \operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} + i_+(CC^* - APA^*),$$
$$p_2 = i_-\begin{bmatrix} B^*B & D^* \\ D & P \end{bmatrix} + \operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix},$$
$$p_3 = \operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} - \operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} + i_-(CC^* - APA^*),$$
$$p_4 = i_+\begin{bmatrix} B^*B & D^* \\ D & \mathrm{P} \end{bmatrix} - \operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} + \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix} - n.$$

In Theorem 13, selecting P as the identity matrix can derive the necessary and sufficient conditions for (1) to have some special solutions, which are presented in the following corollary.

Corollary 1 (Unitary and (strict) contraction solutions for (1) over $\mathbb{C}$). *[26] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{m\times q}$. Suppose that the system (1) has general solutions.*

(*a*) *Then, (1) has unitary matrix solutions if and only if*

$$n \geq m,\ CC^* = AA^*,\ B^*B \geq D^*D,\ i_+(B^*B - D^*D) \leq n - m,$$
$$\operatorname{rank}\begin{bmatrix} D^* & B^* \\ AP & C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} D^*A^* & B^* \\ APA^* & C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & AP \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B^* \\ C \end{bmatrix}.$$

(*b*) *The system (1) has strict contraction matrix solutions if and only if*

$$i_-(CC^* - AA^*) \geq \operatorname{rank}(A),\ i_+(B^*B - D^*D) \geq \operatorname{rank}(B).$$

(*c*) *The system (1) has contraction matrix solutions when $CC^* \leq AA^*$ and*

$$\operatorname{rank}\begin{bmatrix} D^* & B^* \\ A & C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B^*C^* & B^* \\ AA^* & C \end{bmatrix} \leq \operatorname{rank}\begin{bmatrix} BB^*B & BD^* \\ CB & A \end{bmatrix} - i_-(B^*B - D^*D).$$

The left (right) minimal and maximal solutions to (1) are discussed as follows.

Theorem 14 (Left (right) minimal and maximal solutions for (1) over $\mathbb{C}$). *[26] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{m\times q}$ with $\operatorname{rank}(A) < m$ and $\operatorname{rank}(B) < n$.*

(*a*) *There exists a solution X to (1) such that X is the left minimal solution if and only if*

$$\operatorname{rank}\begin{bmatrix} C \\ B^* \end{bmatrix} = \operatorname{rank}(B).$$

Under this circumstance, the left minimal solution is $X = DB^\dagger$.

(*b*) *There exists a solution X to (1) such that X is the right minimal solution if and only if*

$$\operatorname{rank}[D, A^*] = \operatorname{rank}(A).$$

Under these circumstances, the right minimal solution is $X = A^\dagger C$.

(*c*) *There does not exist right or left maximal solution X to (1).*

In a similar manner, Yao derived the maximal and minimal ranks and inertias of $Q - XPX^*$, where X is a general solution of the system (1), with P and Q being Hermitian.

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Theorem 15 (Extreme rank and inertia of $Q - XPX^*$ for X satisfying (1) over $\mathbb{C}$)**.** *[27] For $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times q}$ and Hermitian $P, Q \in \mathbb{C}^{n\times n}$, assume that (1) has a solution. Denote the set of all solutions to (1) by S. Then,*

(a)

$$\max_{X\in S} \operatorname{rank}(Q - XPX^*) = \min\left\{ n + \operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - \operatorname{rank}(P), \right.$$
$$\left. 2n + \operatorname{rank}(AQA^* - CPC^*) - 2\operatorname{rank}(A), \operatorname{rank}\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^{\pm} & O \end{bmatrix} - 2\operatorname{rank}(B) \right\}.$$

(b)

$$\min_{X\in S} \operatorname{rank}(Q - XPX^*) = 2n + 2\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - 2\operatorname{rank}(A) - 2\operatorname{rank}(B) - \operatorname{rank}(P)$$
$$+ \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\}.$$

(c)

$$\max_{X\in S} i_{\pm}(Q - XPX^*) = \min\left\{ n + i_{\pm}(AQA^* - CPC^*) - \operatorname{rank}(A),\ i_{\pm}\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^* & O \end{bmatrix} - \operatorname{rank}(B) \right\}.$$

(d)

$$\min_{X\in S} i_{\pm}(Q - XPX^*) = n + \operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_{\pm}(P) + \max\{s_{\pm}, t_{\pm}\}.$$

In which,

$$s_{\pm} = -n + r(A) - i_{\mp}(P) + i_{\pm}(AQA^* - CPC^*) - \operatorname{rank}\begin{bmatrix} CP & AQA^* \\ B^* & D^*A^* \end{bmatrix},$$
$$t_{\pm} = -n + \operatorname{rank}(A) - i_{\mp}(P) + i_{\pm}\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^* & O \end{bmatrix} - \operatorname{rank}\begin{bmatrix} O & P & B \\ AQ & CP & O \\ D^* & B^* & O \end{bmatrix}.$$

Theorem 15 can derive the positive definiteness of $Q - XPX^*$ in the following corollary.

Corollary 2 (Positive definiteness of $Q - XPX^*$ for X satisfying (1) over $\mathbb{C}$)**.** *[27] Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times q}$ and Hermitian $P, Q \in \mathbb{C}^{n\times n}$. Assume that the system (1) is consistent. Let $s_{\pm}$ and $t_{\pm}$ be as defined in Theorem 15. Then, we have the following statements.*

(a) *The system (1) has a solution such that $Q - XPX^* \geq O$ if and only if*

$$n + \operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_-(P) + s_- \leq 0,$$
$$n + \operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_-(P) + t_- \leq 0.$$

(b) *The system (1) has a solution such that $Q - XPX^* \leq O$ if and only if*

$$n+\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_+(P) + s_+ \leq 0,$$

$$n+\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_+(P) + t_+ \leq 0.$$

(c) *The system* (1) *has a solution such that* $Q - XPX^* > O$ *precisely when*

$$i_+(AQA^* - CPC^\dagger) - \operatorname{rank}(A) \geq 0,\ i_+\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^+ & B^+ & O \end{bmatrix} - \operatorname{rank}(B) \geq n.$$

(d) *The system* (1) *has a solution such that* $Q - XPX^* < O$ *precisely when*

$$i_-(AQA^* - CPC^*) - \operatorname{rank}(A) \geq 0,\ i_-\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^+ & O \end{bmatrix} - \operatorname{rank}(B) \geq n.$$

(e) *All general solutions of* (1) *satisfy* $Q - XPX^* \geq O$ *if and only if*

$$n + i_-(AQA^* - CPC^*) - \operatorname{rank}(A) = 0 \ \text{ or } \ i_-\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^* & O \end{bmatrix} - \operatorname{rank}(B) = 0.$$

(f) *All general solutions of* (1) *satisfy* $Q - XPX^* \leq O$ *if and only if*

$$n + i_+(AQA^* - CPC^*) - \operatorname{rank}(A) = 0 \text{ or } i_+\begin{bmatrix} Q & O & D \\ O & -P & B \\ D^* & B^* & O \end{bmatrix} - \operatorname{rank}(B) = 0.$$

(g) *All general solutions of* (1) *satisfy* $Q - XPX^* > O$ *precisely when*

$$n+\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_+(P) + s_+ = n$$

or

$$n+\operatorname{rank}(B)\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_+(P) + t_+ = n.$$

(h) *All general solutions of* (1) *satisfy* $Q - XPX^* < O$ *precisely when*

$$n+\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_-(P) + s_- = n$$

or

$$n+\operatorname{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - \operatorname{rank}(A) - \operatorname{rank}(B) - i_-(P) + t_- = n.$$

(i) *The system* (1) *has a solution such that* $Q = XPX^* \geq O$ *if and only if*

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$$2n+2\text{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - 2\text{rank}(A) - 2\text{rank}(B) - \text{rank}(P) + s_+ + s_- \leq 0,$$

$$2n+2\text{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - 2\text{rank}(A) - 2\text{rank}(B) - \text{rank}(P) + t_+ + t_- \leq 0,$$

$$2n+2\text{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - 2\text{rank}(A) - 2\text{rank}(B) - \text{rank}(P) + s_+ + t_- \leq 0,$$

$$2n+2\text{rank}\begin{bmatrix} O & P & P \\ AQ & CP & O \\ -D^* & O & B^* \end{bmatrix} - 2\text{rank}(A) - 2\text{rank}(B) - \text{rank}(P) + s_- + t_+ \leq 0.$$

At the end of this section, based on Theorem 4, we present the maximal rank and inertia of the Hermitian solutions to (3), which has an additional inequality constraint $MXM^* \geq N$, where $M \in \mathbb{C}^{m\times n}$ and Hermitian $N \in \mathbb{C}^{m\times m}$ are given matrices.

Theorem 16 (Maximal rank and inertia of the Hermitian solutions of (3) over $\mathbb{C}$)**.** *[28] Let $A, C \in \mathbb{C}^{p\times n}, B, D \in \mathbb{C}^{n\times q}, M \in \mathbb{C}^{m\times n}$ and Hermitian $N \in \mathbb{C}^{m\times m}$. Assume that*

$$P = \begin{bmatrix} A \\ B^* \end{bmatrix}, \; Q = \begin{bmatrix} C \\ D^* \end{bmatrix}, \; T = \begin{bmatrix} CA^* & CB \\ D^*A^* & D^*B \end{bmatrix},$$
$$\widehat{M} = M\mathcal{L}_P, \widehat{P} = \mathcal{L}_P\widehat{M}^\dagger, \; \widehat{N} = N - M\left[P^\dagger Q + Q^*\left(P^\dagger\right)^* - P^\dagger T\left(P^\dagger\right)^*\right]M^*,$$
$$X_0 = P^\dagger Q + Q^* P^{\dagger *} - P^\dagger T P^{\dagger *} + \widehat{P}\widehat{N}\left[I - \mathcal{R}_{\widehat{M}}\widehat{N}^\dagger \mathcal{R}_{\widehat{M}}\widehat{N}\right]\widehat{P}^*.$$

Denote the set of all Hermitian solutions to the system (3) by S.

(a) The maximal rank of $X \in S$ is

$$\max_{X\in S} \text{rank}(X) = \min\{s_1, s_2, s_3, s_4, s_5\},$$

where

$$s_1 = n + \text{rank}(Q) - \text{rank}(P), s_2 = n - \text{rank}(P) + \text{rank}(\widehat{P}\,Q^*),$$
$$s_3 = 2n - \text{rank}\begin{bmatrix} M \\ P \end{bmatrix} - \text{rank}(P) + \text{rank}\begin{bmatrix} \widehat{M}\widehat{M}^\dagger & 0 \\ QM^* & QP^* \end{bmatrix},$$
$$s_4 = 2n - \text{rank}\begin{bmatrix} M \\ P \end{bmatrix} + \text{rank}(QM^*\; QP^*) - \text{rank}(P), s_5 = 2n + \text{rank}(QP^*) - 2\,\text{rank}(P).$$

(b) The maximal inertia of $X \in S$ is

$$\max_{X\in S} i_\pm(X) = \min\left\{ n - \text{rank}\begin{bmatrix} M \\ P \end{bmatrix} + i_\pm \begin{bmatrix} 0 & \widehat{M}\widehat{M}^\dagger & 0 \\ \widehat{M}\widehat{M}^\dagger & MX_0M^* & MQ^* \\ 0 & QM^* & QP^* \end{bmatrix}, n + i_\pm(QP^*) - \text{rank}(P) \right\}.$$

2.3. Generalized (Anti-)Reflexive Solutions

The general (anti-)reflexive matrices have wide applications in fields such as engineering and science [32]. A matrix $X \in \mathbb{C}^{n\times n}$ is called (anti-)reflexive with respect to the nontrivial generalized reflection matrix P if $X = (-)PXP$, or equivalently $PX = sXP$, where $s = (-)1$, and P is the nontrivial

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generalized reflection matrix satisfying $P^* = P = P^{-1} \neq I$. $X \in \mathbb{C}^{n\times n}$ is called generalized (anti-)reflexive if $PX = sXGPG^*$, where G is a given unitary matrix of order n, or equivalently $X = (-)PXQ$, where P and Q are nontrivial generalized reflection matrices.

Qiu et al. considered the (anti-)reflexive solutions to system (1), presenting the related results for the (anti-)reflexive solutions and the generalized (anti-)reflexive solutions, along with the corresponding least norm solutions.

Theorem 17 ((Anti-)reflexive solutions for (1) over $\mathbb{C}$). *[20] For given $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$ and the nontrivial generalized reflection matrix $P \in \mathbb{C}^{m\times n}$, let*

$$A_1 = A(I+P),\ A_2 = A(I-P),\ B_1 = (I+sP)B,\ B_2 = (I-sP)B,$$
$$C_1 = C(I+sP),\ C_2 = C(I-sP),\ D_1 = (I+P)D,\ D_2 = (I-P)D.$$

(a) The system $AX = C, XB = D$ has an (anti-)reflexive solution if and only if

$$BC = AD,\ CPB = sAPD,\ C_i = A_iA_i^{\dagger}C_i,\ D_i = D_iB_i^{+}B_i,\ i = 1,2$$

for $s = (-)1$.

(b) In the meantime, the (anti-)reflexive solution is given by

$$X = \sum_{i=1}^{2}(A_i^{+}C_i + \mathcal{L}_{A_i}D_iB_i^{+}) + (\mathcal{L}_{A_1} + \mathcal{L}_{A_2} - I)F(\mathcal{R}_{B_1} + \mathcal{R}_{B_2} - I),$$

with (anti-)reflexive $F \in \mathbb{C}^{m\times n}$.

(c) The least norm (anti-)reflexive solution is expressed as

$$X = \sum_{i=1}^{2}(A_i^{+}C_i + \mathcal{L}_{A_i}D_iB_i^{+}).$$

The relevant conclusions for generalized (anti-)reflexive solutions are presented below.

Theorem 18 (Generalized (anti-)reflexive solutions for (1) over $\mathbb{C}$). *[20–23] For given $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$, the nontrivial generalized reflection matrix $P, Q \in \mathbb{C}^{m\times n}$, let*

$$A_1 = A(I+P),\ A_2 = A(I-P),\ B_1 = (I+sQ)B,\ B_2 = (I-sQ)B,$$
$$C_1 = C(I+sQ),\ C_2 = C(I-sQ),\ D_1 = (I+P)D,\ D_2 = (I-P)D.$$

(a) The system (1) has a generalized (anti-)reflexive solution if and only if

$$CB = AD,\ CQB = sAPD,\ C_i = A_iA_i^{+}C_i,\ D_i = D_iB_i^{+}B_i,\ i = 1,2$$

for $s = (-)1$.

(b) In the meantime, the generalized (anti-)reflexive solution is given by

$$X = \sum_{i=1}^{2}(A_i^{+}C_i + \mathcal{L}_{A_i}D_iB_i^{+}) + (\mathcal{L}_{A_1} + \mathcal{L}_{A_2} - I)F(\mathcal{R}_{B_1} + \mathcal{R}_{B_2} - I), \quad (8)$$

where $F \in \mathbb{C}^{n\times n}$ is generalized (anti-)reflexive.

(c) The least norm generalized (anti-)reflexive solution is expressed as

$$X = \sum_{i=1}^{2}(A_i^{+}C_i + \mathcal{L}_{A_i}D_iB_i^{+}).$$

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(d) Let symbols Φ_r and Φ_a represent the set of all generalized reflexive and anti-reflexive solutions of the system (1), respectively. For given $E \in \mathbb{C}^{n\times n}$, when select

$$F = \frac{1}{2}(E + PEQ)$$

in (8), then (8) is the unique solution of approximation problem $\min\limits_{X\in\Phi_r} ||X - E||$. If

$$F = \frac{1}{2}(E - PEQ),$$

then (8) is the unique solution of $\min\limits_{X\in\Phi_a} ||X - E||$.

Theorem 17 and 18 have presented the generalized (anti-)reflexive solutions of (1). However, matrix decomposition is a more widely used method for solving this special type of solution, which will be introduced in the next chapter.

2.4. Re-nnd and Re-pd Solutions

For a matrix $A \in \mathbb{C}^{n\times n}$, the real part of A is defined as $H(A) = \frac{1}{2}(A + A^*)$. A matrix A is referred to as real nonnegative definite (Re-nnd) if $H(A)$ is positive semi-definite , and A is called real positive definite (Re-pd) if $H(A)$ is positive definite.

Between 2011 and 2014, Xiong, Qin, and Liu explored the Re-nnd and Re-pd solutions to the system (1) [29,30].

Theorem 19 (Re-nnd solutions for (1) over $\mathbb{C}$)**.** *[30] For $A, C \in \mathbb{C}^{p\times m}, B, D \in \mathbb{C}^{n\times q}$, suppose that each equation in (1) has a Re-nnd solution. If the system (1) has a solution, then*

(a) there exists a Re-nnd solution if and only if

$$\operatorname{rank}\begin{bmatrix} A & C \\ B^* & -D^* \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & CA^* \\ B^* & -D^*A^* \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & CB \\ B^* & -D^*B \end{bmatrix}.$$

(b) all the solutions are Re-nnd solutions if and only if $\operatorname{rank}(A) = n$ *or* $\operatorname{rank}(B) = n$.

Theorem 20 (Re-pd solutions for (1) over $\mathbb{C}$)**.** *[30] For $A, C \in \mathbb{C}^{p\times m}, B, D \in \mathbb{C}^{n\times q}$, assume that each equation in (1) has a Re-pd solution. If the system (1) has a solution, then*

(a) there exists a Re-pd solution.

(b) all the solutions are Re-pd solutions if and only if

$$\operatorname{rank}\begin{bmatrix} A & C \\ B^* & -D^* \end{bmatrix} - \min\left\{\operatorname{rank}\begin{bmatrix} A & CA^* \\ B^* & -D^*A^* \end{bmatrix} - \operatorname{rank}(A),\ \operatorname{rank}\begin{bmatrix} A & CB \\ B^* & -D^*B \end{bmatrix} - \operatorname{rank}(B)\right\} = n.$$

Remark 4. *[29] When the system (1) has a Re-nnd (Re-pd) solution, one of the Re-nnd (Re-pd) solutions is given by*

$$X = A^{\dagger}C - (A^{\dagger}C)^* + A^{\dagger}AC^*(A^{\dagger})^* + (I_n - A^{\dagger}A)Y(I_n - A^{\dagger}A),$$

for some Re-nnd (Re-pd) matrix $Y \in \mathbb{C}^{n\times n}$.

In this chapter, we introduced the generalized inverse method for solving the system (1), along with the conclusions regarding special solutions, such as Hermitian, nonnegative, generalized (anti-)reflexive, Re-nnd, and Re-pd solutions. Additionally, we focused on the maximal and minimal ranks and inertias of the solutions to system (1), the conditions under which these extremal values are achieved, and the expression of solutions that satisfy inequality constraints. In Chapter 4, a deeper exploration of these special solutions will be presented using matrix decomposition techniques.

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3. The System (1) over Hilbert Spaces, Hilbert C^*-Modules and Rings

The research mentioned above on matrices can be extended to more general cases, such as Hilbert spaces, Hilbert C^*-modules, and rings. However, there are several limitations when extending to these cases, leading to fewer studies compared to those on matrices. These studies primarily focus on the Hermitian and positive cases, with some scholars also investigating reducible solutions.

Dajić and Koliha were the first to study the system (1) for bounded linear operators between Hilbert spaces with the restriction that A and B have closed ranges. They provided conditions for the existence of general, Hermitian, and positive solutions and obtained formulas for the general form of these solutions [33]. Later, they extended these results from rings to rectangular matrices and operators between complex Hilbert spaces via embedding [34]. In 2008, Xu considered conditions under Hilbert C^*-modules [35]. In 2016, (anti-)reflexive solutions over rings were examined using the inner inverse [36]. In 2021, Radenković et al. reconsidered the system (1) in the context of Hilbert C^*-modules using orthogonally complemented projections, providing alternative expressions [37]. Subsequently, Zhang et al. discussed positive and real positive solutions to (1) over Hilbert spaces using the reduced solution to the system $AX = C$ and $B^*X = D^*$, where the ranges of A and C may not be closed [38,39].

A Hilbert space is a complete inner-product space. A Hilbert C^*-module is a natural generalization of a Hilbert space, obtained by replacing the field of scalars $\mathbb{C}$ with a C^*-algebra. Since finite-dimensional spaces, Hilbert spaces, and C^*-algebras can all be regarded as Hilbert C^*-modules, matrix equations can be studied in a unified manner within the framework of Hilbert C^*-modules. The scope of rings is even broader. In the following statements, "ring" refers to an associative ring R with a unit element $1 \neq 0$.

Hereinafter, we introduce the notations and definitions used in this chapter:

Let H, K, and L denote complex Hilbert spaces, $\mathcal{B}(H,K)$ represent the set of all bounded linear operators between H and K, and $\mathcal{B}(H)$ be the set of all bounded linear operators over H. For $A \in \mathcal{B}(H,K)$, let $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\overline{\mathcal{R}(A)}$ represent the range, the null space, and the closure of the range of the operator A, respectively. An operator $A \in \mathcal{B}(H,K)$ is said to be regular if there exists an operator $A^- \in \mathcal{B}(K,H)$ such that $AA^-A = A$. A^- is referred to as the inner inverse of A. It is well known that A is regular if and only if A has a closed range. For M is a closed subspace of H, P_M denotes the orthogonal projection onto M.

The Hilbert C^*-module is analogous to a Hilbert space, except that its inner product is not scalar-valued, but takes values in a C^*-algebra. Therefore, we continue to use the same notation for the Hilbert module as is used for Hilbert spaces. On Hilbert C^*-modules, a closed submodule M of H is said to be orthogonally complemented in H if $H = M \oplus M^\perp$, where $M^\perp = \{x \in H \mid \langle x, y\rangle = 0, \text{ for any } y \in M\}$. In this case, the projection from H onto M is denoted by P_M.

For an arbitrary ring R with involution $x \mapsto x^*$, an element $a \in R$ is Hermitian if $a = a^*$. If there exists $b \in R$ such that $aba = a$, then a is said to be regular (or inner invertible), and b is called the inner inverse of a, denoted a^-.

Initially, the general solution of (1) over rings is presented.

Theorem 21 (General solutions for (1) over a ring)**.** *[34,36] Let $a, b, c, d \in R$ such that a and b are regular elements. Then, the following statements are equivalent.*

(a) There exists a solution $x \in R$ of the system of equations $ax = c$, $xc = d$.

(b) $cb = ad$, $c = aa^-c$, and $d = db^-b$.

Moreover, if (a) or (b) is satisfied, then any solution of $ax = c$, $xc = d$ can be expressed as

$$x = a^-c + (1 - a^-a)db^- + (1 - a^-a)f(1 - bb^-)$$

for any $f \in R$.

Remark 5. *In [34], the authors further extended the results from elements in the ring to matrices over R, as well as to bounded linear operators between complex Banach or Hilbert spaces by constructing an embedding.*

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Next, the expression for the general solution in Hilbert C^*-modules is provided through orthogonal complements.

Theorem 22 (General solutions for (1) over Hilbert C^*-modules)**.** *[37] Let H, K, L be Hilbert C^*-modules, $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$ such that $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B)}$ are orthogonally complemented. Then, the system (1) has a general solution if and only if*

$$\mathcal{R}(C) \subseteq \mathcal{R}(A),\ \mathcal{R}(D^*) \subseteq \mathcal{R}(B^*),\ A^\dagger AD = A^\dagger CB.$$

In such case, the general solution has the form of

$$X = A^\dagger C + (I - A^\dagger A)[(B^*)^\dagger D^*(I - A^\dagger A)]^* + (I - A^\dagger A)Z[I - (B^*)^\dagger B^*],$$

where $Z \in \mathcal{B}(H)$ is arbitrary.

Remark 6. *Actually, $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B)}$ being orthogonally complemented implies that A and B are regular and have closed ranges. Additionally, Theorem 22 uses the Moore-Penrose inverse in place of the inner inverse. The definition of the Moore-Penrose inverse is similar to that in matrices, and therefore will not be elaborated further.*

In 2023, Zhang et al. extended this result to infinite-dimensional Hilbert spaces without the requirement that the corresponding operators A and B have closed ranges, using reduced matrices.

Theorem 23 (General solutions for (1) over a Hilbert space)**.** *[38] Let $A, B, C, D \in \mathcal{B}(H)$, and $P = P_{\overline{\mathcal{R}(A^*)}}$. Then, the system (1) has a solution if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{R}(D^*) \subseteq \mathcal{R}(B^*)$, and $AD = CB$. In this case, the general solution can be represented by*

$$X = F - (I - P)H^* + (I - P)Z(I - P_{\overline{\mathcal{R}(B)}}),$$

where F is the reduced solution of $AX = C$, H is the reduced solution of $B^*X = D^*$, $Z \in \mathcal{B}(H)$ is arbitrary. Specifically, if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, the general solution can be represented by

$$X = A^\dagger C + DB^\dagger - A^\dagger ADB^\dagger + (I - A^\dagger A)Z(I - BB^\dagger).$$

The situations of Hermitian solutions, positive solutions, real positive solutions, and reflexive solutions will be introduced in the following sections.

3.1. Hermitian Solutions

The Hermitian solution of (1) over Hilbert space was first studied by Dajić and Koliha.

Theorem 24 (Hermitian solutions for (1) over a Hilbert space)**.** *[33,34] Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$, and the operators A and B have closed ranges. Assume $M = B^*(I - A^- A)$ have a closed range, $T = D^* - B^* A^- C$. Let A^-, B^-, and M^- represent the inner inverses of A, B, and M, respectively. Then, the system (1) has a Hermitian solution $X \in \mathcal{B}(H)$ if and only if*

$$AA^- C = C,\ AD = CB,\ (I - MM^-)D^* = (I - MM^-)B^* A^- C, \tag{9}$$

and AC^* and B^*D are Hermitian. The general Hermitian solution is given by

$$\begin{aligned} X = {} & A^- C + (I - A^- A)M^- T + (I - A^- A)(I - M^- M)[A^- C + (I - A^- A)M^- T]^* \\ & + (I - A^- A)(I - M^- M)U(I - M^- M)^*(I - A^- A)^*, \end{aligned} \tag{10}$$

where $U \in \mathcal{B}(H)$ is Hermitian.

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Remark 7. *In [34], the authors presented an alternative form of (9) and (10):*

$$\mathcal{N}(B) \subset \mathcal{N}(D),\ AD = CB,\ (I - NN^{\dagger})C = (I - NN^{\dagger})A(B^{-})^{*}D^{*}$$

and

$$\begin{aligned} X = {} & B^{-*}D^{*} + (I - BB^{-})^{*}N^{-}Q + (I - BB^{-})^{*}(I - N^{-}N)[B^{-*}D^{*} + (I - BB^{-})N^{-}Q]^{*} \\ & + (I - BB^{-})^{*}(I - N^{-}N)V(I - N^{-}N)^{*}(I - BB^{-}), \end{aligned}$$

where $N = A(I - BB^{\dagger})$, $Q = C - AB^{-*}D^{*}$, and $V \in \mathcal{B}(H)$ is Hermitian. Furthermore, Theorem 24 holds true in C^{*}-Hilbert modules and rings with involution as well [34,35].

By using the projection operator, Theorem 24 can be restated in another form over Hilbert C^{*}-modules.

Theorem 25 (General solutions for (1) over Hilbert C^{*}-modules)**.** *[37] Let H, K, L be Hilbert C^{*}-modules, and $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$ such that $\overline{\mathcal{R}(A^{*})}$ and $\overline{\mathcal{R}(B)}$ are orthogonally complemented. Let $M = B^{*}(I - A^{\dagger}A)$, and assume that $\overline{\mathcal{R}(M^{*})}$ is orthogonally complemented. Then, the system (1) has a Hermitian solution if and only if*

$$\mathcal{R}(C) \subseteq \mathcal{R}(A),\ \mathcal{R}(S) \subseteq \mathcal{R}(M),\ A^{\dagger}AD = A^{\dagger}CB,$$

and CA^{*} and SM^{*} are Hermitian, where $S = (D^{*} - B^{*}A^{\dagger}C)(I - A^{\dagger}A)$. In this case, the Hermitian solution has the form

$$\begin{aligned} X = {} & A^{\dagger}C + (I - A^{\dagger}A)(A^{\dagger}C)^{*} + (I - A^{\dagger}A)(D'')^{*}(I - A^{\dagger}A) \\ & + (I - A^{\dagger}A)D''(I - M^{\dagger}M)(I - A^{\dagger}A) + (I - A^{\dagger}A)(I - M^{\dagger}M)Z(I - M^{\dagger}M)(I - A^{\dagger}A), \end{aligned}$$

where $D'' = M^{\dagger}S$ and $Z \in \mathcal{B}(H)$ is an arbitrary Hermitian matrix.

3.2. Positive Solutions

For the cases where the system (1) has a positive solution over Hilbert space, two different descriptions are presented as following.

Theorem 26 (Positive solutions for (1) over a Hilbert space)**.** *[33] Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$,*

$$\hat{A} = \begin{bmatrix} A \\ B^{*} \end{bmatrix},\ \hat{C} = \begin{bmatrix} C \\ D^{*} \end{bmatrix},\ Q = \begin{bmatrix} CA^{*} & CB \\ (AD)^{*} & D^{*}B \end{bmatrix}.$$

Assume that $\hat{A}$ and Q are regular. The system (1) has a positive solution $X \in \mathcal{B}(H)$ if and only if Q is positive and $\mathcal{R}(\hat{C}) \subseteq \mathcal{R}(Q)$. The general positive solution is given by

$$X = \hat{C}^{*}Q^{-}\hat{C} + (I - \hat{A}^{-}\hat{A})T(I - \hat{A}^{-}\hat{A})^{*},$$

where $T \in \mathcal{B}(H)$ is an arbitrary positive matrix.

Theorem 27 (Positive solutions for (1) over a Hilbert space)**.** *[33,35] If $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$, $M = B^{*}(I - A^{-}A)$, and $G = D^{*} - (CB)^{*}(CA^{*})^{-}C$, such that A, B, M, CA^{*} and GB are regular, $\mathcal{R}(CB) \subseteq \mathcal{R}(CA^{*})$. Then, (1) has a positive solution if and only if*

$$CA^{*} = AC^{*},\ D^{*}B = B^{*}D,\ CB = AD,\ \mathcal{R}(C) = \mathcal{R}(CA^{*}),\ \mathcal{R}(G) = \mathcal{R}(GB),$$

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and CA^* and F are positive. The general positive solution is given by

$$X = C^*(CA^*)^- C + G^* F^- G + (I - A^- A)(I - M^- M)T(I - M^- M)^*(I - A^- A)^*,$$

where $T \in \mathcal{B}(H)$ is positive.

Remark 8. *Dajić and Koliha extended Theorems 26 and 27 to strongly $*$-reducing rings with involution, which is an extension of the C^*-Hilbert modules [34].*

Zhang et al. presented the existence and the general form of the positive solutions of (1) without the restriction on the closed range over Hilbert space.

Theorem 28 (Positive solutions using projection operators for (1) over a Hilbert space)**.** *[38] Let A, B, C, and D be operators in $\mathcal{B}(H)$. Let $P = P_{\mathcal{R}(A^*)}$ and $Q = P_{\mathcal{R}((I-P)B)}$. The system (1) has positive solutions if and only if the following conditions hold.*
(a) $\mathcal{R}(D^*) \subseteq \mathcal{R}(B^*)$, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, and $AD = CB$.
(b) $\mathcal{R}(G) \subseteq \mathcal{R}((GP)^{\frac{1}{2}})$ and $CA^* \geq 0$.
(c) $(D^* - B^*G - B^*H^*H)(I-P)C \geq 0$, $\mathcal{R}((D^* - B^*G)(I-P)) \subseteq \mathcal{R}(C^*(I-P))$, and $\mathcal{R}(K) \subseteq \mathcal{R}((KQ)^{\frac{1}{2}})$.
In which G, H, and K are the reduced solutions of $AX = C$, $(GP)^{\frac{1}{2}}X = G(I-P)$, and $B^*(I-P)X = (D^* - B^*G - B^*(I-P)H^*H)(I-P)$, respectively. Than, the positive solution is given by

$$\begin{aligned} X = {} & G + (I-P)G^* + (I-P)H^*H(I-P) + (I-P)K^*(I-P) \\ & + (I-P)K(I-P-Q) + (I-P-Q)L^*L(I-P-Q) + (I-P-Q)Z(I-P-Q), \end{aligned}$$

for any positive operator $Z \in \mathcal{B}(H)$, where L is the reduced solution of $(KQ)^{\frac{1}{2}}X = K(I-Q)$.

The conclusions for the Hilbert C^*-modules can be directly obtained using the projection operator.

Theorem 29 (Positive solutions using projection operators for (1) over Hilbert C^*-modules)**.** *[37] Let H, K, L be Hilbert C^*-modules, $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$ such that $\overline{\mathcal{R}(A^*)}$ and $\overline{\mathcal{R}(B)}$ are orthogonally complemented. Denote $P = A^\dagger A$, $M = B^*(I-P)$, $D' = A^\dagger C$, $S' = [D^* - B^*D' - M(D')^*(D'P)^\dagger D'](I-P)$ and $D'' = M^\dagger S'$. Assume that $\overline{\mathcal{R}(M^*)}$ is orthogonally complemented, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{R}(D') = \mathcal{R}(D'P)$, along with $CA^* \in \mathcal{L}(H)$ is positive. Then, the system (1) has a positive solution if and only if*

$$PD = D'B, \; \mathcal{R}(S') \subseteq \mathcal{R}(M), \; \mathcal{R}(D'') = \mathcal{R}(D''M^\dagger M),$$

and $S'M^* \in \mathcal{L}(H)$ is positive. In such case, the general solution has the form

$$X = X_0 + (I-P)(I - M^\dagger M)Z(I - M^\dagger M)(I-P),$$

where

$$\begin{aligned} X_0 &= D' + (I-P)(D')^* + (I-P)(D')^*(D'P)^\dagger D'(I-P) + (I-P)Z_0(I-P), \\ Z_0 &= D'' + (I - M^\dagger M)(D'')^* + (I - M^\dagger M)(D'')^*(D''M^\dagger M)^\dagger D''(I - M^\dagger M), \end{aligned}$$

with $Z \in \mathcal{B}(H)$ is arbitrary positive.

3.3. Re-pd Solutions

The general form of the real positive solutions of (1) without the restriction on the closed range over Hilbert spaces was also provided by Zhang et al.

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Theorem 30 (Re-pd solutions for (1) over a Hilbert space)**.** *[39] Let $A, B, C, D \in \mathcal{B}(H)$, $P = P_{\overline{\mathcal{R}(A^*)}}$, and $Q = P_{\mathcal{R}((I-P)B)}$. The system (1) has a real positive solution if the following conditions are satisfied.*

(a) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{R}(D^*) \subseteq \mathcal{R}(B^*)$, and $AD = CB$.

(b) CA^*, $(D^* + B^*F)(I-P)B$ are real positive operators.

(c) $\mathcal{R}((D^* + B^*F)(I-P)) \subseteq \mathcal{R}(B^*(I-P))$, where F is the reduced solution of $AX = C$. In this case, one of the real positive solutions can be represented as

$$X = F - (I-P)F^* + (I-P)H^*(I-P) - (I-P)H(I-P-Q) + (I-P-Q)Z(I-P-Q),$$

where H is the reduced solution of $B^*(I-P)X = (D^* + B^*F)(I-P)$, $Z \in \mathcal{B}(H)$ is arbitrary Re-pd.

3.4. (Anti-)Reflexive Solutions

Načevska used algebraic methods in rings with involution to obtain a generalization of (anti-)reflexive solutions over complex matrices [36].

An element $w \in R$ is said to be a generalized reflection element if $w = w^*$ and $w^2 = 1$. For $w, v \in R$ being generalized reflection elements, an element $x \in R$ is called a generalized reflexive element (with respect to w and v) if $x = wxv$, denoted by $R_r(w,v)$, as well x is called a generalized anti-reflexive element (with respect to w and v) if $x = -wxv$, denoted by $R_{ar}(w,v)$.

For $a, b, c, d \in R$, these elements can be decomposed using projections as follows [36]:

$$\begin{aligned} a &= a_1 + a_2 \in R_r(1,w) \oplus R_{ar}(1,w), \\ b &= b_1 + b_2 \in R_r(v,1) \oplus R_{ar}(v,1), \\ c &= c_1 + c_2 \in R_r(1,v) \oplus R_{ar}(1,v), \\ d &= d_1 + d_2 \in R_r(w,1) \oplus R_{ar}(w,1). \end{aligned} \tag{11}$$

Define

$$\hat{a_1} = \frac{1}{2}(a_1^- + wa_1^-),\ \hat{a_2} = \frac{1}{2}(a_2^- - wa_2^-),\ \hat{b_1} = \frac{1}{2}(b_1^- + b_1^- v),\ \hat{b_2} = \frac{1}{2}(b_2^- - b_2^- v). \tag{12}$$

Theorem 31 (Reflexive solutions for (1) over a ring)**.** *[36] Let $a, b, c, d \in \mathcal{R}$ and (11) hold such that a_1, a_2, b_1 and b_2 are regular elements. Then, the following statements are equivalent.*

(a) There exists a solution $x \in R_r(w,v)$ of the system (1).

