Article

Finite-Time Stabilization of Homogeneous Non-Lipschitz Systems

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Abstract: This paper focuses on the problem of finite-time stabilization of homogeneous, non-Lipschitz systems with dilations. A key contribution of this paper is the design of a virtual recursive *Hölder*, non-Lipschitz state feedback which renders the non-Lipschitz systems in the special case dominated by a lower-triangular nonlinear system, finite-time stable. The proof is based on a recursive design algorithm developed recently, to construct the virtual *Hölder* continuous, finite-time stabilizer as well as a C^1 positive definite and proper Lyapunov function that guarantees finite-time stability of the non-Lipschitz non-linear systems.

Keywords: Finite-time control; nonlinear system; non-Lipschitzian dynamics; lyapunov function

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1. Introduction

In the 1990's, thanks to the differential geometric approach, several techniques appeared to design a systematic feedback insuring global asymptotic stabilization and robust control of systems [11,19,26]. Among them, a Lyapunov-based recursive design technique, called backstepping, which consisted on a recursive adding a power integrator. Originally proposed by Coran and Praly [12], this technique presented certain limitations due to necessary conditions imposed to the system [10,19,23]. Recently, a synthesis derived technique, called desingularizing method [27], based on the homogeneous system theory [2,12,17,22], was investigated by [28,35] in designing a recursive algorithm providing the construction of a state feedback controllers and a Lyapunov function. The importance of system homogeneity stems from the fact that it has useful properties like equivalence between local asymptotic stability and global asymptotic stability for linear homogeneous systems. Furthermore, different results and characteristics were obtained in the past decades, studying nonlinear homogeneous systems [1,26,30]. Therefore, for simplicity, as well as applications in mind, we focused on continuous homogeneous lower-triangular nonlinear systems.

In the works [5,15], finite-time stabilizers were derived by using the theory of homogeneous systems. Moreover, due to the use of a homogeneous approximation only local finite-time stabilization results can be established for specific case of nonlinear systems such as the homogeneous lower-triangular nonlinear systems with almost applications on two- to three-dimensional control systems [4–6,8]. Motivated by the above discussion, our concern was to develop a finite-time controller, relying on the Lyapunov theory for finite-time stability, achieving global finite-time stabilization for non-Lipshitz, n-dimensional homogeneous lower-triangular nonlinear systems. Therefore, we proposed a new constructive methodology based on adding a power integrator, inspired by recent works [9,28,31–33], with a convergence speed improvement suggested in [34] and the idea of homogeneous-based Lyapunov functions.

Recently, in the literature, the authors in [18,36] developed systematic algorithms to design both $non - Hölder$ and *Hölder* continuous state feedback and a C^1 control Lyapunov function, to achieve global finite-time stabilization for a class of uncertain non-linear systems. A $non - Hölder$, controller and a C^0 control Lyapunov function was proposed to globally finite-time stabilize a class of non-linear cascaded system in [29]. In the higher dimension case such as in [16], a *Hölder*, continuous state feedback control laws achieved local finite-time stabilization for triangular systems and for certain classes of nonlinear systems. The fact is that the issue of global finite-time stabilization of n-dimensional nonlinear systems can be achieved by *Hölder*, however a state feedback is still remaining unknown and unanswered.

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The novelty of our contribution comes then, from a new tool which is based on a methodical algorithm. This tool evolves a recursive mathematical relation between the *Hölder* exponents and the dilation coefficients, using the convexity property derived from our assumptions. Moreover, we show, using a step by step resized and reconstructed subsystems, how to explicitly design a C^0 virtual, *Hölder*, non-Lipschitz state feedback control law. This control law is able to stabilizing, in finite-time, a lower-triangular homogeneous non-linear system as well as C^1 homogeneous-based Lyapunov functions, in the form that we will give later when we will formulate our main Theorem with useful hypothesis.

To the best knowledge of the authors, there is no study on this new *Hölder* controller, nor the C^1 Lyapunov function form defined by our Theorem. As an application, a feedback control law is designed based on this C^1 Lyapunov function for three main systems: the single, double and triple integrators.

This paper is organized as follows. In section 2, we introduce the concept of finite-time stability of continuous autonomous systems, state the new Theorem and give the most important definitions and Lemmas. In section 3, we demonstrate, with some detailed proofs, the different steps for the construction of the algorithm. Finally, for proving the convergence in finite-time with the feedback law established in this work, the effectiveness of the proposed algorithm is shown in section 4 with a design example and a computer simulation.

2. Preliminary Results

2.1. Finite-Time Stability

This subsection gives some basic concepts and terminologies related to the notion of finite-time stability and the corresponding Lyapunov stability theory. We also recall the Lyapunov theorem of finite-time stability which gives a necessary and sufficient condition for non-Lipschitz continuous autonomous systems to be finite-time stable [4–6,20,21].

Definition 1 (Bhat and Bernstein 2000 [6]) Consider the non-Lipschitz continuous autonomous system on the open neighborhood D of the origin $x = 0$, $f : D \to \mathbb{R}^n$, such that

$$\begin{cases} \dot{x} = f(x)\,, & x \in \mathbb{R}^n \\ f(0) = 0. \end{cases} \tag{1}$$

The equilibrium $x = 0$ of the system (1) is finite-time convergent if there are an non empty opened neighborhood U of the origin and a function $T_x : U \setminus \{0\} \to (0, \infty)$, such that every solution trajectory $x(t, x_0)$ of (1) starting from the initial point $x_0 \in U \setminus \{0\}$ is well defined and unique in forward time for $t \in [0, T_x(x_0))$ and $\lim_{t \to T_x(x_0)} x(t, x_0) = 0$. T is called the settling-time of the initial state x_0. The equilibrium of (1) is finite-time stable if it is Lyapunov stable and finite-time convergent. If $U = D = \mathbb{R}^n$, the origin is a globally finite-time stable equilibrium.

The reader can note that only non smooth or non-Lipschitz continuous autonomous systems have the property of finite-time stability convergence, whereby the solution trajectory of a non-Lipschitzian system reaches a Lyapunov stable equilibrium state (*the origin* in finite time). Let us introduce the concept of finite-time stabilizability by the next definition.

Definition 2 (Finite-time stabilizability) The controlled system

$$\begin{cases} \dot{x} = f(x,u)\,, & (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \\ f(0,0) = 0 \end{cases}$$

is stabilizable in finite-time if there exists a continuous feedback law $x \to u(x)$ vanishing at the origin $0_{\mathbb{R}^n}$ such that 0 is stable in finite-time for the closed-loop system $\dot{x} = f(x, u(x))$.

The following Theorem provides sufficient conditions for the origin of the system (1) to be a finite-time stable equilibrium.

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Theorem 1. *(Finite-time stability) Consider the non-Lipschitz continuous autonomous system (1). Suppose there is* C^1 *function* $V(x)$ *defined on a neighborhood* $\tilde{U} \subset \mathbb{R}^n$ *of the origin, and real numbers* $c > 0$ *and* $0 < \alpha < 1$*, such that*

- $V(x)$ *is positive definite on* $\tilde{U}$ *and*
- $\dot{V}(x) + c\, V^{\alpha}(x) \leqslant 0, \ \forall x \in \tilde{U}$.

Then, the origin of system (1) is locally finite-time stable. The settling time, depending on the initial state $x(0) = x_0$*, satisfies, for all* x_0 *in some open neighborhood of the origin,*

$$T_x(x_0) \leqslant \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)} \tag{2}$$

If $\tilde{U} = \mathbb{R}^n$ *and* $V(x)$ *is also radially unbounded, the origin of system (1) is globally finite-time stable.*

2.2. Finite-Time Stabilizing Feedback

In this section, we state a Theorem in order to provide a recursive algorithm to design a *Hölderian* continuous state feedback, under certain conditions, which render the homogeneous non-Lipschitz closed-loop system globally finite-time stabilizable. For more convenience, we define $\mathbb{Q}_{odd}$ as the set of all rational numbers, whose numerators and denominators are all positive and odd integers.

