1   *Article*

2 Dispersion Operators Algebra and Linear Canonical
3 Transformations

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13   **Abstract:** This work intends to present a study on relations between a Lie algebra called dispersion
14   operators algebra, linear canonical transformation and a phase space representation of quantum
15   mechanics that we have introduced and studied in previous works. The paper begins with a brief
16   recall of our previous works followed by the description of the dispersion operators algebra which
17   is performed in the framework of the phase space representation. Then, linear canonical
18   transformations are introduced and linked with this algebra. A multidimensional generalization of
19   the obtained results is given.

20   **Keywords:** Dispersion operator; Lie algebra; Linear Canonical Transformation; Quantum theory;
21   Phase space representation

22

23 1. Introduction

24   The present work can be considered as a part of a series of studies concerning phase space, linear
25 canonical transformation and quantum theory that we have started and performed in our previous
26 works [1],[2]. Through history and scientific literature, it can be remarked that the description of
27 phase space in quantum theory and related problems like study of canonical transformations are
28 among of the most interesting subjects. We may quote many works since the beginning of quantum
29 physics and until nowadays; for instance we have [3-14]. A well known approach to tackle these
30 problems is based on the utilization of the Wigner distribution but other approaches may be also
31 considered. Our work is in this framework.

32   Through all the paper, we use the natural system of units in which the light speed c and the
33 reduced Planck constant $\hbar$ are set to unity ($c = 1$, $\hbar = 1$). We use also bold faced letter to denote
34 operators and normal letter for the eigenvalues. The matricial and tensorial notations used in the
35 section 4 are those of the reference [15].

36   The main result that we have obtained from [1] was the establishment of a phase space
37 representation of quantum mechanics which takes into account the uncertainty relation. It is
38 based on the introduction of quantum states, denoted $|n, X, P, \Delta p\rangle$, defined by the means values X, P
39 and statistical dispersions $(\Delta x_n)^2 = (2n + 1)(\Delta x)^2, (\Delta p_n)^2 = (2n + 1)(\Delta p)^2$ of coordinate x and
40 momentum p. Δx and Δp satisfying the relation $(\Delta x)(\Delta p) = \frac{1}{2}$.

41   For the sake of simplicity of writing, we will use the notation $a = \Delta x$, $\not{b} = \Delta p$, $\mathcal{A} = (a)^2 =$
42 $(\Delta x)^2$, and $\mathcal{B} = (\not{b})^2 = (\Delta p)^2$. For instance, the state $|n, X, P, \Delta p\rangle$ will be denoted by $|n, X, P, \not{b}\rangle$. The
43 wave functions corresponding to a state $|n, X, P, \not{b}\rangle = |n, X, P, \Delta p\rangle$ respectively in coordinate and
44 momentum representation are the Harmonic Hermite-Gaussian functions denoted by
45

46
$$\langle x|n, X, P, \not{b}\rangle = \varphi_n(x, X, P, \not{b}) = \varphi_n(x, X, P, \Delta p)$$

47   and their Fourier transform denoted by

48
$$\langle p|n,X,P,\textit{\textbf{b}}\rangle = \tilde{\varphi}_n(p,X,P,\textit{\textbf{b}}) = \tilde{\varphi}_n(p,X,P,\Delta p)$$

49   These functions were introduced and used in our previous works [1], [2], [16]. Explicitly we have

50
$$\langle x|n,X,P,\textit{\textbf{b}}\rangle = \varphi_n(x,X,P,\textit{\textbf{b}}) = \frac{H_n(\frac{x-X}{\sqrt{2\mathcal{A}}})}{\sqrt{2^n n!\sqrt{2\pi\mathcal{A}}}}\,e^{-\frac{(x-X)^2}{4\mathcal{A}}+iPx} \tag{1}$$