(b) $c_1b_1 = a_1d_1, c_1 = a_1\hat{a_1}c_1, d_1 = d_1\hat{b_1}b_1, c_2b_2 = a_2d_2, c_2 = a_2\hat{a_2}c_2$, and $d_2 = d_2\hat{b_2}b_2$.

Moreover, if (a) or (b) is satisfied, then any generalized reflexive solution of (1) can be expressed as

$$x = x_0 + efg,$$

where

$$\begin{aligned} x_0 &= \hat{a_1}c_1 + (1 - \hat{a_1}a_1)d_1\hat{b_1} + \hat{a_2}c_2 + \left(1 - \hat{b_2}b_2\right)d_2\hat{b_2}, \\ e &= 1 - \hat{a_1}a_1 - \hat{a_2}a_2,\ g = 1 - b_1\hat{b_1} - c_2\hat{b_2}, \end{aligned}$$

for $\hat{a_1}, \hat{a_2}, \hat{c_1}, \hat{c_2}$ given by (12) and arbitrary $f \in R_r(w,v)$.

Theorem 32 (Anti-reflexive solutions for (1) over a ring)**.** *[36] Let $a, b, c, d \in R$ and (11) hold such that a_1, a_2, b_1, b_2 are regular elements. Then, the following statements are equivalent.*

(a) There exists a solution $x \in R_{ar}(w,v)$ of the system (1).

(b) $c_2b_2 = a_1d_1, c_2 = a_1\hat{a_1}c_2, d_2 = d_2\hat{b_1}b_1, c_1b_1 = a_2d_2, c_1 = a_2\hat{a_2}c_1$ and $d_1 = d_1\hat{b_2}c_2$.

Moreover, if (a) or (b) is satisfied, then any generalized anti-reflexive solution of (1) can be expressed as

$$x = x_0 + efg,$$

where

$$x_0 = \hat{a}_1 c_2 + (1 - \hat{a}_1 a_1) d_1 \hat{b}_2 + \hat{a}_2 c_1 + (1 - \hat{a}_2 a_2) d_2 \hat{b}_1,$$
$$e = 1 - \hat{a}_1 a_1 - \hat{a}_2 a_2, \; g = 1 - b_1 \hat{b}_1 - b_2 \hat{b}_2,$$

for $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ *given by* (12) *and arbitrary* $f \in R_{ar}(w, v)$.

This chapter mainly introduces the system (1) over Hilbert spaces, Hilbert C^*-modules, and rings, focusing on tools such as inner inverses, orthogonal complements, and reducibility.

4. Matrix Decomposition Methods for Solving (1)

Matrix decomposition techniques, such as eigenvalue decomposition (EVD), singular value decomposition (SVD), QR decomposition, LU decomposition, and others, play a crucial role in solving matrix equations. They are especially important in finding special solutions for system (1), such as mirror-symmetric, skew-symmetric, orthogonal symmetric, unitary, $\{P, Q, k\}$-reflexive, (Hermitian) R-conjugate, (R, S)-conjugate, and Hamiltonian solutions, as well as the corresponding least squares solutions. These types of solutions, which are difficult to calculate using generalized inverses alone, can be efficiently computed using matrix decompositions. This chapter will introduce the related methods and results.

We introduce the eigenvalue decomposition (EVD). Let A be an $n \times n$ matrix with n linearly independent eigenvectors q_i for $i = 1, \cdots, n$. Then, A can be factored as

$$A = Q \Lambda Q^*,$$

where Q is the square $n \times n$ matrix whose i-th column is the eigenvector q_i of A, and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, with $\Lambda_{ii} = \lambda_i$. However, it is important to note that not all square matrices are diagonalizable. In such cases, a more practical form of decomposition is the singular value decomposition.

The singular value decomposition (SVD) of a given matrix $A \in \mathbb{C}^{m \times n}$ of rank k is

$$A = U \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} V^*,$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_k)$, with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.

In 1981, Paige and Saunders extended the B-singular value decomposition in [40] to the generalized singular value decomposition (GSVD) for two matrices with the same number of columns. This is a powerful tool for solving equations. The GSVD can be expressed as the following theorem.

Theorem 33 (The generalized singular value decomposition). *[41] Let* $A \in \mathbb{C}^{m \times n}$ *and* $B \in \mathbb{C}^{p \times n}$ *be two matrices with the same number of columns. Denote* $k = \mathrm{rank}\left[A^{\mathrm{T}}, B^{\mathrm{T}}\right]$. *There exist unitary matrices* U *and* V*, and a nonsingular matrix* Q*, such that*

$$A = U \Sigma_A Q, \; B = V \Sigma_B Q,$$

where

$$\Sigma_A = \left[\begin{array}{ccc|c} I & & & \\ & S_A & & O \\ & & O & \end{array}\right], \; \Sigma_B = \left[\begin{array}{ccc|c} O & & & \\ & S_B & & O \\ & & I & \end{array}\right],$$

with $S_A = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_s)$, $S_B = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_s), 1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0, 1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_s > 0$, *and* $\alpha_i^2 + \beta_i^2 = 1$ *for* $i = 1, 2, \cdots, s$.

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It is important to note that the GSVD has various representations, which can not be exhaustively listed. In the subsequent discussion, alternative forms of the GSVD will be presented.

Next, we present the necessary and sufficient conditions for the system (1) and the expression of the general solutions through the SVD.

Theorem 34 (General solutions using the SVD for (1) over $\mathbb{C}$)**.** *For $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$, $\operatorname{rank}(A) = r_1$ and $\operatorname{rank}(B) = r_2$, the SVD of A and C expressed as*

$$A = U_1 \begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} V_1^*,\ B = U_2 \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix} V_2^*,$$

where $\Sigma_A = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{r_1})$ and $\Sigma_B = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{r_2})$. Denote

$$\hat{C} = U_1^* C U_2 = \begin{matrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} & \begin{matrix} r_1 \\ p-r_1 \end{matrix} \\ \begin{matrix} r_1 & n-r_1 \end{matrix} & \end{matrix} \quad \text{and } \hat{D} = V_1^* D V_2 = \begin{matrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}. & \begin{matrix} r_2 \\ m-r_2 \end{matrix} \\ \begin{matrix} r_2 & q-r_2 \end{matrix} & \end{matrix}$$

Then, the system (1) is consistent if and only if $C_{12}, C_{22}, D_{12}, D_{22}$ are zero matrices and $\Sigma_A^{-1} C_{11} = D_{11} \Sigma_B^{-1}$. The general solution can be expressed as

$$X = V_1 \begin{bmatrix} \Sigma_A^{-1} C_{11} & \Sigma_A^{-1} C_{12} \\ D_{21} \Sigma_B^{-1} & Z \end{bmatrix} U_2^*,$$

where $Z \in \mathbb{C}^{(n-r_1)\times(m-r_2)}$ is arbitrary.

Remark 9. *Theorem 34 is a special form of the system $AXB = E, FXG = H$, which is considered by the GSVD in [42].*

4.1. Various symmetric solutions

In this section, we consider the least squares forms of various symmetric solutions to the system (1) over $\mathbb{R}$, including (k, p)-mirror(skew-)symmetric solutions, symmetric solutions, and bi-(anti-)symmetric solutions. The existence conditions and expressions for symmetric solutions in subspaces are also be discussed.

In 2006, Li et al. considered the least squares (k, p)-mirror-symmetric solutions through the SVD of matrices over $\mathbb{R}$ [43].

A (k, p)-mirror matrix $W_{(k,p)}$ is defined by

$$W_{(k,p)} = \begin{bmatrix} O & O & J_k \\ O & I_p & O \\ J_k & O & O \end{bmatrix},$$

where J_k is the k-square backward identity matrix with ones along the secondary diagonal and zeros elsewhere. A matrix $A \in \mathbb{R}^{(2k+p)\times(2k+p)}$ is called a (k, p)-mirror-symmetric matrix if and only if

$$A = W_{(k,p)} A W_{(k,p)}.$$

We denote the set of all (k, p)-mirror(skew-)symmetric matrices by $MS_{(k,p)}$.

The results about least squares (k, p)-mirror(skew-)symmetric solutions are as follows.

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Theorem 35 (Least squares (k,p)-mirror(skew-)symmetric solutions for (1) over $\mathbb{R}$). *[43] Let $A, C \in \mathbb{R}^{h\times(2k+p)}$, $B, D \in \mathbb{R}^{(2k+p)\times l}$ and*

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & O & -J_k \\ O & \sqrt{2}I_p & O \\ J_k & O & I_k \end{bmatrix}.$$

Denote

$$AK = [A_1, A_2],\ CK = [C_1, C_2],\ K^{\mathrm{T}}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ K^{\mathrm{T}}D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, C_1 \in \mathbb{R}^{h\times(k+p)}$, $B_1, D_1 \in \mathbb{R}^{(k+p)\times l}$, $A_2, C_2 \in \mathbb{R}^{h\times k}$, $B_2, D_2 \in \mathbb{R}^{k\times l}$. Denote the SVDs of A_1, A_2, B_1, B_2 as

$$A_1 = U_1\begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix}V_1^{\mathrm{T}} = [U_{11}, U_{12}]\begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix}\begin{bmatrix} V_{11}^{\mathrm{T}} \\ V_{12}^{\mathrm{T}} \end{bmatrix} = U_{11}\Sigma_1 V_{11}^{\mathrm{T}},$$

$$A_2 = U_2\begin{bmatrix} \Sigma_2 & O \\ O & O \end{bmatrix}V_2^{\mathrm{T}} = [U_{21}, U_{22}]\begin{bmatrix} \Sigma_2 & O \\ O & O \end{bmatrix}\begin{bmatrix} V_{21}^{\mathrm{T}} V_{22}^{\mathrm{T}} \end{bmatrix} = U_{21}\Sigma_2 V_{21}^{\mathrm{T}},$$

$$B_1 = P_1\begin{bmatrix} \tau_1 & O \\ O & O \end{bmatrix}Q_1^{\mathrm{T}} = [P_{11}, P_{12}]\begin{bmatrix} \tau_1 & O \\ O & O \end{bmatrix}\begin{bmatrix} Q_{11}^{\mathrm{T}} Q_{12}^{\mathrm{T}} \end{bmatrix} = P_{11}\tau_1 Q_{11}^{\mathrm{T}},$$

$$B_2 = P_2\begin{bmatrix} \tau_2 & O \\ O & O \end{bmatrix}Q_2^{\mathrm{T}} = [P_{21}, P_{22}]\begin{bmatrix} \tau_2 & O \\ O & O \end{bmatrix}\begin{bmatrix} Q_{21}^{\mathrm{T}} \\ Q_{22}^{\mathrm{T}} \end{bmatrix} = P_{21}\tau_2 Q_{21}^{\mathrm{T}},$$

with $\Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_{r_1})$, $\Sigma_2 = \mathrm{diag}(\delta_1, \cdots, \delta_{r_2})$, $\tau_1 = \mathrm{diag}(\alpha_1, \cdots, \alpha_{s_1})$, $\tau_2 = \mathrm{diag}(\beta_1, \cdots, \beta_{s_2})$, $r_1 = \mathrm{rank}(A_1)$, $r_2 = \mathrm{rank}(A_2)$, $s_1 = \mathrm{rank}(B_1)$, $s_2 = \mathrm{rank}(B_2)$.

(a) Then, the general solution $X \in \mathbb{R}^{(2k+p)\times(2k+p)}$ for the problem $\|A_1X - C_1\|^2 + \|XB_1 - D_1\|^2 = \min$ can be expressed as

$$X = V_1\begin{bmatrix} \Phi * [\Sigma_1 U_{11}^{\mathrm{T}} C_1 P_{11} + V_{11}^{\mathrm{T}} D_1 Q_{11}\tau_1] & \Sigma_1^{-1}U_{11}^{\mathrm{T}} C_1 P_{12} \\ V_{12}^{\mathrm{T}} D_1 Q_{11}\tau_1^{-1} & X_{22} \end{bmatrix}P_1^{\mathrm{T}},$$

where $\Phi = (\varphi_{ij})$, $\varphi_{ij} = \frac{1}{\sigma_i^2+\alpha_j^2}$, $1 \le i \le r_1$, $1 \le j \le s_1$, and $X_{22} \in \mathbb{R}^{(2k+p-r_1)\times(2k+p-s_1)}$ is arbitrary.

(b) Then, $\|AX - C\|^2 + \|XB - D\|^2 = \min$ has a solution $X \in MS_{(k,p)}$. Moreover, the general solution can be given by

$$X = K\begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix}K^{\mathrm{T}},$$

where

$$X_1 = V_1\begin{bmatrix} \Phi * (\Sigma_1 U_{11}^{\mathrm{T}} B_1 P_{11} + V_{11}^{\mathrm{T}} D_1 Q_{11}\tau_1) & \Sigma_1^{-1}U_{11}^{\mathrm{T}} B_1 P_{12} \\ V_{12}^{\mathrm{T}} D_1 Q_{11}\tau_1^{-1} & X_{22} \end{bmatrix}P_1^{\mathrm{T}},$$

$$X_2 = V_2\begin{bmatrix} \Phi' * (\Sigma_2 U_{21}^{\mathrm{T}} B_{21}^{\mathrm{T}} B_2 P_{21} + V_{21}^{\mathrm{T}} D_2 Q_{21}\tau_2) & \Sigma_2^{-1}U_{21}^{\mathrm{T}} B_2 P_{22} \\ V_{22}^{\mathrm{T}} D_2 Q_{21}\tau_2^{-1} & X'_{22} \end{bmatrix}P_2^{\mathrm{T}},$$

where $\Phi = (\varphi_{ij})$, $\varphi_{ij} = \frac{1}{\sigma_i^2+\alpha_j^2}$, $1 \le i \le r_1$, $1 \le j \le s_1$, $\Phi' = (\varphi'_{ij})$, $\varphi'_{ij} = \frac{1}{\delta_i^2+\eta_j^2}$, $1 \le i \le r_2$, $1 \le j \le s_2$, $X_{22} \in \mathbb{R}^{(k-r_2)\times(k-s_2)}$ and $X'_{22} \in R^{(k-r_2)\times(k-s_2)}$ are arbitrary.

(c) Let S_E represent the solution set of $\|AX - C\|^2 + \|XB - D\|^2 = \min$ with $X \in MS_{(k,p)}$. For given $E \in \mathbb{R}^{(2k+p)\times(2k+p)}$, $||X - E|| = \min$ has a unique solution $\tilde{X} \in S_E$. Moreover, $\tilde{X}$ can be expressed as

$$\tilde{X} = K\begin{bmatrix} \tilde{X}_1 & O \\ O & \tilde{X}_2 \end{bmatrix}K^{\mathrm{T}},$$

where

$$\tilde{X}_1 = V_1 \begin{bmatrix} \Phi * (\Sigma_1 U_{11}^{\mathrm{T}} B_1 P_{11} + V_{11}^{\mathrm{T}} D_1 Q_{11} \tau_1) & \Sigma_1^{-1} U_{11}^{\mathrm{T}} B_1 P_{12} \\ V_{12}^{\mathrm{T}} D_1 Q_{11} \tau_1^{-1} & V_{12}^{\mathrm{T}} X_{11}^* P_{12} \end{bmatrix} P_1^{\mathrm{T}},$$
$$\tilde{X}_2 = V_2 \begin{bmatrix} \Phi' * (\Sigma_2 U_{21}^{\mathrm{T}} B_2 P_{21} + V_{21}^{\mathrm{T}} D_2 Q_{21} \tau_2) & \Sigma_2^{-1} U_{21}^{\mathrm{T}} B_2 P_{22} \\ V_{22}^{\mathrm{T}} D_2 Q_{21} \tau_2^{-1} & V_{22}^{\mathrm{T}} E_{22} P_{22} \end{bmatrix} P_2^{\mathrm{T}},$$
$$X_{11}^* = \frac{1}{2} \begin{bmatrix} I_k & O & J_k \\ O & \sqrt{2} I_p & O \end{bmatrix} X_1^* \begin{bmatrix} I_k & O \\ O & \sqrt{2} I_p \\ J_k & O \end{bmatrix},$$
$$X_1^* = \frac{1}{2}\left(X^* + W_{(k,p)} X^* W_{(k,p)}\right),\ E_{22} = \frac{1}{2}[-J_k, O, I_k] X_1^* \begin{bmatrix} -J_k \\ O \\ I_k \end{bmatrix}.$$

with Φ *and* Φ' *being given in* (b).

In 2010, Yuan considered the least squares solutions of the linear equation system (1) with a different form of Theorem 35 (a) and the least squares symmetric solutions [44].

Theorem 36 (Least squares symmetric solutions for (1) over $\mathbb{R}$)**.** *[44] Assume that* $A \in \mathbb{R}^{p\times m}, B \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{p\times n}$ *and* $D \in \mathbb{R}^{m\times q}$. *Let the SVDs of the matrices* A, B *be given by*

$$A = U \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} V^{\mathrm{T}},\ B = P \begin{bmatrix} \Omega & O \\ O & O \end{bmatrix} Q^{\mathrm{T}},$$

where $U = [U_1, U_2], V = [V_1, V_2], P = [P_1, P_2], Q = [Q_1, Q_2]$ *are all orthogonal matrices and the partitions are compatible with the sizes of* $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_{r_1})$ *and* $\Omega = \mathrm{diag}(\omega_1, \cdots, \omega_{r_2})$, $r_1 = \mathrm{rank}(A)$, $r_2 = \mathrm{rank}(B)$.

(a) *The least squares solution set of* (1) *can be expressed as*

$$S_1 = \left\{ X \,\middle|\, X = V \begin{bmatrix} \Phi * [V_1^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_1] & (\Sigma^{-1})^2 V_1^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_2 \\ V_2^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_1 (\Omega^{-1})^2 & X_{22} \end{bmatrix} P^{\mathrm{T}} \right\},$$

where $X_{22} \in \mathbb{R}^{(n-r_1)\times(n-r_2)}$ *is an arbitrary matrix. The unique least norm least squares solution can be expressed as*

$$\widehat{X} = V \begin{bmatrix} \Phi * [V_1^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_1] & (\Sigma^{-1})^2 V_1^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_2 \\ V_2^{\mathrm{T}} (A^{\mathrm{T}} C + D B^{\mathrm{T}}) P_1 (\Omega^{-1})^2 & O \end{bmatrix} P^{\mathrm{T}}.$$

(b) *Consider the condition* $m = n$. *Let the EVD of the matrix* $A^{\mathrm{T}} A + B B^{\mathrm{T}}$ *be given by*

$$A^{\mathrm{T}} A + B B^{\mathrm{T}} = W \begin{bmatrix} \Gamma & O \\ O & O \end{bmatrix} W^{\mathrm{T}},$$

where $W = [W_1, W_2]$ *is an orthogonal matrix and the partition is compatible with the size of* $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_l)$, $l = \mathrm{rank}(A^{\mathrm{T}} A + B B^{\mathrm{T}})$. *Then, the least squares symmetric solution set of* (1) *can be expressed as*

$$S_2 = \left\{ X \in \mathbb{R}^{n\times n} \,\middle|\, X = W \begin{bmatrix} \Psi_* (W_1^{\mathrm{T}} H W_1) & \Gamma^{-1} W_1^{\mathrm{T}} H W_2 \\ W_2^{\mathrm{T}} H W_1 \Gamma^{-1} & X_{22} \end{bmatrix} W^{\mathrm{T}} \right\},$$

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where $H = C^{\mathrm{T}}A + A^{\mathrm{T}}C + DB^{\mathrm{T}} + BD^{\mathrm{T}}$ and $X_{22} \in \mathbb{R}^{(n-l)\times(n-l)}$ is an arbitrary symmetric matrix. The unique least norm least squares symmetric solution can be expressed as

$$\widehat{X} = W\begin{bmatrix} \Psi_*(W_1^{\mathrm{T}}HW_1) & \Gamma^{-1}W_1^{\mathrm{T}}HW_2 \\ W_2^{\mathrm{T}}HW_1\Gamma^{-1} & O \end{bmatrix}W^{\mathrm{T}}.$$

In 2014, Ke and Ma derived the generalized bi-(skew-)symmetric solutions of the system (1) with the corresponding least squares solution [31].

For a symmetric orthogonal matrix $P \in \mathbb{R}^{n\times n}$, a matrix $X \in \mathbb{R}^{n\times n}$ is called a generalized bi-(skew-)symmetric matrix if and only if $(-X) = X^{\mathrm{T}} = PXP$.

Theorem 37 (Least squares bi-(skew-)symmetric solutions for (1) over $\mathbb{R}$)**.** *[31] Assume that $A, C \in \mathbb{R}^{p\times n}$, $B, D \in \mathbb{R}^{n\times q}$, and $P \in \mathbb{R}^{n\times n}$ is a symmetric orthogonal matrix. Let P be decomposed as*

$$P = U\begin{bmatrix} I_r & O \\ O & -I_{n-r} \end{bmatrix}U^{\mathrm{T}},$$

where U is a symmetric orthogonal matrix. Let the partitions of AU, CU, B^TU, and D^TU be

$$AU = [A_1, A_2],\ CU = [C_1, C_2],\ B^{\mathrm{T}}U = [B_1^{\mathrm{T}}, B_2^{\mathrm{T}}],\ D^{\mathrm{T}}U = [D_1^{\mathrm{T}}, D_2^{\mathrm{T}}],$$

with $A_1, C_1 \in \mathbb{R}^{p\times r}$, $A_2, C_2 \in \mathbb{R}^{p\times(n-r)}$, $C_1, D_1 \in \mathbb{R}^{r\times q}$, $C_2, D_2 \in \mathbb{R}^{(p-r)\times q}$, respectively.

(a) Denote that $K_1 = B_1^T\mathcal{L}_{A_1}$, $N_1 = \mathcal{R}_{B_1}A_1^T$, $Q_1 = D_1^T - B_1^TA_1^\dagger C_1 - K_1D_1B_1^\dagger$, $\overline{Q} = C_1^T - A_1^\dagger C_1A_1^T - \mathcal{L}_{A_1}D_1B_1^\dagger A_1^T - \mathcal{L}_{A_1}K_1^\dagger Q_1N_1$, $K_2 = B_2^T\mathcal{L}_{A_2}$, $N_2 = \mathcal{R}_{B_2}A_2^T$, $Q_2 = D_2^T - B_2^TA_2^\dagger C_2 - K_2D_2B_2^\dagger$, $\widetilde{Q} = C_2^\dagger - A_2^\dagger C_2A_2^T - \mathcal{L}_{A_2}D_2B_2^\dagger A_2^T - \mathcal{L}_{A_2}K_2^\dagger Q_2N_2$. Then, (1) has bi-symmetric solutions if and only if equations

$$\begin{gathered} K_1K_1^\dagger Q_1\mathcal{R}_{B_1} = Q_1,\ \overline{Q}\mathcal{L}_{N_1} = O,\ \mathcal{R}_{\mathcal{L}_{A_1}}\mathcal{L}_{K_1}\overline{Q} = O, \\ K_2K_2^\dagger Q_2\mathcal{R}_{B_2} = Q_2,\ \widetilde{Q}\mathcal{L}_{N_2} = O,\ \mathcal{R}_{\mathcal{L}_{A_2}}\mathcal{L}_{K_2}\widetilde{Q} = O, \\ A_1D_1 = C_1B_1,\ A_1A_1^\dagger C_1 = C_1,\ D_1B_1^\dagger B_1 = D_1,\ A_2D_2 = C_2B_2, A_2A_2^\dagger C_2 = C_2, D_2B_2^\dagger B_2 = D_2 \end{gathered}$$

hold. Under such circumstance, the bi-symmetric solutions can be expressed as

$$X = U\begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix}U^{\mathrm{T}},$$

where

$$\begin{aligned} X_{11} &= \frac{1}{2}(A_1^\dagger C_1 + \mathcal{L}_{A_1}D_1C_1^\dagger + \mathcal{L}_{A_1}K_1^\dagger Q_1\mathcal{R}_{B_1} + \overline{Q}N_1^\dagger\mathcal{R}_{B_1} + \mathcal{L}_{A_1}\mathcal{L}_{K_1}Z_1\mathcal{R}_{N_1}\mathcal{R}_{B_1}) \\ &\quad + \frac{1}{2}(C_1^{\mathrm{T}}A_1^{\mathrm{T}\dagger} + B_1^{\mathrm{T}\dagger}D_1^{\mathrm{T}}\mathcal{L}_{A_1} + \mathcal{R}_{B_1}Q_1^{\mathrm{T}}K_1^{\dagger\mathrm{T}}\mathcal{L}_{A_1} + \mathcal{R}_{B_1}N_1^{\mathrm{T}\dagger}\overline{Q}^{\mathrm{T}} + \mathcal{R}_{B_1}\mathcal{R}_{N_1}Z_1^{\mathrm{T}}\mathcal{L}_{K_1}\mathcal{L}_{A_1}), \\ X_{22} &= \frac{1}{2}(A_2^\dagger C_2 + \mathcal{L}_{A_2}D_2B_2^\dagger + \mathcal{L}_{A_2}K_2^\dagger Q_2\mathcal{R}_{B_2} + \widetilde{Q}N_2^\dagger\mathcal{R}_{B_2} + \mathcal{L}_{A_2}\mathcal{L}_{K_2}Z_2\mathcal{R}_{N_2}\mathcal{R}_{C_2}). \end{aligned}$$

(b) Let $K = B_1^T\mathcal{L}_{A_1}$, $N = \mathcal{L}_{A_2}B_2$, $Q_1 = -D_2^T - B_1^TA_1^\dagger C_1 + KC_1^TA_2^{\dagger T}$, $Q = D_1 - A_1^\dagger C_2B_2 + \mathcal{L}_{A_1}C_1^TA_2^{\dagger T}B_2 - \mathcal{L}_{A_1}K_1^\dagger Q_1N$. Then, (1) has bi-skew-symmetric solutions if and only if

$$\begin{gathered} KK^\dagger Q_1\mathcal{R}_{A_2^T} = Q_1,\ Q\mathcal{L}_N = 0,\ \mathcal{R}_{\mathcal{L}_{A_1}}\mathcal{L}_KQ = 0, \\ A_1C_1^{\mathrm{T}} = -C_2A_2^{\mathrm{T}},\ A_1A_1^\dagger C_2 = C_2,\ D_2B_1^\dagger B_1 = D_2,\ A_2A_2^\dagger C_1 = C_1,\ D_1B_2^\dagger B_2 = D_1 \end{gathered}$$

hold. Under such circumstance, the bi-skew-symmetric solutions can be expressed as

$$X = U\begin{bmatrix} O & X_{12} \\ -X_{12}^{\mathrm{T}} & O \end{bmatrix}U^{\mathrm{T}},$$

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where

$$X_{12} = A_1^{\dagger}C_2 - \mathcal{L}_{A_1}C_1^{\mathrm{T}}A_2^{\mathrm{T}\dagger} + \mathcal{L}_{A_1}K^{\dagger}Q_1\mathcal{R}_{A_2^{\mathrm{T}}} + QN^{\dagger}\mathcal{R}_{A_2^{\mathrm{T}}} + \mathcal{L}_{A_1}\mathcal{L}_K Z\mathcal{R}_N\mathcal{R}_{A_2^{\mathrm{T}}}.$$

Additionally, let the SVDs of $[A_1^{\mathrm{T}}, B_1]$ *and* $[A_2^{\mathrm{T}}, B_2]$ *be*

$$[A_1^{\mathrm{T}}, B_1] = V\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}W^{\mathrm{T}},\ [A_2^{\mathrm{T}}, B_2] = T\begin{bmatrix}\Omega & 0\\ 0 & 0\end{bmatrix}Q^{\mathrm{T}},$$

where $V = [V_1, V_2]$, $W = [W_1, W_2]$, $T = [T_1, T_2]$, *and* $Q = [Q_1, Q_2]$ *are all orthogonal matrices and the partitions are compatible with the sizes of* $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_s)$, $\Omega = \mathrm{diag}(\omega_1, \cdots, \omega_t)$, $s = \mathrm{rank}[A_1^{\mathrm{T}}, B_1]$, $t = \mathrm{rank}[A_2^{\mathrm{T}}, B_2]$.

(c) *The least squares bi-symmetric solutions of* (1) *can be expressed as*

$$X = U\begin{bmatrix}X_{11} & O\\ O & X_{22}\end{bmatrix}U^{\mathrm{T}},$$

where

$$X_{11} = V\begin{bmatrix}\Phi_1[V_1^{\mathrm{T}}(C_1^{\mathrm{T}}D_1)W_1\Sigma + \Sigma W_1^{\mathrm{T}}(C_1^{\mathrm{T}}D_1)^{\mathrm{T}}V_1] & \Sigma^{-1}W_1^{\mathrm{T}}(C_1^{\mathrm{T}}D_1)^{\mathrm{T}}V_2\\ V_2^{\mathrm{T}}(C_1^{\mathrm{T}}D_1)W_1\Sigma^{-1} & G_1\end{bmatrix}V^{\mathrm{T}},$$
$$X_{22} = T\begin{bmatrix}\Phi_2[T_1^{\mathrm{T}}(C_2^{\mathrm{T}}D_2)Q_1\Omega + \Omega Q_1^{\mathrm{T}}(C_2^{\mathrm{T}}D_2)^{\mathrm{T}}T_1] & \Omega^{-1}Q_1^{\mathrm{T}}(C_2^{\mathrm{T}}D_2)^{\mathrm{T}}T_2\\ T_2^{\mathrm{T}}(C_2^{\mathrm{T}}D_2)Q_1\Omega^{-1} & G_2\end{bmatrix}T^{\mathrm{T}},$$

$\Phi_1 = (\varphi_{ij}) \in \mathbb{R}^{s\times s}$, $\varphi_{ij} = \frac{1}{\sigma_i^2+\sigma_j^2}$, $1 \le i,j \le s$, $\Phi_2 = (\varphi_{ij}) \in \mathbb{R}^{t\times t}$, $\varphi_{ij} = \frac{1}{\omega_i^2+\omega_j^2}$, $1 \le i,j \le t$, $G_1 \in \mathbb{R}^{(r-s)\times(r-s)}$ *and* $G_2 \in \mathbb{R}^{(n-r-t)\times(n-r-t)}$ *are arbitrary symmetric matrices.*

(d) *The least squares bi-skew-symmetric solutions of* (1) *can be expressed as*

$$X = U\begin{bmatrix}O & X_{12}\\ -X_{12}^{\mathrm{T}} & O\end{bmatrix}U^{\mathrm{T}},$$

where

$$X_{12} = V\begin{bmatrix}\Phi(V_1^{\mathrm{T}}X_0T_1) & (\Sigma^{-1})^2V_1^{\mathrm{T}}X_0T_2\\ V_2^{\mathrm{T}}X_0T_1(\Omega^{-1})^2 & G\end{bmatrix}T^{\mathrm{T}}$$

with $X_0 = A_1^{\mathrm{T}}C_2 - C_1^{\mathrm{T}}A_2 - B_1D_2^{\mathrm{T}} + D_1B_2^{\mathrm{T}}$, $\Phi = (\varphi_{ij}) \in \mathbb{R}^{s\times t}$, $\varphi_{ij} = \frac{1}{\sigma_i^2+\omega_j^2}$, $1 \le i \le s$, $1 \le j \le t$, *and* $G \in \mathbb{R}^{(r-s)\times(n-r-t)}$ *being an arbitrary matrix.*

Hu and Yuan have considered the symmetric solutions of (1) on a subspace Ω [45]. Let $S\mathbb{R}_{\Omega}^{n\times n}$ be the set of all $n \times n$ symmetric matrices on subspace Ω, where

$$\Omega = \{z \in \mathbb{R}^n \mid Gz = O, G \in \mathbb{R}^{n\times n}\}.$$

The necessary and sufficient conditions for the system (1) to have a solution in $S\mathbb{R}_{\Omega}^{n\times n}$ and also an expression for the solution X are obtained. Additionally, the associated optimal approximation problem to a given matrix $E \in \mathbb{R}^{n\times n}$ is discussed, and the optimal solution is elucidated.

Theorem 38 ($S\mathbb{R}_{\Omega}$ solutions for (1))**.** *[45] Given* $A, C \in \mathbb{R}^{p\times n}$ *and* $B, D \in \mathbb{R}^{n\times q}$*. Assume that the SVD of* G *is given by*

$$G = U_0\begin{bmatrix}\Sigma & O\\ O & O\end{bmatrix}V_0^{\mathrm{T}},$$

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where $\Sigma = \mathrm{diag}(\gamma_1, \cdots, \gamma_s) > 0$, $s = \mathrm{rank}(G)$, $U_0 = [U_{01}, U_{02}]$, $V_0 = [V_{01}, V_{02}]$ *are orthogonal matrices with* $U_{01}, V_{01} \in \mathbb{R}^{n\times s}$ *and* $U_{02}, V_{02} \in \mathbb{R}^{n\times(n-s)}$. *Let*

$$\begin{aligned}
&AV_0 = A_1,\ CV_0 = B_1,\ V_0^{\mathrm{T}}B = B_1,\ V_0^{\mathrm{T}}D = D_1,\\
&P_1 = [I_s, O],\ P_2 = [O, I_{n-s}],\ Q_1 = \begin{bmatrix} I_s \\ O \end{bmatrix},\ Q_2 = \begin{bmatrix} O \\ I_{n-s} \end{bmatrix},\\
&P_1\mathcal{L}_{A_1} = A_2,\ \mathcal{R}_{B_1}Q_1 = B_2,\ P_2\mathcal{L}_{A_1} = A_3,\ \mathcal{R}_{B_1}Q_2 = B_3,\\
&G = P_2A_1^{\dagger}C_1Q_2 + A_3D_1B_1^{\dagger}Q_2 - (P_2A_1^{\dagger}C_1Q_2 + A_3D_1C_1^{\dagger}Q_2)^{\mathrm{T}},\\
&L = \mathcal{L}_{B_3}A_3A_3^{\dagger},\ P = -\frac{1}{2}G(I_{n-s} - A_3A_3^{\dagger}) - \frac{1}{2}G(Q + Q^{\mathrm{T}})A_3A_3^{\dagger},\\
&Q = 2L^{\dagger}\mathcal{L}_{B_3}G + (I_{n-s} - L^{\dagger}\mathcal{L}_{B_3})GL^{\dagger}L.
\end{aligned}$$

(*a*) *The system* (1) *is solvable over* $S\mathbb{R}_{\Omega}^{n\times n}$ *if and only if*

$$\begin{aligned}
&A_1A_1^{\dagger}C_1^{\dagger} = C_1,\ D_1B_1B_1^{\dagger} = D_1,\ A_1D_1 = C_1B_1,\\
&\mathcal{R}_{A_3}G\mathcal{R}_{A_3} = 0,\ \mathcal{L}_{B_3}G\mathcal{L}_{B_3} = 0,\ [A_3, B_3]^{\dagger}[A_3, B_3]^{\dagger^{\mathrm{T}}}A_3 = A_3.
\end{aligned}$$

In which cases, the solution set S_E *can be expressed as*

$$S_E = \left\{ X \in \mathbb{R}^{n\times n} \middle| X = V_0 \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} V_0^{\mathrm{T}} \right\},$$

where

$$\begin{aligned}
X_{11} &= P_1A_1^{\dagger}C_1Q_1 + A_2D_1B_1^{\dagger}Q_1 + A_2A_3^{\dagger}PB_3^{\dagger}B_2 + A_2A_3^{\dagger}\mathcal{L}_LW\mathcal{L}_{L^{\mathrm{T}}}A_3A_3^{\dagger}B_3^{\dagger}B_2\\
&\quad + A_2ZB_2 - A_2A_3^{\dagger}A_3ZB_3^{\dagger}B_2,\\
X_{12} &= P_1A_1^{\dagger}C_1Q_2 + A_2D_1B_1^{\dagger}Q_2 + A_2A_3^{\dagger}PB_3^{\dagger}B_3 + A_2A_3^{\dagger}\mathcal{L}_LW\mathcal{L}_{L^{\mathrm{T}}}A_3A_3^{\dagger}B_3^{\dagger}B_3\\
&\quad + A_2ZB_3 - A_2A_3^{\dagger}A_3ZB_3,\\
X_{21} &= P_2A_1^{\dagger}C_1Q_1 + A_3D_1B_1^{\dagger}Q_1 + A_3A_3^{\dagger}PB_3^{\dagger}C_2 + A_3A_3^{\dagger}\mathcal{L}_LW\mathcal{L}_{L^{\mathrm{T}}}A_3A_3^{\dagger}B_3^{\dagger}B_2\\
&\quad + A_3ZB_2 - A_3A_3^{\dagger}A_3ZB_3^{\dagger}B_2,\\
X_{22} &= P_2A_1^{\dagger}C_1Q_2 + A_3D_1B_1^{\dagger}Q_2 + A_3A_3^{\dagger}PB_3^{\dagger}B_3 + A_3A_3^{\dagger}\mathcal{L}_LW\mathcal{L}_{L^{\mathrm{T}}}A_3A_3^{\dagger}B_3^{\dagger}B_3,
\end{aligned} \tag{13}$$

W, Z *are arbitrary matrices with* $W = W^{\mathrm{T}}$.