Theorem 2. *We consider the class of non-linear homogeneous systems in the lower triangular form*

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) \\ \dot{x}_2 = x_3 + f_2(x_1, x_2) \\ \vdots \\ \dot{x}_n = u + f_n(x_1, \cdots, x_n) \end{cases} \tag{3}$$

We assume that $\hat{f}(x) = (f_1(x), \cdots, f_n(x))^T$*, such that* $f_i(x) = f_i(x_1, \cdots, x_n)$*, are* C^1 *functions vanishing at the origin and homogeneous of degree* r_i*,* $\forall\ i \in \{1, \cdots, n\}$*. Let* $(v_i - \alpha r_{i+1})_{i \in \{0, \cdots, n-1\}}$ *be a decreasing sequence in* $[0, 1]$*, such that*

$$0 \leqslant v_{n-1} - \alpha r_n \leqslant \cdots \leqslant v_1 - \alpha r_2 \leqslant v_0 - \alpha r_1 \leqslant 1 \tag{4}$$

where $\alpha \in [0, 1]$ *the Hölder exponent of the Lyapunov function associated to the system (1),* $v_i \in \mathbb{Q}_{odd}$ *such that* $v_i \in [0, 1]$*, for* $i = 0, \cdots, n-1$ *and* $r = (r_1, \cdots, r_n) \in (\mathbb{Q}_{odd})^n$ *the dilation coefficient of the system (1), satisfying*

$$r_1 \leqslant \frac{v_0}{\alpha}, r_2 \leqslant min\{\frac{v_0}{\alpha}, \frac{v_1}{\alpha}\}, \cdots, r_n \leqslant min\{\frac{v_0}{\alpha}, \frac{v_1}{\alpha}, \cdots, \frac{v_{n-1}}{\alpha}\} \tag{5}$$

and the iterative relation

$$w_0 + \alpha r_1 = w_1 + \alpha r_2 = \cdots = w_{n-1} + \alpha r_n \tag{6}$$

where $w_i \in \mathbb{Q}_{odd}$ *such that* $w_i \in [0, 1]$ *and* $w_i \leqslant v_i$*, for* $i = 0, \cdots, n-1$*. Then, there exist a* C^0 *feedback controller* $u = u(x)$ *with* $u(0) = 0$ *which renders the origin of the closed-loop system finite-time stable.*

We introduce the next Lemmas, that will be used in the proof of Theorem 2:

Lemma 3. *[3,18] Let* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1 *function with* $f_i(0) = 0$*. Then, there exists a smooth non-negative function* $\gamma_i(x_1, \cdots, x_i)$ *such that*

$$|f_i(x_1, \cdots, x_i)| \leqslant (|x_1| + \cdots + |x_i|)\gamma_i(x_1, \cdots, x_i). \tag{7}$$

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Lemma 4. *[3,9] For $a, b, c \in \mathbb{R}$, such that $0 < a \leqslant b \leqslant c$, then we have the inequality for all $x \in \mathbb{R}$*

$$|x|^b \leqslant |x|^a + |x|^c = |x|^a(1 + |x|^{c-a}). \tag{8}$$

The next Lemmas are a direct consequence of the Young's inequality [9].

Lemma 5. *For any positive real numbers x_i, $i = 1, \cdots, n$ and $0 < b \leqslant 1$, the following inequality holds*

$$(|x_1| + \cdots + |x_n|)^b \leqslant max(n^{p-1}, 1)(|x_1|^b + \cdots + |x_n|^b).$$

Lemma 6. *Let a and b be any real numbers and $\sigma \in (0, 1]$, then*

$$|a - b| \leqslant 2^{1-\sigma}|(a)^{\frac{1}{\sigma}} - (b)^{\frac{1}{\sigma}}|^{\sigma}. \tag{9}$$

Lemma 7. *Let c, d be positive real numbers and $\gamma(x, y)$ a real-valued function, then*

$$|x|^c|y|^d \leqslant \frac{c\,\gamma(x, y)\,|x|^{c+d}}{c+d} + \frac{d\,\gamma^{-\frac{c}{d}}(x, y)\,|y|^{c+d}}{c+d}. \tag{10}$$

3. Proof of Theorem 2

In order to prove Theorem 2, a modified backstepping procedure is used which simultaneously enable the construction of a C^1 positive definite and proper control Lyapunov function, as well as a non-Lipschitz finite-time C^0 feedback control law rendering the closed-loop system (3), finite-time stable. Different Lemmas and demonstrated propositions will be used during the progression of the proof.

It should be noted that the constructive proof is given by induction and the structure is similar to [3,18,26, 28]. For the first step of the induction, we choose the C^1 Lyapunov function which is positive definite, proper and with a *Hölder* exponent

$$V_1(x_1) = \int_{x_1^*}^{x_1} (s^{\frac{1}{v_0}} - x_1^{*\frac{1}{v_0}})^{w_0} ds \tag{11}$$

with the convention that $x_1^* = 0$. Using (11) and Lemma 3, we can find, by a simple derivative computation, a smooth function $\gamma_1(x_1)$ such that

$$\dot{V}_1(x_1) = (x_1)^{\frac{w_0}{v_0}}(x_2 + f_1(x_1)) \leqslant (x_1)^{\frac{w_0}{v_0}}(x_2 - x_2^*) + (x_1)^{\frac{w_0}{v_0}} x_2^* + (x_1)^{\frac{w_0}{v_0}} |x_1|\gamma_1(x_1). \tag{12}$$

Using Lemma 4, we can deduce as $0 < v_0 - \alpha r_1 \leqslant v_1 - \alpha r_2 < 1 \leqslant c$ that

$$|x_1| \leqslant |x_1|^{\frac{\alpha r_1}{v_0}}(1 + |x_1|^{c - \frac{\alpha r_1}{v_0}}). \tag{13}$$

Therefore we can write

$$\dot{V}_1(x_1) \leqslant (x_1)^{\frac{w_0}{v_0}}(x_2 - x_2^*) + (x_1)^{\frac{w_0}{v_0}} x_2^* + (x_1)^{\frac{w_0}{v_0}} |x_1|^{\frac{\alpha r_1}{v_0}}(1 + |x_1|^{c - \frac{\alpha r_1}{v_0}})\gamma_1(x_1). \tag{14}$$

By tacking a smooth non-negative function $\tilde{\gamma}_1(x_1)$ such that

$$(1 + |x_1|^{c - \frac{\alpha r_1}{v_0}})\gamma_1(x_1) \leqslant \tilde{\gamma}_1(x_1)$$

we have then $\dot{V}_1(x_1) \leqslant (x_1)^{\frac{w_0}{v_0}}(x_2 - x_2^*) + (x_1)^{\frac{w_0}{v_0}} x_2^* + (x_1)^{\frac{w_0 + \alpha r_1}{v_0}} \tilde{\gamma}_1(x_1)$. If we choose to take the virtual control x_2^* as follows,

$$x_2^* = -x_1^{\frac{\alpha r_1}{v_0}}(n + \tilde{\gamma}_1(x_1)) = -x_1^{\frac{\alpha r_1}{v_0}} \varphi_1^{\alpha r_1}(x_1). \tag{15}$$

with $\varphi_1(x_1)$ is a smooth positive function, it yields

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$$\dot{V}_1(x_1) \leqslant (x_1)^{\frac{w_0}{v_0}}(x_2 - x_2^*) - n(x_1)^{\frac{w_0+\alpha r_1}{v_0}}. \tag{16}$$

Inductive assumption: Suppose that at the $(k-1)$th step, there is a C^1 proper and positive definite Lyapunov function $V_{k-1}(x_1, \cdots, x_{k-1})$ for the system (3) and a set of C^0 virtual controllers $x_1^*, \cdots, x_k^*$ defined by the form

$$\begin{array}{ll} x_1^* = 0 & \eta_1 = x_1^{\frac{1}{v_0}} - x_1^{*\frac{1}{v_0}} \\ x_2^* = -\eta_1\varphi_1(x_1) & \eta_2 = x_2^{\frac{1}{v_1}} - x_2^{*\frac{1}{v_1}} \\ x_3^* = -\eta_2\varphi_2(x_1, x_2) & \eta_3 = x_3^{\frac{1}{v_2}} - x_3^{*\frac{1}{v_2}} \\ \vdots & \vdots \\ x_k^* = -\eta_{k-1}\varphi_{k-1}(x_1, x_2, \cdots, x_{k-1}), & \eta_k = x_k^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}} \end{array} \tag{17}$$

where $\varphi_i(x_1, \cdots, x_i), \forall i = 1, \cdots, n$ are smooth positive functions, such that

$$\dot{V}_{k-1}(x_1, \cdots, x_{k-1}) \leqslant -(n-k+2)(\textstyle\sum_{i=1}^{k-1} \eta_i^{w_{i-1}+\alpha r_i}) + \eta_{k-1}^{w_{k-2}}(x_k - x_k^*). \tag{18}$$

To prove the induction at the kth step, we consider the Lyapunov function defined by

$$V_k(x_1, \cdots, x_k) = V_{k-1}(x_1, \cdots, x_{k-1}) + W_k(x_1, \cdots, x_k). \tag{19}$$

where $W_k(x_1, \cdots, x_k) = \int_{x_k^*}^{x_k} (s^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}})^{w_{k-1}} ds$.