51
$$\langle p|n,X,P,\textit{\textbf{b}}\rangle = \tilde{\varphi}_n(p,X,P,\textit{\textbf{b}}) = \frac{H_n(\frac{p-P}{\sqrt{2\mathcal{B}}})}{\sqrt{2^n n!\sqrt{2\pi\mathcal{B}}}}\,e^{-\frac{(p-P)^2}{4\mathcal{B}}-iX(p-P)} \tag{2}$$

52   As usual H_n is the Hermite polynomial of degree n. $\Delta x = a$, $\Delta p = \textit{\textbf{b}}$, $\mathcal{A} = (\Delta x)^2 = (a)^2 =$ and
53    $\mathcal{B} = (\Delta p)^2 = (\textit{\textbf{b}})^2$ satisfying the relations

54
$$(\Delta x)(\Delta p) = a\textit{\textbf{b}} = \frac{1}{2} \tag{3}$$

55
$$(\Delta x)^2(\Delta p)^2 = (a)^2(\textit{\textbf{b}})^2 = \mathcal{A}\mathcal{B} = \frac{1}{4} \tag{4}$$

56   The state $|n,X,P,\textit{\textbf{b}}\rangle$is the eigenstate of the coordinate and momentum dispersion operators $\Sigma_p$and
57    $\Sigma_x$respectively with the eigenvalues$(2n+1)(\textit{\textbf{b}})^2$ and $(2n+1)(a)^2$. If we denote respectively $\textbf{\textit{p}}$
58
59   and $\textbf{\textit{x}}$the momentum and coordinate operators [17] we have [1]

60
$$\Sigma_p = \frac{1}{2}\left[\frac{(\textbf{\textit{p}}-P)^2}{(\textit{\textbf{b}})^2} + \frac{(\textbf{\textit{x}}-X)^2}{(a)^2}\right](\textit{\textbf{b}})^2 = \frac{1}{2}[(\textbf{\textit{p}}-P)^2 + 4(\mathcal{B})^2(\textbf{\textit{x}}-X)^2] \tag{5}$$

61
$$\Sigma_x = \frac{1}{2}\left[\frac{(\textbf{\textit{p}}-P)^2}{(\textit{\textbf{b}})^2} + \frac{(\textbf{\textit{x}}-X)^2}{(a)^2}\right](a)^2 = \frac{1}{2}[4(\mathcal{A})^2(\textbf{\textit{p}}-P)^2 + (\textbf{\textit{x}}-X)^2] \tag{6}$$

62
$$\Sigma_p|n,X,P,\textit{\textbf{b}}\rangle = (2n+1)(\textit{\textbf{b}})^2|n,X,P,\textit{\textbf{b}}\rangle = (2n+1)\mathcal{B}|n,X,P,\textit{\textbf{b}}\rangle \tag{7}$$

63
$$\Sigma_x|n,X,P,\textit{\textbf{b}}\rangle = (2n+1)(a)^2|n,X,P,\textit{\textbf{b}}\rangle = (2n+1)\mathcal{A}|n,X,P,\textit{\textbf{b}}\rangle \tag{8}$$

64   In our work [2], it was remarked that a link may be established between linear canonical
65   transformation and the phase space representation of quantum mechanics. In the present work, we
66   show that this link can be described properly with the introduction of a Lie algebra that we may call
67   dispersion operators algebra. This Lie algebra is generated by the dispersion operators and some
68   other operators related to them. We have remarked during the design of the present work that
69   operators analogous to these operators have been already introduced and studied previously in
70   various works on linear canonical transformation [18]. As mentioned, our main contribution in this
71   paper is the exploitation of some properties of these operators in the introduction of the dispersion
72   operators algebra to describe properly the relations that can be established between this algebra, the
73   phase space representation and linear canonical transformation. We introduce also a generalization
74   of the results for the case of multidimensional theory.
75