(*b*) *For* $E \in \mathbb{R}^{n\times n}$, *let*

$$V_0^{\mathrm{T}}EV_0 = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

where $E_{11} \in \mathbb{R}^{s\times s}$, $E_{22} \in \mathbb{R}^{(n-s)\times(n-s)}$. *The optimal problem* $||E - X|| = \min_{X\in S_E}$ *has the unique solution* $\widehat{X}$ *admitting*

$$\widehat{X} = V_0 \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} V_0^{\mathrm{T}},$$

where X_{11}, X_{12}, X_{21} *and* X_{22} *are given by* (13) *with* $W = W^{\mathrm{T}}$, *and* Z *is determined by solving the unique solution of* (44) *in* [45].

4.2. Various Orthogonality Solutions

This section introduces various orthogonal solutions to the system (1) over $\mathbb{R}$. Wang et al. extended the conditions for various symmetric solutions to the equations $AX = C$ or $XB = D$ to the system (1), providing a series of conclusions. Qiu et al. constructed special matrices and used their EVDs to derive the corresponding conclusions [46,47].

Wang et al. have considered the orthogonality, (skew-)symmetric orthogonality, and least squares (skew-)symmetric orthogonality solutions, as well as the necessary and sufficient conditions for (1) to have these solutions and their corresponding expressions, respectively [46].

Theorem 39 (Orthogonality solutions for (1) over $\mathbb{R}$)**.** *[46] Given $A, C \in \mathbb{R}^{n\times m}$ and $B, D \in \mathbb{R}^{m\times n}$.*

(a) Suppose the GSVD of A and C is

$$A = U\begin{bmatrix}\Sigma & O\\ O & O\end{bmatrix}V^T,\ C = U\begin{bmatrix}\Sigma & O\\ O & O\end{bmatrix}Q^T,$$

where $\Sigma = \mathrm{diag}(\delta_1, \cdots, \delta_l)$, $l = \mathrm{rank}(A) = \mathrm{rank}(C)$, $U = [U_1, U_2] \in \mathbb{R}^{n\times n}, V = [V_1, V_2], Q = [Q_1, Q_2] \in \mathbb{R}^{m\times m}$ are orthogonal, $U_1 \in \mathbb{R}^{n\times l}$, $V_1, Q_1 \in \mathbb{R}^{m\times l}$, $U_2 \in \mathbb{R}^{n\times(n-l)}$, V_2, $Q_2 \in \mathbb{R}^{m\times(m-l)}$. Denote

$$Q^T B = \begin{bmatrix}B_1\\ B_2\end{bmatrix},\ V^T D = \begin{bmatrix}D_1\\ D_2\end{bmatrix},$$

where $B_1, D_1 \in \mathbb{R}^{l\times n}$ and $B_2, D_2 \in \mathbb{R}^{(m-l)\times n}$. Let the GSVD of B_2 and D_2 be

$$B_2 = \widetilde{U}\begin{bmatrix}\Pi & O\\ O & O\end{bmatrix}\widetilde{V}^T,\ D_2 = \widetilde{W}\begin{bmatrix}\Pi & O\\ O & O\end{bmatrix}\widetilde{V}^T,$$

where $\widetilde{U}, \widetilde{W} \in \mathbb{R}^{(m-l)\times(m-l)}, \widetilde{V} \in \mathbb{R}^{n\times n}$ are orthogonal, $\Pi \in \mathbb{R}^{k\times k}$is diagonal. Then, the system (1) has orthogonal solutions if and only if

$$AA^T = CC^T,\ B_1 = D_1,\ D_2^T D_2 = B_2^T B_2.$$

In which case, the orthogonal solutions can be expressed as

$$X = \widehat{V}\begin{bmatrix}I_{k+l} & O\\ O & G'\end{bmatrix}\widehat{Q}^T,$$

where

$$\widehat{V} = V\begin{bmatrix}I_l & O\\ O & \widetilde{W}\end{bmatrix} \in \mathbb{R}^{m\times m},\ \widehat{Q} = Q\begin{bmatrix}I_l & O\\ O & \widetilde{U}\end{bmatrix} \in \mathbb{R}^{m\times m}$$

are orthogonal, and $G' \in \mathbb{R}^{(m-k-l)\times(m-k-l)}$ is an arbitrary orthogonal matrix.

(b) Let the GSVD of B and D be

$$B = U\begin{bmatrix}\Pi & O\\ O & O\end{bmatrix}V^T,\ D = W\begin{bmatrix}\Pi & O\\ O & O\end{bmatrix}V^T,$$

where $\Pi = \mathrm{diag}(\sigma_1, \cdots, \sigma_k)$, $k = \mathrm{rank}(B) = \mathrm{rank}(D)$, $U = [U_1, U_2], W = [W_1, W_2] \in \mathbb{R}^{m\times m}$ and $V = [V_1, V_2] \in \mathbb{R}^{n\times n}$ are orthogonal, $U_1, W_1 \in \mathbb{R}^{m\times k}$, $V_1 \in \mathbb{R}^{n\times k}, U_2, W_2 \in \mathbb{R}^{m\times(m-k)}$, $V_2 \in \mathbb{R}^{n\times(n-k)}$. Partition

$$AW = [A_1, A_2],\ CU = [C_1, C_2],$$

where $A_1, C_1 \in \mathbb{R}^{n\times k}, A_2, C_2 \in \mathbb{R}^{n\times(m-k)}$. Assume the GSVD of A_2 and B_2 is

$$A_2 = \widetilde{U}\begin{bmatrix}\Sigma & O\\ O & O\end{bmatrix}\widetilde{V}^T,\ C_2 = \widetilde{U}\begin{bmatrix}\Sigma & O\\ O & O\end{bmatrix}\widetilde{Q}^T,$$

where $\widetilde{V}, \widetilde{Q} \in \mathbb{R}^{(m-k)\times(m-k)}, \widetilde{U} \in \mathbb{R}^{n\times n}$ *are orthogonal, and* $\Sigma \in \mathbb{R}^{l'\times l'}$ *is diagonal. Then, the system* (1) *has orthogonal solutions if and only if*

$$D^TD = B^TB,\ A_1 = C_1,\ A_2A_2^T = C_2C_2^T.$$

In which case, the orthogonal solutions can be expressed as

$$X = \widehat{W}\begin{bmatrix} I_{k+l'} & O \\ O & H' \end{bmatrix}\widehat{U}^T,$$

where

$$\widehat{W} = W\begin{bmatrix} I_k & O \\ O & \widetilde{W} \end{bmatrix} \in \mathbb{R}^{m\times m},\ \widehat{U} = U\begin{bmatrix} I_k & O \\ O & \widetilde{U} \end{bmatrix} \in \mathbb{R}^{m\times m},$$

with arbitrary orthogonal $H' \in \mathbb{R}^{(m-k-l')\times(m-k-l')}$.

Remark 10. *Some researchers have also investigated the correction of the coefficient matrices when the system* (1) *is inconsistent under orthogonal constraints [48], specifically focusing on the optimization problem*

$$\min_{X,E_1,E_2} (\|E_1\|^2 + \|E_2\|^2)$$

subject to the constraints

$$\begin{cases} (A+E_1)X = C, \\ X(B+E_2) = D, \\ XX^{\mathrm{T}} = I. \end{cases}$$

Theorem 40 (Symmetric orthogonality solutions for (1) over $\mathbb{R}$)**.** *[46] Given* $A, C \in \mathbb{R}^{n\times m}, B, D \in \mathbb{R}^{m\times n}$. *Notations* $Q, D_2, R_2, \widetilde{U}, \widetilde{W}$ *are defined in Theorem 39.*

(*a*) *Let the symmetric orthogonal solutions of the matrix equation* $AX = C$ *be described as in*

$$X = \widetilde{V}\begin{bmatrix} I_{2l-r} & O \\ O & G \end{bmatrix}\widetilde{Q}^T,$$

where $\widetilde{Q} \in \mathbb{R}^{m\times m}$ *and* $\widetilde{V} \in \mathbb{R}^{m\times m}$ *are orthogonal,* $G \in \mathbb{R}^{(m-2l+r)\times(m-2l+r)}$ *is an arbitrary symmetric orthogonal matrix. Partition*

$$\widetilde{Q}^TB = \begin{bmatrix} B_1' \\ B_2' \end{bmatrix},\ \widetilde{V}^TD = \begin{bmatrix} D_1' \\ D_2' \end{bmatrix},$$

where $B_1', D_1' \in \mathbb{R}^{(2l-r)\times n}, B_2', D_2' \in \mathbb{R}^{(m-2l+r)\times n}$. *Then, the system* (1) *has symmetric orthogonal solutions if and only if*

$$AA^T = CC^T,\ AC^T = CA^T,\ B_1' = D_1',\ D_2'^TD_2' = B_2'^TB_2',\ D_2'^TB_2' = B_2'^TD_2'.$$

In which case, the solutions can be expressed as

$$X = \widehat{V}\begin{bmatrix} I & O \\ O & G'' \end{bmatrix}\widehat{Q}^T,$$

where

$$\widehat{V} = V\begin{bmatrix} I_{2l-r} & O \\ O & \widetilde{W} \end{bmatrix},\ \widehat{Q} = Q\begin{bmatrix} I_{2l-r} & O \\ O & \widetilde{U}, \end{bmatrix}$$

and $G'' \in \mathbb{R}^{(m-2l+r-2l'+r')\times(m-2l+r-2l'+r')}$ *is arbitrary symmetric orthogonal.*

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(b) Let the symmetric orthogonal solutions of the matrix equation $XB = D$ be described as

$$X = \widetilde{M}\begin{bmatrix} I_{2k-r} & O \\ O & G \end{bmatrix}\widetilde{N}^T,$$

where $\widetilde{M}, \widetilde{N} \in \mathbb{R}^{m\times m}$ are orthogonal and $G \in \mathbb{R}^{(m-2k+r)\times(m-2k+r)}$ is symmetric orthogonal. Partition

$$A\widetilde{M} = [M_1, M_2],\ C\widetilde{N} = [N_1, N_2],$$

where $M_1, N_1 \in \mathbb{R}^{n\times(2k-r)}, M_2, N_2 \in \mathbb{R}^{n\times(m-2k+r)}$. Then, the system (1) has symmetric orthogonal solutions if and only if

$$D^TD = B^TB,\ D^TB = B^TD,\ M_1 = N_1,\ M_2M_2^T = N_2N_2^T,\ M_2N_2^T = N_2M_2^T.$$

In which case, the solutions can be expressed as

$$X = \widehat{M}\begin{bmatrix} I & O \\ O & H'' \end{bmatrix}\widehat{N}^T,$$

where

$$\widehat{M} = \widetilde{M}\begin{bmatrix} I_{2k-r} & O \\ O & \widetilde{W}_1 \end{bmatrix},\ \widehat{N} = \widetilde{N}\begin{bmatrix} I_{2k-r} & O \\ O & \widetilde{U}_1 \end{bmatrix},$$

and $H'' \in \mathbb{R}^{(m-2k+r-2k'+r')\times(m-2k+r-2k'+r')}$ is an arbitrary symmetric orthogonal matrix.

Theorem 41 (Skew-symmetric orthogonality solutions for (1) over $\mathbb{R}$)**.** *[46] Given $A, C \in \mathbb{R}^{n\times 2m}, B, D \in \mathbb{R}^{2m\times n}$.*

(a) Suppose the matrix equation $AX = C$ has skew-symmetric orthogonal solutions with the form

$$X = \widetilde{V}_1\begin{bmatrix} I & O \\ O & H \end{bmatrix}\widetilde{Q}_1^T,$$

where $H \in \mathbb{R}^{2r\times 2r}$ is arbitrary skew-symmetric orthogonal, $\widetilde{V}_1, \widetilde{Q}_1 \in \mathbb{R}^{2m\times 2m}$ is orthogonal. Partition

$$\widetilde{Q}_1^TB = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},\ \widetilde{V}_1^TD = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where $Q_1, V_1 \in \mathbb{R}^{(2m-2r)\times n}, Q_2, V_2 \in \mathbb{R}^{2r\times n}$. Then, the system (1) has skew-symmetric orthogonal solutions if and only if

$$AA^T = CC^T,\ AC^T = -CA^T,\ Q_1 = V_1,\ Q_2^TQ_2 = V_2^TV_2,\ Q_2^TV_2 = -V_2^TQ_2.$$

In which case, the solutions can be expressed as

$$X = \widehat{V}_1\begin{bmatrix} I & O \\ O & J' \end{bmatrix}\widehat{Q}_1^T,$$

where

$$\widehat{V} = \widetilde{V}_1\begin{bmatrix} I_{2m-2r} & O \\ O & \widetilde{W} \end{bmatrix},\ \widehat{Q} = \widetilde{Q}_1\begin{bmatrix} I_{2m-2r} & O \\ O & \widetilde{U} \end{bmatrix},$$

and $J' \in \mathbb{R}^{2k\times 2k}$ is an arbitrary skew-symmetric orthogonal matrix.

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(b) Suppose the matrix equation $XB = D$ has skew-symmetric orthogonal solutions with the form

$$X = \widetilde{W} \begin{bmatrix} I & O \\ O & K \end{bmatrix} \widetilde{U}^T,$$

where $K \in \mathbb{R}^{2p \times 2p}$ is arbitrary skew-symmetric orthogonal, $\widetilde{W}, \widetilde{U} \in \mathbb{R}^{2m \times 2m}$ are orthogonal. Partition

$$A\widetilde{W} = [W_1, W_2],\ C\widetilde{U} = [U_1, U_2],$$

where $W_1, U_1 \in \mathbb{R}^{n \times (2m-2p)}, W_2, U_2 \in \mathbb{R}^{n \times 2p}$. Then, the system (1) has skew-symmetric orthogonal solutions if and only if

$$D^T D = B^T B,\ D^T B = -B^T D,\ W_1 = U_1,\ W_2 W_2^T = U_2 U_2^T,\ U_2 W_2^T = -W_2 U_2^T.$$

In which case, the solutions can be expressed as

$$X = \widehat{W}_1 \begin{bmatrix} I & O \\ O & J'' \end{bmatrix} \widehat{U}_1^T,$$

where

$$\widehat{W}_1 = \widetilde{W} \begin{bmatrix} I_{2m-2p} & O \\ O & \widetilde{W}_1 \end{bmatrix},\ \widehat{U}_1 = \begin{bmatrix} I_{2m-2p} & O \\ O & \widetilde{U}_1^T \end{bmatrix} \widetilde{U},$$

and $J'' \in \mathbb{R}^{2q \times 2q}$ is an arbitrary skew-symmetric orthogonal matrix.

Theorem 42 (Least squares (skew-)symmetric orthogonality solutions for (1) over $\mathbb{R}$)**.** *[46] Given $A, C \in \mathbb{R}^{n \times 2m}$ and $B, D \in \mathbb{R}^{2m \times n}$, denote*

$$A^{\mathrm{T}} C + DB^{\mathrm{T}} = K,\ \frac{1}{2}(K^{\mathrm{T}} - K) = T,\ \frac{1}{2}(K^{\mathrm{T}} + K) = N,$$

Let the EVDs of T and N be

$$T = E \begin{bmatrix} \Lambda & O \\ O & O \end{bmatrix} E^{\mathrm{T}},\ N = M \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} M^{\mathrm{T}},$$

with $\Lambda = \mathrm{diag}(\Lambda_1, \cdots, \Lambda_l)$, $\Lambda_i = \begin{bmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{bmatrix}$, $\alpha_i > 0, i = 1, \cdots, l$, $2l = \mathrm{rank}(T)$, $\Sigma = \begin{bmatrix} \Sigma^{\dagger} & O \\ O & \Sigma^{-} \end{bmatrix}$, $\Sigma^{\dagger} = \mathrm{diag}(\lambda_1 \cdots, \lambda_s)$, $\lambda_1 \geq \cdots \geq \lambda_s > 0$, $\Sigma^{-} = \mathrm{diag}(\lambda_{s+1} \cdots, \lambda_t)$, $\lambda_t \leq \cdots \leq \lambda_{s+1} < 0$, $t = \mathrm{rank}(N)$.

(a) Then, the least squares symmetric orthogonal solutions of the system (1) can be expressed as

$$X = M \begin{bmatrix} \bar{I} & O \\ O & L \end{bmatrix} M^{\mathrm{T}},$$

with $\bar{I} = \begin{bmatrix} I_s & O \\ O & -I_{t-s} \end{bmatrix}$ and $L \in \mathbb{R}^{(n-t) \times (n-t)}$ being an arbitrary symmetric orthogonal matrix.

(b) Then, the least squares skew-symmetric orthogonal solutions of the system (1) can be expressed as

$$X = E \begin{bmatrix} \hat{I} & O \\ O & G \end{bmatrix} E^{\mathrm{T}},$$

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with $\hat{I} = \mathrm{diag}(\hat{I}_1, \cdots, \hat{I}_l)$, $\hat{I}_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, *for* $i = 1, \cdots, l$ *and* $G \in \mathbb{R}^{(2m-2l)\times(2m-2l)}$ *is an arbitrary skew-symmetric orthogonal matrix.*

Remark 11. *Theorems 39, 40, 41, 42 actually treat the system* $AX = C, XB = D$ *as an extension of the single equation* $AX = C$ *or* $XB = D$*. The proof of these theorems is based on the perspective that one of the equations has a corresponding solution.*

Qiu et al. consider the least squares orthogonality, symmetric orthogonality, symmetric idempotence, and their corresponding P-commuting matrix solutions of (1).

Theorem 43 (Least squares orthogonality, symmetric orthogonality and symmetric idempotence solutions for (1) over $\mathbb{R}$)**.** *[47] Assume that* $A, C \in \mathbb{R}^{p\times n}$*,* $B, D \in \mathbb{R}^{n\times q}$*.*

(a) Let $W_1 = A^{\mathrm{T}}C + DB^{\mathrm{T}}$*. Denote the SVD of* W_1 *is*

$$W_1 = U\begin{bmatrix} \Sigma_r & O \\ O & O \end{bmatrix}V^{\mathrm{T}},$$

where $U, V \in \mathbb{R}^{n\times n}$ *are orthogonal matrices,* $\Sigma_r = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$*,* $r = \mathrm{rank}(W_1)$*. Then, the least squares orthogonal solutions to (1) satisfying*

$$X = U\begin{bmatrix} I_r & O \\ O & G \end{bmatrix}V^{\mathrm{T}},$$

where $G \in \mathbb{R}^{(n-r)\times(n-r)}$ *is arbitrary orthogonal.*

(b) Denote $W_1 = A^{\mathrm{T}}C + DB^{\mathrm{T}}$ *and* $W_2 = W_1 + W_1^{\mathrm{T}}$ *with* W_2 *is an orthogonal matrix. Let the EVD of* W_2 *be given by*

$$W_2 = U\mathrm{diag}(\lambda_1 I_{l_1}, \cdots, \lambda_t I_{l_t})U^{\mathrm{T}},$$

where $U \in \mathbb{R}^{n\times n}$ *is an orthogonal matrix,* $\sum_{j=1}^{t} l_j = n$*. Then, the least squares symmetric orthogonal solutions to (1) are expressed as*

$$X = U\begin{bmatrix} -I_{n_s-l_s} & & \\ & G_{l_s} & \\ & & I_{n-n_s} \end{bmatrix}U^{\mathrm{T}},$$

where $G_{l_s} \in \mathbb{R}^{l_s\times l_s}$ *is arbitrary symmetric orthogonal.*

(c) Denote $W_3 = A^{\mathrm{T}}A + BB^{\mathrm{T}} - 2(A^{\mathrm{T}}C + DB^{\mathrm{T}})$ *and* $W_4 = W_3 + W_3^{\mathrm{T}}$*. Let the EVD of* W_4 *be given by*

$$W_4 = U\mathrm{diag}(\lambda_1 I_{l_1}, \cdots, \lambda_t I_{l_t})U^{\mathrm{T}},$$

where $U \in \mathbb{R}^{n\times n}$*,* $\sum_{j=1}^{t} l_j = n$*. Then, the least squares symmetric idempotent solutions to (1) are expressed as*

$$X = U\begin{bmatrix} I_{n_s-l_s} & & \\ & G_{l_s} & \\ & & I_{n-n_s} \end{bmatrix}U^{\mathrm{T}},$$

with arbitrary symmetric idempotent $G_{l_s} \in \mathbb{R}^{l_s\times l_s}$*.*

Additionally, we generalize the corresponding P-commuting constraints, where $P \in \mathbb{R}^{n\times n}$ is a given symmetric matrix. Let the EVD of P be

$$P = H\mathrm{diag}(\tilde{\lambda}_1 I_{k_1}, \cdots, \tilde{\lambda}_p I_{k_p})H^{\mathrm{T}},$$

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where $k_1 + \cdots + k_p = n$ and $H \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. A matrix X commutes with P, (i.e. $PX = XP$), if and only if

$$X = H\text{diag}(X_1, \cdots, X_p)H^{\text{T}}, \tag{14}$$

where $X_i \in \mathbb{R}^{k_i\times k_i}$ for $i = 1, \cdots, p$.

Theorem 44 (Least squares orthogonality, symmetric orthogonality and symmetric idempotence solutions commuting with P for (1) over $\mathbb{R}$)**.** *[47] Assume that $A, C \in \mathbb{R}^{p\times n}$, $B, D \in \mathbb{R}^{n\times q}$.*

(a) Suppose that $\bar{W} = H^{\text{T}}W_1H$ with $W_1 = A^{\text{T}}B + DC^{\text{T}}$. Partition the matrix $\bar{W} = (W_{ij})$ conforming to (14), where $W_{ij} \in \mathbb{R}^{k_i\times k_j}$. Let the SVD of the matrix W_{ii} be

$$W_{ii} = U_i\text{diag}(\Sigma_{i_i}^{(i)}, O_{k_i-\xi_i})V_i^{\text{T}},$$

where $U_i \in \mathbb{R}^{k_i\times k_i}$, $V_i \in \mathbb{R}^{k_j\times k_j}$, $\Sigma_{r_i}^{(i)} = \text{diag}(\sigma_i^{(i)}, \cdots, \sigma_{ii}^{(i)})$, $r_i = \text{rank}(W_{ii})$. Then, the least squares orthogonal solutions commuting with P to (1) are

$$X = H\text{diag}(X_1, \cdots, X_p)H^{\text{T}},$$

where X_i satisfies

$$X_i = U_i\begin{bmatrix} I_{r_i} & \\ & G_{k_k-r_i}\end{bmatrix}V_i^{\text{T}},$$

with arbitrary orthogonal $G_{k_i-r_i} \in \mathbb{R}^{k_i-r_i}$.

(b) Denote by $\bar{W} = H^{\text{T}}W_1H$ with $W_1 = A^{\text{T}}B + DC^{\text{T}}$, and partition the matrix $\bar{W} = (W_{ij})$ conforming to (14), where $W_{ij} \in \mathbb{R}^{k_i\times k_j}$. Let the EVD of the matrix $W_{ii} + W_{ii}^{\text{T}}$ be

$$W_{ii} + W_{ii}^{\text{T}} = U_i\text{diag}(\lambda_1^{(i)}I_{l_1^{(i)}}, \cdots, \lambda_{t_i}^{(i)}I_{l_{t_i}^{(i)}})U_i^{\text{T}},$$

where $U_i \in \mathbb{R}^{k\times k}$ is orthogonal, $k_i = \sum_{j=1}^{t_i} l_j^{(i)}$,. The least squares symmetric orthogonal solutions commuting with P to (1) are

$$X = H\text{diag}(X_1, \cdots, X_p)H^{\text{T}},$$

where X_i satisfies

$$X_i = U_i\begin{bmatrix} -I_{n_{s_i-1}^{(m)}} & & \\ & G_{l_{s_i}^{(i)}} & \\ & & I_{k_i-n_{s_i}^{(i)}}\end{bmatrix}U_i^{\text{T}},$$

$G_{l_{s_i}^{(i)}}$ is an arbitrary symmetric orthogonal matrix with order $l_{s_i}^{(i)}$.

(c) Denote by $\widetilde{W} = H^{\text{T}}W_2H$ with $W_2 = A^{\text{T}}A + CC^{\text{T}} - 2(A^{\text{T}}B + DC^{\text{T}})$, and partition the matrix $\widetilde{W} = (W_{ij})$ conforming to (14), where $W_{ij} \in \mathbb{R}^{k_i\times k_j}$. Let the EVD of the matrix $W_{ii} + W_{ii}^{\text{T}}$ be

$$W_{ii} + W_{ii}^{\text{T}}(W_{ii}) = U_i\text{diag}(\lambda_1^{(i)}I_{l_1^{(i)}}, \cdots, \lambda_{t_i}^{(i)}I_{l_{t_i}}^{(i)})U_i^{\text{T}},$$

where $U_i \in \mathbb{R}^{k_i\times k_i}$ is an orthogonal matrix, $k_i = \sum_{j=1}^{t_i} l_j^{(i)}$. The least squares symmetric idempotent solutions commuting with P to (1) are

$$X = H\text{diag}(X_1, \cdots, X_p)H^{\text{T}},$$

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where X_i satisfies

$$X_i = U_i \begin{bmatrix} I_{n^{(i)}_{s_i-1}} & & \\ & G_{l^{(i)}_{s_i}} & \\ & & O_{k_i - n^{(i)}_{s_i}} \end{bmatrix} U_i^{\mathrm{T}},$$

with arbitrary symmetric idempotent $G_{l_{s_i}} \in \mathbb{R}^{l^{(i)}_i \times x^{(i)}_{s_i}}$.

4.3. Unitary Solutions

This section presents the solvability conditions for the system (1) with the constraint $X^*X = I_p$. These conditions are derived by applying the EVD and SVD of matrices. The general solutions to these matrix equations are also provided. Furthermore, the associated optimal approximation problems for the given matrices are discussed, and the optimal approximate solutions are derived [49].

Theorem 45 (Unitary solutions for (1) over $\mathbb{C}$)**.** *[49] Suppose that $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times q}$ with $m \geq n$, and the SVD of DB^* is given by*

$$DB^* = U_1 \begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} P_1^*,$$

where $\Sigma_1 = \mathrm{diag}(\sigma_1^{(1)}, \cdots, \sigma_{r_1}^{(1)})$, $r_1 = \mathrm{rank}(DB^*)$, $U_1 = [U_{11}, U_{12}] \in \mathbb{C}^{m\times m}$, $P_1 = [P_{11}, P_{12}] \in \mathbb{C}^{n\times n}$ with $U_{11} \in \mathbb{C}^{m\times r_1}$, $P_{11} \in \mathbb{C}^{n\times r_1}$. Let the matrices A_i and C_i, $i = 1,2$, be given by $AU_1 = [A_1, A_2]$, $CP_1 = [C_1, C_2]$, and the SVD of A_2 be

$$A_2 = P_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} U_2^* = P_{21}\Sigma_2 U_{21}^*,$$

where $\Sigma_2 = \mathrm{diag}(\sigma_1^{(2)}, \cdots, \sigma_{r_2}^{(2)})$, $r_2 = \mathrm{rank}(A_2)$, $P_2 = [P_{21}, P_{22}] \in \mathbb{C}^{p\times p}$, $U_2 = [U_{21}, U_{22}] \in \mathbb{C}^{(m-r_1)\times(m-r_1)}$ with $P_{21} \in \mathbb{C}^{p\times r_2}$ and $U_{21} \in \mathbb{C}^{(m-r_1)\times r_2}$.

(a) Then, (1) is solvable for unitary matrices if and only if

$$\begin{gathered} B^*B = D^*D,\ A_1 = C_1,\ A_2A_2^\dagger C_2 = C_2,\ I_{p-r_1} - C_2^*(A_2A_2^*)^\dagger C_2 \geq O, \\ \mathrm{rank}\big(I_{p-r_1} - C_2^*(A_2A_2^*)^\dagger C_2\big) \leq n - \mathrm{rank}(DB^*) - r_2. \end{gathered} \tag{15}$$

In this case, let the EVD of $I_{p-r_1} - C_2^*(A_2A_2^*)^\dagger C_2$ be given by

$$I_{n-r_1} - C_2^*(A_2A_2^*)^\dagger C_2 = Q_2 \begin{bmatrix} \Lambda_2 & O \\ O & O \end{bmatrix} Q_2^* = Q_{21}\Lambda_2 Q_{21}^*,$$

where $\Lambda_2 = \mathrm{diag}(\lambda_1^{(2)}, \cdots, \lambda_{s_2}^{(2)})$, $s_2 = \mathrm{rank}(I_{p-r_1} - C_2^*(A_2A_2^*)^\dagger C_2)$, $Q_2 = [Q_{21}, Q_{22}] \in \mathbb{C}^{(n-r_1)\times(n-r_1)}$, $Q_{21} \in \mathbb{C}^{(n-r_1)\times s_2}$. The unitary solution set of (1) is

$$S_2 = \left\{ X = U_1 \begin{bmatrix} I_{r_1} & O \\ O & A_2^\dagger C_2 + U_{52}K_2\Lambda_2^{\frac{1}{2}} Q_{22}^* \end{bmatrix} P_1^* \,\middle|\, \forall\, K_2 \in \mathbb{C}^{(n-r_1-r_2)\times s_2} \text{ is unitary} \right\}.$$

(b) Let $E \in \mathbb{C}^{m\times n}$ and partition $U_1^*EP_1$ as in

$$U_1^*EP_1 = \begin{bmatrix} U_{11}^*EP_{41} & U_{41}^*EP_{12} \\ U_{12}^*EP_{41} & U_{42}^*EP_{12} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

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Let the SVD of $U_{52}^*(N_{22} - A_2^{\dagger}C_2)Q_{21}\Lambda_2^{\frac{1}{2}}$ *be*

$$U_{22}^*(N_{22} - A_2^{\dagger}C_2)Q_{21}\Lambda_2^{\frac{1}{2}} = U_3\begin{bmatrix}\Sigma_3 & O\\ O & O\end{bmatrix}P_3^*,$$

where $\Sigma_3 = \mathrm{diag}(\sigma_1^{(3)}, \cdots, \sigma_{r_3}^{(3)})$, $r_3 = \mathrm{rank}(U_{52}^*(N_{22} - A_2^{\dagger}C_2)Q_{21}\Lambda_2^{\frac{1}{2}})$, $U_3 = [U_{31}, U_{32}] \in \mathbb{C}^{(m-r_1-r_2)\times(m-r_1-r_2)}$, $P_3 = [P_{31}, P_{32}] \in \mathbb{C}^{s_2\times s_2}$ *with* $U_{31} \in \mathbb{C}^{(m-r_1-r_2)\times r_3}$ *and* $P_{31} \in \mathbb{C}^{s_2\times r_3}$. *If the conditions* (15) *are satisfied, then the solution of* $||E - X|| = \min_{X\in S_2}$ *is given by*

$$\widehat{X} = U_1\begin{bmatrix}I_{r_1} & O\\ O & A_2^{\dagger}C_2 + U_{22}K_2\Lambda_2^{\frac{1}{2}}Q_{21}^*\end{bmatrix}P_1^*,$$

where

$$K_2 = U_3\begin{bmatrix}I_{r_3} & O\\ O & H_{K_2}\end{bmatrix}P_3^*,$$

with arbitrary unitary $H_{K_2} \in \mathbb{C}^{(m-r_1-r_2-r_3)\times(s_2-r_3)}$.

4.4. Re-nnd and Re-pd solutions and inequality constrains

This section introduces the relevant conclusions regarding the solutions of the system (1) with inequality constraints, as well as the Re-pd and Re-nnd solutions, using the GSVD [50,51].

Recently, Liao et al. considered the system (1) with the inequality constraint $G^*X + X^*G \geq (>)H$
.

Theorem 46 (General solutions with $G^*X + X^*G \geq (>)H$ constraint for (1) over $\mathbb{C}$)**.** *[50] Given matrices* $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{m\times q}$, $G \in \mathbb{C}^{m\times n}$ *and* $H \in \mathbb{C}^{n\times n}$. *Let* $X_0 = A^{\dagger}C + \mathcal{L}_A DB^{\dagger}$, $\bar{H} = G^*X_0 + X_0^*G - H$, $\bar{G} = G^*\mathcal{L}_A$, $V = \bar{H} - K_0$ *and* $P_S = [\bar{G}, \mathcal{R}_B][\bar{G}, \mathcal{R}_B]^{\dagger}$. *The EVDs of* $I_n - P_S$, $P_S - \mathcal{R}_B$ *and* $P_S - \bar{G}\bar{G}^{\dagger}$ *can be given by*

$$I_n - P_S = U\begin{bmatrix}I_g & O\\ O & O\end{bmatrix}U^* = U_1U_1^*,$$
$$P_S - \mathcal{R}_B = E\begin{bmatrix}I_a & O\\ O & O\end{bmatrix}E^* = E_1E_1^*,$$
$$P_S - \bar{G}\bar{G}^{\dagger} = F\begin{bmatrix}I_b & O\\ O & O\end{bmatrix}F^* = F_1F_1^*,$$

where $g = \mathrm{rank}(I_n - P_S)$, $a = \mathrm{rank}(P_S - \mathcal{R}_B)$, $b = \mathrm{rank}(P_S - \bar{G}\bar{G}^{\dagger})$ *and* $U_1 \in \mathbb{C}^{n\times g}$, $E_1 \in \mathbb{C}^{n\times a}$, $F_1 \in \mathbb{C}^{n\times b}$ *are unitary matrices. The GSVD of* E_1 *and* F_1 *is*

$$E_1 = J\Sigma_1P^*,\ F_1 = J\Sigma_2Q^*,$$

where $J \in \mathbb{C}^{n\times n}$ *is a nonsingular matrix,* $P \in \mathbb{C}^{a\times a}$, $Q \in \mathbb{C}^{b\times b}$ *are unitary matrices, and*

$$\Sigma_1 = \begin{bmatrix}I & O\\ O & \Lambda\\ O & O\\ O & O\end{bmatrix}\begin{matrix}a-s\\ s\\ k-a\\ p-k\end{matrix},\quad \Sigma_2 = \begin{bmatrix}O & O\\ \Gamma & O\\ O & I\\ O & O\end{bmatrix}\begin{matrix}a-s\\ s\\ k-a\\ p-k\end{matrix},$$
$$\qquad\quad a-s \quad s \qquad\qquad\qquad s \quad b-s$$

$k = \mathrm{rank}[E_1, F_1] = a + b - s$, *and* $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_s)$, $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_s)$ *with* $1 > \lambda_1 \geq \cdots \geq \lambda_s > 0$, $0 < \gamma_1 \leq \cdots \leq \gamma_s < 1$, $\lambda_i^2 + \gamma_i^2 = 1$, $i = 1, \cdots, s$. *Partition* J^*VJ *into*

$$J^*VJ = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{12}^* & V_{22} & V_{23} & V_{24} \\ V_{13}^* & V_{23}^* & V_{33} & V_{34} \\ V_{14}^* & V_{24}^* & V_{34}^* & V_{44} \end{bmatrix} \begin{matrix} a-s \\ s \\ k-a \\ p-k \end{matrix}.$$
$$\begin{matrix} a-s & s & k-a & p-k \end{matrix}$$

(a) The matrix inequality $G^*X + X^*G \geq H$ *subject to* (1) *has a general solution if and only if the conditions*

$$\begin{aligned} & AA^\dagger C = C,\ DB^\dagger B = B,\ AD = CB, \\ & (I_n - P_S)\bar{H}(I_n - P_S) \geq 0,\ \mathcal{R}((I_n - P_S)\bar{H}) = \mathcal{R}((I_n - P_S)\bar{H}(I_n - P_S)), \\ & \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{bmatrix} \geq O,\ \begin{bmatrix} V_{22} & V_{23} \\ V_{23}^* & V_{33} \end{bmatrix} \geq O \end{aligned}$$

hold. In this case, the general solution of $G^*X + X^*G \geq H$ *subject to* (1) *can be expressed as*

$$X = X_0 + \mathcal{L}_A W \mathcal{R}_B,$$

where W *is*

$$\begin{aligned} & W = \bar{G}^\dagger(\Phi + \mathcal{L}_L S_W \mathcal{L}_L \bar{G}\bar{G}^\dagger)\mathcal{R}_B + M - \bar{G}^\dagger\bar{G}M\mathcal{R}_B, \\ & \Phi = \frac{1}{2}(K - \bar{H})(2I_p - \bar{G}^\dagger) + \frac{1}{2}(\Psi - \Psi^*)\bar{G}\bar{G}^\dagger, \\ & \Psi = 2L^\dagger BB^\dagger(K - \bar{H}) + (I_n - L^\dagger BB^\dagger)(K - \bar{H})L^\dagger L, \\ & K = K_0 + P_S\widehat{H}P_S,\ K_0 = \bar{H}(I_n - P_S)[(I_n - P_S)\widehat{H}(I_n - P_S)]^\dagger(I_n - P_S)\bar{H}, \\ & \widehat{H} = (J^*)^{-1}\begin{bmatrix} S(\widehat{H}_{13}) & S(\widehat{H}_{13})N_1 \\ N_1^*S(\widehat{H}_{13}) & N_2 + N_1^*S(\widehat{H}_{13})N_1 \end{bmatrix}J^{-1},\ S(\widehat{H}_{13}) = \begin{bmatrix} V_{11} & V_{12} & \widehat{H}_{13} \\ V_{12}^* & V_{22} & V_{23} \\ \widehat{H}_{13}^* & V_{23}^* & V_{33} \end{bmatrix}, \\ & \widehat{H}_{13} = V_{12}V_{22}^\dagger V_{23} + (V_{11} - V_{12}V_{22}^\dagger V_{12}^*)^{\frac{1}{2}}N(V_{33} - V_{23}^*V_{22}^\dagger V_{23})^{\frac{1}{2}}, \end{aligned}$$

with $L = BB^\dagger\widetilde{G}^\dagger$, *arbitrary* $M \in \mathbb{C}^{m\times n}$ *and* $N_1 \in \mathbb{C}^{k\times(n-k)}$, *arbitrary anti-Hermitian* $S_W \in \mathbb{C}^{n\times n}$, *arbitrary Hermitian nonnegative definite* $N_2 \in \mathbb{C}^{(n-k)\times(n-k)}$.