The next proposition that we are using, is available for the set of all rationales in $\mathbb{Q}_{odd}$. It was used in [3] for the same technique and one can be referred to [24] for the proof.

Proposition 8. *$W_k(x_1, \cdots, x_k)$ and $V_k(x_1, \cdots, x_k)$ are C^1 functions and we have the following results:*

$$\begin{aligned} \frac{\partial W_k}{\partial x_k} &= (x_k^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}})^{w_{k-1}} = \eta_k^{w_{k-1}} \\ \frac{\partial W_k}{\partial x_j} &= -(w_{k-1})\frac{\partial(x_k^{*\frac{1}{v_{k-1}}})}{\partial x_j} \int_{x_k^*}^{x_k} (s^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}})^{w_{k-1}-1} ds, \ with \ \forall\, j = 1, \cdots, k-1. \end{aligned} \tag{20}$$

We can then deduce from the inequality (18), that

$$\begin{aligned} \dot{V}_k(x_1, \cdots, x_k) &= \dot{V}_{k-1}(x_1, \cdots, x_{k-1}) + \tfrac{\partial W_k}{\partial x_k}\dot{x}_k + \textstyle\sum_{i=1}^{k-1} \tfrac{\partial W_k}{\partial x_i}\dot{x}_i \\ &\leqslant \begin{cases} -(n-k+2)(\sum_{i=1}^{k-1} \eta_i^{w_{i-1}+\alpha r_i}) \\ +\eta_{k-1}^{w_{k-2}}(x_k - x_k^*) \\ +\eta_k^{w_{k-1}}(x_{k+1} + f_k(x_1, \cdots, x_k)) \\ +\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}\dot{x}_i. \end{cases} \end{aligned} \tag{21}$$

In order to refine the previous inequality, we investigate one by one each term of (21). Using Lemma 6 and knowing that $v_{k-1} \in (0, 1]$, we can write the inequality

$$|x_k - x_k^*| \leqslant 2^{1-v_{k-1}} \left| x_k^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}} \right|^{v_{k-1}}. \tag{22}$$

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Then, we deduce using Lemma 7 that

$$\begin{aligned} |x_k - x_k^*||\eta_{k-1}|^{w_{k-2}} &\leqslant 2^{1-v_{k-1}}|\eta_{k-1}|^{w_{k-2}}|\eta_k|^{v_{k-1}} \leqslant \tfrac{1}{4}|\eta_{k-1}|^{w_{k-2}+v_{k-1}} + c_{k1}|\eta_k|^{w_{k-2}+v_{k-1}} \\ &\leqslant \tfrac{1}{4}(|\eta_1|^{w_0+\alpha r_1} + \cdots + |\eta_{k-1}|^{w_{k-2}+\alpha r_{k-1}}) + c_{k2}|\eta_k|^{w_{k-1}+\alpha r_k} \end{aligned} \tag{23}$$

where $c_{k2} > 0$ is an adequate fixed constant. To continue the estimation of the inequality (21), we introduce the next propositions.

Proposition 9. *For $i = 1, \cdots, k-1$, there are C^1 positive functions $\tilde{\gamma}_i(x_1, \cdots, x_i)$, such that*

$$|f_i(x_1, \cdots, x_i)| \leqslant (|\eta_1|^{\alpha r_i} + \cdots + |\eta_i|^{\alpha r_i})\tilde{\gamma}_i(x_1, \cdots, x_i). \tag{24}$$

Proof. Using the fact that $x_i^* = -\eta_{i-1}\varphi_{i-1}(x_1, \cdots, x_{i-1})$, we can write

$$|x_i| = |\eta_i + x_i^{*\frac{1}{v_i}}|^{v_i} \leqslant |\eta_i|^{v_i} + |\eta_{i-1}|\varphi_{i-1}(x_1, \cdots, x_{i-1}). \tag{25}$$

Using Lemma 5 and the fact that $\alpha r_i \in [0,1]$ and $\alpha r_i \leqslant \{v_0, \cdots, v_{i-1}\}$, we have for $i = 1, \cdots, k-1$:

$$\begin{aligned} |f_i(x_1, \cdots, x_i)| &\leqslant [(|\eta_1| + \textstyle\sum_{j=2}^{i} |\eta_j|^{v_i} + |\eta_{j-1}|\varphi_{j-1}(x_1, \cdots, x_{j-1})] \times \gamma_i(x_1, \cdots, x_i) \\ &\leqslant (|\eta_1|^{\alpha r_i} + \cdots + |\eta_i|^{\alpha r_i})\tilde{\gamma}_i(x_1, \cdots, x_i) \end{aligned} \tag{26}$$

where $\tilde{\gamma}_i(x_1, \cdots, x_i)$ is a C^1 chosen positive function satisfying two conditions given later in the following demonstration (see equation (39)).

Proposition 10. *For $i = 1, \cdots, k-1$ there are C^1 positive functions $\bar{\phi}_{i,j,k}(x_1, \cdots, x_k)$, such that*

$$|\frac{\partial(x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i}\dot{x}_i| \leqslant \sum_{j=1}^{k-1} |\eta_j|^{1-v_{i-1}+\alpha r_i}\bar{\phi}_{i,j,k}(x_1, \cdots, x_k). \tag{27}$$

Proof. We have the next estimation, for $i = 1, \cdots, k-1$, using Lemma 3, Lemma 6 and that $\forall\, 1 \leqslant i \leqslant n-1$, $\alpha r_i \leqslant \min\{v_0, \cdots, v_{i-1}\}$ with $1 - v_i \in (0,1]$,

$$\begin{aligned} |\dot{x}_i| &\leqslant |x_{i+1}| + |f_i(x_1, \cdots, x_i)| \leqslant |x_{i+1} - x_{i+1}^*| + |x_{i+1}^*| + (|x_1| + \cdots + |x_i|)\gamma_i(x_1, \cdots, x_i) \\ &\leqslant 2^{1-v_i}|\eta_{i+1}|^{v_i} + |\eta_i\varphi_i(x_1, \cdots, x_i)| + ([|x_1 - x_1^*| + \cdots |x_i - x_i^*|] + [|x_2^*| + \cdots + |x_i^*|]) \times \gamma_i(x_1, \cdots, x_i) \\ &\leqslant 2^{1-v_i}|\eta_{i+1}|^{v_i} + |\eta_i\varphi_i(x_1, \cdots, x_i)| + [\textstyle\sum_{j=0}^{i-1} 2^{1-v_j}|\eta_{j+1}|^{v_j} + \sum_{j=1}^{i-1} |\eta_j|\varphi_j(x_1, \cdots, x_j)] \times \gamma_i(x_1, \cdots, x_i) \\ &\leqslant \textstyle\sum_{j=1}^{i+1} |\eta_j|^{v_i}\phi_{i,j,k}(x_1, \cdots, x_i). \end{aligned} \tag{28}$$

where $\phi_{i,j,k}(x_1, \cdots, x_k)$ is a C^1 chosen positive function. □

For the estimation of the term $\left|\frac{\partial(x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i}\right|$, we use an inductive argument as described in the following.