76   **2. Dispersion Operators Algebra**

77   *2.1. Definitions and properties*

78   Let us consider the three hermitian operators

79
$$\begin{cases} \beth^+ = \Sigma_p = \frac{1}{2}[(\textbf{\textit{p}}-P)^2 + 4(\mathcal{B})^2(\textbf{\textit{x}}-X)^2] \\ \beth^- = \frac{1}{2}[(\textbf{\textit{p}}-P)^2 - 4(\mathcal{B})^2(\textbf{\textit{x}}-X)^2] \\ \beth^\times = \mathcal{B}[(\textbf{\textit{p}}-P)(\textbf{\textit{x}}-X) + (\textbf{\textit{x}}-X)(\textbf{\textit{p}}-P)] \end{cases} \tag{9}$$

80   Using the commutation relation of $\textbf{\textit{x}}$ and $\textbf{\textit{p}}$,$[\textbf{\textit{x}},\textbf{\textit{p}}]_- = i$, we can deduce the following commutation
81   relations

82
$$\begin{cases} [\beth^+,\beth^-]_- = 4i\mathcal{B}\beth^\times \\ [\beth^-,\beth^\times]_- = -4i\mathcal{B}\beth^+ \\ [\beth^\times,\beth^+]_- = 4i\mathcal{B}\beth^- \end{cases} \tag{10}$$

83   Let $\mathfrak{g}$ be the complex vectorial space generated by the linear combination

84
$$\beth = \lambda\beth^+ + \mu\beth^- + \nu\beth^\times$$

85   of the three operators $\beth^+, \beth^-$ and $\beth^\times$ with λ, μ, ν three complex numbers. It can be deduced easily
86   form the relation (10) that $\mathfrak{g}$ is a complex Lie algebra of three dimensions. For any two elements

87
$$\beth_1 = \lambda_1\beth^+ + \mu_1\beth^- + \nu_1\beth^\times$$

88
$$\beth_2 = \lambda_2\beth^+ + \mu_2\beth^- + \nu_2\beth^\times$$

89   of $\mathfrak{g}$, it may be shown that

90
$$\beth_3 = [\beth_1, \beth_2]_- = \lambda_3\beth^+ + \mu_3\beth^- + \nu_3\beth^\times$$

91   with

92
$$\begin{cases} \lambda_3 = -4i\mathcal{B}(\mu_1\nu_2 - \nu_1\mu_2) \\ \mu_3 = 4i\mathcal{B}(\nu_1\lambda_2 - \nu_2\lambda_2) \\ \nu_3 = 4i\mathcal{B}(\lambda_1\mu_2 - \lambda_2\mu_1) \end{cases}$$

93   is also an element of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ may be called dispersion operators algebra.

94   For future use and convenience, we introduce the following operators

95
$$\begin{cases} z^- = \frac{1}{\sqrt{2}}[(p - P) - 2i\mathcal{B}(x - X)] \\ z^+ = \frac{1}{\sqrt{2}}[(p - P) + 2i\mathcal{B}(x - X)] = (z^-)^\dagger \end{cases} \Leftrightarrow \begin{cases} (p - P) = \frac{1}{\sqrt{2}}(z^- + z^+) \\ (x - X) = \frac{i}{2\sqrt{2}\mathcal{B}}(z^- - z^+) \end{cases} \quad (11)$$

96
$$\begin{cases} \varkappa = \frac{(x - X)}{\sqrt{2}(\Delta x)} = \frac{(x - X)}{\sqrt{2}a} = \frac{(x - X)}{\sqrt{2}\mathcal{A}} = \sqrt{2}(\Delta p)(x - X) = \sqrt{2}b(x - X) = \sqrt{2}\mathcal{B}(x - X) \\ \varpi = \frac{(p - P)}{\sqrt{2}(\Delta p)} = \frac{(p - P)}{\sqrt{2}b} = \frac{(p - P)}{\sqrt{2}\mathcal{B}} = \sqrt{2}(\Delta x)(p - P) = \sqrt{2}a(p - P) = \sqrt{2}\mathcal{A}(p - P) \end{cases} \quad (12)$$