(b) The matrix inequality $G^*X + X^*G > H$ *subject to* (1) *has a general solution if and only if the conditions*

$$\begin{aligned} & AA^\dagger C = C,\ DB^\dagger B = B,\ AD = CB, \\ & U_1^*\bar{H}U_1 > O,\ \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{bmatrix} > O,\ \begin{bmatrix} V_{22} & V_{23} \\ V_{23}^* & V_{33} \end{bmatrix} > O \end{aligned}$$

hold. In this case, the general solution of $G^*X + X^*G > H$ *subject to* (1) *can be expressed as*

$$X = X_0 + \mathcal{L}_A W \mathcal{R}_B,$$

where

$$\begin{aligned}
&W = \bar{G}^{\dagger}(\Phi + \mathcal{L}_L S_W \mathcal{L}_L \bar{G}\bar{G}^{\dagger})\mathcal{R}_B + M - \bar{G}^{\dagger}\bar{G}M\mathcal{R}_B,\\
&\Phi = \frac{1}{2}(K - \bar{H})(2I_p - \bar{G}^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)\bar{G}\bar{G}^{\dagger},\\
&\Psi = 2L^{\dagger}BB^{\dagger}(K - \bar{H}) + (I_n - L^{\dagger}BB^{\dagger})(K - \bar{H})L^{\dagger}L,\\
&K = K_0 + P_S\widehat{H}P_S,\ K_0 = \bar{H}(I_n - P_S)[(I_n - P_S)\widehat{H}(I_n - P_S)]^{\dagger}(I_n - P_S)\bar{H},\\
&\hat{H} = (J^*)^{-1}\begin{bmatrix} S(\hat{H}_{13}) & N_1 \\ N_1^* & N_2 + N_1^* S(\hat{H}_{13})^{-1}N_1 \end{bmatrix}J^{-1},\ S(\widehat{H}_{13}) = \begin{bmatrix} V_{11} & V_{12} & \widehat{H}_{13} \\ V_{12}^* & V_{22} & V_{23} \\ \widehat{H}_{13}^* & V_{23}^* & V_{33} \end{bmatrix},\\
&\widehat{H}_{13} = V_{12}V_{22}^{-1}V_{23} + (V_{11} - V_{12}V_{22}^{-1}V_{12}^*)^{\frac{1}{2}}N(V_{33} - V_{23}^*V_{22}^{-1}V_{23})^{\frac{1}{2}},
\end{aligned}$$

with $L = BB^{\dagger}\widetilde{G}^{\dagger}$, *arbitrary* $M \in \mathbb{C}^{m\times n}$ *and* $N_1 \in \mathbb{C}^{k\times(n-k)}$, *arbitrary anti-Hermitian* $S_W \in \mathbb{C}^{n\times n}$, *arbitrary Hermitian nonnegative definite* $N_2 \in \mathbb{C}^{(n-k)\times(n-k)}$.

Yuan et al. expanded upon the above research by deriving necessary and sufficient conditions for the system (1) to have Re-nnd and Re-pd solutions. Additionally, explicit representations of the general Re-nnd and Re-pd solutions are provided when the stated conditions are satisfied.

Theorem 47 (Re-pd and Re-nnd solutions for (1) over $\mathbb{C}$)**.** *[51] For given matrices* $A \in \mathbb{C}^{p\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$ *and* $D \in \mathbb{C}^{n\times q}$, *denote* $A^{\dagger}C + \mathcal{L}_A DB^{\dagger}$, $BB^{\dagger}\mathcal{L}_A$, $\mathcal{R}_L BB^{\dagger}$ *and* $X_0 + X_0^* - K_0$ *by* X_0, L, G *and* J, *respectively. Suppose that the EVDs of* G, $A^{\dagger}A - G$ *and* $BB^{\dagger} - G$ *can be given by*

$$\begin{aligned}
G &= U\begin{bmatrix} I_g & O \\ O & O \end{bmatrix}U^* = U_1U_1^*,\\
A^{\dagger}A - G &= P\begin{bmatrix} I_a & O \\ O & O \end{bmatrix}P^* = P_1P_1^*,\\
BB^{\dagger} - G &= Q\begin{bmatrix} I_b & O \\ O & O \end{bmatrix}Q^* = Q_1Q_1^*,
\end{aligned}$$

where $g = \mathrm{rank}(G)$, $a = \mathrm{rank}(A^{\dagger}A - G)$, $b = \mathrm{rank}(BB^{\dagger} - G)$, $U_1 \in \mathbb{C}^{n\times g}$, $P_1 \in \mathbb{C}^{n\times a}$ *and* $Q_1 \in \mathbb{C}^{n\times b}$. *The GSVD of the matrices* P_1 *and* Q_1 *is*

$$P_1 = \Upsilon\Sigma_1\widetilde{E}^*,\ Q_1 = \Upsilon\Sigma_2\widetilde{F}^*,$$

where $\Upsilon \in \mathbb{C}^{n\times n}$ *is a nonsingular matrix and* $\widetilde{E} \in \mathbb{C}^{a\times a}$, $\widetilde{F} \in \mathbb{C}^{b\times b}$ *are unitary matrices, and*

$$\Sigma_1 = \begin{array}{c}\begin{bmatrix} I & O \\ O & \Lambda \\ O & O \\ O & O \end{bmatrix}\begin{matrix} a-s \\ s \\ k-a \\ n-k \end{matrix}\\ \begin{matrix} a-s & s \end{matrix}\end{array},\ \Sigma_2 = \begin{array}{c}\begin{bmatrix} O & O \\ \Delta & O \\ O & I \\ O & O \end{bmatrix}\begin{matrix} a-s \\ s \\ k-a \\ n-k \end{matrix}\\ \begin{matrix} s & b-s \end{matrix}\end{array},$$

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with $k = \mathrm{rank}[P_1, Q_1] = a + b - s, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_s)$, $\Delta = \mathrm{diag}(\delta_1, \cdots, \delta_s)$, $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_s < 1$, $\lambda_i^2 + \delta_i^2 = 1$, $i = 1, \cdots, s$ *, and the partition of the matrix* $Y^* J Y$ *is the form of*

$$Y^*Y = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{12}^* & J_{22} & J_{23} & J_{24} \\ J_{13}^* & J_{23}^* & J_{33} & J_{34} \\ J_{14}^* & J_{24}^* & J_{34}^* & J_{44} \end{bmatrix} \begin{matrix} a-s \\ s \\ k-a \\ n-k \end{matrix} .$$
$$\quad\; a-s \quad s \quad k-a \quad n-k$$

(*a*) *The system* (1) *has a Re-nnd solution if and only if*

$$\begin{aligned} &AA^\dagger C = C,\ DB^\dagger B = D,\ AD = CB, \\ &G(X_0 + X_0^*)G \geq 0,\ H(G(X_0 + X_0^*)) = H(G(X_0 + X_0^*)G), \\ &\begin{bmatrix} J_{11} & J_{12} \\ J_{12}^* & J_{22} \end{bmatrix} \geq O,\ \begin{bmatrix} J_{22} & J_{23} \\ J_{23}^* & J_{33} \end{bmatrix} \geq O. \end{aligned}$$

In this case, the general Re-nnd solution of (1) *can be expressed as*

$$X = X_0 + \mathcal{L}_A(\Theta + \mathcal{L}_L S_K \mathcal{L}_L \mathcal{L}_A)\mathcal{R}_B,$$

where K_0, Θ, Ψ, H, $M(H_{13})$ *and* H_{13} *are given by*

$$\begin{aligned} &K_0 = (X_0 + X_0^*)G[G(X_0 + X_0^*)G]^\dagger G(X_0 + X_0^*), \\ &\Theta = \frac{1}{2}[(I_n - G)H(I_n - G) - J](2I_n - \mathcal{L}_A) + \frac{1}{2}(\Psi - \Psi^*)\mathcal{L}_A, \\ &\Psi = 2L^\dagger BB^\dagger[(I_n - G)H(I_n - G) - J] + (I_n - L^\dagger BB^\dagger)[(I_n - G)H(I_n - G) - J]L^\dagger L, \\ &H = (Y^*)^{-1}\begin{bmatrix} M(H_{13}) & M(H_{13})S \\ S^* M(H_{13}) & T + S^* M(H_{13})S \end{bmatrix} Y^{-1},\ M(H_{13}) = \begin{bmatrix} J_{11} & J_{12} & H_{13} \\ J_{12}^* & J_{22} & J_{23} \\ H_{13}^* & J_{23}^* & J_{33} \end{bmatrix}, \\ &H_{13} = J_{12}J_{22}^\dagger J_{23} + (J_{11} - J_{12}J_{22}^\dagger J_{12}^*)^{\frac{1}{2}} N (J_{33} - J_{23}^* J_{22}^\dagger J_{23})^{\frac{1}{2}}, \end{aligned}$$

with arbitrary $S \in \mathbb{C}^{k\times(n-k)}$, $S_K \in \mathbb{C}^{n\times n}$ *satisfying* $S_K^* = -S_K$, *arbitrary Hermitian nonnegative definite* $T \in \mathbb{C}^{(n-k)\times(n-k)}$ *and arbitrary contraction* $N \in \mathbb{C}^{(a-s)\times(k-a)}$.

(*b*) *The system* (1) *has a Re-pd solution if and only if*

$$\begin{aligned} &AA^\dagger C = C,\ DB^\dagger B = D,\ AD = CB,\ U_1^*(X_0 + X_0^*)U_1 > O, \\ &\begin{bmatrix} J_{11} & J_{12} \\ J_{12}^* & J_{22} \end{bmatrix} > O,\ \begin{bmatrix} J_{22} & J_{23} \\ J_{23}^* & J_{33} \end{bmatrix} > O. \end{aligned}$$

In this case, the general Re-pd solution of (1) *can be expressed as*

$$X = X_0 + \mathcal{L}_A(\Theta + \mathcal{L}_L S_K \mathcal{L}_L \mathcal{L}_A)\mathcal{R}_B,$$

where K_0, Θ, Ψ, H, $M(H_{13})$ and H_{13} are, respectively, given by

$$K_0 = (X_0 + X_0^*)G[G(X_0 + X_0^*)G]^{\dagger}G(X_0 + X_0^*),$$
$$\Theta = \frac{1}{2}[(I_n - G)H(I_n - G) - J](2I_n - \mathcal{L}_A) + \frac{1}{2}(\Psi - \Psi^*)\mathcal{L}_A,$$
$$\Psi = 2L^{\dagger}BB^{\dagger}[(I_n - G)H(I_n - G) - J] + (I_n - L^{\dagger}BB^{\dagger})[(I_n - G)H(I_n - G) - J]L^{\dagger}L,$$
$$H = (Y^*)^{-1}\begin{bmatrix} M(H_{13}) & S \\ S^* & T + S^*(M(H_{13}))^{-1}S \end{bmatrix} Y^{-1},\ M(H_{13}) = \begin{bmatrix} J_{11} & J_{12} & H_{13} \\ J_{12}^* & J_{22} & J_{23} \\ H_{13}^* & J_{23}^* & J_{33} \end{bmatrix},$$
$$H_{13} = J_{12}J_{22}^{-1}J_{23} + (J_{11} - J_{12}J_{22}^{-1}J_{12}^*)^{\frac{1}{2}}N(J_{33} - J_{23}^*J_{22}^{-1}J_{23})^{\frac{1}{2}},$$

with arbitrary $S \in \mathbb{C}^{k\times(n-k)}$, $S_K \in \mathbb{C}^{n\times n}$ satisfying $S_K^* = -S_K$, arbitrary Hermitian positive definite $T \in \mathbb{C}^{(n-k)\times(n-k)}$ and arbitrary strict contraction $N \in \mathbb{C}^{(a-s)\times(k-a)}$.

4.5. Different Types of Reflexive Solutions

Some scholars considered the (anti-)reflexive solutions of the system (1) and presented the following theorem.

Any nontrivial generalized reflection matrix $P \in \mathbb{C}^{n\times n}$ can be expressed in the form

$$P = U\begin{bmatrix} I_r & O \\ O & -I_{n-r} \end{bmatrix}U^*, \tag{16}$$

where $U = [U_1, U_2]$, with $U_1^*U_2 = O$.

Theorem 48 ((Anti-)reflexive solutions for (1) over $\mathbb{C}$)**.** *[17–19] Let $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$ and the nontrivial generalized reflection matrix $P \in \mathbb{C}^{n\times n}$ be known and $X \in \mathbb{C}^{n\times n}$ be unknown. The EVD of P is given by (16). Let*

$$AU = [A_1, A_2], CU = [B_1, B_2], U^*B = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

(a) The system (1) has an (anti-)reflexive solution with respect to a nontrivial generalized reflection matrix P if and only if

$$\mathcal{R}_{A_i}C_i = O,\ D_i\mathcal{L}_{B_i} = O,\ A_iD_i = C_iB_i,\ i = 1, 2.$$

In this case, the reflexive solution X with respect to P can be expressed as

$$X = U\begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}U^*,$$

the anti-reflexive solution X with respect to P can be expressed as

$$X = U\begin{bmatrix} 0 & H \\ K & 0 \end{bmatrix}U^*,$$

where

$$H = A_1^{\dagger}B_1 + \mathcal{L}_{A_1}D_1C_1^{\dagger} + \mathcal{L}_{A_1}Y_1\mathcal{R}_{C_1},$$
$$K = A_2^{\dagger}B_2 + \mathcal{L}_{A_2}D_2C_2^{\dagger} + \mathcal{L}_{A_2}Y_2\mathcal{R}_{C_2},$$

$Y_1 \in \mathbb{C}^{r\times r}$ and $Y_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ are arbitrary.

(b) For a given matrix $E \in \mathbb{C}^{n\times n}$, let

$$U^*EU = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

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Symbols Φ_r and Φ_a represent the set of all reflexive and anti-reflexive solutions of the system $AX = C, XB = D$. Then, the approximation problem $\min_{X \in \Phi_r} ||X - E||$ has a unique solution

$$\hat{X} = U \begin{bmatrix} Z_1 & O \\ O & Z_2 \end{bmatrix} U^*,$$

where

$$Z_1 = A_1^\dagger B_1 + \mathcal{L}_{A_1} D_1 C_1^\dagger + \mathcal{L}_{A_1}(E_{11} - A_1^\dagger B_1 - \mathcal{L}_{A_1} D_1 C_1^\dagger)\mathcal{R}_{C_1},$$
$$Z_2 = A_2^\dagger B_2 + \mathcal{L}_{A_2} D_2 C_2^\dagger + \mathcal{L}_{A_2}(E_{22} - A_2^\dagger B_2 - \mathcal{L}_{A_2} D_2 C_2^\dagger)\mathcal{R}_{C_2}.$$

The approximation problem $\min_{X \in \Phi_a} ||X - E||$ has a unique solution

$$\hat{X} = U \begin{bmatrix} O & Z_3 \\ Z_4 & O \end{bmatrix} U^*,$$

where

$$Z_3 = A_1^\dagger B_1 + \mathcal{L}_{A_1} D_1 C_1^\dagger + \mathcal{L}_{A_1}(E_{12} - A_1^\dagger B_1 - \mathcal{L}_{A_1} D_1 C_1^\dagger)\mathcal{R}_{C_1},$$
$$Z_4 = A_2^\dagger B_2 + \mathcal{L}_{A_2} D_2 C_2^\dagger + \mathcal{L}_{A_2}(E_{21} - A_2^\dagger B_2 - \mathcal{L}_{A_2} D_2 C_2^\dagger)\mathcal{R}_{C_2}.$$

Zhou and Yang considered the existence conditions of the (anti)-Hermitian reflexive solutions which added the (anti)-Hermitian constraints in the reflexive solutions [52].

Theorem 49 ((Anti-)Hermitian reflexive solutions for (1) over $\mathbb{C}$)**.** *[52] Let $A, C \in \mathbb{C}^{p \times n}$, $B, D \in \mathbb{C}^{n \times q}$ and the nontrivial generalized reflection matrix $P \in \mathbb{C}^{n \times n}$ be known. The EVD of P is given by (16). Denote*

$$AU = [A_1, A_2],\ CU = [C_1, C_2],\ U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, C_1 \in \mathbb{C}^{p \times r}, A_2, C_2 \in \mathbb{C}^{p \times (n-r)}, C_1, D_1 \in \mathbb{C}^{r \times q}, C_2, D_2 \in \mathbb{C}^{(n-r) \times q}$. Let

$$M = (I_r - A_1^* A_1^\dagger)C_1,\ T_1 = A_1^{*\dagger} C_1(I_s - M^\dagger M),\ K_1 = (I_m + T_1 T_1^\dagger)^{-1},$$
$$N = (I_{n-r} - A_2^* A_2^\dagger)C_2,\ T_2 = A_2^{*\dagger} C_2(I_s - N^\dagger N),\ K_2 = (I_m + T_2 T_2^*)^{-1},$$
$$P = (A_1^*\ C_1)(A_1^*\ C_1)^\dagger = A_1^* W_1 + C_1 W_2,\ Q = (A_2^*\ C_2)(A_2^*\ C_2)^\dagger = A_2^* W_3 + C_2 W_4,$$
$$W_1 = K_1 A_1^{*\dagger}(I_r - C_1 M^\dagger),\ W_2 = T_1^* K_1 A_1^{*\dagger}(I_r - C_1 M^\dagger) + M^\dagger,$$
$$W_3 = K_2 A_2^{*\dagger}(I_{n-r} - C_2 N^\dagger),\ W_4 = T_2^* K_2 A_2^{*\dagger}(I_{n-r} - C_2 N^\dagger) + N^\dagger.$$

(a) Then, the system (1) has Hermitian reflexive solutions in $\mathbb{C}^{n \times n}$ if and only if

$$\begin{bmatrix} B_1 A_1^* & B_1 C_1 \\ D_1^* A_1^* & D_1^* C_1 \end{bmatrix} = \begin{bmatrix} A_1 B_1^* & A_1 D_1 \\ C_1^* B_1^* & C_1^* D_1 \end{bmatrix},$$
$$\begin{bmatrix} B_2 A_2^* & B_2 C_2 \\ D_2^* A_2^* & D_2^* C_2 \end{bmatrix} = \begin{bmatrix} A_2 B_2^* & A_2 D_2 \\ C_2^* B_2^* & C_2^* D_2 \end{bmatrix},$$
$$\begin{bmatrix} B_1^*, D_1 \end{bmatrix} = \begin{bmatrix} B_1^* W_1 A_1^* + D_1 W_2 A_2^*, B_1^* W_1 C_1 + D_1 W_2 C_1 \end{bmatrix},$$
$$\begin{bmatrix} B_2^*, D_2 \end{bmatrix} = \begin{bmatrix} B_2^* W_3 A_2^* + D_2 W_4 A_2^*, B_2^* W_3 C_2 + D_2 W_4 C_2 \end{bmatrix}.$$

Moreover, the general Hermitian reflexive solution can be expressed as

$$X = U \begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix} U^*,$$

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where $X_{11} \in \mathbb{C}^{r\times r}, X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *are*

$$\begin{aligned} X_{11} &= X_{10} + (I_r - P)G_1(I_r - P), \\ X_{22} &= X_{20} + (I_{n-r} - Q)G_2(I_{n-r} - Q), \end{aligned}$$

and

$$\begin{aligned} X_{10} &= B_1^* W_1 + D_1 W_2 + (W_1^* B_1 + W_2^* D_1^*)(I_r - P), \\ X_{20} &= B_2^* W_3 + D_2 W_4 + (W_3^* B_2 + W_4^* D_2^*)(I_{n-r} - Q), \end{aligned} \tag{17}$$

with arbitrary Hermitian $G_1 \in \mathbb{C}^{r\times r}$ *and* $G_2 \in \mathbb{C}^{(n-r)\times(n-r)}$.

(b) *Then, the system* (1) *has anti-Hermitian reflexive solutions in* $\mathbb{C}^{n\times n}$ *if and only if*

$$\begin{aligned} \begin{bmatrix} -B_1A_1^* & -B_1C_1 \\ D_1^*A_1^* & D_1^*C_1 \end{bmatrix} &= \begin{bmatrix} A_1B_1^* & -A_1D_1 \\ C_1^*B_1^* & -C_1^*D_1 \end{bmatrix}, \\ \begin{bmatrix} -B_2A_2^* & -B_2C_2 \\ D_2^*A_2^* & D_2^*C_2 \end{bmatrix} &= \begin{bmatrix} A_2B_2^* & -A_2D_2 \\ C_2^*B_2^* & -C_2^*D_2 \end{bmatrix}, \\ \begin{bmatrix} -B_1^*, D_1 \end{bmatrix} &= \begin{bmatrix} -B_1^*W_1A_1^* + D_1W_2A_2^*, -B_1^*W_1C_1 + D_1W_2C_1 \end{bmatrix}, \\ \begin{bmatrix} -B_2^*, D_2 \end{bmatrix} &= \begin{bmatrix} -B_2^*W_3A_2^* + D_2W_4A_2^*, -B_2^*W_3C_2 + D_2W_4C_2 \end{bmatrix}. \end{aligned}$$

Moreover, the general anti-Hermitian reflexive solution can be expressed as

$$X = U \begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix} U^*,$$

where $X_{11} \in \mathbb{C}^{r\times r}, X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *are*

$$\begin{aligned} X_{11} &= X_{10} + (I_r - P)G_1(I_r - P), \\ X_{22} &= X_{20} + (I_{n-r} - Q)G_2(I_{n-r} - Q), \end{aligned}$$

and

$$\begin{aligned} X_{10} &= -B_1^* W_1 + D_1 W_2 - (-W_1^* B_1 + W_2^* D_1^*)(I_r - P), \\ X_{20} &= -B_2^* W_3 + D_2 W_4 - (-W_3^* B_2 + W_4^* D_2^*)(I_{n-r} - Q), \end{aligned} \tag{18}$$

with arbitrary anti-Hermitian $G_1 \in \mathbb{C}^{r\times r}$ *and* $G_2 \in \mathbb{C}^{(n-r)\times(n-r)}$.

(c) *Given matrix* $E \in \mathbb{C}^{n\times n}$. *Let*

$$E = U \begin{bmatrix} E_{11}^* & E_{12} \\ E_{21} & E_{22}^* \end{bmatrix} U^*, \tag{19}$$

where $E_{11} \in \mathbb{C}^{r\times r}, E_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$. *If* (1) *has Hermitian reflexive solutions, then the optimize problem* $||X - E|| = \min$ *has a unique Hermitian reflexive solutions* $\hat{X}$ *of* (1), *which can be represented as*

$$\hat{X} = U \begin{bmatrix} \hat{X}_{11} & O \\ O & \hat{X}_{22} \end{bmatrix} U^*,$$

where

$$\begin{aligned} \hat{X}_{11} &= X_{10} + (I_r - P)\hat{G}_1(I_r - P), \\ \hat{X}_{22} &= X_{20} + (I_{n-r} - Q)\hat{G}_2(I_{n-r} - Q), \\ \hat{G}_1 &= \frac{1}{2}(E_{11} - X_{10} + (E_{11} - X_{10})^*), \\ \hat{G}_2 &= \frac{1}{2}(E_{22} - X_{20} + (E_{22} - X_{20})^*), \end{aligned}$$

with X_{10}, X_{20} *given by* (17).

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(d) For given matrix $E \in \mathbb{C}^{n\times n}$, U^*E^*U is partitioned as (19). If (1) has anti-Hermitian reflexive solutions in, then the optimize problem $||X - E|| = \min$ has a unique anti-Hermitian reflexive solutions $\hat{X}$ of (1) which can be represented as

$$\hat{X} = U\begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & \hat{X}_{22} \end{bmatrix} U^*,$$

where

$$\begin{aligned} \hat{X}_{11} &= X_{10} + (I_r - P)\hat{G}_1(I_r - P), \\ \hat{X}_{22} &= X_{20} + (I_{n-r} - Q)\hat{G}_2(I_{n-r} - Q), \\ \hat{G}_1 &= \frac{1}{2}(E_{11} - X_{10} - (E_{11} - X_{10})^*), \\ \hat{G}_2 &= \frac{1}{2}(E_{22} - X_{20} - (E_{22} - X_{20})^*), \end{aligned}$$

with X_{10}, X_{20} given by (18).

Zhou et al. also considered the least squares (anti)-Hermitian reflexive solutions [53].

Theorem 50 (Least squares (anti-)Hermitian reflexive solutions for (1) over $\mathbb{C}$)**.** *[53] Let $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$ and the nontrivial generalized reflection matrix $P \in \mathbb{C}^{n\times n}$ be known. The EVD of P is given by (16). Denote*

$$AU = [A_1, A_2],\ CU = [C_1, C_2],\ U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

$\Sigma_i (i = 1,2,3,4), P_{ij}, Q_{ij}, S_{ij}, T_{ij} (i = 1,2,\ j = 1,2)$ and $\alpha_i, \beta_j (i = 1,\cdots, r_1, j = 1,\cdots, r_2)$ are given by the SVDs of $(A_1^*, B_1), (A_2^*, B_2)$, $(C_1^*, D_1), (C_2^*, D_2)$, $(-C_1^*, D_1), (-C_2^*, D_2)$:

$$\begin{aligned}
(A_1^*, B_1) &= P_1\begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} Q_1^* = (P_{11}, P_{12})\begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix}\begin{pmatrix} Q_{11}^* \\ Q_{12}^* \end{pmatrix} = P_{11}\Sigma_1 Q_{11}^*, \\
(A_2^*, B_2) &= P_2\begin{bmatrix} \Sigma_2 & O \\ O & O \end{bmatrix} Q_2^* = (P_{21}, P_{22})\begin{bmatrix} \Sigma_2 & O \\ O & O \end{bmatrix}\begin{pmatrix} Q_{21}^* \\ Q_{22}^* \end{pmatrix} = P_{21}\Sigma_2 Q_{21}^*, \\
(C_1^*, D_1) &= S_1\begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} T_1^* = (S_{11}, S_{12})\begin{bmatrix} \Sigma_s & O \\ O & O \end{bmatrix}\begin{pmatrix} T_{11}^* \\ T_{12}^* \end{pmatrix} = S_{11}\Sigma_3 T_{11}^*, \\
(C_2^*, D_2) &= S_2\begin{bmatrix} \Sigma_4 & O \\ O & O \end{bmatrix} T_2^* = (S_{21}, S_{22})\begin{bmatrix} \Sigma_4 & O \\ O & O \end{bmatrix}\begin{pmatrix} T_{21}^* \\ T_{22}^* \end{pmatrix} = S_{21}\Sigma_4 T_{21}^*, \\
(-C_1^*, D_1) &= S_3\begin{bmatrix} \Sigma_5 & O \\ O & O \end{bmatrix} T_3^* = (S_{31}, S_{32})\begin{bmatrix} \Sigma_5 & O \\ O & O \end{bmatrix}\begin{pmatrix} T_{31}^* \\ T_{32}^* \end{pmatrix} = S_{31}\Sigma_5 T_{31}^*, \\
(-C_2^*, D_2) &= S_4\begin{bmatrix} \Sigma_6 & O \\ O & O \end{bmatrix} T_4^* = (S_{41}, S_{42})\begin{bmatrix} \Sigma_6 & O \\ O & O \end{bmatrix}\begin{pmatrix} T_{41}^* \\ T_{42}^* \end{pmatrix} = S_{41}\Sigma_6 T_{41}^*,
\end{aligned}$$

where $P_1 = [P_{11}, P_{12}]$, $S_1 = [S_{11}, S_{12}]$, $S_3 = [S_{31}, S_{32}] \in \mathbb{C}^{r\times r}$, $Q_1 = [Q_{11}, Q_{12}]$, $T_1 = [T_{11}, T_{12}]$, $T_3 = [T_{31}, T_{32}] \in \mathbb{C}^{(m+x)\times(m+x)}$, $\Sigma_1 = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{r_1}), r_1 = \mathrm{rank}[A_1^*, B_1]$, $\Sigma_2 = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_{r_2})$, $r_2 = \mathrm{rank}[A_2^*, B_2]$, $\Sigma_3 = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_{r_3})$, $r_3 = \mathrm{rank}[C_1^*, D_1]$, $\Sigma_4 = \mathrm{diag}(\eta_1, \eta_2, \cdots, \eta_{r_4})$, $r_4 = \mathrm{rank}[C_2^*, D_2]$, $\Sigma_5 = \mathrm{diag}(\zeta_1, \zeta_2, \cdots, \zeta_{r_5})$, $r_5 = \mathrm{rank}[-C_1^*, D_1]$, $\Sigma_6 = \mathrm{diag}(\xi_1, \xi_2, \cdots, \xi_{r6})$, $r_6 = \mathrm{rank}[-C_2^*, D_2]$.

(a) The least squares Hermitian reflexive solutions of the system $AX = C, XB = D$ with respect to P can be expressed as

$$X = U\begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix} U^*,$$

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with $X_{11} \in \mathbb{C}^{r\times r}, X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *being Hermitian, given by*

$$X_{11} = P_1 \begin{bmatrix} \Phi_1 * (P_{11}^* S_{11}\Sigma_3 T_{11}^* Q_{11}\Sigma_1 + \Sigma_1 Q_{11}^* T_{11}\Sigma_3 S_{11}^* P_{11}) & \Sigma_1^{-1} Q_{11}^* T_{11}\Sigma_3 S_{11}^* P_{12} \\ P_{12}^* S_{11}\Sigma_3 T_{11}^* Q_{11}\Sigma_1^{-1} & G_1 \end{bmatrix} P_1^*,$$
$$X_{22} = P_2 \begin{bmatrix} \Phi_2 * (P_{21}^* S_{21}\Sigma_4 T_{21}^* Q_{21}\Sigma_2 + \Sigma_2 Q_{21}^* T_{21}\Sigma_4 S_{21}^* P_{21}) & \Sigma_2^{-1} Q_{21}^* T_{21}\Sigma_4 S_{21}^* P_{22} \\ P_{22}^* S_{21}\Sigma_4 T_{21}^* Q_{21}\Sigma_2^{-1} & G_2 \end{bmatrix} P_2^*,$$

where $G_1 \in \mathbb{C}^{(r-r_1)\times(r-r_1)}$ *and* $G_2 \in \mathbb{C}^{(n-r-r_2)\times(n-r-r_2)}$ *are arbitrary Hermitian matrices,* $\Phi_1 = (\phi_{ij}) \in \mathbb{C}^{r_1\times r_1}$, $\phi_{ij} = \frac{1}{\alpha_1^2+\alpha_j^2}$, $i,j = 1,\cdots,r_1, \Phi_2 = (\phi_{ij}) \in \mathbb{C}^{r_2\times r_2}$, $\phi_{ij} = \frac{1}{\beta_i^2+\beta_j^2}$, $i,j = 1,\cdots,r_2$.

(b) *The least squares anti-Hermitian reflexive solutions of the system* (1) *with respect to P can be expressed as*

$$X = U \begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix} U^*,$$

with $X_{11} \in \mathbb{C}^{r\times r}, X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *being anti-Hermitian, given by*

$$X_{11} = P_1 \begin{bmatrix} \Phi_1 * (P_{11}^* S_{31}\Sigma_5 T_{31}^* Q_{11}\Sigma_1 - \Sigma_1 Q_{11}^* T_{31}\Sigma_5 S_{31}^* P_{11}) & -\Sigma_1^{-1} Q_{11}^* T_{31}\Sigma_5 S_{31}^* P_{12} \\ P_{12}^* S_{31}\Sigma_5 T_{31}^* Q_{11}\Sigma_1^{-1} & G_1 \end{bmatrix} P_1^*,$$
$$X_{22} = P_2 \begin{bmatrix} \Phi_2 * (P_{21}^* S_{41}\Sigma_6 T_{41}^* Q_{21}\Sigma_2 - \Sigma_2 Q_{21}^* T_{41}\Sigma_6 S_{41}^* P_{21}) & -\Sigma_2^{-1} Q_{21}^* T_{41}\Sigma_6 S_{41}^* P_{22} \\ P_{22}^* S_{41}\Sigma_6 T_{41}^* Q_{21}\Sigma_2^{-1} & G_2 \end{bmatrix} P_2^*,$$

where $G_1 \in \mathbb{C}^{(r-r_1)\times(r-r_1)}$ *and* $G_2 \in \mathbb{C}^{(n-r-r_2)\times(n-r-r_2)}$ *are arbitrary anti-Hermitian matrices.*

Given $E \in \mathbb{C}^{n\times n}$. *Let*

$$U^*EU = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

where $E_{11} \in \mathbb{C}^{r\times r}, E_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *and denote*

$$E_{11} = \begin{bmatrix} \tilde{X}_{111} & \tilde{X}_{112} \\ \tilde{X}_{121} & \tilde{X}_{122} \end{bmatrix}, E_{22} = \begin{bmatrix} \tilde{X}_{211} & \tilde{X}_{212} \\ \tilde{X}_{221} & \tilde{X}_{222} \end{bmatrix},$$

where $\tilde{X}_{111} \in \mathbb{C}^{r_1\times r_1}, \tilde{X}_{211} \in \mathbb{C}^{r_2\times r_2}$.

(c) *The optimization problem* $||X - E|| = \min$ *has a unique solution* $\hat{X}$ *which is the least squares Hermitian reflexive solution of* (1) *can be represented as*

$$\hat{X} = U \begin{bmatrix} \hat{X}_{11} & O \\ O & \hat{X}_{22} \end{bmatrix} U^*,$$

where $\hat{X}_{11} \in \mathbb{C}^{r\times r}, \hat{X}_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *are Hermitian, with*

$$\hat{X}_{11} = P_1 \begin{bmatrix} \Phi_1 * (P_{11}^* S_{11}\Sigma_3 T_{11}^* Q_{11}\Sigma_1 + \Sigma_1 Q_{11}^* T_{11}\Sigma_3 S_{11}^* P_{11}) & \Sigma_1^{-1} Q_{11}^* T_{11}\Sigma_3 S_{11}^* P_{12} \\ P_{12}^* S_{11}\Sigma_3 T_{11}^* Q_{11}\Sigma_1^{-1} & \frac{1}{2}(\tilde{X}_{122} + \tilde{X}_{122}^*) \end{bmatrix} P_1^*,$$
$$\hat{X}_{22} = P_2 \begin{bmatrix} \Phi_2 * (P_{21}^* S_{21}\Sigma_4 T_{21}^* Q_{21}\Sigma_2 + \Sigma_2 Q_{21}^* T_{21}\Sigma_4 S_{21}^* P_{21}) & \Sigma_2^{-1} Q_{21}^* T_{21}\Sigma_4 S_{21}^* P_{22} \\ P_{22}^* S_{21}\Sigma_4 T_{21}^* Q_{21}\Sigma_2^{-1} & \frac{1}{2}(X_{222} + \tilde{X}_{222}^*) \end{bmatrix} P_2^*.$$

(*d*) *The optimization problem* $||X - E|| = \min$ *has a unique solution* $\hat{X}$ *which is the least squares anti-Hermitian reflexive solution of* (1) *can be represented as*

$$\hat{X} = U\begin{bmatrix} \hat{X}_{11} & O \\ O & \hat{X}_{22} \end{bmatrix}U^*,$$

where $\hat{X}_{11} \in \mathbb{C}^{r\times r}, \hat{X}_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ *are anti-Hermitian, with*

$$\hat{X}_{11} = P_1\begin{bmatrix} \Phi_1 * (P_{11}^* S_{31}\Sigma_5 T_{31}^* Q_{11}\Sigma_1 - \Sigma_1 Q_{11}^* T_{31}\Sigma_5 S_{31}^* P_{11}) & -\Sigma_1^{-1} Q_{11}^* T_{31}\Sigma_5 S_{31}^* P_{12} \\ P_{12}^* S_{31}\Sigma_5 T_{31}^* Q_{11}\Sigma_1^{-1} & \frac{1}{2}(\bar{X}_{122} - \bar{X}_{122}^*) \end{bmatrix} P_1^*,$$
$$\hat{X}_{22} = P_2\begin{bmatrix} \Phi_2 * (P_{21}^* S_{41}\Sigma_6 T_{41}^* Q_{21}\Sigma_2 - \Sigma_2 Q_{21}^* T_{41}\Sigma_6 S_{41}^* P_{21}) & -\Sigma_2^{-1} Q_{21}^* T_{41}\Sigma_6 S_{41}^* P_{22} \\ P_{22}^* S_{41}\Sigma_6 T_{41}^* Q_{21}\Sigma_2^{-1} & \frac{1}{2}(X_{222} - X_{222}^*) \end{bmatrix} P_2^*.$$

Dong and Wang have presented the system of matrix equations (1) subject to $\{P, Q, k+1\}$-reflexive and anti-reflexive constraints by converting it into two simpler cases: $k = 1$ and $k = 2$. They provide the solvability conditions, the general solution to this system, and the least squares solution when (1) is inconsistent [54].

Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be Hermitian and $\{k+1\}$-potent matrices, that is, $P^{k+1} = P = P^*$ and $Q^{k+1} = Q = Q^*$. A matrix $X \in \mathbb{C}^{m\times n}$ is called $\{P, Q, k+1\}$-(anti-)reflexive if $PXQ = X$ (or $PXQ = -X$). For $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ to be Hermitian, they are $\{k+1\}$-potent matrices if and only if P and Q are idempotent (i.e., $P^2 = P$, $Q^2 = Q$) when k is odd, or tripotent (i.e., $P^3 = P$, $Q^3 = Q$) when k is even. Moreover, there exist $U \in \mathbb{U}^{m\times m}$ and $V \in \mathbb{U}^{n\times n}$ such that

$$P = U\begin{bmatrix} I_p & \\ & O \end{bmatrix}U^*, \quad Q = V\begin{bmatrix} I_q & \\ & O \end{bmatrix}V^*, \tag{20}$$

if k is odd, and

$$P = U\begin{bmatrix} I_r & & \\ & -I_{p-r} & \\ & & O \end{bmatrix}U^*, \quad Q = V\begin{bmatrix} I_s & & \\ & -I_{q-s} & \\ & & O \end{bmatrix}V^*, \tag{21}$$

if k is even, where $p = \text{rank}(P)$ and $q = \text{rank}(Q)$.

Theorem 51 ($\{P, Q, 2\}$-(anti-)reflexive solutions for (1) over $\mathbb{C}$)**.** *[54] Given* $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{n\times k}$, $D \in \mathbb{C}^{m\times k}$. *Let* $P \in \mathbb{C}^{m\times m}$ *and* $Q \in \mathbb{C}^{n\times n}$ *be Hermitian and* $\{k+1\}$*-potent with* $k = 1$. *For* U *and* V *are given in* (20)*, let* A_1, C_1, B_1, D_1 *be defined in*

$$AU = [A_1, A_2],\ CV = [C_1, C_2],\ V^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ U^*D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1 \in \mathbb{C}^{l\times p}$, $C_1 \in \mathbb{C}^{l\times q}$, $B_1 \in \mathbb{C}^{q\times k}$, *and* $D_1 \in \mathbb{C}^{p\times k}$. *Then, we have the following results.*

(*a*) *The system* (1) *is consistent for* $\{P, Q, 2\}$*-reflexive* X *if and only if*

$$A_1 A_1^\dagger C_1 = C_1,\ D_1 B_1^\dagger B_1 = D_1,\ A_1 D_1 = C_1 B_1.$$

In this case, the general solution is

$$X = U\begin{bmatrix} A_1^\dagger C_1 + D_1 B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + (I - A_1^\dagger A_1)U_1(I - B_1 B_1^\dagger) & O \\ O & O \end{bmatrix}V^*,$$

where $U_1 \in \mathbb{C}^{p\times q}$ *is arbitrary.*

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(b) *Let S_X be the set of all $\{P, Q, 2\}$-reflexive solutions to (1) and E be a given matrix in $\mathbb{C}^{m\times n}$. Partition*

$$U^*EV = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

with $E_{11} \in \mathbb{C}^{p\times q}$. Then,

$$\|\widehat{X} - E\| = \min_{X\in S_X} \|X - E\|$$

has an only solution $\widehat{X}$ which can be expressed as

$$\widehat{X} = U \begin{bmatrix} A_1^\dagger C_1 + D_1 B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + (I - A_1^\dagger A_1)E_{11}(I - B_1 B_1^\dagger) & O \\ O & O \end{bmatrix} V^*.$$

(c) *Assume that the SVDs of A_1, B_1 is expressed as*

$$A_1 = W \begin{bmatrix} M_1 & O \\ O & O \end{bmatrix} Z^*,\ B_1 = P \begin{bmatrix} N_1 & O \\ O & O \end{bmatrix} Q^*,$$

where $W = [W_1, W_2] \in \mathbb{C}^{l\times l}$, $Z = [Z_1, Z_2] \in \mathbb{C}^{p\times p}$, $P = [P_1, P_2] \in \mathbb{C}^{q\times q}$, and $Q = [Q_1, Q_2] \in \mathbb{C}^{k\times k}$ are unitary matrices, $M_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_{r_1})$, $r_1 = \mathrm{rank}(M_1)$, $W_1 \in \mathbb{C}^{l\times r_1}$, $Z_1 \in \mathbb{C}^{p\times r_1}$, $N_1 = \mathrm{diag}(\rho_1, \cdots, \rho_{r_2})$, $r_2 = \mathrm{rank}(N_1)$, $P_1 \in \mathbb{C}^{q\times r_2}$, $Q_1 \in \mathbb{C}^{k\times r_2}$. Then, the least norm least squares solution $X \in S_L$ can be expressed as

$$X = U \begin{bmatrix} Z \begin{bmatrix} \Theta(M_1 W_1^* C_1 P_1 + Z_1^* D_1 Q_1 N_1) & M_1^{-1} W_1^* C_1 P_2 \\ Z_2^* D_1 Q_1 N_1^{-1} & Y_4 \end{bmatrix} P^* & O \\ O & O \end{bmatrix} V^*,$$

where $\Theta = (\theta_{ij}) \in \mathbb{R}^{r_1\times r_2}, \theta_{ij} = \frac{1}{\sigma_i^2 + \rho_j^2}$, and $Y_4 \in \mathbb{C}^{(p-r_1)\times(q-r_2)}$ is an arbitrary matrix.

Theorem 52 ($\{P, Q, 3\}$-(anti-)reflexive solutions for (1) over $\mathbb{C}$). *[54] Given $A \in \mathbb{C}^{l\times m}$, $C \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{n\times k}$, $D \in \mathbb{C}^{m\times k}$, $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ are Hermitian and $\{k+1\}$-potent with $k = 2$. For U and V are given in (20), let $A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2$ be defined in*

$$AU = [A_1, A_2, A_3],\ CV = [C_1, C_2, C_3],\ V^*B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},\ U^*D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix},$$

where $A_1 \in \mathbb{C}^{l\times r}$, $A_2 \in \mathbb{C}^{l\times(p-r)}$, $C_1 \in \mathbb{C}^{l\times s}$, $C_2 \in \mathbb{C}^{l\times(q-s)}$, $B_1 \in \mathbb{C}^{s\times k}$, $B_2 \in \mathbb{C}^{(q-s)\times k}$, $D_1 \in \mathbb{C}^{r\times k}$, and $D_2 \in \mathbb{C}^{(p-r)\times k}$.

(a) *Then, the system (1) is consistent for $\{P, Q, 3\}$-reflexive X if and only if*

$$A_1 A_1^\dagger C_1 = C_1,\ D_1 B_1^\dagger B_1 = D_1,\ A_1 D_1 = C_1 B_1,\ A_2 A_2^\dagger C_2 = C_2,\ D_2 B_2^\dagger B_2 = D_2,\ A_2 D_2 = C_2 B_2.$$

In this case, the general solution is

$$X = U \begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & O \end{bmatrix} V^*,$$

where $X_1 = A_1^\dagger C_1 + D_1 B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + (I - A_1^\dagger A_1)U_{11}(I - B_1 B_1^\dagger)$, $X_2 = A_2^\dagger C_2 + D_2 B_2^\dagger - A_2^\dagger A_2 D_2 B_2^\dagger + (I - A_2^\dagger A_2)U_{22}(I - B_2 B_2^\dagger)$, and U_{11}, U_{22} are arbitrary with suitable orders.

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(b) Let S_X be the set of all $\{P, Q, 3\}$-reflexive solutions to (1) and let E be a given matrix in $\mathbb{C}^{m\times n}$. Partition

$$U^*EV = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

with $E_{11} \in \mathbb{C}^{r\times s}$, $E_{22} \in \mathbb{C}^{(p-r)\times(q-s)}$. Then, $\|\widehat{X} - E\| = \min_{X\in S_X} \|X - E\|$ has a unique solution $\widehat{X}$ which can be expressed as

$$\widehat{X} = U \begin{bmatrix} X_1 & 0 & O \\ O & X_2 & O \\ O & O & O \end{bmatrix} V^*,$$

where $X_1 = A_1^\dagger C_1 + D_1 B_1^\dagger - A_1^\dagger A_1 D_1 B_1^\dagger + (I - A_1^\dagger A_1)E_{11}(I - B_1 B_1^\dagger)$ and $X_2 = A_2^\dagger C_2 + D_2 B_2^\dagger - A_2^\dagger A_2 D_2 B_2^\dagger + (I - A_2^\dagger A_2)E_{22}(I - B_2 B_2^\dagger)$.

Remark 12. *The $\{P, Q, 3\}$-reflexive least squares problem can be reduced similarly to Theorem 51 (c), hence the conclusion is omitted.*

4.6. Different Types of Conjugate Solutions

Chang et al. have presented the (R, S)-conjugate solution to the linear equation system (1) [55]. A matrix $A \in \mathbb{C}^{n\times n}$ is called an R-conjugate matrix if it satisfies $\bar{A} = RAR$, where R is a nontrivial involution (i.e. $R^2 = I$, $R \neq I$). A matrix $A \in \mathbb{C}^{m\times n}$ is called an (R, S)-conjugate matrix if it satisfies $\bar{A} = RAS$, where R and S are nontrivial involutions. The sets of R-conjugate and (R, S)-conjugate matrices are denoted by $\mathbb{C}_C^{n\times n}(R)$ and $\mathbb{C}_C^{m\times n}(R, S)$, respectively. For nontrivial involution matrices $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$, there exists

$$R = [P, Q] \begin{bmatrix} I_r & O \\ O & -I_{m-r} \end{bmatrix} \begin{bmatrix} P^{\mathrm{T}} \\ Q^{\mathrm{T}} \end{bmatrix}, \; S = [U, V] \begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \begin{bmatrix} U^{\mathrm{T}} \\ V^{\mathrm{T}}\} \end{bmatrix}.$$

Denote $\Gamma = [P, iQ]$, $F = [U, iV]$. The results for the solutions in $\mathbb{C}_C^{n\times n}(R)$ and $\mathbb{C}_C^{m\times n}(R, S)$ to the system (1) are presented below.

Theorem 53 ((R, S)-conjugate solutions for (1) over $\mathbb{C}$)**.** *[55] Given $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$, nontrivial involutions R and S. Suppose that $A = A_1 + iA_2 \in \mathbb{C}^{p\times m}$ and $B = B_1 + iB_2 \in \mathbb{C}^{n\times q}$, where*

$$A_1 = \frac{1}{2}(A + \bar{A}R), \; A_2 = \frac{1}{2i}(A - \bar{A}R), \; B_1 = \frac{1}{2}(B + S\bar{B}), \; B_2 = \frac{1}{2i}(B - S\bar{B}).$$

Let

$$CF = C_1 + iC_2, \; \Gamma^* D = D_1 + iD_2,$$

where $C_1, C_2 \in \mathbb{R}^{k\times n}, D_1, D_2 \in \mathbb{R}^{m\times q}$. Denote

$$F = \begin{bmatrix} A_1\Gamma \\ A_2\Gamma \end{bmatrix}, \; G = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \; K = [F^* B_1, F^* B_2], \; L = [D_1, D_2].$$

Assume that the SVDs of $F \in \mathbb{R}^{2p\times m}$ and $K \in \mathbb{R}^{n\times 2q}$ are

$$F = W \begin{bmatrix} D_{r_1} & O \\ O & O \end{bmatrix} \widetilde{V}^{\mathrm{T}} = W_1 D_{r_1} V_1^{\mathrm{T}}, \; K = M \begin{bmatrix} D_{r_2} & O \\ O & O \end{bmatrix} \widetilde{U}^{\mathrm{T}} = M_1 D_{r_2} U_1^{\mathrm{T}},$$

where

$$W = [W_1, W_2], \; \widetilde{V} = [V_1, V_2], \; M = [M_1, M_2], \; \widetilde{U} = [U_1, U_2],$$

with $W_1 \in \mathbb{R}^{2p\times r_1}, V_1 \in \mathbb{R}^{m\times r_1}, M_1 \in \mathbb{R}^{n\times r_2}$, $U_1 \in \mathbb{R}^{2q\times r_2}$, $D_{r_1} = \mathrm{diag}\,(\lambda_1,\cdots,\lambda_{r_1})$ *and* $D_{r_2} = \mathrm{diag}(\mu,\cdots,\mu_{r_2})$.

(*a*) *Then, the system* (1) *has a solution in* $\mathbb{C}_C^{m\times n}(R,S)$ *if and only if*

$$FF^{\dagger}G = G,\; LK^{\dagger}K = L,\; FL = GK.$$

In which case, the general (R,S)*-conjugate solution to* (1) *can be represented as*

$$X = \Gamma\left(F^{\dagger}G + V_2V_2^{\mathrm{T}}LK^{\dagger} + V_2NM_2^{\mathrm{T}}\right)F^{*},$$

where $N \in \mathbb{R}^{(m-r_1)\times(n-r_2)}$ *is arbitrary.*

(*b*) *For a given* $E \in \mathbb{C}^{m\times n}$, *let* $E_1 = \frac{1}{2}(E + R\bar{E}S)$. *Denote the* (R,S)*-conjugate solution set of* (1) *is* S_X. *If* S_X *is nonempty, then the approximation problem* $||X - E|| = \min\limits_{X\in S_E}$ *has a unique solution* $\hat{X}$ *is the form of*

$$\hat{X} = \Gamma\left(F^{\dagger}G + V_2V_2^{\mathrm{T}}LK^{\dagger}M_1M_1^{\mathrm{T}} + V_2V_2^{\mathrm{T}}\Gamma^{*}E_1FM_2M_2^{\mathrm{T}}\right)F^{*}.$$

(*c*) *When* (1) *has not an* (R,S)*-conjugate solution. Then, the least squares solution if* (1) *can be expressed as*

$$X = \Gamma\widetilde{V}\begin{bmatrix} \Phi * \left(D_{r_1}W_1^{\mathrm{T}}GM_1 + V_1^{\mathrm{T}}LUD_{r_2}\right) & D_{r_1}^{-1}W_1^{\mathrm{T}}GM_2 \\ V_2^{\mathrm{T}}LU_1D_{r_2}^{-1} & Y_{22}\end{bmatrix}M^{\mathrm{T}}F^{*},$$

where $\Phi = (\phi_{ij}) \in \mathbb{R}^{r_1\times r_2}, \phi = \frac{1}{\lambda_i^2+\mu_j^2}$ *and* $Y_{22} \in \mathbb{R}^{(m-r_1)\times(n-r_2)}$ *is an arbitrary matrix. The unique least squares least norm solution of* (1) *is*

$$\tilde{X} = \Gamma\widetilde{V}\begin{bmatrix} \Phi * \left(D_{r_1}W_1^{\mathrm{T}}GM_1 + V_1^{\mathrm{T}}LUD_{r_2}\right) & D_{r_1}^{-1}W_1^{\mathrm{T}}GM_2 \\ V_2^{\mathrm{T}}LU_1D_{r_2}^{-1} & O\end{bmatrix}M^{\mathrm{T}}F^{*}.$$

Two years later, Chang et al. extended the results in [55] to consider the Hermitian R-conjugate solutions. They provided the necessary and sufficient conditions for the existence of the Hermitian R-conjugate solution to the system of complex matrix equations $AX = C$ and $XB = D$, and presented an expression for the Hermitian R-conjugate solution to this system when the solvability conditions are satisfied. In addition, the solution to an optimal approximation problem was obtained. Furthermore, the least squares Hermitian R-conjugate solution with the least norm for this system was also considered [56].

Theorem 54 (Hermitian R-conjugate solutions for (1) over $\mathbb{C}$)**.** *[56] For given* $A \in \mathbb{R}^{p\times n}, C \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{n\times q}$, *and* $D \in \mathbb{R}^{n\times q}$. *Given a nontrivial symmetric involution matrix* $R \in \mathbb{R}^{n\times n}$, *which can be expressed as*

$$R = [P,Q]\begin{bmatrix} I_r & O \\ O & -I_{n-r}\end{bmatrix}\begin{bmatrix} P^{\mathrm{T}} \\ Q^{\mathrm{T}}\end{bmatrix},$$

where $P \in \mathbb{R}^{n\times r}$ *and* $Q \in \mathbb{R}^{n\times(n-r)}$ *satisfying* $P^{\mathrm{T}}P = I_r$, $Q^{\mathrm{T}}Q = I_{n-r}$, $P^{\mathrm{T}}Q = O$, $Q^{\mathrm{T}}P = O$. *Denote* $\Gamma = [P,\ iQ]$. *Let* $A\Gamma = A_1 + iA_2$, $C\Gamma = C_1 + iC_2$, $\Gamma^{*}B = B_1 + iB_2$, *and* $\Gamma^{*}D = D_1 + iD_2$, *where*

$$A_1 = \frac{1}{2}(A\Gamma + \overline{A\Gamma}),\ A_2 = \frac{1}{2i}(A\Gamma - \overline{A\Gamma}),\ C_1 = \frac{1}{2}(C\Gamma + \overline{C\Gamma}),\ C_2 = \frac{1}{2i}(C\Gamma - \overline{C\Gamma}),$$
$$B_1 = \frac{1}{2}(\Gamma^{*}B + \overline{\Gamma^{*}B}),\ B_2 = \frac{1}{2i}(\Gamma^{*}B - \overline{\Gamma^{*}B}),\ D_1 = \frac{1}{2}(\Gamma^{*}D + \overline{\Gamma^{*}D}),\ D_2 = \frac{1}{2i}(\Gamma^{*}D - \overline{\Gamma^{*}D}).$$

Denote

$$F = \begin{bmatrix} A_1 \\ A_2\end{bmatrix},\ G = \begin{bmatrix} C_1 \\ C_2\end{bmatrix},\ K = [B_1,\ B_2],\ L = [D_1,\ D_2],\ M = \begin{bmatrix} F \\ K^{\mathrm{T}}\end{bmatrix},\ N = \begin{bmatrix} G \\ L^{\mathrm{T}}\end{bmatrix}.$$

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Assume that the SVD of $M \in \mathbb{R}^{(2p+2q)\times n}$ is

$$M = U\left[\begin{array}{cc} M_1 & O \\ O & O \end{array}\right]V^{\mathrm{T}},$$

where $U = [U_1, U_2] \in \mathbb{R}^{(2m+2l)\times(2m+2l)}$ and $V = [V_1, V_2] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $M_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, $r = \mathrm{rank}(M)$, $U_1 \in \mathbb{R}^{(2m+2l)\times r}$, $V_1 \in \mathbb{R}^{n\times r}$.

(a) The system (1) has a Hermitian R-conjugate solution in $\mathbb{C}^{n\times n}$ if and only if

$$MN^{\mathrm{T}} = NM^{\mathrm{T}},\ U_2^{\mathrm{T}}N = O.$$

In that case, (1) has the general Hermitian R-conjugate solution

$$X = \Gamma\left(V_1 M_1^{-1} U_1^{\mathrm{T}} N + V_2 V_2^{\mathrm{T}} N^{\mathrm{T}} U_1 M_1^{-1} V_1^{\mathrm{T}} + V_2 G V_2^{\mathrm{T}}\right)\Gamma^*,$$

where G is a arbitrary $(n-r)\times(n-r)$ symmetric matrix.

(b) For $E \in \mathbb{R}^{n\times n}$, let $E_1 = \frac{1}{2}(\Gamma^* E\Gamma + \Gamma^*\bar{E}\Gamma)$. The system (1) has Hermitian R-conjugate solutions, then the optimal approximation problem $||X - E|| = \min$ has a unique Hermitian R-reflexive solution of (1) as

$$\hat{X} = \Gamma(V_1 M_1^{-1} U_1^{\mathrm{T}} N + V_2 V_2^{\mathrm{T}} N^{\mathrm{T}} U_1 M_1^{-1} V_1^{\mathrm{T}} + V_2 V_2^{\mathrm{T}} E_1 V_2 V_2^{\mathrm{T}})\Gamma^*.$$

(c) The least squares Hermitian R-conjugate solution of (1) can be expressed as

$$X = \Gamma V\begin{bmatrix} M_1^{-1} U_1^{\mathrm{T}} N V_1 & M_1^{-1} U_1^{\mathrm{T}} N V_2 \\ V_2^{\mathrm{T}} N^{\mathrm{T}} U_1 M_1^{-1} & Y_{22} \end{bmatrix} V^{\mathrm{T}}\Gamma^*,$$

where $Y_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ is an arbitrary symmetric matrix.

(d) The least norm least squares Hermitian R-conjugate solution of (1) can be expressed as

$$X = \Gamma V\begin{bmatrix} M_1^{-1} U_1^{\mathrm{T}} N V_1 & M_1^{-1} U_1^{\mathrm{T}} N V_2 \\ V_2^{\mathrm{T}} N^{\mathrm{T}} U_1 M_1^{-1} & O \end{bmatrix} V^{\mathrm{T}}\Gamma^*.$$

4.7. Conjugate Class Solutions

Recall that matrices $X, Y \in \mathbb{C}^{n\times n}$ are in the same *congruence class if there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $X = P^* Y P$.

Zheng first considered the *congruence class of the solutions of (1) [57].

Theorem 55 (*congruence class solutions for (1) over $\mathbb{C}$). *[57] Let $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$. The GSVD of the matrices A and B is given by*

$$A = U_A S_1 T^*, B = T S_2 V_B^*,$$

where $U_A \in \mathbb{C}^{p\times p}$, $V_B \in \mathbb{C}^{q\times q}$ are unitary matrices, $T \in \mathbb{C}^{n\times n}$ is a nonsingular matrix and

$$S_1 = \begin{array}{c} \begin{bmatrix} I & O \\ O & O \end{bmatrix} \begin{array}{l} r_1 \\ m - r_1 \end{array} \\ \begin{array}{cc} r_1 & n - r_1 \end{array} \end{array},$$

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$$S_2 = \begin{bmatrix} \Gamma_1 & O & O \\ O & O & O \\ O & \Gamma_2 & O \\ O & O & O \end{bmatrix} \begin{matrix} r_2^1 \\ r_1 - r_2^1 \\ r_2^2 \\ n - r_1 - r_2^2 \end{matrix}$$
$$\quad r_2^1 \quad r_2^2 \quad k - r_2$$

are block matrices with positive diagonal matrices Γ_1, Γ_2. $r_1 = \mathrm{rank}(A)$, $r_2 = r_2^1 + r_2^2 = \mathrm{rank}(B)$. *Block* $U_A CT$ *and* $T^* DV_B$ *into suitable size as the form of*

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \end{bmatrix}, \; D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \end{bmatrix}.$$

The system (1) *is solvable if and only if*

$$C_{2j} = O, \; j = 1,2,3,4, \; D_{i3} = O, \; i = 1,2, \; C_{11} = D_{11}\Gamma_1^{-1}, \; C_{13} = D_{12}\Gamma_2^{-1}.$$

The general form of a solution of (1) *is*

$$X = T^{-*} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ D_{21}\Gamma_1^{-1} & X_{22} & D_{22}\Gamma_2^{-1} & X_{24} \end{bmatrix} T^{-1},$$

where $X_{22} \in \mathbb{C}^{(n-r_1)\times(r_1 - r_2^1)}$ *and* $X_{24} \in \mathbb{C}^{(n-r_1)\times(n-r_1-r_2^2)}$ *are an arbitrary. Obviously, X is *congruent to*

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ D_{21}\Gamma_1^{-1} & X_{22} & D_{22}\Gamma_2^{-1} & X_{24} \end{bmatrix}.$$

Later, Zhang presented the following result.

Theorem 56 (*congruence class solutions for (1) over $\mathbb{C}$). *[58] Let* $A, C \in \mathbb{C}^{p\times n}$ *and* $B, D \in \mathbb{C}^{n\times q}$. *Assume that the GSVD of A and* B^* *can be expressed as*

$$A = U\Sigma_A P, \; B^* = V\Sigma_B P,$$

where $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{p\times p}$ *are unitary matrices,* $P \in \mathbb{C}^{n\times n}$ *is nonsingular matrix,* $\Sigma_A \in \mathbb{C}^{m\times n}$, $\Sigma_B \in \mathbb{C}^{p\times n}$, $r = \mathrm{rank}[A^*, B]$,

$$\Sigma_A = \begin{bmatrix} I_A & O & O & O \\ O & S_A & O & O \\ O & O & O & O \end{bmatrix},$$
$$\quad t \quad s \quad r-s-t \quad n-r$$

$$\Sigma_A = \begin{bmatrix} O & O & O & O \\ O & S_B & O & O \\ O & O & I_B & O \end{bmatrix},$$
$$\quad t \quad s \quad r-s-t \quad n-r$$

where $S_A = \mathrm{diag}(\alpha_1, \cdots, \alpha_s)$, $S_B = \mathrm{diag}(\beta_1, \cdots, \beta_s)$ *with* $1 > \alpha_1 \geq \cdots \geq \alpha_s > 0, 0 < \beta_1 \leq \cdots \leq \beta_s < 1$, *and* $\alpha_i^2 + \beta_i^2 = 1, i = 1, \cdots, s$. *Denote*

$$U^* C P^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{bmatrix},$$
$$\quad t \quad s \quad r-s-t \quad n-r$$

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$$PDV = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \\ D_{41} & D_{42} & D_{43} \end{bmatrix}.$$
$$\quad p-r-t \quad s \quad r-s-t$$

(a) The system (1) has a solution in $\mathbb{C}^{n\times n}$ if and only if

$$C_{3i} = O,\ D_{i1} = O,\ i = 1,2,3,4,$$
$$C_{12} = D_{12}S_B^{-1},\ C_{13} = D_{13},\ S_A^{-1}C_{22} = D_{22}S_B^{-1},\ S_A^{-1}C_{23} = D_{23}.$$

In that case, the general solutions of (1) are

$$X = P^{-1}\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ X_{31} & D_{32}S_B^{-1} & D_{33} & X_{34} \\ X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44} \end{bmatrix}(P^{-1})^*,$$

where X_{31}, X_{41}, X_{34}, and X_{44} are arbitrary.

*(b) For arbitrary X_{31}, X_{41}, X_{34}, and X_{44}, there exists a solution in $\mathbb{C}^{n\times n}$ of (1), which is *congruent to*

$$Y = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ X_{31} & D_{32}S_B^{-1} & D_{33} & X_{34} \\ X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44} \end{bmatrix}.$$

*(c) There exists a minimum norm solution in $\mathbb{C}^{n\times n}$ of (1), which is *congruent to*

$$Y = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ O & D_{32}S_B^{-1} & D_{33} & O \\ O & D_{42}S_B^{-1} & D_{43} & O \end{bmatrix}.$$

Remark 13. *Theorem 56 differs from Theorem 55 in both the approach to decomposition and the way solutions are expressed. Specifically, Theorem 56 provides an extended formulation that generalizes the results in Theorem 55, offering a more comprehensive method for decomposing the matrix equations and presenting solutions in a broader context.*

The next theorem shows the corresponding least squares and least squares least norm solutions through GSVD.

Theorem 57 (Least squares *congruence class solutions for (1) over $\mathbb{C}$)**.** *[58] Let $A, C \in \mathbb{C}^{p\times n}$, $B, D \in \mathbb{C}^{n\times q}$, and $\mathrm{rank}(A) = \mathrm{rank}(B) = k$. There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and nonsingular matrices $R_A \in \mathbb{C}^{p\times p}$ and $R_B \in \mathbb{C}^{q\times q}$, such that the GSVD of matrix pair $[A^*, B]$ is given as*

$$A^* = U(\Sigma_A, O)R_A^{-1},\ B = U(\Sigma_B, O)R_B^{-1},$$

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where $\Sigma_A, \Sigma_B \in \mathbb{C}^{n\times k}$ *are*

$$\Sigma_A = \begin{bmatrix} I_r & O & O \\ O & G & O \\ O & O & O \\ O & O & O \\ O & S & O \\ O & O & I_t \end{bmatrix}, \Sigma_B = \begin{bmatrix} I_r & O & O \\ O & I_s & O \\ O & O & I_{q-r-s} \\ O & O & O \\ O & O & O \\ O & O & O \end{bmatrix},$$

with $p = r+s+t$, $G = \mathrm{diag}(g_{r+1}, \cdots, g_{r+s})$, $1 > g_{r+1} \geq \cdots \geq g_{r+s} > 0$, $S = \mathrm{diag}(w_{r+1}, \cdots, w_{r+s})$, $0 > w_{r+1} \geq \cdots \geq w_{r+s} > 1$, $G^2 + S^2 = I_s$. *Denote the suitable block matrices with the forms*

$$(R_A)^* CU = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \end{bmatrix},$$

$$U^* D R_B = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \\ D_{51} & D_{52} & D_{53} & D_{54} \\ D_{61} & D_{62} & D_{63} & D_{64} \end{bmatrix}.$$

(a) *The least square solutions to* (1) *are*

$$X = U \begin{bmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16} \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & X_{34} & X_{35} & X_{36} \\ D_{41} & D_{42} & D_{43} & X_{44} & X_{45} & X_{46} \\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26} \\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{bmatrix} U^*,$$

where X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, *and* X_{46} *are arbitrary,* $Y_{5i} = \Phi * (S(GD_{2i} - C_{2i}) + D_{5i})$, $i = 1,2,3$, $\Phi = (\varphi_{jk}) \in \mathbb{C}^{s\times s}$, $\varphi_{jj} = \frac{1}{w_{r+j}^2+1}$, $\varphi_{jk} = 0, j \neq k$.

(b) *For arbitrary* X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, *and* X_{46}, *there exists a least square solution in* $\mathbb{C}^{n\times n}$ *of* (1), *which is *congruent to*

$$Y = \begin{bmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16} \\ D_{21} & D_{22} & D_{23} & O & O & O \\ D_{31} & D_{32} & D_{33} & X_{34} & X_{35} & X_{36} \\ D_{41} & D_{42} & D_{43} & X_{44} & X_{45} & X_{46} \\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26} \\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{bmatrix},$$

where $Y_{5i} = \Phi * (S(GD_{2i} - C_{2i}) + D_{5i})$, $i = 1,2,3$, $\Phi = (\varphi_{jk}) \in \mathbb{C}^{s\times s}$, $\varphi_{jj} = \frac{1}{w_{r+j}^2+1}$, $\varphi_{jk} = 0, j \neq k$.

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*(c) There exists a minimum norm least square solution in $\mathbb{C}^{n\times n}$ of (1), which is *congruent to*

$$Y=\begin{bmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16}\\ D_{21} & D_{22} & D_{23} & O & O & O\\ D_{31} & D_{32} & D_{33} & O & O & O\\ D_{41} & D_{42} & D_{43} & O & O & O\\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26}\\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{bmatrix},$$

where $Y_{5i}=\Phi*(S(GD_{2i}-C_{2i})+D_{5i}),\ i=1,2,3,\ \Phi=(\varphi_{jk})\in\mathbb{C}^{s\times s},\ \varphi_{jj}=\frac{1}{w_{r+j}^2+1},\ \varphi_{jk}=0, j\neq k.$

4.8. (Anti-)Hermitian (Anti-)Hamiltonian Solutions

Hamiltonian matrices play a crucial role in various engineering applications, particularly in solving Riccati equations. Yu et al. studied four extended Hamiltonian solutions of the system (1) [59].

Table 1 outlines the definitions of anti-symmetric orthogonal matrices and (anti-)Hermitian generalized (anti-)Hamiltonian matrices, with $J\in\mathbb{R}^{2n\times 2n}$ representing a non-trivial anti-symmetric orthogonal matrix, and $X\in\mathbb{C}^{2n\times 2n}$ denoting the (anti-)Hermitian generalized (anti-)Hamiltonian matrix.

Table 1. Definition of (anti-) Hermitian generalized (anti-)Hamiltonian matrices

Symbols	Types of matrix	
$ASO\mathbb{R}^{2n\times 2n}$	non-trivial anti-symmetric orthogonal matrix	$J^T=-J=J^{-1}\neq I$
$HH\mathbb{C}^{2n\times 2n}$	Hermitian generalized Hamiltonian	$X=X^*$ and $JXJ=X^*$
$HAH\mathbb{C}^{2n\times 2n}$	Hermitian generalized anti-Hamiltonian	$X=X^*$ and $JXJ=-X^*$
$AHH\mathbb{C}^{2n\times 2n}$	anti-Hermitian generalized Hamiltonian	$X=-X^*$ and $JXJ=X^*$
$AHAH\mathbb{C}^{2n\times 2n}$	anti-Hermitian generalized anti-Hamiltonian	$X=-X^*$ and $JXJ=-X^*$

For $J\in ASO\mathbb{R}^{2n\times 2n}$, the EVD of J can be expressed as

$$J=P\begin{bmatrix} iI_n & O\\ O & -iI_n\end{bmatrix}P^*, \tag{22}$$

where $P\in\mathbb{C}^{2n\times 2n}$ is a unitary matrix [59].

Next, we present the necessary and sufficient conditions for the (anti-)Hermitian generalized (anti-)Hamiltonian solutions to the system (1) along with corresponding expressions. Additionally, for a given $E\in\mathbb{C}^{2n\times 2n}$, we consider the optimization problem $||X-E||=\min$, where X satisfies (1).