For the first step, we note that there exists a C^1 positive function $\tilde{C}_{2,1}(x_1)$ obviously verifying

$$\left|\frac{\partial(x_2^{*\frac{1}{v_1}})}{\partial x_1}\right| = \left|\frac{\partial[\eta_1^{\frac{1}{v_1}}\varphi_1(x_1)^{\frac{1}{v_1}}]}{\partial x_1}\right| \leqslant \tilde{C}_{2,1}(x_1).$$

Inductive assumption For $i = 1, \cdots, k-2$, there exist smooth positive functions $\tilde{C}_{k-1,i}(x_1, \cdots, x_{k-2})$ such that

$$\left|\frac{\partial(x_{k-1}^{*\frac{1}{v_{k-2}}})}{\partial x_i}\right| \leqslant (\sum_{j=1}^{k-2} |\eta_j|^{v_i-\alpha r_i})\tilde{C}_{k-1,i}(x_1, \cdots, x_{k-2}). \tag{29}$$

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We want to prove that for $i = 1, \cdots, k-1$, there are C^1 positive functions $\tilde{C}_{k,i}(x_1, \cdots, x_{k-1})$, such that

$$\left| \frac{\partial (x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i} \right| \leqslant \sum_{j=1}^{k-1} |\eta_j|^{v_i - \alpha r_i} \tilde{C}_{k,i}(x_1, \cdots, x_{k-1}). \tag{30}$$

We also know that $x_k^* = -\eta_{k-1}\varphi_{k-1}(x_1, \cdots, x_{k-1})$, then for $i = 1, \cdots, k-2$, then we can calculate and we obtain

$$\begin{aligned}
|\frac{\partial (x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i}| &\leqslant |\eta_{k-1}^{\frac{1}{v_{k-1}}} \frac{\partial (\varphi_{k-1}^{\frac{1}{v_{k-1}}})}{\partial x_i}| + \frac{1}{v_{k-1}} |\eta_{k-1}|^{\frac{1}{v_{k-1}}-1} |\eta_{k-2}|^{\frac{1}{v_{k-2}}} \varphi_{k-1}^{\frac{1}{v_{k-1}}} |\frac{\partial (\varphi_{k-2}^{\frac{1}{v_{k-1}}})}{\partial x_i}| \\
&+ \frac{1}{v_{k-1}v_{k-2}} |\eta_{k-1}|^{\frac{1}{v_{k-1}}-1} |\eta_{k-2}|^{\frac{1}{v_{k-2}}-1} |\eta_{k-3}|^{\frac{1}{v_{k-3}}} \varphi_{k-1}^{\frac{1}{v_{k-1}}} \varphi_{k-2}^{\frac{1}{v_{k-2}}} |\frac{\partial (\varphi_{k-3}^{\frac{1}{v_{k-3}}})}{\partial x_i}| \\
&+ \frac{1}{v_{k-1}v_{k-2}v_{k-3}} |\eta_{k-1}|^{\frac{1}{v_{k-1}}-1} |\eta_{k-2}|^{\frac{1}{v_{k-2}}-1} |\eta_{k-3}|^{\frac{1}{v_{k-3}}-1} |\eta_{k-4}|^{\frac{1}{v_{k-4}}} \varphi_{k-1}^{\frac{1}{v_{k-1}}} \varphi_{k-2}^{\frac{1}{v_{k-2}}} \varphi_{k-3}^{\frac{1}{v_{k-3}}} |\frac{\partial (\varphi_{k-4}^{\frac{1}{v_{k-4}}})}{\partial x_i}| \\
&+ \frac{1}{v_{k-1}v_{k-2}v_{k-3}} |\eta_{k-1}|^{\frac{1}{v_{k-1}}-1} |\eta_{k-2}|^{\frac{1}{v_{k-2}}-1} |\eta_{k-3}|^{\frac{1}{v_{k-3}}-1} |\eta_{k-4}|^{\frac{1}{v_{k-4}}-1} \varphi_{k-1}^{\frac{1}{v_{k-1}}} \varphi_{k-2}^{\frac{1}{v_{k-2}}} \varphi_{k-3}^{\frac{1}{v_{k-3}}} \varphi_{k-4}^{\frac{1}{v_{k-4}}} |\frac{\partial (x_{k-4}^{*\frac{1}{v_{k-4}}})}{\partial x_i}| \\
&\leqslant \cdots \leqslant |\eta_{k-1}^{\frac{1}{v_{k-1}}} \frac{\partial (\varphi_{k-1}^{\frac{1}{v_{k-1}}})}{\partial x_i}| + |\frac{\partial (x_{i+1}^{*\frac{1}{v_i}})}{\partial x_i}| \times \widetilde{\varphi_{i+1}^{\frac{1}{v_{i+1}}}} \times \widetilde{\eta_{i+1}^{\frac{1}{v_{i+1}}-1}} \times \widetilde{\frac{1}{v_i}} \\
&+ \sum_{j=1}^{k-(i+2)} \widetilde{\varphi_{j+i+1}^{\frac{1}{v_{j+i+1}}}} \times |\frac{\partial \varphi_{i+j}^{\frac{1}{v_{i+j}}}}{\partial x_i}| \times \widetilde{\frac{1}{v_{i+j+1}}} \times \widetilde{\eta_{i+j+1}^{\frac{1}{v_{i+j+1}}-1}} \times \eta_{i+j}^{\frac{1}{v_{i+j}}} \\
&\leqslant \widetilde{\frac{1}{v_i}} \widetilde{\eta_{i+1}^{\frac{1}{v_{i+1}}-1}} \widetilde{\varphi_{i+1}^{\frac{1}{v_{i+1}}}} \times |x_i|^{\frac{1}{v_{i-1}}-1} + \sum_{j=0}^{k-(i+2)} \widetilde{\varphi_{j+i+1}^{\frac{1}{v_{j+i+1}}}} \times |\frac{\partial \varphi_{i+j}^{\frac{1}{v_{i+j}}}}{\partial x_i}| \times \widetilde{\frac{1}{v_{i+j+1}}} \times \widetilde{\eta_{i+j+1}^{\frac{1}{v_{i+j+1}}-1}} \times \eta_{i+j}^{\frac{1}{v_{i+j}}} \\
&\leqslant \widetilde{\frac{1}{v_i}} \widetilde{\eta_{i+1}^{\frac{1}{v_{i+1}}-1}} \widetilde{\varphi_{i+1}^{\frac{1}{v_{i+1}}}} \times |x_i^{\frac{1}{v_{i-1}}} - x_i^{*\frac{1}{v_{i-1}}} + x_i^{*\frac{1}{v_{i-1}}}|^{1-v_{i-1}} \\
&+ \sum_{j=0}^{k-(i+2)} \widetilde{\varphi_{j+i+1}^{\frac{1}{v_{j+i+1}}}} \times |\frac{\partial \varphi_{i+j}^{\frac{1}{v_{i+j}}}}{\partial x_i}| \times \widetilde{\frac{1}{v_{i+j+1}}} \times \widetilde{\eta_{i+j+1}^{\frac{1}{v_{i+j+1}}-1}} \times \eta_{i+j}^{\frac{1}{v_{i+j}}} \\
&\leqslant \widetilde{\frac{1}{v_i}} \widetilde{\eta_{i+1}^{\frac{1}{v_{i+1}}-1}} \widetilde{\varphi_{i+1}^{\frac{1}{v_{i+1}}}} \times (|\eta_i| + |\eta_{i-1}^{\frac{1}{v_{i-1}}} | \varphi_{i-1}^{\frac{1}{v_{i-1}}})^{1-v_{i-1}} \\
&+ \sum_{j=0}^{k-(i+2)} \widetilde{\varphi_{j+i+1}^{\frac{1}{v_{j+i+1}}}} \times |\frac{\partial \varphi_{i+j}^{\frac{1}{v_{i+j}}}}{\partial x_i}| \times \widetilde{\frac{1}{v_{i+j+1}}} \times \widetilde{\eta_{i+j+1}^{\frac{1}{v_{i+j+1}}-1}} \times \eta_{i+j}^{\frac{1}{v_{i+j}}} .
\end{aligned} \tag{31}$$

where we have adopted for more convenience, the notation

$$\begin{cases}
\widetilde{\varphi_j^{\frac{1}{v_j}}} = \varphi_j^{\frac{1}{v_j}} . \varphi_{j-1}^{\frac{1}{v_{j-1}}} \cdots \varphi_{k-1}^{\frac{1}{v_{k-1}}} \\
\widetilde{\eta_j^{\frac{1}{v_j}-1}} = \eta_j^{\frac{1}{v_j}-1} \eta_{j-1}^{\frac{1}{v_{j-1}}-1} \cdots \eta_{k-1}^{\frac{1}{v_{k-1}}-1} \\
\widetilde{\frac{1}{v_j}} = \frac{1}{v_j v_{j-1} \cdots v_{k-1}}
\end{cases}$$