97
$$\begin{cases} \mathbf{z}^- = \frac{z^-}{\sqrt{2\mathcal{B}}} = \frac{1}{\sqrt{2}}\left[\frac{(p - P)}{\sqrt{2\mathcal{B}}} - i\frac{(x - X)}{\sqrt{2\mathcal{A}}}\right] = \frac{1}{\sqrt{2}}[\varpi - i\varkappa] \\ \mathbf{z}^+ = \frac{z^+}{\sqrt{2\mathcal{B}}} = \frac{1}{\sqrt{2}}\left[\frac{(p - P)}{\sqrt{2\mathcal{B}}} + i\frac{(x - X)}{\sqrt{2\mathcal{A}}}\right] = \frac{1}{\sqrt{2}}[\varpi - i\varkappa] \end{cases} \quad (13)$$

98   The following relations can be deduced

99
$$\begin{cases} \beth^+ = \mathcal{B}[(\varpi)^2 + (\varkappa)^2] = \mathcal{B}(\mathbf{z}^-\mathbf{z}^+ + \mathbf{z}^+\mathbf{z}^-) \\ \beth^- = \mathcal{B}[(\varpi)^2 - (\varkappa)^2] = \mathcal{B}[(\mathbf{z}^-)^2 + (\mathbf{z}^+)^2] \\ \beth^\times = \mathcal{B}(\varpi\varkappa + \varkappa\varpi) = i\mathcal{B}[(\mathbf{z}^-)^2 - (\mathbf{z}^+)^2] \end{cases} \quad (14)$$

100   We may remark that the sets

101
$$\{(p)^2, (x)^2, px + xp\}, \{(\varpi)^2, (\varkappa)^2, \varpi\varkappa + \varkappa\varpi\}$$

102
$$\{z^-z^+ + z^+z^-, (z^-)^2, (z^+)^2\}, \{\mathbf{z}^-\mathbf{z}^+ + \mathbf{z}^+\mathbf{z}^-, (\mathbf{z}^-)^2, (\mathbf{z}^+)^2\}$$

103   are also four basis of the dispersion operators algebra $\mathfrak{g}$.

104   We have the following commutation relations

105
$$[z^-, z^+]_- = 2\mathcal{B} \quad (15)$$

106
$$[\varkappa, \varpi]_- = \left[\frac{(x - X)}{\sqrt{2}a}, \frac{(p - P)}{\sqrt{2}b}\right]_- = [(x - X), (p - P)]_- = [x, p]_- = i \quad (16)$$

107
$$[\mathbf{z}^-, \mathbf{z}^+] = \frac{1}{2\mathcal{B}}[z^-, z^+] = 1 \quad (17)$$

108
$$\begin{cases} [(\varkappa)^2, \boldsymbol{p}]_- = 2i\varkappa \\ [\boldsymbol{p}\varkappa, \boldsymbol{p}]_- = i\boldsymbol{p} \\ [\varkappa\boldsymbol{p}, \boldsymbol{p}]_- = i\boldsymbol{p} \end{cases} \tag{18}$$

109
$$\begin{cases} [(\boldsymbol{p})^2, \varkappa]_- = -2i\boldsymbol{p} \\ [\boldsymbol{p}\varkappa, \varkappa]_- = -i\varkappa \\ [\varkappa\boldsymbol{p}, \varkappa]_- = -i\varkappa \end{cases} \tag{19}$$

110
$$\begin{cases} [(\boldsymbol{p})^2, (\varkappa)^2]_- = -2i(\boldsymbol{p}\varkappa + \varkappa\boldsymbol{p}) \\ [(\boldsymbol{p})^2, \boldsymbol{p}\varkappa]_- = -2i(\boldsymbol{p})^2 \\ [(\varkappa)^2, \boldsymbol{p}\varkappa]_- = 2i(\varkappa)^2 \end{cases} \tag{20}$$