Theorem 58 ($HH\mathbb{C}$ solutions for (1) over $\mathbb{C}$). *[59] Given $A,C\in\mathbb{C}^{p\times 2n}, B,D\in\mathbb{C}^{2n\times q}$, let the decomposition of $J\in ASO\mathbb{R}^{2n\times 2n}$ be (22). Partition*

$$AP=[A_1,A_2],\ CP=[C_1,C_2],\ P^*B=\begin{bmatrix}B_1\\B_2\end{bmatrix},\ P^*D=\begin{bmatrix}D_1\\D_2\end{bmatrix}, \tag{23}$$

where $A_1,\ A_2,\ C_1,\ C_2\in\mathbb{C}^{p\times n}$ and $B_1,\ B_2,\ D_1,\ D_2\in\mathbb{C}^{n\times q}$. Denote

$$A'=\begin{bmatrix}A_1\\B_1^*\end{bmatrix},\ C'=\begin{bmatrix}C_2\\D_2^*\end{bmatrix},\ B'=[A_2^*,B_2],\ D'=[C_1^*,D_1].$$

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(a) *Then, the system* (1) *has a solution* $X \in HH\mathbb{C}^{2n\times 2n}$ *if and only if*

$$A'(A')^{\dagger}C' = C',\ D'(B')^{\dagger}B' = D',\ A'D' = C'B',$$

in which case the Hermitian generalized Hamiltonian solution to (1) *can be expressed as*

$$X = P\begin{bmatrix} O & X_{12} \\ X_{12}^{*} & O \end{bmatrix}P^{*},$$

where

$$X_{12} = (A')^{\dagger}C' + D'(B')^{\dagger} - (A')^{\dagger}A'D'(B')^{\dagger} + \mathcal{L}_{A'}W\mathcal{R}_{B'}$$

and $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

(b) *For a given* $E \in \mathbb{C}^{2n\times 2n}$, *let*

$$P^{*}EP = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},\ E_{11} \in \mathbb{C}^{n\times n}, E_{22} \in \mathbb{C}^{n\times n}.$$

Assume that the system (1) *has a solution* $X \in HH\mathbb{C}^{2n\times 2n}$. *Then, the optimization problem* $||X - E|| = \min$ *has a unique solution* $X \in HH\mathbb{C}^{2n\times 2n}$ *of* (1) *if and only if*

$$\mathcal{L}_{A'}\left(\frac{1}{2}(E_{12} + (E_{21})^{*}) - X_0\right)\mathcal{R}_{B'} = \frac{1}{2}(E_{12} + (E_{21})^{*}) - X_0,$$

in which case the unique solution X *can be expressed as*

$$E = P\begin{bmatrix} O & \overline{X_0} \\ (\overline{X_0})^{*} & O \end{bmatrix}P^{*},$$

where

$$\overline{X_0} = \frac{1}{2}(E_{12} + (E_{21})^{*}),\ X_0 = (A')^{\dagger}C' + D'(B')^{\dagger} - (A')^{\dagger}A'D'(B')^{\dagger}.$$

(c) *Denote*

$$A'' = [A_1^{*}, B_1],\ C'' = [C_2^{*}, D_2].$$

Let the SVDs of A'' *and* C'' *be given by*

$$A'' = P_1\begin{bmatrix} \Gamma & O \\ O & O \end{bmatrix}Q_1^{*},\ C'' = U_1\begin{bmatrix} \Lambda & O \\ O & O \end{bmatrix}V_1^{*},$$

where $P_1 = [P_{11}, P_{12}]$, $Q_1 = [Q_{11}, Q_{12}]$, $U_1 = [U_{11}, U_{12}]$, $V_Q = [V_{11}, V_{12}]$, $\Gamma = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_{t_1})$, $t_1 = \mathrm{rank}(A'')$, $\Lambda = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_{t_2})$, $t_2 = \mathrm{rank}(C'')$. *Then, the least squares Hermitian generalized Hamiltonian solution to* (1) *can be described as*

$$X = P\begin{bmatrix} O & X_{12} \\ X_{12}^{*} & O \end{bmatrix}P^{*},$$

where

$$X_{12} = P_1\begin{bmatrix} \phi * (P_{11}^{*}D'V_{11}\Lambda + \Gamma Q_{11}^{*}(C')^{*}U_{12}) & \Gamma^{-1}Q_{11}^{*}(C')^{*}U_{12} \\ P_{12}^{*}D'V_{11}\Lambda^{-1} & X_{22}' \end{bmatrix}U_1^{*},$$

with $\phi = (\phi_{ij}) \in \mathbb{C}^{t_1\times t_2}$, $\phi_{ij} = \frac{1}{\delta_i^2 + \gamma_j^2}$, *and arbitrary* $X_{22}' \in \mathbb{C}^{(n-t_1)\times(n-t_2)}$.

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Theorem 59 ($HAH\mathbb{C}$ solutions for (1) over $\mathbb{C}$)**.** *[59] Given $A, C \in \mathbb{C}^{p\times 2n}$, $B, D \in \mathbb{C}^{2n\times q}$, let the decomposition of $J \in ASO\mathbb{R}^{2n\times 2n}$ be (22). The matrices AP, CP, P^*B, and P^*D respectively have the partitions as in (23). Denote*

$$\overline{A} = \begin{bmatrix} A_1 \\ B_1^* \end{bmatrix},\ \overline{C} = \begin{bmatrix} C_1 \\ D_1^* \end{bmatrix},\ \overline{B} = \begin{bmatrix} A_2 \\ B_2^* \end{bmatrix},\ \overline{D} = \begin{bmatrix} C_2 \\ D_2^* \end{bmatrix}.$$

Let the SVDs of $\overline{A}$ and $\overline{B}$ be

$$\overline{A} = U\begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix}V^*,\ \overline{B} = Q\begin{bmatrix} \Pi & O \\ O & O \end{bmatrix}R^*,$$

where $U, Q \in \mathbb{C}^{(m+q)\times k}$, $V, R \in \mathbb{C}^{n\times n}$, $\Sigma = \mathrm{diag}(\alpha_1, \cdots, \alpha_r)$, $r = \mathrm{rank}(\overline{A})$, $\Pi = \mathrm{diag}(\beta_1, \cdots, \beta_s)$, $s = \mathrm{rank}(\overline{B})$. Set

$$U^*\overline{C}V = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} \\ \overline{C}_{21} & \overline{C}_{22} \end{bmatrix},\ Q^*\overline{D}R = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} \\ \overline{D}_{21} & \overline{D}_{22} \end{bmatrix},$$

where $\overline{C}_{11} \in \mathbb{C}^{r\times r}$, $\overline{D}_{11} \in \mathbb{C}^{s\times s}$, $\overline{C}_{22} \in \mathbb{C}^{(m+q-r)\times(k-r)}$, $\overline{D}_{22} \in \mathbb{C}^{(m+q-s)\times(k-s)}$.

(a) Then, (1) has a solution $X \in HAH\mathbb{C}^{2n\times 2n}$ if and only if

$$\overline{A}\overline{A}^{\dagger}\overline{C} = \overline{C},\ \overline{A}(\overline{C})^* = \overline{C}(\overline{A})^*,\ \overline{C}_{21} = O,\ \overline{C}_{22} = O,$$
$$\overline{B}\overline{B}^{\dagger}\overline{D} = \overline{D},\ \overline{B}(\overline{D})^* = \overline{D}(\overline{B})^*,\ \overline{D}_{21} = O,\ \overline{D}_{22} = O,$$

in which case the Hermitian generalized anti-Hamiltonian solution to (1) can be described as

$$X = P\begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix}P^*,$$

where

$$X_{11} = V\begin{bmatrix} \Sigma^{-1}\overline{C}_{11} & \Sigma^{-1}\overline{C}_{12} \\ (\overline{C}_{12})^*\Sigma^{-1} & \overline{X}_{22} \end{bmatrix}V^*,\ X_{22} = R\begin{bmatrix} \Pi^{-1}\overline{D}_{11} & \Pi^{-1}\overline{D}_{12} \\ (\overline{D}_{12})^*\Pi^{-1} & \widehat{X}_{22} \end{bmatrix}R^*,$$

$\overline{X}_{22} \in \mathbb{C}^{(k-r)\times(k-r)}$ and $\widehat{X}_{22} \in \mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary Hermitian matrices.

(b) For a given $E \in \mathbb{C}^{2n\times 2n}$ and the system (1) has a solution $X \in HAH\mathbb{C}^{2n\times 2n}$, let

$$\frac{1}{2}(P^*E + E^*P) = \begin{bmatrix} E'_{11} & E'_{12} \\ (E'_{12})^* & E'_{22} \end{bmatrix},\ V^*E'_{11}V = \begin{bmatrix} \widehat{X}^0_{11} & \widehat{X}^0_{12} \\ (\widehat{X}^0_{12})^* & \widehat{X}^0_{22} \end{bmatrix},\ R^*E'_{22}R = \begin{bmatrix} \widehat{X}''_{11} & \widehat{X}''_{12} \\ (\widehat{X}''_{12})^* & \widehat{X}''_{22} \end{bmatrix},$$

where $E'_{11} \in \mathbb{C}^{n\times n}$, $E'_{22} \in \mathbb{C}^{n\times n}$, $\widehat{X}^0_{11} \in \mathbb{C}^{r\times r}$, $\widehat{X}^0_{22} \in \mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}''_{11} \in \mathbb{C}^{s\times s}$, $\widehat{X}''_{22} \in \mathbb{C}^{(k-s)\times(k-s)}$ are Hermitian. Then, the optimization problem $||X - E|| = \min$ has a unique solution $X^0 \in HAH\mathbb{C}^{2n\times 2n}$ of (1) as

$$X^0 = P\begin{bmatrix} X^0_{11} & O \\ O & X^0_{22} \end{bmatrix}P^*,$$

where

$$X^0_{11} = V\begin{bmatrix} \Sigma^{-1}\overline{C}_{11} & \Sigma^{-1}\overline{C}_{12} \\ (\overline{C}_{12})^*\Sigma^{-1} & \widehat{X}^0_{22} \end{bmatrix}V^*,\ X^0_{22} = R\begin{bmatrix} \Pi^{-1}\overline{D}_{11} & \Pi^{-1}\overline{D}_{12} \\ (\overline{D}_{12})^*\Pi^{-1} & \widehat{X}''_{22} \end{bmatrix}R^*.$$

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Theorem 60 ($AHH\mathbb{C}$ solutions for (1) over $\mathbb{C}$). *[59] Given $A, C \in \mathbb{C}^{p\times 2n}$, $B, D \in \mathbb{C}^{2n\times q}$, let the decomposition of $J \in ASO\mathbb{R}^{2n\times 2n}$ be (22). The matrices AP, CP, P^*B, and P^*D respectively have the partitions as in (23). Denote*

$$\widetilde{A} = \begin{bmatrix} A_1 \\ B_1^* \end{bmatrix}, \widetilde{C} = \begin{bmatrix} C_1 \\ -D_1^* \end{bmatrix}, \widetilde{B} = \begin{bmatrix} A_2 \\ C_2^* \end{bmatrix}, \widetilde{D} = \begin{bmatrix} C_2 \\ -D_2^* \end{bmatrix}.$$

Let the SVDs of $\widetilde{A}$ and $\widetilde{B}$ be, respectively,

$$\widetilde{A} = U\begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix}V^*, \ \widetilde{B} = Q\begin{bmatrix} \Pi & O \\ O & O \end{bmatrix}R^*,$$

where $U, Q \in \mathbb{C}^{(m+q)\times k}$, $V, R \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma = \mathrm{diag}(\alpha_1, \cdots, \alpha_r)$, $r = \mathrm{rank}(\widetilde{A})$, $\Pi = \mathrm{diag}(\beta_1, \cdots, \beta_s)$, $s = \mathrm{rank}(\widetilde{C})$. Set

$$U^*\widehat{C}V = \begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} \\ \widehat{C}_{21} & \widehat{C}_{22} \end{bmatrix}, \ Q^*\widehat{D}R = \begin{bmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & \widehat{D}_{22} \end{bmatrix},$$

where $\widehat{C}_{11} \in \mathbb{C}^{r\times r}$, $\widehat{D}_{11} \in \mathbb{C}^{s\times s}$, $\widehat{C}_{22} \in \mathbb{C}^{(m+q-r)\times(k-r)}$, $\widehat{D}_{22} \in \mathbb{C}^{(m+q-s)\times(k-s)}$.

(a) Then, (1) has a solution $X \in AHH\mathbb{C}^{2n\times 2n}$ if and only if

$$\widehat{A}\widehat{A}^{\dagger}\widehat{C} = \widehat{C}, \ \widehat{A}\widehat{C}^* = -\widehat{C}\widehat{A}^*, \ \widehat{C}_{21} = O, \ \widehat{C}_{22} = O,$$
$$\widehat{B}\widehat{B}^{\dagger}\widehat{D} = \widehat{D}, \ \widehat{B}\widehat{D}^* = -\widehat{D}\widehat{C}^*, \ \widehat{D}_{21} = O, \ \widehat{D}_{22} = O,$$

in which case the anti-Hermitian generalized Hamiltonian solution to (1) can be described as

$$X = P\begin{bmatrix} X_{11} & O \\ O & X_{22} \end{bmatrix}P^*,$$

where

$$X_{11} = V\begin{bmatrix} \Sigma^{-1}\widehat{C}_{11} & \Sigma^{-1}\widehat{C}_{12} \\ -\widehat{C}_{12}^*\Sigma^{-1} & \overline{X}_{22} \end{bmatrix}V^*, \ X_{22} = R\begin{bmatrix} \Pi^{-1}\widehat{D}_{11} & \Pi^{-1}\widehat{D}_{12} \\ -\widehat{D}_{12}^*\Pi^{-1} & \widehat{X}_{22} \end{bmatrix}R^*,$$

and $\overline{X}_{22} \in \mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}_{22} \in \mathbb{C}^{(k-s)\times(k-s)}$ are arbitrary anti-Hermitian.

(b) For a given $E \in \mathbb{C}^{2n\times 2n}$, let

$$\frac{1}{2}(P^*E - E^*P) = \begin{bmatrix} E'_{11} & E'_{12} \\ -(E'_{12})^* & E'_{22} \end{bmatrix}, \ V^*E'_{11}V = \begin{bmatrix} \widehat{X}^0_{11} & \widehat{X}^0_{12} \\ -(\widehat{X}^0_{12})^* & \widehat{X}^0_{22} \end{bmatrix}, \ R^*E'_{22}R = \begin{bmatrix} \widehat{X}''_{11} & \widehat{X}''_{12} \\ -(\widehat{X}''_{12})^* & \widehat{X}''_{22} \end{bmatrix},$$

where $E'_{11} \in \mathbb{C}^{n\times n}$, $E'_{22} \in \mathbb{C}^{n\times n}$, $\widehat{X}^0_{11} \in \mathbb{C}^{r\times r}$, $\widehat{X}^0_{22} \in \mathbb{C}^{(k-r)\times(k-r)}$, $\widehat{X}''_{11} \in \mathbb{C}^{s\times s}$, $\widehat{X}''_{22} \in \mathbb{C}^{(k-s)\times(k-s)}$ are anti-Hermitian. Assume that the system (1) has a solution $X \in AHH\mathbb{C}^{2n\times 2n}$. Then, the optimization problem $||X - E|| = \min$ has a unique solution $\tilde{X} \in AHH\mathbb{C}^{2n\times 2n}$ of (1) satisfying

$$\tilde{X} = P\begin{bmatrix} X^0_{11} & O \\ O & X^0_{22} \end{bmatrix}P^*,$$

where

$$X^0_{11} = V\begin{bmatrix} \Sigma^{-1}\widehat{B}_{11} & \Sigma^{-1}\widehat{B}_{12} \\ -\widehat{B}_{12}^*\Sigma^{-1} & \widehat{X}^0_{22} \end{bmatrix}V^*, \ X^0_{22} = R\begin{bmatrix} \Pi^{-1}\widehat{D}_{11} & \Pi^{-1}\widehat{D}_{12} \\ -\widehat{D}_{12}^*\Pi^{-1} & \widehat{X}''_{22} \end{bmatrix}R^*.$$

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Theorem 61 ($AHAH\mathbb{C}$ solutions for (1) over $\mathbb{C}$). *[59] Given $A, C \in \mathbb{C}^{p\times 2n}, B, D \in \mathbb{C}^{2n\times q}$, let the decomposition of $J \in ASO\mathbb{R}^{2n\times 2n}$ be (22). The matrices AP, CP, P^*B, and P^*D respectively have the partitions as in (23). Denote*

$$\widetilde{A} = \begin{bmatrix} A_1 \\ B_1^* \end{bmatrix},\ \widetilde{C} = \begin{bmatrix} C_2 \\ -D_2^* \end{bmatrix},\ \widetilde{B} = [A_2^*, B_2],\ \widetilde{D} = [-C_1^*, D_1].$$

(a) Then, (1) has a solution $X \in AHAH\mathbb{C}^{2n\times 2n}$ if and only if

$$\widetilde{A}\widetilde{A}^\dagger\widetilde{C} = \widetilde{C},\ \widetilde{D}\widetilde{B}^\dagger\widetilde{B} = \widetilde{D},\ \widetilde{A}\widetilde{D} = \widetilde{C}\widetilde{B},$$

in which case the anti-Hermitian generalized anti-Hamiltonian solution to (1) can be expressed as

$$X = P\begin{bmatrix} O & X_{12} \\ -X_{12}^* & O \end{bmatrix}P^*,$$

where

$$X_{12} = \widetilde{A}^\dagger\widetilde{C} + \widetilde{D}\widetilde{B}^\dagger - \widetilde{A}^\dagger\widetilde{A}\widetilde{D}\widetilde{B}^\dagger + \mathcal{L}_{\widetilde{A}}Z\mathcal{R}_{\widetilde{B}}$$

and $Z \in \mathbb{C}^{n\times n}$ is arbitrary.

(b) For a given $E \in \mathbb{C}^{2n\times 2n}$, let

$$P^*EP = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},\ E_{11} \in \mathbb{C}^{n\times n}, E_{22} \in \mathbb{C}^{n\times n}.$$

If the system (1) has a solution $X \in AHAH\mathbb{C}^{2n\times 2n}$, the optimization problem $||X - E|| = \min$ has a unique solution $\hat{X} \in AHAH\mathbb{C}^{2n\times 2n}$ of (1) if and only if

$$\mathcal{L}_{A'}\left(\frac{1}{2}(E_{12} - (E_{21})^*) - X_0\right)\mathcal{R}_{C'} = \frac{1}{2}(E_{12} - (E_{21})^*) - X_0,$$

in which case the unique solution X can be expressed as

$$\hat{X} = P\begin{bmatrix} O & \overline{X_0} \\ -(\overline{X_0})^* & O \end{bmatrix}P^*,$$

where

$$\overline{X_0} = \frac{1}{2}(E_{12} - (E_{21})^*),\ X_0 = (\widetilde{A})^\dagger\widetilde{C} + \widetilde{D}(\widetilde{B})^\dagger - (\widetilde{A})^\dagger\widetilde{A}\widetilde{D}(\widetilde{B})^\dagger.$$

(c) Denote

$$A'' = [A_1^*, B_1],\ C'' = [C_2^*, -D_2],\ B'' = [A_2^*, B_2],\ D'' = [-C_1^*, D_1].$$

Let the SVDs of A'' and B'' be as given in

$$A'' = P_1\begin{bmatrix} \Gamma & O \\ O & O \end{bmatrix}Q_1^*,\ B'' = U_1\begin{bmatrix} \Lambda & O \\ O & O \end{bmatrix}V_1^*,$$

where $P_1 = [P_{11}, P_{12}]$, $Q_1 = [Q_{11}, Q_{12}]$, $U_1 = [U_{11}, U_{12}]$, $V_Q = [V_{11}, V_{12}]$, $\Gamma = \mathrm{diag}(\delta_1, \cdots, \delta_{t_1})$, $t_1 = \mathrm{rank}(A'')$, $\Lambda = \mathrm{diag}(\gamma_1, \cdots, \gamma_{t_2})$, $t_2 = \mathrm{rank}(B'')$. Then, the least squares Hermitian generalized Hamiltonian solution to (1) can be described as

$$X = P\begin{bmatrix} O & X_{12} \\ -X_{12}^* & O \end{bmatrix}P^*,$$

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where

$$X_{12} = P_1 \begin{bmatrix} \Phi * \left(P_{11}^* D' V_{11} \Lambda + \Gamma Q_{11}^* (C')^* U_{12}\right) & \Gamma^{-1} Q_{11}^* (C')^* U_{12} \\ P_{12}^* D' V_{11} \Lambda^{-1} & X_{22}' \end{bmatrix} U_1^*,$$

with $\Phi = (\phi_{ij})$, $\phi_{ij} = \frac{1}{\delta_i^2 + \gamma_j^2}$, $1 \le i \le t_1, 1 \le j \le t_2$ *and* $X_{22}' \in \mathbb{C}^{(n-t_1)\times(n-t_2)}$ *is arbitrary.*

In this chapter, we introduce matrix decomposition methods for solving special solutions of system (1), including various symmetric solutions, orthogonal solutions over the real field, unitary solutions over the complex field, inequality-constrained solutions, real-positive definite and real-semi-positive definite solutions, reflexive solutions, various conjugate solutions, and Hamiltonian-type solutions.

5. The System (1) over Dual Numbers

In 1873, Clifford introduced dual numbers for studying non-Euclidean geometry [60]. The set of dual numbers is typically denoted by

$$\mathbb{D} = \{a = a_1 + \epsilon a_2 \mid a_1, a_2 \in \mathbb{R}, \epsilon \neq 0, \epsilon^2 = 0\}.$$

For two dual numbers $a = a_1 + \epsilon a_2$ and $b = b_1 + \epsilon b_2$, the arithmetic operations for dual numbers are defined as follows:

(a) Equality : $a = b \Leftrightarrow a_1 = b_1, a_2 = b_2$.
(b) Addition : $a + b = (a_1 + b_1) + \epsilon(a_2 + b_2)$.
(c) Multiplication : $ab = a_1 b_1 + \epsilon(a_1 b_2 + a_2 b_1)$.

A matrix whose elements are dual numbers is called a dual matrix. Specifically, the set of all $m \times n$ real dual matrices is given by

$$\mathbb{D}^{m\times n} = \{A = A_1 + \epsilon A_2 \mid A_1, A_2 \in \mathbb{R}^{m\times n}\}.$$

The operational rules for dual matrices follow those of dual numbers. Dual matrices have significant applications in kinematic analysis and robotics. The solution of systems of linear dual equations is a crucial task in various fields, such as synthesis problems and sensor calibration [61].

Recently, the existence of general solutions and corresponding expressions, along with minimal norm solutions, for (1) over dual numbers has been investigated. We present the following theorem.

Theorem 62. *[62] Assume that dual matrices* $A = A_1 + \epsilon A_2$, $B = B_1 + \epsilon B_2$, $C = C_1 + \epsilon C_2$ *and* $D = D_1 + \epsilon D_2$, *where* $A_i \in \mathbb{R}^{p\times m}$, $B_i \in \mathbb{R}^{n\times q}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{m\times q}$ $(i = 1, 2)$. *Suppose that the SVDs of the matrices* A_1 *and* B_1 *are*

$$A_1 = P \begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix} Q^{\mathrm{T}}, \ B_1 = U \begin{bmatrix} \Omega & O \\ O & O \end{bmatrix} V^{\mathrm{T}},$$

where $\Sigma = \mathrm{diag}(\gamma_1, \cdots, \gamma_{r_1})$, $r_1 = \mathrm{rank}(A_1)$, $P = [P_1, P_2] \in \mathbb{R}^{p\times p}$, $Q = [Q_1, Q_2] \in \mathbb{R}^{m\times m}$, $\Omega = \mathrm{diag}(\beta_1, \cdots, \beta_{r_2})$, $r_2 = \mathrm{rank}(B_1)$, $U = [U_1, U_2] \in \mathbb{R}^{n\times n}$, $V = [V_1, V_2] \in \mathbb{R}^{q\times q}$ *with* $P_1 \in \mathbb{R}^{m\times r_1}$, $Q_1 \in \mathbb{R}^{p\times r_1}$, $U_1 \in \mathbb{R}^{q\times r_2}$ *and* $V_1 \in \mathbb{R}^{n\times r_2}$. *Let the partitions of the matrices* $U^{\mathrm{T}} B_2 V$, $P^{\mathrm{T}} A_2 Q$, $P^{\mathrm{T}} C_1 U$, $P^{\mathrm{T}} C_2 U$, $Q^{\mathrm{T}} D_1 V$ *and* $Q^{\mathrm{T}} D_2 V$ *be given by*

$$U^{\mathrm{T}} B_2 V = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}, \ P^{\mathrm{T}} A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \ P^{\mathrm{T}} C_1 U = \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & C_{14} \end{bmatrix},$$

$$P^{\mathrm{T}} C_2 U = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \ Q^{\mathrm{T}} D_1 V = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{13} & \tilde{D}_{14} \end{bmatrix}, \ Q^{\mathrm{T}} D_2 V = \begin{bmatrix} \tilde{D}_{21} & \tilde{D}_{22} \\ \tilde{D}_{23} & \tilde{D}_{24} \end{bmatrix}.$$

(a) Then, (1) *is solvable over dual numbers if and only if*

$$\begin{aligned}&\mathcal{R}_{A_{24}}(C_{24}-A_{23}\Sigma^{-1}C_{12})=O,\mathcal{R}_{A_1}(C_2B_2-A_2D_2)\mathcal{L}_{B_1}=O,\\&\mathcal{R}_{A_1}(C_2B_1-A_2D_1)=O,\ \mathcal{R}_{A_1}C_1=O,\ D_1\mathcal{L}_{B_1}=O,\ (C_1B_2-A_1D_2)\mathcal{L}_{B_1}=O,\\&C_2B_1-A_2D_1=A_1D_2-C_1B_2,\ (\tilde{D}_{24}-\tilde{D}_{13}\Omega^{-1}B_{22})\mathcal{L}_{B_{24}}=O,\ C_1B_1=A_1D_1.\end{aligned}\tag{24}$$

In this case, the solution set of (1) *over dual numbers can be expressed as*

$$\begin{aligned}X_1&=Q\begin{bmatrix}\Sigma^{-1}C_{11} & \Sigma^{-1}C_{12}\\ \tilde{D}_{13}\Omega^{-1} & J_4+\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}\end{bmatrix}U^{\mathrm{T}},\\X_2&=Q\begin{bmatrix}\Sigma^{-1}(C_{21}-A_{21}\Sigma^{-1}C_{11}-A_{22}\tilde{D}_{13}\Omega^{-1}) & J_5-\Sigma^{-1}A_{22}\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}\\ J_6-\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}B_{23}\Omega^{-1} & Z_4\end{bmatrix}U^{\mathrm{T}},\end{aligned}$$

where

$$\begin{aligned}J_4&=A_{24}^{\dagger}(C_{24}-A_{23}\Sigma^{-1}C_{12})+\mathcal{L}_{A_{24}}(\tilde{D}_{24}-\tilde{D}_{13}\Omega^{-1}B_{22})B_{24}^{\dagger},\\J_5&=\Sigma^{-1}C_{22}-\Sigma^{-1}A_{21}\Sigma^{-1}C_{12}-\Sigma^{-1}A_{22}J_4,\\J_6&=\tilde{D}_{23}\Omega^{-1}-\tilde{D}_{13}\Omega^{-1}B_{21}\Omega^{-1}-J_4B_{23}\Omega^{-1}.\end{aligned}$$

and W_3, Z_4 *are arbitrary matrices.*

(b) If the conditions (24) *are satisfied, then the solution of* $||\hat{X}_1||^2+||\hat{X}_2||^2=\min$ *with* $\hat{X}$ *being the dual number solution of* (1) *is given by* $\hat{X}=\hat{X}_1+\epsilon\hat{X}_2$, *where*

$$\begin{aligned}\hat{X}_1&=Q\begin{bmatrix}\Sigma^{-1}C_{11} & \Sigma^{-1}C_{12}\\ \tilde{D}_{13}\Omega^{-1} & J_4+\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}\end{bmatrix}U^{\mathrm{T}},\\\hat{X}_2&=Q\begin{bmatrix}\Sigma^{-1}J_4 & J_5-\Sigma^{-1}A_{22}\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}\\ J_6-\mathcal{L}_{A_{24}}W_3\mathcal{R}_{B_{24}}B_{23}\Omega^{-1} & O\end{bmatrix}U^{\mathrm{T}}.\end{aligned}$$

W_3 *satisfying*

$$\tilde{\Pi}\mathrm{vec}(W_3)=\mathrm{vec}(J_7),$$

where

$$\begin{aligned}J_7&=2\mathcal{L}_{A_{24}}A_{22}^{\mathrm{T}}\Sigma^{-1}J_5\mathcal{R}_{B_{24}}-2\mathcal{L}_{A_{24}}J_4\mathcal{R}_{B_{24}}+2\mathcal{L}_{A_{24}}J_6\Omega^{-1}B_{23}^{\mathrm{T}}\mathcal{R}_{B_{24}}.\\\tilde{\Pi}&=\mathcal{R}_{B_{24}}\otimes(\mathcal{L}_{A_{24}}+2L_2^{\mathrm{T}}L_2)+(\mathcal{R}_{B_{24}}+2L_1^{\mathrm{T}}L_1)\otimes\mathcal{L}_{A_{24}}.\end{aligned}$$

Remark 14. *In 2024, Fan presented an alternative form of Theorem 62 using the Moore-Penrose inverse instead of block matrices, which also requires the SVD form of* A_1 *and* C_1 *[63]. In fact, Theorem 62, when applied with the SVD, provides the specific form of the Moore-Penrose inverse, which may be more efficient in practical computations.*

In this chapter, we introduced the solution of the system (1) in terms of dual quaternions. A more general form involving dual quaternions will be discussed in the next chapter.

6. The System (1) over Quaternions

Since Hamilton's discovery of quaternions in 1843 [64], they have become a widely used tool for representing concepts across algebra, analysis, topology, and physics. Additionally, quaternion matrices have garnered significant attention in fields such as computer science, quantum physics, signal processing, and color image processing [65,66].

In 1849, Cockle introduced split quaternions [67]. The algebra of split quaternions is a 4-dimensional Clifford algebra that is associative and noncommutative, but it has zero divisors, nilpotent

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elements, and nontrivial idempotents. As a result, the algebraic structure of split quaternions, denoted as $\mathbb{H}_s$, is more complex than that of real quaternions $\mathbb{H}$. Despite this complexity, the unique algebraic properties of split quaternions make them a valuable tool in quantum mechanics and geometry [68,69].

In 1873, Clifford extended the concepts of dual numbers and dual quaternions [60]. Dual quaternions have since become widely used in robot kinematics and unmanned aerial vehicle formation control due to their ability to represent the motion of rigid bodies in 3D space [70–73]. Similarly, the dual split quaternion can also be defined.

The system (1) over quaternion algebra, split quaternion algebra, and dual quaternion algebra has also been the focus of several scholars. Comparatively, since the algebraic structure of quaternions is well-understood, and the definitions of generalized inverses and rank have been extended to quaternion matrices, the system (1) has been thoroughly studied over quaternions. Since 2005, when Wang first proposed the general solution to the extended form of the system

$$\begin{cases} A_1X = C_1, \\ A_2X = C_2, \\ A_3XB_3 = C_3, \\ A_4XB_4 = C_4 \end{cases} \tag{25}$$

of the system (1), relevant results for the bi-symmetric, centro-symmetric, symmetric and skew-antisymmetric, (P, Q)-reflexive solutions and reducible solutions have been successively presented [74–77]. The study of the split quaternion matrix equation typically relies on the real or complex representation of the split quaternion, or vectorization operators. However, recent work by Jiang on split quaternion matrix SVD and generalized inverses has enabled the consideration of more diverse approaches [78,79]. Dual quaternions are more intricate, and only Xie has explored the system (1) over dual quaternions [90]. Yang et al. also considered the results of the system (1) over dual split quaternion tensors [94]. The following details are introduced.

6.1. The System (1) over Quaternions

Denote the set of all real quaternions by

$$\mathbb{H} = \{a = a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where i, j, and k are the quaternion units. For $a \in \mathbb{H}$, $\bar{a} = a_0 - a_1 i - a_2 j - a_3 k$ is the conjugate of a. The set of all $m \times n$ quaternion matrices is denoted by $\mathbb{H}^{m\times n}$. For a quaternion matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, the transpose conjugate of A is expressed as $A^* = (\bar{a}_{ji})$. The Moore-Penrose inverse of A is denoted as $A^\dagger$ satisfying the same equations in the definition of complex Moore-Penrose. The two orthogonal projectors $\mathcal{L}_A$ and $\mathcal{R}_A$ are defined as

$$\mathcal{L}_A = I - A^\dagger A,\ \mathcal{R}_A = I - AA^\dagger,$$

The rank of A, denoted by $\operatorname{rank}(A)$, is defined as the dimension of $\mathcal{R}(A)$, where $\mathcal{R}(A)$ is the column right space of A [80,81].

Using the results of the system (25) and the properties of the rank of quaternion matrix equations, the general solution of system (1) over quaternions is presented below.

Theorem 63 (General solutions for (1) over $\mathbb{H}$)**.** *[74] For given $A \in \mathbb{H}^{p\times m}$, $C \in \mathbb{H}^{p\times n}$, $B \in \mathbb{H}^{n\times q}$ and $D \in \mathbb{H}^{m\times q}$, then there exists three conditions, one of which is equivalent to (1) is consistence over $\mathbb{H}$:*

(a) $\mathcal{R}_A C = O,\ D\mathcal{L}_B = O,\ AD = CB,$

(b) $AA^\dagger C = C,\ DB^\dagger B = D,\ AD = CB,$

(c) $\operatorname{rank}[A, C] = \operatorname{rank}(A),\ \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} = \mathrm{D},\ AD = CB.$

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In this case, the general solution of (1) *can be form of*

$$X = A^{\dagger}C + \mathcal{L}_A DB^{\dagger} + \mathcal{L}_A Y \mathcal{R}_B,$$

where Y *is an arbitrary matrix over* $\mathbb{H}$ *with appropriate order.*

Based on Theorem 63, Kyrchei investigated the row-column determinant expression of the solution of system (1) over quaternions [82].

Let S_n denote the symmetric group on $\{1, 2, \cdots, n\}$. For a quaternion matrix $A \in \mathbb{H}^{n\times n}$, the row and column determinants are defined as follows:

(a) Row determinant: The i-th row determinant of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ for all $i = 1, \cdots, n$ is defined as

$$\mathrm{rdet}_i A = \sum_{\sigma\in S_n} (-1)^{n-r} (a_{ik_1} a_{ik_1+1} \cdots a_{ik_1+l_1}) \cdots (a_{ik_r} a_{ik_r+1} \cdots a_{ik_r+l_{k_r}}),$$

where $\sigma = (i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_{k_r}})$, $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_s} < i_{k_{s+1}}$ for all $t = 2, \cdots, r$ and $s = 1, \cdots, l_t$.

(b) Colimn determinant: The j-th column determinant of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ for all $j = 1, \cdots, n$ is defined as

$$\mathrm{cdet}_j A = \sum_{\tau\in S_n} (-1)^{n-r} (a_{j_{k_r+l_r}} \cdots a_{j_{k_r+1}} a_{j_{k_r}}) \cdots (a_{j_{k_1+l_1}} \cdots a_{j_{k_1+1}} a_{j_{k_1}} a_{j_{k_1}} a_{j_{k_1}}),$$

where $\tau = (j_{k_r+l_r} \cdots j_{k_r}) \cdots (j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2})(j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j_{k_1} j_{k_1}),$, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_s} < j_{k_{s+1}}$ for all $t = 2, \cdots, r$ and $s = 1, \cdots, l_t$.

For $1 \le k \le n$, let $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta = \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ with $1 \le k \le \min\{m, n\}$. The collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} = \{\alpha = (\alpha_1, \ldots, \alpha_k) \mid 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$. For a fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} = \{\alpha \in L_{r,m} \mid i \in \alpha\}$, $J_{r,n}\{j\} = \{\beta \in L_{r,n} \mid j \in \beta\}$. Assume $A = (a_{ij}) \in \mathbb{H}^{n\times n}$. Let A^{α}_{α} be a principal submatrix of A whose rows and columns are indexed by α. If $A \in \mathbb{H}^{n\times n}$ is Hermitian, then $|A|^{\alpha}_{\alpha}$ denotes the corresponding principal minor of $\det A$. Let $a_{.j}$ be the j-th column and $a_{i.}$ be the i-th row of A. Suppose $A_{.j}(b)$ denotes the matrix obtained from A by replacing its j-th column with the column b, and $A_{i.}(b)$ denotes the matrix obtained from A by replacing its i-th row with the row b.

Theorem 64 (General solutions using row and column determinants for (1) over $\mathbb{H}$)**.** *[82] Let* $A = (a_{ij}) \in \mathbb{H}^{p\times m}$, $B = (b_{ij}) \in \mathbb{H}^{n\times q}$, $C = (c_{ij}) \in \mathbb{H}^{p\times n}$, $D = (d_{ij}) \in \mathbb{H}^{m\times q}$, $A^{\dagger} = (a^{\dagger}_{ij}) \in \mathbb{H}^{n\times m}$, $B^{\dagger} = (b^{\dagger}_{ij}) \in \mathbb{H}^{s\times r}$, $\mathcal{L}_A = I - A^{\dagger}A = (l_{ij}) \in \mathbb{H}^{n\times n}$*. Denote* $A^*C = \hat{C} = (\hat{c}_{ij}) \in \mathbb{H}^{n\times r}$ *and* $\mathcal{L}_A DB^* = \hat{D} = (\hat{d}_{ij}) \in \mathbb{H}^{n\times r}$*. Assume that* $p < m$ *and* $q < n$*. Quaternion matrix* $X^0 = (x^0_{ij}) \in \mathbb{H}^{n\times s}$ *as the solution of* (1) *has the following determinantal representation.*

(a) *If* $\mathrm{rank}(A) = k \le p < m$ *and* $\mathrm{rank}(B) = t \le q < n$*, then*

$$x^0_{ij} = \frac{\sum_{\beta\in J_{k,m}\{i\}} \mathrm{cdet}_i((A^*A)_{.i}(\hat{c}_{.j}))^{\beta}_{\beta}}{\sum_{\beta\in J_{k,m}} |A^*A|^{\beta}_{\beta}} + \frac{\sum_{\alpha\in I_{t,n}\{j\}} \mathrm{rdet}_j((BB^*)_{j.}(\hat{d}_{i.}))^{\alpha}_{\alpha}}{\sum_{\alpha\in I_{t,n}} |BB^*|^{\alpha}_{\alpha}}.$$

(b) *If* $\mathrm{rank}(A) = m$ *and* $\mathrm{rank}(B) = n$*, then*

$$x^0_{ij} = \frac{\mathrm{cdet}_i(A^*A)_{.i}(\hat{c}_{.j})}{\det(A^*A)} + \frac{\mathrm{rdet}_j(BB^*)_{j.}(\hat{d}_{i.})}{\det(BB^*)}.$$

(c) *If* $\operatorname{rank}(A) = k \leq p < m$ *and* $\operatorname{rank}(B) = n$, *then*

$$x_{ij}^0 = \frac{\sum_{\beta \in J_{k,m}\{i\}} \operatorname{cdet}_i((A^*A)_{\cdot i}(\hat{c}_{\cdot j}))_\beta^\beta}{\sum_{\beta \in J_{k,m}} |A^*A|_\beta^\beta} + \frac{\operatorname{rdet}_j(BB^*)_{j\cdot}(\hat{d}_{i\cdot})}{\det(BB^*)}.$$

(d) *If* $\operatorname{rank}(A) = m$ *and* $\operatorname{rank}(B) = t \leq q < n$, *then*

$$x_{ij}^0 = \frac{\operatorname{cdet}_i(A^*A)_{\cdot i}(\hat{c}_{\cdot j})}{\det(A^*A)} + \frac{\sum_{\alpha \in I_{t,n}\{j\}} \operatorname{rdet}_j((BB^*)_{j\cdot}(\hat{d}_{i\cdot}))_\alpha^\alpha}{\sum_{\alpha \in I_{t,n}} |BB^*|_\alpha^\alpha}.$$

The following presents some special forms of symmetric solutions over quaternions and related results.