For the last step, we prove that (29) holds for $i = k-1$. Recalling that x_{k-1}^* doesn't depend on x_{k-1}, we calculate the derivative form

$$\begin{aligned}
|\frac{\partial (x_k^{*\frac{1}{v_{k-1}}})}{\partial x_{k-1}}| &\leqslant |\eta_{k-1}^{\frac{1}{v_{k-1}}} \frac{\partial (\varphi_{k-1}^{\frac{1}{v_{k-1}}}(x_1, \cdots, x_{k-1}))}{\partial x_{k-1}}| + |\varphi_{k-1}^{\frac{1}{v_{k-1}}}(x_1, \cdots, x_{k-1}) \frac{\partial (\eta_{k-1}^{\frac{1}{v_{k-1}}})}{\partial x_{k-1}}| \\
&= |\eta_{k-1}^{\frac{1}{v_{k-1}}} \frac{\partial (\varphi_{k-1}^{\frac{1}{v_{k-1}}}(x_1, \cdots, x_{k-1}))}{\partial x_{k-1}}| + \frac{1}{v_{k-1}} \varphi_{k-1}^{\frac{1}{v_{k-1}}} |\eta_{k-1}|^{\frac{1}{v_{k-1}}-1} |\frac{\partial (x_{k-1}^{*\frac{1}{v_{k-2}}})}{\partial x_{k-1}}| \\
&\leqslant (\sum_{j=1}^{k-1} |\eta_j|^{v_i - \alpha r_i}) \tilde{C}_{k,k-1}(x_1, \cdots, x_{k-1})
\end{aligned} \tag{32}$$

with $\tilde{C}_{k,k-1}(x_1, \cdots, x_{k-1})$ is an adequate chosen smooth positive function.

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Finally, using equations (28) and (31) with the fact that $1 \geqslant 1-(v_{i-1}-\alpha r_i) \in \mathbb{Q}_{odd}$, we deduce the next inequality for $i = 1, \cdots, k-1$,

$$\begin{aligned}|\frac{\partial \overset{*}{x}_k^{\frac{1}{v_{k-1}}}}{\partial x_i}\dot{x}_i| \leqslant & \underbrace{[\frac{\widetilde{1}}{v_i}\eta_{i+1}^{\widetilde{\frac{1}{v_{i+1}}-1}}\varphi_{i+1}^{\widetilde{\frac{1}{v_{i+1}}}} \times (|\eta_i| + |\eta_{i-1}|^{\frac{1}{v_{i-1}}}\varphi_{i-1}^{\frac{1}{v_{i-1}}})^{1-v_{i-1}}] \times [\sum_{j=1}^{i+1}|\eta_j|^{v_i}\phi_{i,j,k}(x_1,\cdots,x_k)]}_{A} \\ & \underbrace{+[\sum_{j=0}^{k-(i+2)}\varphi_{j+i+1}^{\widetilde{\frac{1}{v_{j+i+1}}}} \times |\frac{\partial\varphi_{i+j}^{\frac{1}{v_{i+j}}}}{\partial x_i}| \times \frac{\widetilde{1}}{v_{i+j+1}} \times \eta_{i+j+1}^{\widetilde{\frac{1}{v_{i+j+1}}-1}} \times \eta_{i+j}^{\frac{1}{v_{i+j}}}] \times [\sum_{j=1}^{i+1}|\eta_j|^{v_i}\phi_{i,j,k}(x_1,\cdots,x_k)]}_{B}.\end{aligned}$$

We estimate the parts of the equations A and B by the following

$$\begin{aligned}A & \leqslant \frac{\widetilde{1}}{v_i}\eta_{i+1}^{\widetilde{\frac{1}{v_{i+1}}-1}}\varphi_{i+1}^{\widetilde{\frac{1}{v_{i+1}}}} \times \Sigma_{j=1}^{i+1}[(|\eta_i| + |\eta_{i-1}|^{\frac{1}{v_{i-1}}}\varphi_{i-1}^{\frac{1}{v_{i-1}}})^{1-v_{i-1}}]|\eta_j|^{v_i}\phi_{i,j,k}(x_1,\cdots,x_k)] \\ & \leqslant \Sigma_{j=1}^{i+1}[(|\eta_i| + |\eta_{i-1}|\varphi_{i-1})^{1-v_{i-1}+\alpha r_i} + |\eta_j|^{1-v_i+\alpha r_i}]\breve{\breve{\phi}}_{i,j,k}(x_1,\cdots,x_k) \\ & \leqslant \Sigma_{j=1}^{i+1}|\eta_{i-1}|^{1-v_{i-1}+\alpha r_i}\breve{\phi}_{i,j,k}(x_1,\cdots,x_k) \text{ and}\end{aligned} \tag{33}$$

$$B \leqslant \Sigma_{j=0}^{k-(i+2)}|\eta_{i+j}|^{1-v_i+\alpha r_i}\tilde{\phi}_{i,j,k}(x_1,\cdots,x_{i+1}). \tag{34}$$

Then, by adding (33) and (34), we can write :

$$\left|\frac{\partial \overset{*}{x}_k^{\frac{1}{v_{k-1}}}}{\partial x_i}\dot{x}_i\right| \leqslant \Sigma_{j=1}^{k-1}(|\eta_j|^{1-v_{i-1}+\alpha r_i}\bar{\phi}_{i,j,k}(x_1,\cdots,x_k)) \tag{35}$$

where $\breve{\breve{\phi}}_{i,j,k}(x_1,\cdots,x_k)$, $\breve{\phi}_{i,j,k}(x_1,\cdots,x_k)$,$\tilde{\phi}_{i,j,k}(x_1,\cdots,x_k)$ and $\bar{\phi}_{i,j,k}(x_1,\cdots,x_k)$ are positive smooth functions. □

Using both Lemma 3 and 7, we estimate the next term of the inequality (21):

$$\begin{aligned}|\eta_k^{w_{k-1}}f_k(x_1,\cdots,x_k)| & \leqslant |\eta_k^{w_{k-1}}|(|x_1| + \cdots + |x_k|)\gamma_k(x_1,\cdots,x_k) \\ & \leqslant |\eta_k^{w_{k-1}}|[(|x_1| + |x_2 - x_2^*| + \cdots + |x_k - x_k^*|) + (|x_2^*| + \cdots + |x_k^*|)] \times \gamma_k(x_1,\cdots,x_k).\end{aligned} \tag{36}$$

Using the inequality $|x_{i+1} - x_{i+1}^*| \leqslant 2^{1-v_i}|x_{i+1}^{\frac{1}{v_i}} - \overset{*}{x}_{i+1}^{\frac{1}{v_i}}|^{v_i} = 2^{1-v_i}|\eta_i|^{v_i}$. We obtain,

$$|\eta_k^{w_{k-1}}f_k(x_1,\cdots,x_k)| \leqslant |\eta_k^{w_{k-1}}|[\Sigma_{j=0}^{k-1}2^{1-v_j}|\eta_{j+1}|^{v_j} + \Sigma_{j=1}^{k-1}|\eta_j|^{v_j}\varphi_j^{v_j}(x_1,\cdots,x_j)] \times \gamma_k(x_1,\cdots,x_k). \tag{37}$$

Knowing that $\forall i = 1,\cdots,n-1,\ r_i \leqslant \frac{1}{\alpha}min\{v_0,\cdots,v_i\}$, we have

$$\begin{aligned}|\eta_k^{w_{k-1}}f_k(x_1,\cdots,x_k)| & \leqslant |\eta_k^{w_{k-1}}|[\Sigma_{j=0}^{k-1}2^{1-v_j}|\eta_{j+1}|^{\alpha r_k}(1 + |\eta_{j+1}^{1-\alpha r_k}|) \\ & + \Sigma_{j=1}^{k-1}|\eta_j|^{\alpha r_k}\varphi_j^{v_j}(x_1,\cdots,x_j)(1 + |\eta_j|^{1-\alpha r_k})] \times \gamma_k(x_1,\cdots,x_k).\end{aligned} \tag{38}$$

We impose in addition, when constructing to the smooth positive function $\tilde{\gamma}_k(x_1,\cdots,x_k)$, to satisfy the two conditions below:

$$\begin{cases}\tilde{\gamma}_k(x_1,\cdots,x_k) \geqslant 2^{2-\alpha r_k}(1 + |\eta_{j+1}|^{1-\alpha r_k})\gamma_k(x_1,\cdots,x_k) \text{ and} \\ \tilde{\gamma}_k(x_1,\cdots,x_k) \geqslant 2(1 + |\eta_j|^{1-\alpha r_k})\varphi_j^{v_j}(x_1,\cdots,x_j)\gamma_k(x_1,\cdots,x_k).\end{cases} \tag{39}$$