111   We may define the operators

112
$$\begin{cases} \beth^+ = \dfrac{\beth^+}{4\mathcal{B}} = \dfrac{1}{4}((\boldsymbol{p})^2 + (\varkappa)^2) = \dfrac{1}{4}(\boldsymbol{z}^-\boldsymbol{z}^+ + \boldsymbol{z}^+\boldsymbol{z}^-) \\[2mm] \beth^- = \dfrac{\beth^-}{4\mathcal{B}} = \dfrac{1}{4}((\boldsymbol{p})^2 + (\varkappa)^2) = \dfrac{1}{4}((\boldsymbol{z}^-)^2 + (\boldsymbol{z}^+)^2) \\[2mm] \beth^\times = \dfrac{\beth^\times}{4\mathcal{B}} = \dfrac{1}{4}(\boldsymbol{p}\varkappa + \varkappa\boldsymbol{p}) = \dfrac{i}{4}((\boldsymbol{z}^-)^2 - (\boldsymbol{z}^+)^2) \end{cases} \tag{21}$$

113   The set $\{\beth^+, \beth^-, \beth^\times\}$ is also a basis of the dispersion operator algebra $\mathfrak{g}$. It may be deduced easily
114   from the commutation relation of $\beth^+, \beth^-$ and $\beth^\times$ that $\beth^+, \beth^-$ and $\beth^\times$ satisfy to the following
115   commutation relations

116
$$\begin{cases} [\beth^+, \beth^-]_- = i\beth^\times \\ [\beth^-, \beth^\times]_- = -i\beth^+ \\ [\beth^\times, \beth^+]_- = i\beth^- \end{cases} \tag{22}$$

117   And we have

118
$$\begin{cases} [\beth^+, \boldsymbol{p}]_- = \dfrac{1}{2}i\varkappa \\[2mm] [\beth^-, \boldsymbol{p}]_- = -\dfrac{1}{2}i\varkappa \\[2mm] [\beth^\times, \boldsymbol{p}]_- = \dfrac{1}{2}i\boldsymbol{p} \end{cases} \tag{23}$$

119
$$\begin{cases} [\beth^+, \varkappa]_- = -\dfrac{1}{2}i\boldsymbol{p} \\[2mm] [\beth^-, \varkappa]_- = -\dfrac{1}{2}i\boldsymbol{p} \\[2mm] [\beth^\times, \varkappa]_- = -\dfrac{1}{2}i\varkappa \end{cases} \tag{24}$$

120   *2.2. Representation of the dispersion operators algebra over the state space of a particle*

121   To define a representation of the dispersion operators algebra $\mathfrak{g}$ over the state space $\mathcal{E}$ of a particle,
122   we have to find the matrix representation of the three operators $\beth^+, \beth^-$ and $\beth^\times$ in a basis of $\mathcal{E}$. It is
123   obvious that an adequate basis is the basis $\{|n, X, P, \ell\rangle\}$ composed by the eigenstates of the
124   momentum dispersion operator $\beth^+$

125
$$\beth^+|n, X, P, \ell\rangle = (2n+1)\mathcal{B}|n, X, P, \ell\rangle \tag{25}$$

126    $\beth^+$is represented by a diagonal matrix with elements equal to $(2n+1)\mathcal{B}$.

127   Now we have to find the expressions of $\beth^-|n, X, P, \ell\rangle$ and $\beth^\times|n, X, P, \ell\rangle$. We consider the relation

128
$$\begin{cases} ℶ^+ = \mathcal{B}(\pmb{z}^-\pmb{z}^+ + \pmb{z}^+\pmb{z}^-) \\ ℶ^- = \mathcal{B}[(\pmb{z}^-)^2 + (\pmb{z}^+)^2] \\ ℶ^\times = i\mathcal{B}[(\pmb{z}^-)^2 - (\pmb{z}^+)^2] \end{cases} \tag{26}$$