Let $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, $A^* = (\bar{a}_{ji}) \in \mathbb{H}^{n\times m}$, $A^\# = (a_{m-i+1,n-j+1}) \in \mathbb{H}^{m\times n}$, where $\bar{a}_{ji}$ is the conjugate of the quaternion a_{ji}.

(a) The matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is called symmetric if $A = A^*$.

(b) The matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is called bisymmetric if $a_{ij} = a_{n-i+1,n-j+1} = \bar{a}_{ji}$.

(c) The matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$ is called centrosymmetric if $A = A^\#$.

(d) The matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$ is called symmetric and skew-antisymmetric if $A = A^* = -A^\#$.

(e) The matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$ is called (P,Q)-(skew)symmetric if $A = PAQ$ with $P^2 = I$, $Q^2 = I$, $P \in \mathbb{H}^{m\times m}$, $Q \in \mathbb{H}^{n\times n}$.

Next, we will sequentially present the conclusions regarding the above special solutions of the system (1) over quaternions.

Theorem 65 (Bisymmetric solutions for (1) over $\mathbb{H}$)**.** *[74] Let* $A, C \in \mathbb{H}^{p\times n}$, $B, D \in \mathbb{H}^{n\times q}$, *and* $V = (v_{ij}) \in \mathbb{R}^{k\times k}$, *where* $v_{ij} = 1$ *when* $i + j = k + 1$ *and* $v_{ij} = 0$ *otherwise. Denote*

$$U = \begin{bmatrix} I_k & -V_k \\ V_k & I_k \end{bmatrix},$$

when $n = 2k$, *or*

$$U = \begin{bmatrix} I_k & O & -V_k \\ O & 1 & O \\ V_k & O & I_k \end{bmatrix},$$

when $n = 2k + 1$. *There exists block matrices*

$$AU^{-1} = [A_1, A_2],\ CU^* = [C_1, C_2],\ U^{-*}B = \begin{bmatrix} B_1 \\ B_3 \end{bmatrix},\ UD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, A_2, C_1, C_2 \in \mathbb{H}^{p\times k}$ *and* $B_1, B_2, D_1, D_2 \in \mathbb{H}^{k\times q}$ *when* $n = 2k$; $A_1, C_1 \in \mathbb{H}^{p\times k}$, $A_2, C_2 \in \mathbb{H}^{p\times (k+1)}$, $B_1, D_1 \in \mathbb{H}^{k\times q}$ *and* $B_2, D_2 \in \mathbb{H}^{(k+1)\times q}$ *when* $n = 2k + 1$. *Let* $i = 1, 2$,

$$\begin{aligned}
&S_i = B_i^* \mathcal{L}_{A_i},\ G_i = \mathcal{R}_{S_i} B_i^*,\ T_i = \mathcal{L}_{A_i}\mathcal{L}_{S_i},\ N_i = \mathcal{R}_{B_i} A_i^*, \\
&\Psi_i = [D_i B_i^\dagger - A_i^\dagger C_i - \mathcal{L}_{A_i} S_i^* B_i^*((D_i B_i^\dagger)^* - A_i^\dagger C_i)]B_i, \\
&\Phi_i = S_i^* B_i^*[(D_i B_i^\dagger)^* - A_i^\dagger C_i] + \mathcal{L}_{S_i} T_i^* \Psi_i B_i^\dagger,\ Q_i = C_i^* - A_i^\dagger C_i A_i^* - \mathcal{L}_{A_i}\Phi_i A_i^*.
\end{aligned}$$

Then, the system (1) *has a bisymmetric solution over quaternions if and only if*

$$T_i T_i^\dagger \Psi_i = \Psi_i, \mathcal{R}_{T_i} Q_i = O,\ D_i B_i^\dagger B_i = D_i,\ A_i A_i^\dagger C_i = C_i,\ G_i[(D_i B_i^\dagger)^* - A_i^\dagger C_i] = O,$$

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in which case, the general bisymmetric solution can be expressed as

$$X = U^{-1}\begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix}U^{-*},$$

where

$$X_i = \frac{1}{2}(Y_i + Y_i^*),$$

$$Y_i = A_i^{\dagger}C_i + \mathcal{L}_{A_i}S_i^*B_i^*[(D_iB_i^{\dagger})^* - A_i^{\dagger}C_i] + \Psi_iB_i^{\dagger} + Q_iN_i^{\dagger}\mathcal{R}_{B_i} + T_iW_i\mathcal{R}_{N_i}\mathcal{R}_{B_i},$$

with W_i is an arbitrary matrix over $\mathbb{H}$ with compatible dimension.

Remark 15. *In 2015, Yuan et al. considered the least squares η-bi-Hermitian solution for another linear system $(AXB, CXD) = (E, F)$ [83].*

Theorem 66 (Cetrosymmetric solutions for (1) over $\mathbb{H}$)**.** *[74] For $A, C \in \mathbb{H}^{p\times n}$ and $B, D \in \mathbb{H}^{n\times q}$, denote*

$$\begin{aligned}
&S = A^{\#}\mathcal{L}_A,\ T = \mathcal{L}_A\mathcal{L}_S,\ G = \mathcal{R}_SA^{\#},\ N = \mathcal{R}_BB^{\#},\ P = \mathcal{R}_{\mathcal{L}_A\mathcal{L}_S}\mathcal{L}_T\mathcal{L}_A\mathcal{L}_S,\\
&\Psi = [DB^{\dagger} - A^{\dagger}C - \mathcal{L}_AS^{\dagger}(AA^{\dagger}C)^{\#} + \mathcal{L}_AS^{\dagger}(A^{\#}A^{\dagger}C)]B,\\
&\Phi = S^{\dagger}A^{\#}[(A^{\dagger}C)^{\#} - (A^{\dagger}C)] + \mathcal{L}_ST^{\dagger}\Psi B^{\dagger},\ Q = D^{\#} - A^{\dagger}CB^{\#} - M\Phi B^{\#}.
\end{aligned}$$

Then, the system (1) has a centrosymmetric solution if and only if

$$\begin{aligned}
&TT^{\dagger}\Psi = \Psi,\ \mathcal{R}_P\mathcal{R}_{\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T}Q = O,\ \mathcal{R}_{\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T}Q\mathcal{L}_N = O,\\
&DB^{\dagger}B = D,\ AA^{\dagger}C = C,\ G[(A^{\dagger}C)^{\#} - (A^{\dagger}C)] = O.
\end{aligned}$$

In that case, the centrosymmetric solution can be expressed as

$$X = \frac{1}{2}(X_1 + X_1^{\#}),$$

where

$$\begin{aligned}
X_1 = &A^{\dagger}C + \mathcal{L}_AS^{\dagger}A^{\#}[(A^{\dagger}C)^{\#} - (A^{\dagger}C)] + \mathcal{L}_A\mathcal{L}_ST^{\dagger}\Psi B^{\dagger} + \mathcal{L}_A\mathcal{L}_S(\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T)^{\dagger}QB^{\#\dagger}\\
&+ \mathcal{L}_A\mathcal{L}_SP^{\dagger}\mathcal{R}_{\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T}QN^{\dagger}\mathcal{R}_B - \mathcal{L}_A\mathcal{L}_ST(\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T)^{\dagger}\mathcal{L}_A\mathcal{L}_SP^{\dagger}\mathcal{R}_{\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T}QB^{\#\dagger}\\
&+ \mathcal{L}_A\mathcal{L}_S\mathcal{L}_TZ - \mathcal{L}_A\mathcal{L}_ST(\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T)^{\dagger}\mathcal{L}_A\mathcal{L}_S\mathcal{L}_TZ(BB^{\dagger})^{\#} + \mathcal{L}_{A_1}\mathcal{L}_SW\mathcal{R}_B\\
&- \mathcal{L}_A\mathcal{L}_ST(\mathcal{L}_A\mathcal{L}_S\mathcal{L}_T)^{\dagger}M\mathcal{L}_SPWNB^{\#\dagger} - \mathcal{L}_A\mathcal{L}_SP^{\dagger}PWNN^{\dagger}\mathcal{R}_B,
\end{aligned}$$

with arbitrary W, Z.

Theorem 67 (Symmetric and skew-antisymmetric solutions for (1) over $\mathbb{H}$)**.** *[75] Let $A, C \in \mathbb{H}^{p\times n}$, $B, D \in \mathbb{H}^{n\times q}$, $V = (v_{ij}) \in \mathbb{R}^{k\times k}$, where $v_{ij} = 1$ when $i + j = k + 1$ and $v_{ij} = 0$ otherwise. Denote*

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ V_k & -V_k \end{bmatrix}$$

when $n = 2k$, or

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & O & I_k \\ O & \sqrt{2} & O \\ V_k & O & -V_k \end{bmatrix}$$

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when $n = 2k+1$. There exists block matrices

$$AU^{-1} = [A_1, A_2],\ CU^* = [C_1, C_2],\ U^{-*}B = \begin{bmatrix} B_1 \\ B_3 \end{bmatrix},\ UD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, A_2, C_1, C_2 \in \mathbb{H}^{p\times k}$ and $B_1, B_2, D_1, D_2 \in \mathbb{H}^{k\times q}$ when $n = 2k$; $A_1, C_1 \in \mathbb{H}^{p\times k}$, $A_2, C_2 \in \mathbb{H}^{p\times(k+1)}$, $B_1, D_1 \in \mathbb{H}^{k\times q}$ and $B_2, D_2 \in \mathbb{H}^{(k+1)\times q}$ when $n = 2k+1$. Let

$$\begin{aligned} &S = B_2^*\mathcal{L}_{A_2},\ G = \mathcal{R}_S B_2^*, T = \mathcal{L}_{A_2}\mathcal{L}_S,\ N = \mathcal{R}_{B_1}A_1^*, \\ &\psi = [D_2B_1^\dagger - A_2^\dagger C_1 - \mathcal{L}_{A_2}S^\dagger B_2^*((D_1B_2^\dagger)^* - A_2^\dagger C_1)]B_1, \\ &\phi = S^\dagger B_2^*[(D_1B_2^\dagger)^* - A_2^\dagger C_1] + \mathcal{L}_S T^\dagger \psi B_1^\dagger,\ Q = C_2^* - A_2^\dagger C_1 A_1^* - \mathcal{L}_{A_2}\phi A_1^*. \end{aligned}$$

Then, the system (1) has symmetric and skew-antisymmetric solutions if and only if

$$\begin{aligned} &TT^\dagger\psi = \psi,\ \mathcal{R}_T Q = O,\ D_2B_1^\dagger B_1 = D_2,\ C_2^*(A_1^*)^\dagger A_1^* = C_2^*, \\ &A_2A_2^\dagger C_1 = C_1,\ B_2^*(B_2^*)^\dagger D_1^* = D_1^*,\ G((D_1B_2^\dagger)^* - A_2^\dagger C_1) = O. \end{aligned}$$

In which case, the general symmetric and skew-antisymmetric solution can be expressed as

$$X = U\begin{bmatrix} O & Y^* \\ Y & O \end{bmatrix}U^*,$$

where

$$Y = A_2^\dagger C_1 + \mathcal{L}_{A_2}S^\dagger B_2^*((D_1B_2^\dagger)^* - A_2^\dagger C_1) + \psi B_1^\dagger + QN^\dagger \mathcal{R}_{B_1} + TW\mathcal{R}_N\mathcal{R}_{B_1}$$

with W is an arbitrary matrix over $\mathbb{H}$ with compatible dimension.

Theorem 68 ((P, Q)-(skew-)symmetric solution for (1) over $\mathbb{H}$). *[76] Let $A \in \mathbb{H}^{p\times m}$, $C \in \mathbb{H}^{p\times n}$, $B \in \mathbb{H}^{n\times q}$, $D \in \mathbb{H}^{m\times q}$, and $P \in \mathbb{H}^{m\times m}, Q \in \mathbb{H}^{n\times n}$ satisfying $P^2 = I, Q^2 = I$. The EVDs of P and Q can be written as the form of*

$$P = U^{-1}\begin{bmatrix} I_{r_1} & O \\ O & -I_{m-r_1} \end{bmatrix}U,\quad Q = V^{-1}\begin{bmatrix} I_{r_2} & O \\ O & -I_{n-r_2} \end{bmatrix}V,$$

where U and V are invertible. Denote

$$AU^{-1} = [A_1, A_2],\ CV^{-1} = [C_1, C_2], VB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ UD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $A_1, C_1 \in \mathbb{H}^{p\times r_1}$, $B_1, D_1 \in \mathbb{H}^{r_2\times q}$. Then, the system (1) has a (P, Q)-(skew-)symmetric solution if and only if

$$\mathcal{R}_{A_i}C_i = O,\ D_i\mathcal{L}_{B_i} = O,\ A_iD_i = C_iB_i, i = 1,2,$$

or equivalently

$$A_iD_i = C_iB_i,\ \operatorname{rank}[A_i, C_i] = \operatorname{rank}(A_i),\ \operatorname{rank}\begin{bmatrix} C_i \\ D_i \end{bmatrix} = \operatorname{rank}(C_i),\ i = 1,2.$$

(*a*) *The general (P, Q)-symmetric solution of (1) can be expressed as*

$$X = U^{-1}\begin{bmatrix} A_1^\dagger C_1 + \mathcal{L}_{A_1}D_1B_1^\dagger + \mathcal{L}_{A_1}Y_1\mathcal{R}_{B_1} & O \\ O & A_2^\dagger C_2 + \mathcal{L}_{A_2}D_2B_2^\dagger + \mathcal{L}_{A_2}Y_2\mathcal{R}_{B_2} \end{bmatrix}V,$$

where Y_1, Y_2 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

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(b) The general (P,Q)-skew-symmetric solution of (1) can be expressed as

$$X = U^{-1}\begin{bmatrix} O & A_1^{\dagger}C_1 + \mathcal{L}_{A_1}D_1B_1^{\dagger} + \mathcal{L}_{A_1}Y_1\mathcal{R}_{B_1} \\ A_2^{\dagger}C_2 + \mathcal{L}_{A_2}D_2B_2^{\dagger} + \mathcal{L}_{A_2}Y_2\mathcal{R}_{B_2} & O \end{bmatrix} V,$$

where Y_1, Y_2 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Subsequently, we introduce the extreme rank (P,Q)-(skew-)symmetric solutions of the system (1) over quaternions.

Theorem 69 (Extreme rank (P,Q)-(skew-)symmetric solutions for (1) over $\mathbb{H}$)**.** *[76] Suppose that the system (1) has a (P,Q)-(skew-)symmetric solutions X and $\Omega(\hat{\Omega})$ is the set of all (P,Q)-(skew-)symmetric solutions of (1). Denote*

$$T_1 = \mathcal{R}_{\mathcal{L}_{A_1}}(A_1^{\dagger}C_1 + \mathcal{L}_{A_1}D_1B_1^{\dagger}),\ S_1 = (A_1^{\dagger}C_1 + \mathcal{L}_{A_1}D_1B_1^{\dagger})\mathcal{L}_{\mathcal{R}_{B_1}},$$
$$T_2 = \mathcal{R}_{\mathcal{L}_{A_2}}(A_2^{\dagger}C_2 + \mathcal{L}_{A_2}D_2B_2^{\dagger}),\ S_2 = (A_2^{\dagger}C_2 + \mathcal{L}_{A_2}D_2B_2^{\dagger})\mathcal{L}_{\mathcal{R}_{B_2}}.$$

(a) The maximal rank of $X \in \Omega$ is

$$\max_{X\in\Omega} r(X) = \min\{r_1 + \text{rank}(C_1) - \text{rank}(A_1), r_2 + \text{rank}(D_1) - \text{rank}(B_1)\}$$
$$+ \min\{m - r_1 + \text{rank}(C_2) - \text{rank}(A_2), n - r_2 + \text{rank}(D_2) - \text{rank}(B_2)\}.$$

The corresponding general expression of X is

$$X = U^{-1}\begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix} V,$$

where,

$$X_i = (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}(A_i^{\dagger}C_i + \mathcal{L}_{A_i}D_iB_i^{\dagger})(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger} - K_i,\ i = 1,2,$$

and K_i is chosen such that $\text{rank}(\mathcal{R}_{S_i}\mathcal{L}_{A_i}K_i\mathcal{R}_{B_i}\mathcal{L}_{T_i}) = \min\{\text{rank}(\mathcal{R}_{S_i}\mathcal{L}_{A_i}), \text{rank}(\mathcal{R}_{B_i}\mathcal{L}_{T_i})\}$.

(b) The minimal rank of $X \in \Omega$ is

$$\min_{X\in\Omega} \text{rank}(X) = \text{rank}(C_1) + \text{rank}(C_2) + \text{rank}(D_1) + \text{rank}(D_2) - \text{rank}(C_1B_1) - \text{rank}(C_2B_2).$$

The corresponding general expression of X is

$$X = U^{-1}\begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix} V,$$

where,

$$X_i = (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}(A_i^{\dagger}C_i + \mathcal{L}_{A_i}D_iB_i^{\dagger})(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger} + W_i + (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}\mathcal{L}_{A_i}W_i\mathcal{R}_{B_i}(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger},$$

for $i = 1,2$ and W_i is an arbitrary quaternion matrix with appropriate sizes.

(c) The maximal rank of the (P,Q)-skewsymmetric solution of (1) is

$$\max_{X\in\hat{\Omega}} \text{rank}(X) = \min\{m - r_1 + \text{rank}(C_1) - \text{rank}(A_1), n - r_2 + \text{rank}(D_1) - \text{rank}(B_1)\}$$
$$+ \min\{r_1 + \text{rank}(C_2) - \text{rank}(A_2), r_2 + \text{rank}(D_2) - \text{rank}(B_2)\}.$$

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The general expression of X attaining the maximal rank can be expressed as

$$X = U^{-1}\begin{bmatrix} O & X_1 \\ X_2 & O \end{bmatrix} V,$$

where

$$X_i = (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}(A_i^{\dagger}C_i + \mathcal{L}_{A_i}D_iB_i^{\dagger})(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger} - K_i,\ i = 1, 2,$$

and K_i is chosen such that $\operatorname{rank}(\mathcal{R}_{S_i}\mathcal{L}_{A_i}K_i\mathcal{R}_{B_i}\mathcal{L}_{T_i}) = \min\{\operatorname{rank}(\mathcal{R}_{S_i}\mathcal{L}_{A_i}), \operatorname{rank}(\mathcal{R}_{B_i}\mathcal{L}_{T_i})\}$.

(d) The minimal rank of the (P, Q)-skewsymmetric solution of (1) is

$$\min_{X\in\hat{\Omega}} \operatorname{rank}(X) = \operatorname{rank}(C_1) + \operatorname{rank}(C_2) + \operatorname{rank}(D_1) + \operatorname{rank}(D_2) - \operatorname{rank}(A_1D_1) - \operatorname{rank}(A_2D_2),$$

or

$$\min_{X\in\hat{\Omega}} \operatorname{rank}(X) = \operatorname{rank}(C_1) + \operatorname{rank}(C_2) + \operatorname{rank}(D_1) + \operatorname{rank}(D_2) - \operatorname{rank}(C_1B_1) - \operatorname{rank}(C_2B_2).$$

The general expression of X attaining the minimal rank can be expressed as

$$X = U^{-1}\begin{bmatrix} O & X_1 \\ X_2 & O \end{bmatrix} V,$$

where

$$X_i = (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}(A_i^{\dagger}C_i + \mathcal{L}_{A_i}D_iB_i^{\dagger})(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger} + K_i + (-\mathcal{R}_{S_i}\mathcal{L}_{A_i})^{\dagger}\mathcal{L}_{A_i}K_i\mathcal{R}_{B_i}(\mathcal{R}_{B_i}\mathcal{L}_{T_i})^{\dagger}$$

for $i = 1, 2$, and K_i is an arbitrary quaternion matrix with appropriate size.

At the end of this section, we introduce the reducible solution of the system (1) over quaternions. A matrix $A \in \mathbb{H}^{n\times n}$ is called reducible, if there exists a permutation matrix K such that

$$A = K\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix} K^{-1},$$

where A_1 and A_3 are square matrices of order at least 1 over $\mathbb{H}$. Moreover, if the order of A_3 is k $(1 \leq k < n)$, we call A to be k-reducible with respect to the permutation matrix K.

Theorem 70 (Reducible solutions for (1) over $\mathbb{H}$)**.** *[77] Let $A, C \in \mathbb{H}^{p\times n}$, $B, D \in \mathbb{H}^{n\times q}$ be known, $X \in \mathbb{H}^{n\times n}$ unknown, $K \in \mathbb{H}^{n\times n}$ be a permutation matrix, $1 \leq k < n$. Denote*

$$AK = [A_1\ A_3],\ CK = [C_1\ C_2],\ K^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ K^{-1}D = \begin{bmatrix} C_3 \\ C_4 \end{bmatrix},$$

where $A_1, C_1 \in \mathbb{H}^{p\times(n-k)}$, $A_3, C_2 \in \mathbb{H}^{p\times k}$, $B_1, C_3 \in \mathbb{H}^{(n-k)\times q}$, $B_2, C_4 \in \mathbb{H}^{k\times q}$. Assume that M, N, P, Q, E, F, G, S and T are defined as

$$\begin{aligned}
&E_2 = \mathcal{R}_{A_3}A_1,\ F_1 = B_2\mathcal{L}_{B_1},\ M = A_1^{\dagger}N,\ N = A_1\mathcal{L}_{E_2},\ P = \mathcal{R}_{F_1}B_2,\ Q = PB_2^{\dagger},\\
&E = A_1^{\dagger}A_1C_3 - A_1^{\dagger}C_1B_1 - A_1^{\dagger}A_1E_2^{\dagger}C_2B_2 - A_1^{\dagger}NC_3F_1^{\dagger}B_2,\\
&F = C_2B_2B_2^{\dagger} - A_3C_4B_2^{\dagger} - A_1E_2^{\dagger}C_2B_2B_2^{\dagger} - NC_3F_1^{\dagger}B_2B_2^{\dagger},\\
&G = N^{\dagger}FQ^{\dagger} - M^{\dagger}EP^{\dagger},\ S = \mathcal{L}_M + \mathcal{L}_N,\ T = \mathcal{R}_P + \mathcal{R}_Q.
\end{aligned} \tag{26}$$

Then, the system (1) has a k-reducible solution $X \in \mathbb{H}^{n\times n}$ with the permutation matrix K if and only if one of the following two statements holds.

(*a*)

$$\begin{aligned}
&\mathcal{R}_{A_1}C_1 = O,\ C_4\mathcal{L}_{B_2} = O,\ \mathcal{R}_{(A_1A_1^*+A_3A_3^*)}C_2 = O,\ C_3\mathcal{L}_{(B_1^*B_1+B_2^*B_2)} = O,\\
&\mathcal{R}_{A_3}(A_1C_3 - C_2B_2)\mathcal{L}_{B_1} = O,\ A_1C_3 - C_2B_2 - C_1B_1 + A_3C_4 = O,\\
&\mathcal{R}_{A_3}(A_1C_3 - C_2B_2 - C_1B_1) = O,\ (C_1B_1 - A_1C_3)\mathcal{L}_{B_2} = O,\\
&(A_1C_3 - C_2B_2 + A_3C_4)\mathcal{L}_{B_1} = O,\ \mathcal{R}_{A_1}(A_3C_4 - C_2B_2) = O,
\end{aligned}$$

(*b*)

$$\begin{aligned}
&A_1C_3 - C_2B_2 - C_1B_1 + A_3C_4 = O,\\
&\text{rank}[A_1, C_1] = \text{rank}(A_1), \text{rank}[A_1, A_3, C_2] = \text{rank}[A_1, A_3],\\
&\text{rank}\begin{bmatrix} B_2 \\ C_4 \end{bmatrix} = \text{rank}(B_2),\ \text{rank}\begin{bmatrix} B_1 \\ B_2 \\ C_3 \end{bmatrix} = \text{rank}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\\
&\text{rank}\begin{bmatrix} A_1C_3 - C_2B_2 & A_3 \\ B_1 & O \end{bmatrix} = \text{rank}(A_3) + \text{rank}(B_1),\\
&\text{rank}[A_3, A_1C_3 - C_2B_2 - C_1B_1] = \text{rank}(A_3),\ \text{rank}[A_1, A_3C_4 - C_2B_2] = \text{rank}(A_1),\\
&\text{rank}\begin{bmatrix} C_1B_1 - A_1C_3 \\ B_2 \end{bmatrix} = \text{rank}(B_2),\ \text{rank}\begin{bmatrix} B_1 \\ A_1C_3 - C_2B_2 + A_3C_4 \end{bmatrix} = \text{rank}(B_1).
\end{aligned}$$

In that case, the k-reducible solution X of system (1) with respect to K can be expressed as

$$X = K\begin{bmatrix} X_1 & X_2 \\ O & X_3 \end{bmatrix}K^{-1},$$

where

$$\begin{aligned}
X_1 &= A_1^{\dagger}C_1 + \mathcal{L}_{A_1}(C_3 - X_2B_2)B_1^{\dagger} + \mathcal{L}_{A_1}V_1\mathcal{R}_{B_1},\\
X_2 &= E_2^{\dagger}C_2 + \mathcal{L}_{E_2}C_3^{\dagger}F_1 + \mathcal{L}_{E_2}D\mathcal{R}_{F_1} - \mathcal{L}_{E_2}\mathcal{L}_M\widehat{U}_2\mathcal{R}_Q\mathcal{R}_{F_1} + \mathcal{L}_{E_2}\mathcal{L}_N\widehat{U}_3\mathcal{R}_P\mathcal{R}_{F_1}\\
&\quad + \mathcal{L}_{E_2}\mathcal{L}_M(I - S^{\dagger})\mathcal{L}_M\widehat{V}_1\mathcal{R}_{F_1} + \mathcal{L}_{E_2}\mathcal{L}_MS^{\dagger}\mathcal{L}_N\widehat{V}_2\mathcal{R}_{F_1}\\
&\quad + \mathcal{L}_{E_2}\widehat{W}_1\mathcal{R}_P(I - T^{\dagger})\mathcal{R}_P\mathcal{R}_{F_1} + \mathcal{L}_{E_2}\widehat{W}_2\mathcal{R}_QT^{\dagger}\mathcal{R}_P\mathcal{R}_{F_1},\\
X_3 &= C_4B_2^{\dagger} + A_3^{\dagger}(C_2 - A_1X_2)\mathcal{R}_{B_2} + \mathcal{L}_{A_3}U_2\mathcal{R}_{B_2}.
\end{aligned}$$

with U_1, U_2, $\widehat{U}_2$, $\widehat{U}_3$, $\widehat{V}_1$, $\widehat{V}_2$, $\widehat{W}_1$ and $\widehat{W}_2$ being arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Remark 16. *Due to the definition of reducible matrices, considering the reducible solution of the system (1) is actually equivalent to considering the general solution of a more complex system*

$$\begin{cases} A_1X_1 = C_1, \\ AX_1B_1 + X_2B_2 = C_3, \\ A_1X_2 + A_3X_3B = C_2, \\ X_3B_3 = C_4. \end{cases}$$

The proof process of Theorem 70 follows this approach as well.

6.2. The System (1) over Split Quaternions

The set of real quaternions form a noncommutative division algebra. In 1849, Cockle introduced split quaternions:

$$\mathbb{H}_s = \{q = q_0 + q_1i + q_2j + q_3k : q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where

$$i^2 = -j^2 = -k^2 = -1, ij = -ji = k, jk = -kj = -i, ki = -ik = j.$$

Split quaternions have zero factors, which gives $\mathbb{H}_s$ a more complex algebraic structure than $\mathbb{H}$. Solving the split quaternion matrix equation mainly relies on real representation and complex representation, with the real representation having better structure-preserving properties and performing better in numerical examples.

Si et al. designed several real representations of the split quaternion matrix to establish sufficient and necessary conditions for the existence of the general, η-(anti-)conjugate, and η-(anti-)Hermitian solutions. Further, they derived expressions of the corresponding solutions when the system is solvable [84].

For any matrix $A \in \mathbb{H}_s^{m\times n}$, it can be represented uniquely as $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. The three corresponding η-conjugates ($\eta \in \{i, j, k\}$) are defined as

$$\begin{aligned} A^i &= i^{-1}Ai = A_1 + A_2 i - A_3 j - A_4 k, \\ A^j &= j^{-1}Aj = A_1 - A_2 i + A_3 j - A_4 k, \\ A^k &= k^{-1}Ak = A_1 - A_2 i - A_3 j + A_4 k. \end{aligned}$$

Let $A^* = A_1^{\mathrm{T}} - A_2^{\mathrm{T}} i - A_3^{\mathrm{T}} j - A_4^{\mathrm{T}} k$ be the usual conjugate transpose of A. Then the three other η-conjugate transposes ($\eta \in \{i, j, k\}$) of A are defined as follows:

$$\begin{aligned} A^{i*} &= -iA^*i = A_1^{\mathrm{T}} - A_2^{\mathrm{T}} i + A_3^{\mathrm{T}} j + A_4^{\mathrm{T}} k, \\ A^{j*} &= jA^*j = A_1^{\mathrm{T}} + A_2^{\mathrm{T}} i - A_3^{\mathrm{T}} j + A_4^{\mathrm{T}} k, \\ A^{k*} &= kA^*k = A_1^{\mathrm{T}} + A_2^{\mathrm{T}} i + A_3^{\mathrm{T}} j - A_4^{\mathrm{T}} k. \end{aligned}$$

For $A \in \mathbb{H}_s^{n\times n}$, $\eta \in \{i, j, k\}$, A is called η-(anti-)Hermitian if $A^{\eta *} = (-)A$ [85].

Given $A \in \mathbb{H}_s^{m\times n}$, $A = A_1 + A_2 i + A_3 j + A_4 k$, $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, we define the following four real representations of A:

(a)

$$A^\sigma = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix}, \tag{27}$$

(b)

$$A^{\sigma_i} = U_m A^\sigma = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & -A_1 & A_2 \\ -A_4 & -A_3 & -A_2 & -A_1 \end{bmatrix}, \; U_m = \begin{bmatrix} I_m & O & O & O \\ O & I_m & O & O \\ O & O & -I_m & O \\ O & O & O & -I_m \end{bmatrix},$$

(c)

$$A^{\sigma_j} = V_m A^\sigma = \begin{bmatrix} A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \\ A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \end{bmatrix}, \; V_m = \begin{bmatrix} O & O & I_m & O \\ O & O & O & I_m \\ I_m & O & O & O \\ O & I_m & O & O \end{bmatrix},$$

(d)

$$A^{\sigma_k} = W_m A^\sigma = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & -A_1 & A_4 & -A_3 \\ -A_3 & A_4 & -A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix}, \; W_m = \begin{bmatrix} I_m & O & O & O \\ O & -I_m & O & O \\ O & O & -I_m & O \\ O & O & O & I_m \end{bmatrix}.$$

Using the real representations above, reference [84] presents the following results.

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Theorem 71 (General solutions for (1) over $\mathbb{H}_s$)**.** *[84] Consider matrices $A \in \mathbb{H}_s^{p\times m}$, $C \in \mathbb{H}_s^{p\times n}$, $B \in \mathbb{H}_s^{n\times q}$, and $D \in \mathbb{H}_s^{m\times q}$. Then there exist two equivalent statements for the system (1) that has a solution $X \in \mathbb{H}_s^{m\times n}$.*

(a) The system of real matrix equations

$$(A^\sigma Y, YB^\sigma) = (C^\sigma, D^\sigma) \tag{28}$$

has a solution $Y \in \mathbb{R}^{4m\times 4n}$.

(b) $C^\sigma B^\sigma = A^\sigma D^\sigma,\ C^\sigma = A^\sigma (A^\sigma)^\dagger C^\sigma,\ D^\sigma = D^\sigma (B^\sigma)^\dagger B^\sigma$.

If the system (28) is consistent, then

$$X = \frac{1}{8}(I_n\ I_n i\ I_n j\ I_n k)(Y + P_n Y P_r^{\mathrm{T}} + Q_n Y Q_r^{\mathrm{T}} + R_n Y R_r^{\mathrm{T}})\begin{bmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{bmatrix},$$

where

$$Y = (A^\sigma)^\dagger C^\sigma + \mathcal{L}_{A^\sigma} D^\sigma (B^\sigma)^\dagger + \mathcal{L}_{A^\sigma} F \mathcal{R}_{B^\sigma},$$

for arbitrary $F \in \mathbb{R}^{4m\times 4n}$.

Theorem 72 (η-conjugate solutions for (1) over $\mathbb{H}_s$)**.** *[84] Let $A \in \mathbb{H}_s^{p\times m}$, $C \in \mathbb{H}_s^{p\times n}$, $B \in \mathbb{H}_s^{n\times q}$, and $D \in \mathbb{H}_s^{m\times q}$. Then, the following statements are equivalent:*

(a) The system of split quaternion matrix equations (1) has a solution $X = (-)X^\eta \in \mathbb{H}_s^{m\times n}$, $\eta \in \{i, j, k\}$.

(b) The system of real matrix equations (28) has a $\begin{cases} (U_n, U_r),\ \eta = i \\ (V_n, V_r),\ \eta = j \\ (W_n, W_r),\ \eta = k \end{cases}$ *generalized (anti-)reflexive solution $Y \in \mathbb{R}^{4m\times 4n}$.*

(c)

$$\begin{gathered} C_1^\sigma B_1^\sigma = A_1^\sigma D_1^\sigma,\ C_1^\sigma = A_1^\sigma (A_1^\sigma)^\dagger C_1^\sigma,\ D_1^\sigma = D_1^\sigma (B_1^\sigma)^\dagger B_1^\sigma, \\ C_2^\sigma B_2^\sigma = A_2^\sigma D_2^\sigma,\ C_2^\sigma = A_2^\sigma (A_2^\sigma)^\dagger C_2^\sigma,\ D_2^\sigma = D_2^\sigma (B_2^\sigma)^\dagger B_2^\sigma \end{gathered}$$

$$\left(\text{or} \begin{array}{c} C_2^\sigma B_2^\sigma = A_1^\sigma D_1^\sigma,\ C_1^\sigma = A_2^\sigma (A_2^\sigma)^\dagger C_1^\sigma,\ D_1^\sigma = D_1^\sigma (B_2^\sigma)^\dagger B_2^\sigma, \\ C_1^\sigma B_1^\sigma = A_2^\sigma D_2^\sigma,\ C_2^\sigma = A_1^\sigma (A_1^\sigma)^\dagger C_2^\sigma,\ D_2^\sigma = D_2^\sigma (B_1^\sigma)^\dagger B_1^\sigma \end{array} \right)$$

hold, where

$$\begin{array}{cccc} A^\sigma = A_1^\sigma + A_2^\sigma, & U_n (A_1^\sigma)^* = (A_1^\sigma)^*, & U_n (A_2^\sigma)^* = -(A_2^\sigma)^*, & (A_1^\sigma (A_2^\sigma)^* = O, \\ C^\sigma = C_1^\sigma + C_2^\sigma, & U_r (C_1^\sigma)^* = (C_1^\sigma)^*, & U_r (C_2^\sigma)^* = -(C_2^\sigma)^*, & C_1^\sigma (C_2^\sigma)^* = O, \\ B^\sigma = B_1^\sigma + B_2^\sigma, & U_r B_1^\sigma = B_1^\sigma, & U_r B_2^\sigma = -B_2^\sigma, & (B_2^\sigma)^* B_1^\sigma = O, \\ D^\sigma = D_1^\sigma + D_2^\sigma, & U_n D_1^\sigma = D_1^\sigma, & U_n D_2^\sigma = -D_2^\sigma, & (D_2^\sigma)^* D_1^\sigma = O. \end{array}$$

when $\eta = i$;

$$\begin{array}{cccc} A^\sigma = A_1^\sigma + A_2^\sigma, & V_n (A_1^\sigma)^* = (A_1^\sigma)^*, & V_n (A_2^\sigma)^* = -(A_2^\sigma)^*, & A_1^\sigma (A_2^\sigma)^* = O, \\ C^\sigma = C_1^\sigma + C_2^\sigma, & V_r (C_1^\sigma)^* = (C_1^\sigma)^*, & V_r (C_2^\sigma)^* = -(C_2^\sigma)^*, & C_1^\sigma (C_2^\sigma)^* = O, \\ B^\sigma = B_1^\sigma + B_2^\sigma, & V_r B_1^\sigma = B_1^\sigma, & V_r B_2^\sigma = -B_2^\sigma, & (B_2^\sigma)^* B_1^\sigma = O, \\ D^\sigma = D_1^\sigma + D_2^\sigma, & V_n (D_1^\sigma) = D_1^\sigma, & V_n D_2^\sigma = -D_2^\sigma, & (D_2^\sigma)^* D_1^\sigma = O. \end{array}$$

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when $\eta = j$;

$$\begin{array}{llll} A^{\sigma} = A_1^{\sigma} + A_2^{\sigma}, & W_n(A_1^{\sigma})^* = (A_1^{\sigma})^*, & W_n(A_2^{\sigma})^* = -(A_2^{\sigma})^*, & A_1^{\sigma}(A_2^{\sigma})^* = O, \\ C^{\sigma} = C_1^{\sigma} + C_2^{\sigma}, & W_r(C_1^{\sigma})^* = (C_1^{\sigma})^*, & W_r(C_2^{\sigma})^* = -(C_2^{\sigma})^*, & C_1^{\sigma}(C_2^{\sigma})^* = O, \\ B^{\sigma} = B_1^{\sigma} + B_2^{\sigma}, & W_r B_1^{\sigma} = B_1^{\sigma}, & W_r B_2^{\sigma} = -B_2^{\sigma}, & (B_2^{\sigma})^* B_1^{\sigma} = O, \\ D^{\sigma} = D_1^{\sigma} + D_2^{\sigma}, & W_n(D_1^{\sigma}) = D_1^{\sigma}, & W_n D_2^{\sigma} = -D_2^{\sigma}, & (D_2^{\sigma})^* D_1^{\sigma} = O. \end{array}$$

when $\eta = k$.