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Therefore, we have for $j = 0, \cdots, k-1$

$$\begin{array}{ll} |\eta_k^{w_{k-1}} f_k(x_1, \cdots, x_k)| & \leqslant |\eta_k^{w_{k-1}}| \sum_{j=1}^{k} |\eta_j|^{\alpha r_k} \tilde{\gamma}_k(x_1, \cdots, x_k) \\ & \leqslant \frac{1}{4} \sum_{j=1}^{k} |\eta_j^{\alpha r_k + w_{k-1}}| + |\eta_k|^{\alpha r_k + w_{k-1}} \nu_k(x_1, \cdots, x_k) \\ & \leqslant \frac{1}{4} \sum_{j=1}^{k} |\eta_j^{\alpha r_j + w_{j-1}}| + |\eta_k|^{\alpha r_k + w_{k-1}} \nu_k(x_1, \cdots, x_k) \end{array} \tag{40}$$

where $\nu_k(x_1, \cdots, x_k)$ is a smooth positive function. Finally for the last term of (21), there exists a smooth positive function $\xi_k(x_1, \cdots, x_k)$ such that

$$\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \dot{x}_i \leqslant \frac{1}{4} (\sum_{i=1}^{k-1} \eta_i^{w_{i-1} + \alpha r_i}) + \eta_k^{w_{k-1} + \alpha r_k} \xi_k(x_1, \cdots, x_k). \tag{41}$$

To prove this claim, we start with the next inequality:

$$\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \dot{x}_i \leqslant (w_{k-1}) \sum_{i=1}^{k-1} |\frac{\partial (x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i} \dot{x}_i| \, |\int_{x_k^*}^{x_k} (s^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}})^{w_{k-1}-1} ds|. \tag{42}$$

Using the fact that $w_{k-1} \in [0, 1]$ and Lemma 6, we can write

$$\begin{array}{ll} (w_{k-1}) | \int_{x_k^*}^{x_k} (s^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}})^{w_{k-1}-1} ds| & \leqslant |x_k - x_k^*| |x_k^{\frac{1}{v_{k-1}}} - x_k^{*\frac{1}{v_{k-1}}}|^{w_{k-1}-1} \\ & \leqslant 2^{1-v_{k-1}} |\eta_k|^{w_{k-1}-1} |\eta_k|^{v_{k-1}} = |\eta_k|^{w_{k-1}+v_{k-1}-1}. \end{array} \tag{43}$$

Combining the result obtained in (43), with Proposition 10 and Lemma 7 yields

$$\begin{array}{ll} \sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \dot{x}_i & \leqslant |\eta_k|^{w_{k-1}+v_{k-1}-1} \sum_{i=1}^{k-1} |\frac{\partial (x_k^{*\frac{1}{v_{k-1}}})}{\partial x_i} \dot{x}_i| \\ & \leqslant \sum_{i=1}^{k-1} (\sum_{j=1}^{k-1} |\eta_k|^{w_{k-1}+v_{k-1}-1} |\eta_j|^{1-v_{i-1}+\alpha r_i}) \bar{\phi}_{i,j,k}(x_1, \cdots, x_k) \\ & \leqslant |\eta_k|^{w_{k-1}+v_{k-1}-1} (\sum_{j=1}^{k-1} |\eta_j|^{1-v_{i-1}+\alpha r_i}) (\sum_{i=1}^{k-1} \bar{\phi}_{i,j,k}(x_1, \cdots, x_k)) \\ & \leqslant \frac{1}{4} \sum_{i=1}^{k-1} |\eta_i|^{w_{i-1}+\alpha r_i} + |\eta_k|^{v_{k-1}+\alpha r_k} \xi_k(x_1, \cdots, x_k). \end{array} \tag{44}$$

where $\xi_k(x_1, \cdots, x_k)$ is a positive smooth function. Substituting the estimates (23), (40) and (41) into (21) yields:

$$\begin{array}{ll} \dot{V}_k(x_1, \cdots, x_k) & \leqslant -(n-k+1)(\sum_{i=1}^{k-1} \eta_i^{w_{i-1}+\alpha r_i}) + \eta_k^{w_{k-1}} x_{k+1}^* + \eta_k^{w_{k-1}} (x_{k+1} - x_{k+1}^*) \\ & +\eta_k^{w_{k-1}+\alpha r_k} (c_{k2} + \nu_k(x_1, \cdots, x_k) + \xi_k(x_1, \cdots, x_k)). \end{array}$$

A plausible choice of a continuous *Hölderian* virtual controller x_{k+1}^* is then given by:

$$x_{k+1}^* = -\eta_k^{\alpha r_k} [(n-k+1) + c_{k1} + \xi_k(x_1, \cdots, x_k) + \nu_k(x_1, \cdots, x_k)] = -\eta_k^{\alpha r_k} \varphi_k^{\alpha r_k}(x_1, \cdots, x_k).$$

Then, the correspondent Lyapunov function satisfies:

$$\dot{V}_k(x_1, \cdots, x_k) \leqslant -(n-k+1)[\sum_{i=1}^{k} \eta_i^{w_{i-1}+\alpha r_i}] + \eta_k^{w_{k-1}} (x_{k+1} - x_{k+1}^*).$$

which completes the proof of the inductive step. Using the inductive argument above, we conclude that at the nth step, there exists a non-Lipschitz continuous state feedback control law of the form

$$u = x_{n+1}^* = -\eta_n^{\alpha r_n} \varphi_n^{\alpha r_n}(x_1, \cdots, x_n)$$

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where $\varphi_n(x_1,\cdots,x_n)$ is a C^1 positive function and a C^1 positive definite and proper Lyapunov function $V_n(x_1,\cdots,x_n)$ constructed by the inductive procedure verifying the inequality:

$$\dot{V}_n(x_1,\cdots,x_n) \leqslant -\sum_{i=1}^{n} \eta_i^{w_{i-1}+\alpha r_i} = -\sum_{i=1}^{n} (x_i^{\frac{1}{v_{i-1}}} - x_i^{*\frac{1}{v_{i-1}}})^{w_{i-1}+\alpha r_i}.$$

Using Theorem 1, one can verify that the system (3) is globally finite-time stable [18]. As a consequence and under the necessary and sufficient conditions of Theorem 2, the closed-loop homogeneous nonlinear system in the form (3) is finite-time stable.

4. Simulations of the controller

Different results of finite-time controller designing have been already obtained with different techniques and methodologies [5,7,18,20,21,34]. Since the proposed Theorem gives a new form for the Lyapunov function, the effectiveness of the algorithm developped in this paper is demonstrated with a design example and a computer simulation for the three commun cases: the single, double and triple integrators. For the first step $n = 1$ which defined the single integrator, we choose to work with the C^1 positive definite Lyapunov function $V_1(x) = \frac{1}{2}x^2$. Also, we choose to take $\alpha = \frac{1}{3}$, $v_0 = 1$, $w_0 = \frac{1}{3}$ and $r_1 = 1$, which leads to a virtual control with the following form:

$$\begin{aligned} & x_2^* = -x_1^{\frac{1}{3}}(1 + x_1^{\frac{5}{3}})^{\frac{1}{3}} \text{ such that} \\ & \dot{V}_1(x_1) \leqslant -x_1^{\frac{1}{3}}(x_2 + x_1^{\frac{1}{3}}(1 + x_1^{\frac{5}{3}})^{\frac{1}{3}}) - x_1^{\frac{2}{3}}. \end{aligned} \tag{45}$$

It should be noted that the feedback law $u = x_2^*$ stabilizes the simple integrator $\dot{x}_1 = u$, as shown in figure 1, with the initial condition $x_1(0) = 0.5$.

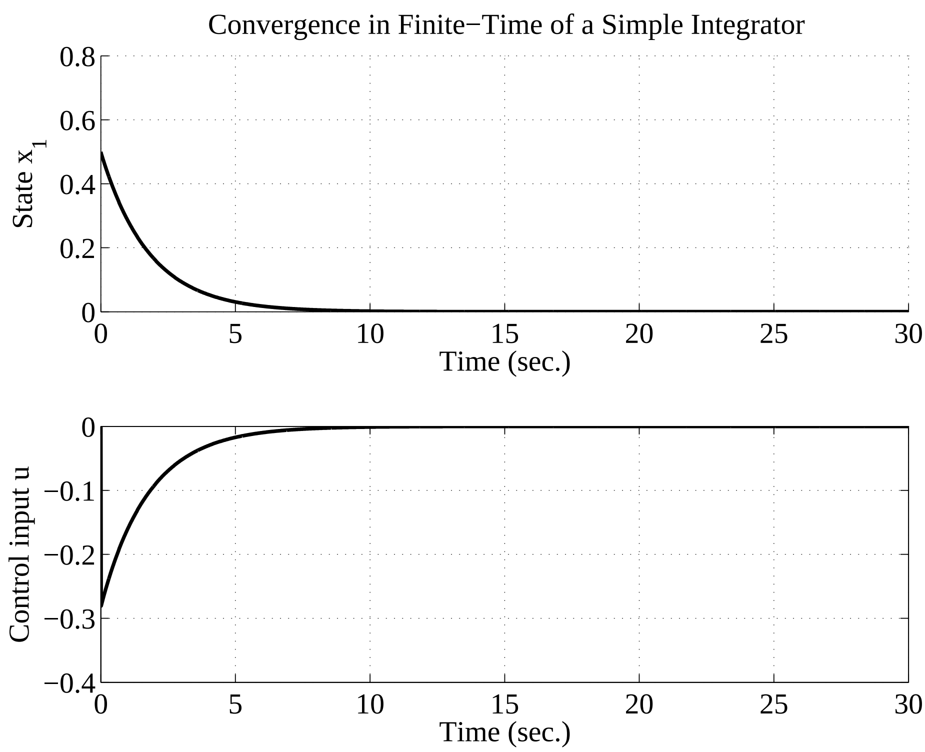

Figure 1. Initial condition response of a finite-time stabilized simple integrator

In the following steps, we notice that our main Theorem (2), gives us two frame relations to help us to find v_1 and v_2 as they have to satisfy:

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$$\begin{cases} w_1 = w_0 + \alpha(r_1 - r_2) \leqslant v_1 \leqslant v_0 + \alpha(r_2 - r_1) \text{ and} \\ w_2 = w_0 + \alpha(r_2 - r_3) \leqslant v_2 \leqslant v_0 + \alpha(r_3 - r_1). \end{cases} \tag{46}$$

For the second step $n = 2$, we choose to take $r_2 = \frac{1}{3}$, $v_1 = \frac{10}{11}$ and $w_1 = \frac{5}{9}$. The corresponding virtual control x_3^*, is given by the form:

$$\begin{cases} x_3^* = -x_2^{\frac{11}{10}} - (x_1 + x_1^{\frac{8}{3}})^{\frac{11}{30}} \\ \text{under} \left| \begin{array}{l} \dot{V}_2(x_1, x_2) \leqslant -[x_1^{\frac{2}{3}} + |x_2^{\frac{11}{10}} - x_2^{*\frac{11}{10}}|^{\frac{2}{3}}] + |x_2^{\frac{11}{10}} - x_2^{*\frac{11}{10}}|^{\frac{5}{9}}(x_2 - x_2^*) \text{ with} \\ V_2(x_1, x_2) = \frac{x_1^2}{2} + \int_{x_2^*}^{x_2} (s^{\frac{11}{10}} - x_2^{*\frac{11}{10}})^{\frac{5}{9}} \mathrm{d}s. \end{array} \right. \end{cases} \tag{47}$$

As shown in the figure 2, this feedback law stabilizes as well the double integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, in response to the initial conditions $x_1(0) = 1.5,\ x_2(0) = 2$.

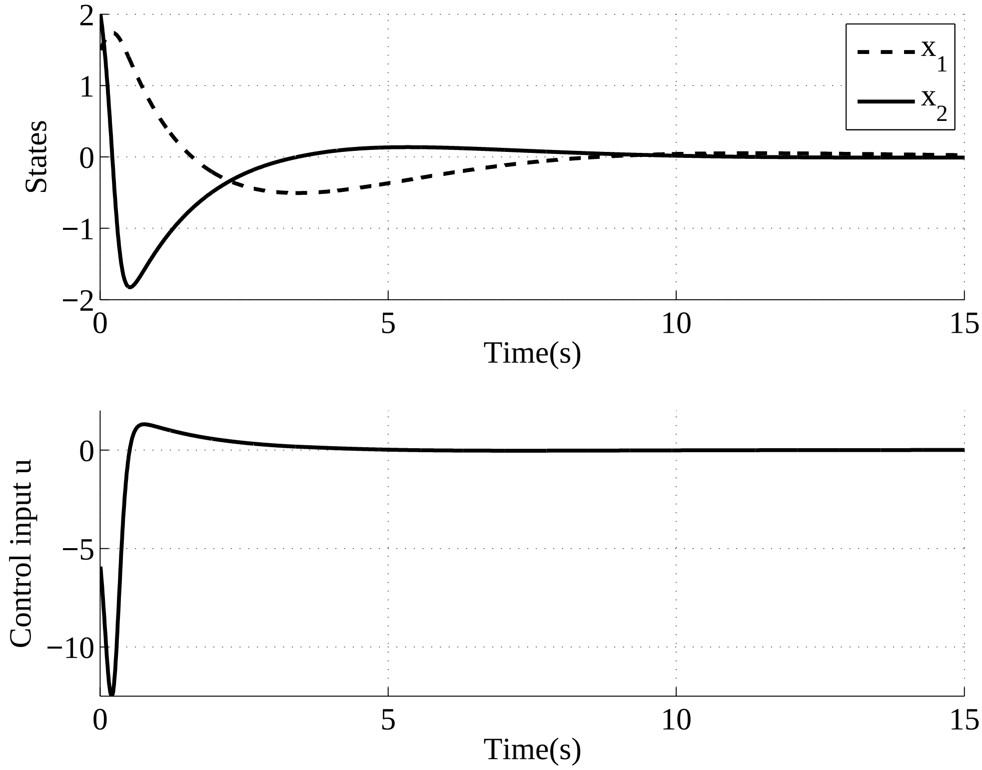

Figure 2. Initial condition response of a finite-time stabilized double integrator

For the final step $n = 3$, we choose to take $r_3 = \frac{5}{7}$, $v_2 = \frac{2}{9}$ and $w_2 = \frac{13}{63}$. The corresponding virtual control x_4^*, is given by the form:

$$\begin{cases} x_4^* = -(x_2^{1.1} + (x_1 + x_1^{\frac{8}{3}})^{4.5}) - x_3^{1.1} \\ \text{under} \left| \begin{array}{l} \dot{V}_3(x_1, x_2, x_3) \leqslant -[x_1^{\frac{2}{3}} + |x_2^{\frac{11}{10}} - x_2^{*\frac{11}{10}}|^{\frac{2}{3}} + |x_3^{\frac{9}{2}} - x_3^{*\frac{9}{2}}|^{\frac{28}{63}}] + |x_3^{\frac{9}{2}} - x_2^{*\frac{9}{2}}|^{\frac{13}{63}}(x_3 - x_3^*) \text{ with} \\ V_3(x_1, x_2, x_3) = \frac{x_1^2}{2} + V_2(x_1, x_2) + \int_{x_3^*}^{x_3} (s^{\frac{9}{2}} - x_2^{*\frac{9}{2}})^{\frac{13}{63}} \mathrm{d}s. \end{array} \right. \end{cases} \tag{48}$$

Figure 3 shows that the triple integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$ is stabilized in response to the initial conditions $x_1(0) = 0.5,\ x_2(0) = 0,\ x_3(0) = -0.5$.