If the system (1) is consistent, then

$$X = \frac{1}{8}(I_n, I_n i, I_n j, I_n k)(Y + P_n Y P_r^{\mathrm{T}} + Q_n Y Q_r^{\mathrm{T}} + R_n Y R_r^{\mathrm{T}}) \begin{bmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{bmatrix},$$

where $Y = Y_0 + EFG$,

$$\begin{aligned} &Y_0 = (A_1^{\sigma})^{\dagger}(C_1)^{\sigma} + \mathcal{L}_{A_1^{\sigma}} D_1^{\sigma}(B_1^{\sigma})^{\dagger} + (A_2^{\sigma})^{\dagger} C_2^{\sigma} + \mathcal{L}_{A_2^{\sigma}} D_2^{\sigma}(B_2^{\sigma})^{\dagger}, \\ &(or \quad Y_0 = (A_1^{\sigma})^{\dagger}(C_2)^{\sigma} + \mathcal{L}_{A_2^{\sigma}} D_2^{\sigma}(B_1^{\sigma})^{\dagger} + (A_2^{\sigma})^{\dagger} C_1^{\sigma} + \mathcal{L}_{A_1^{\sigma}} D_1^{\sigma}(B_1^{\sigma})^{\dagger},) \\ &E = I_{4n} - (A_1^{\sigma})^{\dagger} A_1^{\sigma} - (A_2^{\sigma})^{\dagger} A_2^{\sigma},\ G = I_{4r} - B_1^{\sigma}(B_1^{\sigma})^{\dagger} - B_2^{\sigma}(B_2^{\sigma})^{\dagger}. \end{aligned}$$

with arbitrary $F \in \mathbb{R}^{4m\times 4n}$ is a $\begin{cases} (U_n, U_r),\ \eta = i \\ (V_n, V_r),\ \eta = j \\ (W_n, W_r),\ \eta = k \end{cases}$ *generalized (anti-)reflexive matrix.*

Theorem 73 (η-Hermitian solutions for (1) over $\mathbb{H}_s$)**.** *[84] If $A, C \in \mathbb{H}_s^{p\times m}$, $B, D \in \mathbb{H}_s^{n\times q}$, the following statements are equivalent:*

(a) The system of split quaternion matrix equations (1) has a solution $X = (-)X^{\eta *} \in \mathbb{H}_s^{n\times n}$, $\eta \in \{i, j, k\}$.

(b) The system of real matrix equation

$$\begin{cases} (A^{\sigma}Y, YB^{\sigma}) = (C^{\sigma}, D^{\sigma}),\ \eta = i, \\ (A^{\sigma_k} W_n Y, Y W_n B^{\sigma_k}) = (C^{\sigma_k}, D^{\sigma_k}),\ \eta = j, \\ (A^{\sigma_j} V_n Y, Y V_n B^{\sigma_j}) = (C^{\sigma_j}, D^{\sigma_j}),\ \eta = k, \end{cases}$$

has a (skew-)symmetric solution $Y \in \mathbb{R}^{4n\times 4n}$.

(c)

$$\begin{cases} C^{\sigma}B^{\sigma} = A^{\sigma}D^{\sigma},\ C^{\sigma} = A^{\sigma}(A^{\sigma})^{\dagger}C^{\sigma},\ D^{\sigma} = D^{\sigma}(B^{\sigma})^{\dagger}B^{\sigma},\ \eta = i, \\ C^{\sigma_k} W_n B^{\sigma_k} = A^{\sigma_k} W_n D^{\sigma_k},\ C^{\sigma_k} = A^{\sigma_k} W_n (A^{\sigma_k} W_n)^{\dagger} C^{\sigma_k},\ D^{\sigma_k} = D^{\sigma_k}(W_n B^{\sigma_k})^{\dagger} W_n B^{\sigma_k},\ \eta = j, \\ C^{\sigma_j} V_n B^{\sigma_j} = A^{\sigma_j} V_n D^{\sigma_j},\ C^{\sigma_j} = A^{\sigma_j} V_n (A^{\sigma_j} V_n)^{\dagger} C^{\sigma_j},\ D^{\sigma_j} = D^{\sigma_j}(V_n B^{\sigma_j})^{\dagger} V_n B^{\sigma_j},\ \eta = k, \end{cases}$$

and M_{η} is a symmetric matrix, where

$$M_{\eta} = \begin{cases} \begin{bmatrix} C^{\sigma}(A^{\sigma})^{\mathrm{T}} & C^{\sigma}B^{\sigma} \\ (D^{\sigma})^{\mathrm{T}}(A^{\sigma})^{\mathrm{T}} & (D^{\sigma})^{\mathrm{T}}B^{\sigma} \end{bmatrix},\ \eta = i, \\ \begin{bmatrix} C^{\sigma_k}(A^{\sigma_k}W_n)^{\mathrm{T}} & C^{\sigma_k}W_n B^{\sigma_k} \\ (D^{\sigma_k})^{\mathrm{T}}(A^{\sigma_k}W_n)^{\mathrm{T}} & (D^{\sigma_k})^{\mathrm{T}}W_n B^{\sigma_k} \end{bmatrix},\ \eta = j, \\ \begin{bmatrix} C^{\sigma_j}(A^{\sigma_j}V_n)^{\mathrm{T}} & C^{\sigma_j}V_n B^{\sigma_j} \\ (D^{\sigma_j})^{\mathrm{T}}(A^{\sigma_j}V_n)^{\mathrm{T}} & (D^{\sigma_j})^{\mathrm{T}}V_n B^{\sigma_j} \end{bmatrix},\ \eta = k. \end{cases}$$

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In this case, the general η-(anti-)Hermitian solution to the system (1) can be expressed as

$$X = \frac{1}{8}(I_n, I_n i, I_n j, I_n k)(Y + P_n Y P_n^{\mathrm{T}} + Q_n Y Q_n^{\mathrm{T}} + R_n Y R_n^{\mathrm{T}}) \begin{bmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{bmatrix},$$

where

$$Y = E_\eta^\dagger F_\eta \pm F_\eta^{\mathrm{T}} (E_\eta^\dagger)^{\mathrm{T}} \mp E_\eta^\dagger M_\eta (E_\eta^\dagger)^{\mathrm{T}} + \mathcal{L}_{E_\eta} U (\mathcal{L}_{E_\eta})^{\mathrm{T}},$$

and $U = \pm U^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ is an arbitrary matrix,

$$E_\eta = \begin{cases} \begin{bmatrix} A^\sigma \\ (B^\sigma)^{\mathrm{T}} \end{bmatrix}, \ \eta = i, \\ \begin{bmatrix} A^{\sigma_k} W_n \\ (W_n B^{\sigma_k})^{\mathrm{T}} \end{bmatrix}, \ \eta = j, \\ \begin{bmatrix} A^{\sigma_j} V_n \\ (V_n B^{\sigma_j})^{\mathrm{T}} \end{bmatrix}, \ \eta = k, \end{cases} \quad F_\eta = \begin{cases} \begin{bmatrix} C^\sigma \\ (D^\sigma)^{\mathrm{T}} \end{bmatrix}, \ \eta = i, \\ \begin{bmatrix} C^{\sigma_k} \\ (D^{\sigma_k})^{\mathrm{T}} \end{bmatrix}, \ \eta = j, \\ \begin{bmatrix} C^{\sigma_j} \\ (D^{\sigma_j})^{\mathrm{T}} \end{bmatrix}, \ \eta = k. \end{cases}$$

Remark 17. *More complex linear matrices or even tensor equations can be solved by means of complex representations or semi-tensor products of split quaternions, as detailed in references [86–89].*

6.3. The System (1) over Dual Quaternions

The collection of dual quaternions is expressed as

$$\mathbb{DQ} = \{c = c_0 + c_1\epsilon : c_0, c_1 \in \mathbb{H}, \epsilon^2 = 0\},$$

where c_0 and c_1 represent the standard part and the infinitesimal part of c, respectively [91]. We denote $\mathbb{DQ}^{m \times n}$ as the set of all $m \times n$ matrices over $\mathbb{DQ}$.

For the general solution of the system (1) over dual quaternions, Xie et al. recently presented the following theorem.

Theorem 74 (General solutions for (1) over $\mathbb{DQ}$). *[90] Let $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{p \times m}$, $B = B_0 + B_1\epsilon \in \mathbb{DQ}^{n \times q}$, $C = C_0 + C_1\epsilon \in \mathbb{DQ}^{p \times n}$ and $D = D_0 + D_1\epsilon \in \mathbb{DQ}^{m \times q}$ be given. Set*

$$\begin{aligned} &C_{11} = C_1 - A_1(A_0^\dagger C_0 + \mathcal{L}_{A_0} D_0 B_0^\dagger), \ D_{11} = D_1 - (A_0^\dagger C_0 + \mathcal{L}_{A_0} D_0 B_0^\dagger) B_1, \ A_{11} = A_1 \mathcal{L}_{A_0}, \\ &B_{11} = \mathcal{R}_{B_0} B_1, \ A_2 = \mathcal{R}_{A_0} A_{11}, \ C_2 = \mathcal{R}_{B_0}, \ B_2 = \mathcal{R}_{A_0} C_{11}, \ A_3 = \mathcal{L}_{A_0}, \ C_3 = B_{11} \mathcal{L}_{B_0}, \\ &B_3 = D_{11} \mathcal{L}_{B_0}, \ A_{00} = A_3 \mathcal{L}_{A_2}, \ C_{00} = \mathcal{R}_{C_2} C_3, \ B_{00} = B_3 - A_3 A_2^\dagger B_2 C_2^\dagger C_3, \ D_{00} = \mathcal{R}_{A_{00}} A_3, \\ &\Phi = A_2^\dagger B_2 C_2^\dagger + \mathcal{L}_{A_2} A_{00}^\dagger B_{00} C_3^\dagger - \mathcal{L}_{A_2} A_0^\dagger A_3 D_{00}^\dagger \mathcal{R}_{A_{00}} B_{00} C_3^\dagger + D_{00}^\dagger \mathcal{R}_{A_{00}} B_{00} C_0^\dagger \mathcal{R}_{C_2}. \end{aligned}$$

Then, the system (1) is consistent if and only if

$$\begin{aligned} &\mathcal{R}_{A_0} C_0 = 0, \ D_0 \mathcal{L}_{B_0} = 0, \ \mathcal{R}_{A_2} B_2 = 0, \\ &B_2 \mathcal{L}_{C_2} = 0, \ \mathcal{R}_{A_4} B_3 = 0, \ B_3 \mathcal{L}_{C_3} = 0, \ \mathcal{R}_{A_{40}} B_{00} \mathcal{L}_{C_{00}} = 0, \end{aligned}$$

or equivalently,

$$\operatorname{rank}\begin{bmatrix} A_0, C_0 \end{bmatrix} = \operatorname{rank}(A_0), \ \operatorname{rank}\begin{bmatrix} B_0 \\ D_0 \end{bmatrix} = r(B_0),$$
$$\operatorname{rank}\begin{bmatrix} A_0 & C_1 & A_1 \\ 0 & C_0 & A_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_0 & A_1 \\ 0 & A_0 \end{bmatrix}, \ \operatorname{rank}\begin{bmatrix} A_0, A_1 D_0 - C_1 B_0 \end{bmatrix} = \operatorname{rank}(A_0),$$
$$\operatorname{rank}\begin{bmatrix} B_0 \\ C_0 B_1 - A_0 D_1 \end{bmatrix} = \operatorname{rank}(B_0), \ \operatorname{rank}\begin{bmatrix} D_1 & D_0 \\ B_1 & B_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_0 & O \\ B_1 & B_0 \end{bmatrix},$$
$$\operatorname{rank}\begin{bmatrix} C_1 B_1 - A_1 D_1 & A_0 & C_1 B_0 - A_1 D_0 \\ B_0 & O & O \\ C_0 B_1 - A_0 D_1 & O & O \end{bmatrix} = \operatorname{rank}(A_0) + \operatorname{rank}(B_0).$$

and equations $A_0 D_0 = C_0 B_0$, $A_0 D_1 - C_0 B_1 = C_1 B_0 - A_1 D_0$ *hold. In such circumstances, the general solution of the system* (1) *can be expressed as* $X = X_0 + X_1 \epsilon$, *where*

$$X_0 = A_0^\dagger C_0 + \mathcal{L}_{A_0} D_0 B_0^\dagger + \mathcal{L}_{A_0} U \mathcal{R}_{B_0},$$
$$X_1 = A_1^\dagger (C_{11} - A_{11} U \mathcal{R}_{B_0}) + \mathcal{L}_{A_1}(D_{11} - \mathcal{L}_{A_0} U B_{11}) B_0^\dagger + \mathcal{L}_{A_0} U_1 \mathcal{R}_{B_0},$$
$$U = \Phi + \mathcal{L}_{A_2} \mathcal{L}_{A_{00}} + U_3 \mathcal{R}_{C_{00}} \mathcal{R}_{C_2} + \mathcal{L}_{A_4} U_5 \mathcal{R}_{C_2} + \mathcal{L}_{A_3} U_2 \mathcal{R}_{C_2},$$

and $U_i (i = 1, 5)$ *are arbitrary.*

6.4. The System (1) over Dual Split Quaternions

Yang et al. studied the system (1) over the dual split quaternion tensor and provided the general solution as well as the existence conditions and expressions for the η-Hermitian solution [94].

For a multidimensional array tensor $\mathcal{A} = (a_{i_1 \cdots i_M})_{1 \le i_k \le I_k (k=1,\ldots,M)}$ with $I_1 \times I_2 \times \cdots \times I_M$ entries, the general inverse of $\mathcal{A}$ can also be extended from the general inverse of a matrix [92], denoted as $\mathcal{A}^\dagger$. Let $\mathbb{H}_s^{I_1 \times \cdots \times I_M}$ represent the sets of the order M tensors with $I_1 \times \cdots \times I_M$ dimensions over the split quaternion algebra $\mathbb{H}_s$. The identity tensor $\mathcal{I} = (d_{t_1 \cdots t_M t_1 \cdots t_M}) \in \mathbb{H}_s^{T_1 \times \cdots \times T_M \times T_1 \times \cdots \times T_M}$ has all zero entries, except for the elements $d_{t_1 \cdots t_M t_1 \cdots t_M} = 1$. $\mathcal{O}$ denotes the zero tensor whose elements are all zero. Define $\mathcal{L}_A = \mathcal{I} - A^\dagger A$, $\mathcal{R}_A = \mathcal{I} - A A^\dagger$.

The sets of dual split quaternion and dual split quaternion tensors are represented as follows [93]:

$$\mathbb{DQ}_s = \{q = q_0 + q_1 \epsilon : q_0, q_1 \in \mathbb{H}_s, \epsilon \neq 0, \epsilon^2 = 0\},$$
$$\mathbb{DQ}_s^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} = \{Q = Q_0 + Q_1 \epsilon : Q_0, Q_1 \in \mathbb{H}_s^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}, \epsilon \neq 0, \epsilon^2 = 0\}.$$

Let $\mathcal{X} \in \mathbb{DQ}_s^{I_1 \times \cdots \times I_M \times L_1 \times \cdots \times L_N}$ and $\mathcal{Y} \in \mathbb{DQ}_s^{L_1 \times \cdots \times L_N \times K_1 \times \cdots \times K_S}$. Then, we can define the Einstein product of tensors $\mathcal{X}$ and $\mathcal{Y}$ through the operation $*_N$ as

$$(\mathcal{X} *_N \mathcal{Y})_{i_1 \cdots i_M k_1 \cdots k_S} = \sum_{l_1 \cdots l_N} x_{i_1 \cdots i_M l_1 \cdots l_N} y_{l_1 \cdots l_N k_1 \cdots k_S} \in \mathbb{DQ}_s^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_S}.$$

The real representations of the split quaternion tensor are of the same form as (27), with the only difference being the use of tensor notation.

For the system

$$\begin{cases} \mathcal{A} *_N \mathcal{X} = \mathcal{C}, \\ \mathcal{X} *_S \mathcal{B} = \mathcal{D}, \end{cases} \tag{29}$$

the following conclusions hold.

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Theorem 75 (General solutions for (29) over $\mathbb{DQ}_s$)**.** *[94] Suppose that* $\mathcal{A} = \mathcal{A}_{00} + \mathcal{A}_{01}\epsilon \in \mathbb{DQ}_s^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_N}$, $\mathcal{B} = \mathcal{B}_{00} + \mathcal{B}_{01}\epsilon \in \mathbb{DQ}_s^{K_1\times\cdots\times K_S\times L_1\times\cdots\times L_T}$, $\mathcal{C} = \mathcal{C}_{00} + \mathcal{C}_{01}\epsilon \in \mathbb{DQ}_s^{I_1\times\cdots\times I_M\times K_1\times\cdots\times K_S}$*, and* $\mathcal{D} = \mathcal{D}_{00} + \mathcal{D}_{01}\epsilon \in \mathbb{DQ}_s^{I_1\times\cdots\times I_N\times L_1\times\cdots\times L_T}$*. Denote*

$$
\begin{aligned}
&\mathcal{A}_0 = \mathcal{A}_{00}^{\sigma},\ \mathcal{A}_1 = \mathcal{A}_{01}^{\sigma}, \mathcal{B}_0 = \mathcal{B}_{00}^{\sigma},\ \mathcal{B}_1 = \mathcal{B}_{01}^{\sigma}, \mathcal{C}_0 = \mathcal{C}_{00}^{\sigma},\ \mathcal{C}_1 = \mathcal{C}_{01}^{\sigma}, \mathcal{D}_0 = \mathcal{D}_{00}^{\sigma},\ \mathcal{D}_1 = \mathcal{D}_{01}^{\sigma},\\
&\mathcal{A}_{11} = \mathcal{A}_1 *_N \mathcal{L}_{\mathcal{A}_0},\ \mathcal{B}_{11} = \mathcal{R}_{\mathcal{B}_0 *_S}\mathcal{B}_1,\ \mathcal{C}_{11} = \mathcal{C}_1 - \mathcal{A}_1 *_N (\mathcal{A}_0^{\dagger} *_M \mathcal{B}_0 + \mathcal{L}_{\mathcal{A}_0 *_N}\mathcal{D}_0 *_T \mathcal{C}_0^{\dagger}),\\
&\mathcal{D}_{11} = \mathcal{D}_1 - (\mathcal{A}_0^{\dagger} *_M \mathcal{C}_0 + \mathcal{L}_{\mathcal{A}_0} *_N \mathcal{D}_0 *_T \mathcal{B}_0^{\dagger}) *_S \mathcal{B}_1,\ \mathcal{A}_2 = \mathcal{R}_{\mathcal{A}_0} *_M \mathcal{A}_{11},\ \mathcal{C}_2 = \mathcal{R}_{\mathcal{C}_0},\\
&\mathcal{B}_2 = \mathcal{R}_{\mathcal{A}_0} *_M \mathcal{C}_{11}, \mathcal{A}_3 = \mathcal{L}_{\mathcal{A}_0}, \mathcal{B}_3 = \mathcal{B}_{11} *_T \mathcal{L}_{\mathcal{B}_0},\ \mathcal{B}_3 = \mathcal{D}_{11} *_T \mathcal{L}_{\mathcal{B}_0},\ \mathcal{A}_4 = \mathcal{A}_3 *_N \mathcal{L}_{\mathcal{A}_2},\\
&\mathcal{C}_4 = \mathcal{R}_{\mathcal{C}_2} * S\mathcal{C}_3,\ \mathcal{B}_4 = \mathcal{B}_3 - \mathcal{A}_3 *_N \mathcal{A}_2^{\dagger} *_M \mathcal{B}_2 *_S \mathcal{C}_2^{\dagger} *_S \mathcal{C}_3, \mathcal{D}_4 = \mathcal{R}_{\mathcal{A}_4 *_N}\mathcal{A}_3,\\
&\mathcal{F} = \mathcal{L}_{\mathcal{A}_2} *_N \mathcal{A}_4^{\dagger} *_N \mathcal{B}_4 *_T \mathcal{C}_3^{\dagger} - \mathcal{L}_{\mathcal{A}_2} *_N \mathcal{A}_4^{\dagger} *_N \mathcal{A}_3 *_N \mathcal{D}_4^{\dagger} *_N \mathcal{R}_{\mathcal{A}_4} *_N \mathcal{B}_4 *_T \mathcal{C}_3^{\dagger}\\
&\qquad + \mathcal{D}_4^{\dagger} *_N \mathcal{R}_{\mathcal{A}_4} *_N \mathcal{B}_4 *_T \mathcal{C}_4^{\dagger} *_S \mathcal{R}_{\mathcal{C}_2} + \mathcal{A}_2^{\dagger} *_M \mathcal{B}_2 *_S \mathcal{C}_2^{\dagger}.
\end{aligned}
$$

Then, the following descriptions are equivalent:

(a) *The system of dual split quaternion tensor equation* (29) *is solvable.*

(b) *The system of tensor equations:*

$$
\begin{cases}
\mathcal{A}_0 *_N \mathcal{X}_0 = \mathcal{C}_0,\\
\mathcal{X}_0 *_S \mathcal{B}_0 = \mathcal{D}_0,\\
\mathcal{A}_0 *_N \mathcal{X}_1 + \mathcal{A}_1 *_N \mathcal{X}_0 = \mathcal{C}_1,\\
\mathcal{X}_0 *_S \mathcal{B}_1 + \mathcal{X}_1 *_S \mathcal{B}_0 = \mathcal{D}_1
\end{cases}
\tag{30}
$$

is consistent.

(c)

$$
\begin{aligned}
&\mathcal{R}_{\mathcal{A}_0 *_M}\mathcal{C}_0 = O,\ \mathcal{D}_0 *_T \mathcal{L}_{\mathcal{B}_0} = O,\ \mathcal{A}_0 *_N \mathcal{D}_0 = \mathcal{C}_0 *_S \mathcal{B}_0,\\
&\mathcal{A}_0 *_N \mathcal{D}_1 - \mathcal{C}_0 *_S \mathcal{B}_1 = \mathcal{C}_1 *_S \mathcal{B}_0 - \mathcal{A}_1 *_N \mathcal{D}_0,\ \mathcal{R}_{\mathcal{A}_2} *_M \mathcal{B}_2 = 0,\\
&\mathcal{B}_2 *_S \mathcal{L}_{\mathcal{B}_2} = O,\ \mathcal{R}_{\mathcal{A}_3} *_N \mathcal{B}_3 = 0, \mathcal{B}_3 *_T \mathcal{L}_{\mathcal{C}_3} = O,\ \mathcal{R}_{\mathcal{A}_4} *_N \mathcal{B}_4 *_T \mathcal{L}_{\mathcal{B}_4} = O.
\end{aligned}
$$

Based on these circumstances, the general solution of the system (29) *can be represented as* $\mathcal{X} = \mathcal{X}_{00} + \mathcal{X}_{01}\epsilon$*, where*

$$
\begin{aligned}
&\mathcal{X}_{00} = \frac{1}{8}[\mathcal{I}_{\mathrm{N}}, i\mathcal{I}_{\mathrm{N}}, j\mathcal{I}_{\mathrm{N}}, k\mathcal{I}_{\mathrm{N}}] *_{\mathrm{N}} \left(\mathcal{X}_0 + \mathcal{R}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_0 *_S \mathcal{R}_S^T + \mathcal{S}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_0 *_S \mathcal{S}_S^T + \mathcal{T}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_0 *_S \mathcal{T}_S^T\right) *_S \begin{bmatrix} \mathcal{I}_S \\ i\mathcal{I}_S \\ j\mathcal{I}_s \\ k\mathcal{I}_S \end{bmatrix},\\
&\mathcal{X}_{01} = \frac{1}{8}[\mathcal{I}_{\mathrm{N}}, i\mathcal{I}_{\mathrm{N}}, j\mathcal{I}_{\mathrm{N}}, k\mathcal{I}_{\mathrm{N}}] *_{\mathrm{N}} \left(\mathcal{X}_1 + \mathcal{R}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_1 *_s \mathcal{R}_S^T + \mathcal{S}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_1 *_S \mathcal{S}_S^T + \mathcal{T}_{\mathrm{N}} *_{\mathrm{N}} \mathcal{X}_1 *_S \mathcal{T}_S^T\right) *_S \begin{bmatrix} \mathcal{I}_S \\ i\mathcal{I}_S \\ j\mathcal{I}_S \\ k\mathcal{I}_S \end{bmatrix},\\
&\mathcal{X}_0 = \mathcal{A}_0^{\dagger} *_M \mathcal{B}_0 + \mathcal{L}_{\mathcal{A}_0 *_N}\mathcal{D}_0 *_T \mathcal{C}_0^{\dagger} + \mathcal{L}_{\mathcal{A}_0} *_{\mathrm{N}} \mathcal{W} *_s \mathcal{R}_{\mathcal{C}_0},\\
&\mathcal{X}_1 = \mathcal{A}_0^{\dagger} *_M (\mathcal{B}_{11} - \mathcal{A}_{11} *_{\mathrm{N}} \mathcal{W} *_s \mathcal{R}_{\mathcal{C}_0}) + \mathcal{L}_{\mathcal{A}_0 *_N}(\mathcal{D}_{11} - \mathcal{L}_{\mathcal{A}_0} *_N \mathcal{W}_{*_S}\mathcal{C}_{11}) *_T \mathcal{C}_0^{\dagger} + \mathcal{L}_{\mathcal{A}_0} *_N \mathcal{W}_1 * S\mathcal{R}_{\mathcal{C}_0},\\
&\mathcal{W} = \mathcal{F} + \mathcal{L}_{\mathcal{A}_2} *_N \mathcal{L}_{\mathcal{A}_4} *_N \mathcal{W}_2 + \mathcal{W}_3 * 5\mathcal{R}_{\mathcal{B}_4} * S\mathcal{R}_{\mathcal{B}_2} + \mathcal{L}_{\mathcal{A}_2} * N\mathcal{W}_4 * S\mathcal{R}_{\mathcal{B}_3} + \mathcal{L}_{\mathcal{A}_3} * N\mathcal{W}_5 * S\mathcal{R}_{\mathcal{B}_2},
\end{aligned}
$$

with arbitrary $\mathcal{W}_i(i = \overline{1,5})$.

Remark 18. *The (η-)Hermitian solutions for the system* (29) *can be derived by selecting different types of real representations of split quaternion tensors [94].*

This chapter presents the general solution of the system (1) over quaternions, including the determinant expression for the general solution. It also covers bi-symmetric solutions, centrosymmetric solutions, symmetric and skew-symmetric solutions, (P, Q)-(skew-)symmetric solutions, extreme rank (P, Q)-(skew-)symmetric solutions, and reducible solutions. Additionally, the general solution over split quaternions, η-(anti-)conjugate solutions, and η-(anti-)Hermitian solutions are discussed. The

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general solution over dual quaternion matrices and split dual quaternion tensors are also examined. The existence conditions and corresponding expressions for these solutions are provided.

7. Applications

The system (1) has broad applications across various fields. This chapter focuses on its use in encrypting and decrypting color images and videos.

In image processing, the system (1) can be applied to various tasks such as image transformation, filtering, and reconstruction. Matrix equations are used to model the transformation or processing of an image, where A and B represent certain image transformations, X is the unknown matrix to be solved, and C and D represent the image before and after processing, or certain features of the image.

In color image, a pure imaginary quaternion can represent the three color channels—red, green, and blue—using i, j, k, thus effectively representing a pixel. By utilizing quaternion matrices or dual quaternion matrices, which can represent even more information, we can process color images in a highly efficient manner. This approach allows us to simultaneously process multiple color channels of the image.

We present two examples of using the system (1). The first example involves using dual quaternions for encrypting and decrypting images, as shown in Figure 1 [90]. The original, encrypted, and decrypted images are displayed in Figure 2.

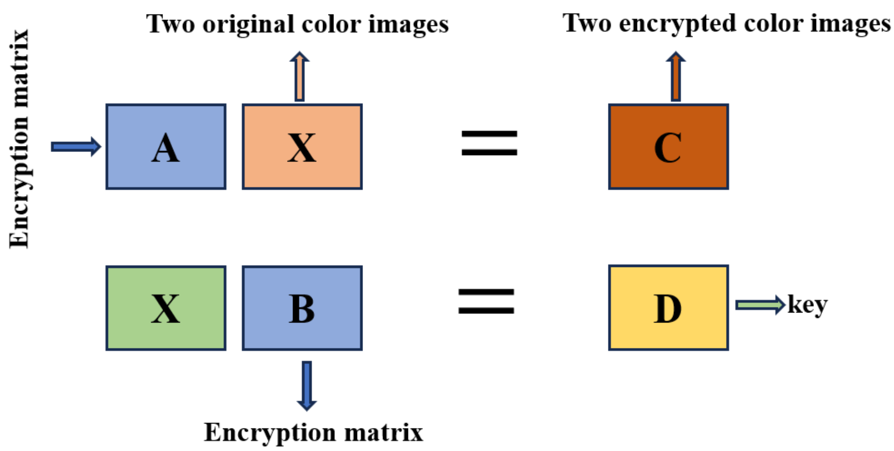

Figure 1. Scheme

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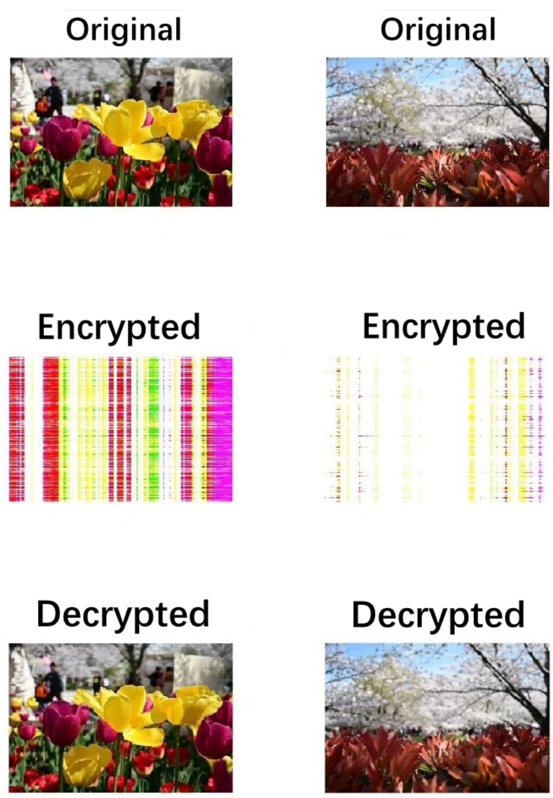

Figure 2. The original, encrypted, and decrypted color images.

The other example demonstrates the application of the dual split quaternion tensor equations in color video processing [94]. The basic framework is the same as in image processing, but the tensor, as a high-dimensional matrix, can directly represent video. The results for several frames are shown here, as illustrated in Figure 3.

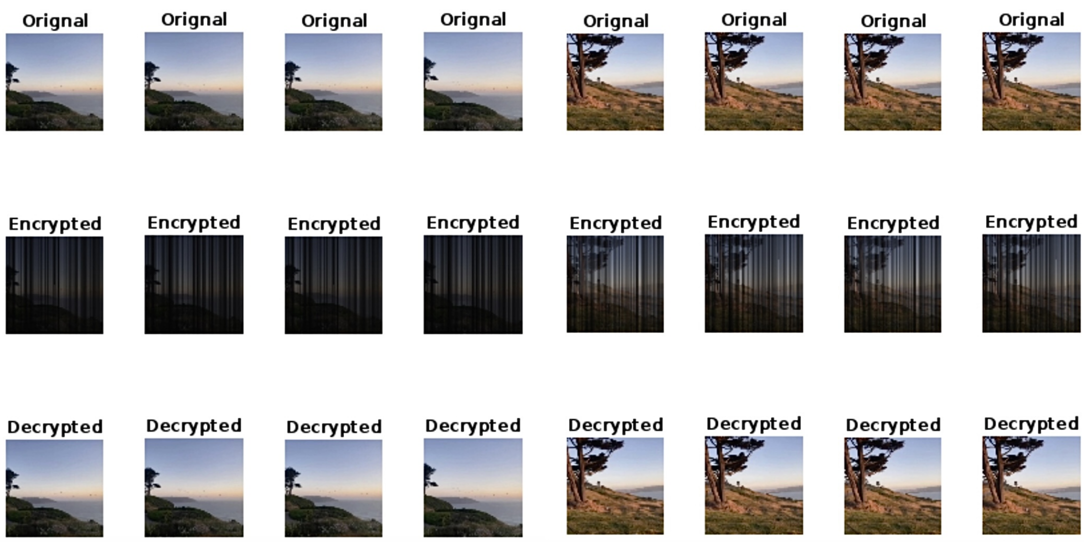

Figure 3. The original, encrypted, and decrypted images of randomly selected slices from color videos.

These two examples demonstrate that by using the solution method of the system (1), color images and videos can be effectively encrypted and decrypted, ensuring the security and accuracy of the communication process.

8. Conclusion

This paper presents a comprehensive review of the system (1) , emphasizing its essential role in a wide range of applications. The discussion includes generalized inverse methods for obtaining both general and specialized solutions, such as Hermitian solutions, nonnegative definite solutions, and maximal and minimal rank solutions. The theory is further extended to more advanced algebraic structures, including Hilbert spaces, Hilbert C^*-modules, and general rings, where specialized solving techniques can be applied. Matrix decomposition methods, such as eigenvalue decomposition, singular value decomposition, and generalized singular value decomposition, are explored for their effectiveness in solving the linear matrix equation systems. Additionally, the paper addresses solutions within specialized algebraic structures like dual numbers and various quaternions. At the end, examples of applications of the system (1) in electronic networks and color image processing are presented. This review aims to comprehensively summarize the research on various solutions to the system (1) across different algebraic structures. However, the differing research perspectives and the vast amount of literature may have resulted in some references being overlooked. Nonetheless, this does not detract from the primary value of the survey.

Future research may focus on addressing the computational challenges associated with large-scale matrix systems, as generalized inverses and matrix decomposition techniques can be computationally intensive. Therefore, finding numerical solutions to the system (1) is an important research direction. Inspired by [95], leveraging neural networks and other methods to explore these solutions could be a promising approach. Moreover, despite the widespread use of tensors in many fields due to their high-dimensional properties, exploration of the system (1) within the tensor framework has been relatively restricted. Consequently, continuing to study the system (1) within the context of tensors presents an exciting opportunity for future research. Lastly, it is worth noting that, given the extensive applications of dual quaternions, studying various special solutions to the system (1) in the context of dual quaternions, dual generalized commutative quaternions, and dual split quaternions—such as minimum norm solutions, Hermitian solutions, and reflexive solutions—presents an important development direction that warrants future attention.

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