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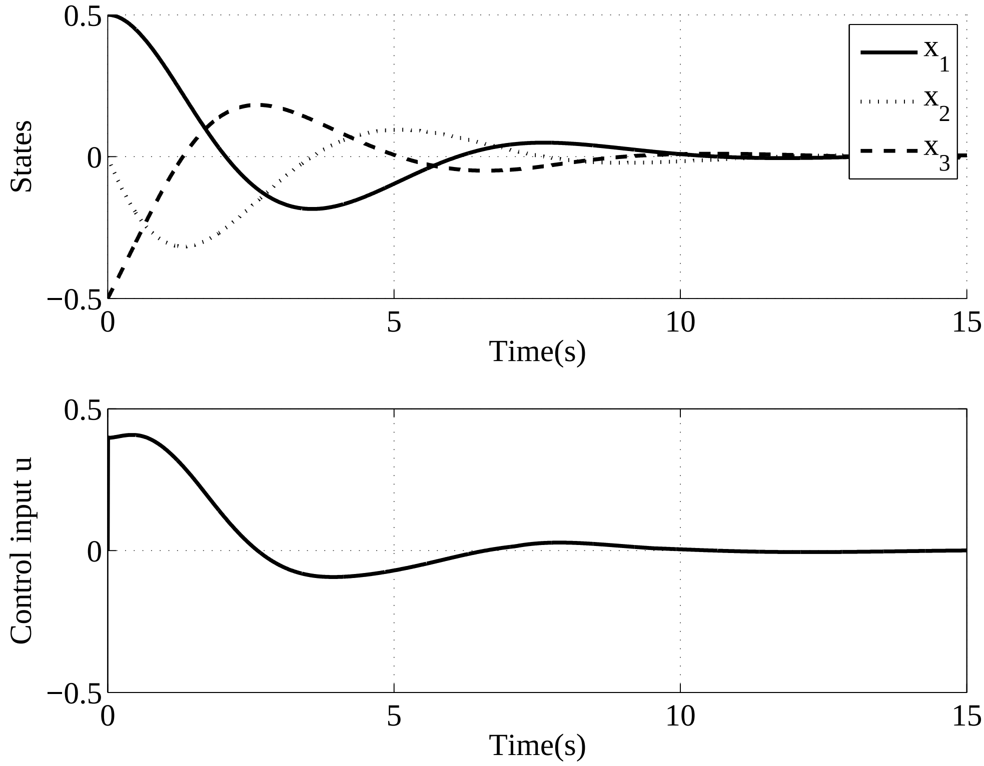

Figure 3. Initial condition response of a finite-time stabilized triple integrator

5. Conclusion

In this paper, we have extended the notion of finite-time stability to homogeneous non-Lipschitz systems, especially in the case of the lower-triangular non-linear systems, using a systematic recursive algorithm, achieving the design of virtual continuous non-Lipschitz finite-time stabilizing controllers as well as C^1 control Lyapunov functions, under appropriate conditions. The advantage of this algorithm is that it uses a recursive relation between the dilation coefficients and the *Hölder*-exponents, to determine step by step, the virtual *Hölder* feedback and the C^1 Lyapunov function, performing the finite-time stabilization task of the considered class of systems.

Finally, we have demonstrated the effectiveness and convenience of the proposed procedure illustrated by computer simulations on a simple, double and a triple integrator models, insuring finite-time stability and Lyapunov functions derterminations.

In Further works, we will be interested by more developed class of non-linear systems such as fractional systems, unknown parameters systems and ones subject to disturbance. Since some applications could be found in robotic control field, we will focus on the Human-Robot interaction model as shown in [14] and on the interesting problematic of finite-time stability frontier under critical values of impedance like damping and mass parameters, especially in the case where the states of the system are subject to disturbance.

References

1. Andreini A, Bacciotti A, Stefani G (1988) Global stabilizability of homogeneous vector fields of odd degree, Systems Control Letters(10):251–256
2. Bacciotti A (1992) Local stabilizability of nonlinear control systems. Singapore: World Scientific
3. Back J, Cheong S G, Shim H, Seo J H (2007) Nonsmooth feedback stabilizer for strict-feedback nonlinear systems that may not be linearizable at the origin. Springer-Verlag Systems & Control Letters 56(11):742–752
4. Bhat S P, Bernstein D S (1997) Finite-time stability of homogeneous systems. In Proceedings of the American Control Conference vol. 4: 2513–2514

5. Bhat S P, Bernstein D S (1998) Continuous finite-time stabilization of the translational and rotational double intergrator. IEEE Transactions on Automatic Control 43(5):678–682
6. Bhat S P, Bernstein D S (2000) Finite-time stability of continuous autonomous systems. SIAM Jouranal on Control and Optimization 38(3):751–766
7. Bhat S P, Bernstein D S (2005) Geometric homogeneiy with applicatons to finite-time stability. Springer-Verlag Mathematics of Control, Signals, and Systems 17(2):101–127
8. Bhat S P, Bernstein D S (1995) Lyapunov analysis of finite-time differential equations. American Control Conference 3(3):1831–1832
9. Bo Yang (2006), Output Feedback Control for Nonlinear Sytems with Unstabilizable and Undetectable Linearization. Thesis, Case Western Reserve University
10. Byrnes C I, Isidori A (1989) New results and examples in nonlinear feedback stabilization. Elsevier Systems & Control Letters 12(5):437–442
11. Byrnes C I, Isidori A (1991) Asymptotic stabilization of minimum phase nonlinear systems. IEEE Transaction on Automatic Control 36(10):1122–1137
12. Coron J M, Praly L (1991) Adding an integrator for the stabilization problem,Systems Control Lett.,17 : 89–104
13. Ding S, Li S, Li Q (2009) Stability analysis for a second order continuous finite-time control system subject to a disturbance, Springer-Verlag Journal of Control Theory and Applications 7(3):271–276
14. Duchaine V, Gosselin C M (2008) Investigation of Human-Robot interaction stability using Lyapunov theory. IEEE International Conference in Robotics and Automation, Pasadena, USA : 2189 - 2194
15. Hong Y, Huang J, Xu Y (2001) On an output feedback finite-time stabilization problem. IEEE Transactions on Automatic Control 46:305–309
16. Hong Y, (2002) Finite-time stabilization and stabilizability of a class of controllable systems. System and Control Letters 46:231–236
17. Hermes H (1991). Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. in: S. Elaydi (Ed.),Differential equations stability and control.Marcel Dekker, New York: 249–260
18. Huang X, Lin W, Yang B (2005) Global finite-time stabilization of a class of uncertain nonlinear systems. Springer-Verlag Automatica 41(5):881–888
19. Isidori A (1999) Nonlinear control systems 2. Springer-Verlag, New York
20. Jammazi C (2008) Finite-time partial stabilizability of chained systems. C. R. Acad. Sci. Paris, Ser. I 346: 975–980
21. Jammazi C (2010) On a sufficient condition for finite-time partial stability and stabilization: Applications. IMA Journal of Mathematical Control and information 27(1):29–56
22. Kawski M (1990) Homogeneous stabilizing feedback laws. Control Theory and Advanced Technology, 6: 497–516
23. Lin W, Qian C (2000) Adding one power integrator: A tool for global stabilization of higher order lower-triangular system. Elsevier Systems & Control Letters 39(3):339–351
24. Nakamura H, Yamashita Y, Nishitani H (2005) Homogeneous eigenvalue analysis of homogeneous systems. 16th IFAC World Congress.
25. Nakamura H, Yamashita Y, Nishitani H (2007) Stabilization of homogeneous systems using implicit control Lyapunov functions. 7th IFAC Symposium on Nonlinear Control Systems, Pretoria, South Africa, vol. 7(1):561–566
26. Palendo J, Qian C (2005) A generalized framework for global output feedback stabilization of genuinely nonlinear systems. 44th IEEE Conference on Decision and Control: 2646–2651
27. Praly L, Andrea-Novel B, Coron J M (1991) Lyapunov design of stabilizing controllers for cascaded systems. IEEE Transaction on Automatic Control 36(10):1177-1181
28. Qian C, Lin W (2001) Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. Elsevier Systems & Control Letters 42(3):185–200
29. Qian C, Lin W (2001) A continuous feedback approach to global strong stabilization of nonlinear systems. IEEE Transaction on Automatic Control 46(7):1061–1079
30. Qian C (2005) A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems. In Proceedings of 2005 American Control Conference
31. Qian C, Lin W (2000) Non-Smooth stabilizers for nonlinear systems with uncontrollable unstable linearization. Proceeding of the 39th IEEE Conference on Decision and Control, Sydney, Australia vol. 2:1655-1660
32. Qian C, Lin W (2001) A continuous feedback approach to global strong stabilization of nonlinear systems. IEEE Transactions on Automatic Control, 46: 1061–1079

33. Qian C, Lin W (2001) Non-Lipschitz continuous stabilizer for nonlinear systems with uncontrollable unstable linearization. System and Control Letters, 42: 185–200
34. Shen Y, Huang Y (2010) Global finite-time stabilisation for a class of nonlinear systems, Taylor & Francis, International Journal of Systems Science: 1–6
35. Shihong D, Shihua L, Wei Xing Z (2012) Nonsmooth stabilization of a class of non linear cascaded systems. IEEE Trans. Automat. vol.48(10): 2597–2606
36. Xianqing H, Wei L, Bo Y (2005) Global finite-time stabilization of a class of uncertain nonlinear systems. IEEE Trans. Automat.A 41: 881–888