129 Let us first search for the expression of $\pmb{z}^-|n,X,P,\ell\rangle$ and $\pmb{z}^+|n,X,P,\ell\rangle$. From the commutation
130 relation $[\pmb{z}^-,\pmb{z}^+]_- = 1$ we can deduce the relation

131
$$ℶ^+ = \mathcal{B}(\pmb{z}^-\pmb{z}^+ + \pmb{z}^+\pmb{z}^-) = \mathcal{B}(2\pmb{z}^+\pmb{z}^- + 1) = \mathcal{B}(2\pmb{z}^-\pmb{z}^+ - 1) \tag{27}$$

132 and the commutation relations

133
$$\begin{cases} [ℶ^+,\pmb{z}^-]_- = -2\mathcal{B}\pmb{z}^- \\ [ℶ^+,\pmb{z}^+]_- = 2\mathcal{B}\pmb{z}^+ \end{cases} \tag{28}$$

134 Then from the relations (27) and (28), it may be deduced after lengthy but straightforward
135 calculations that

136
$$\begin{cases} \pmb{z}^-|n,X,P,\ell\rangle = \sqrt{n}|n-1,X,P,\ell\rangle \\ \pmb{z}^+|n,X,P,\ell\rangle = \sqrt{n+1}|n+1,X,P,\ell\rangle \end{cases} \tag{29}$$

137 So

138
$$\begin{cases} (\pmb{z}^-)^2|n,X,P,\ell\rangle = \pmb{z}^-\sqrt{n}|n-1,X,P,\ell\rangle = \sqrt{n(n-1)}|n-2,X,P,\ell\rangle \\ (\pmb{z}^+)^2|n,X,P,\ell\rangle = \pmb{z}^+\sqrt{n+1}|n-1,X,P,\ell\rangle = \sqrt{(n+1)(n+2)}|n+2,X,P,\ell\rangle \end{cases} \tag{30}$$

139 As

140
$$\begin{cases} ℶ^- = \mathcal{B}[(\pmb{z}^-)^2 + (\pmb{z}^+)^2] \\ ℶ^\times = i\mathcal{B}[(\pmb{z}^-)^2 - (\pmb{z}^+)^2] \end{cases} \tag{31}$$

141 we obtain for the representation of the three operators $ℶ^+, ℶ^-$ and $ℶ^\times$ in the basis $\{|n,X,P,\ell\rangle\}$ of the
142 state space of a particle

143
$$\begin{cases} ℶ^+|n,X,P,\ell\rangle = (2n+1)\mathcal{B}|n,X,P,\ell\rangle \\ ℶ^-|n,X,P,\ell\rangle = \sqrt{n(n-1)}\mathcal{B}|n-2,X,P,\ell\rangle + \sqrt{(n+1)(n+2)}\mathcal{B}|n+2,X,P,\ell\rangle \\ ℶ^\times|n,X,P,\ell\rangle = i\left[\sqrt{n(n-1)}\mathcal{B}|n-2,X,P,\ell\rangle - \sqrt{(n+1)(n+2)}\mathcal{B}|n+2,X,P,\ell\rangle\right] \end{cases} \tag{32}$$

144 We may also write these relations in the form

145
$$\begin{cases} ℶ^+|n,X,P,\ell\rangle = (2n+1)\mathcal{B}|n,X,P,\ell\rangle \\ (ℶ^- - iℶ^\times)|n,X,P,\ell\rangle = 2\sqrt{n(n-1)}\mathcal{B}|n-2,X,P,\ell\rangle \\ (ℶ^- + iℶ^\times)|n,X,P,\ell\rangle = 2\sqrt{(n+1)(n+2)}\mathcal{B}|n+2,X,P,\ell\rangle \end{cases} \tag{33}$$

146

147 **3. Linear Canonical Transformations**

148 *3.1. Definitions and properties*

149 In quantum mechanics, a linear canonical transformation can be defined as a linear transformation
150 mixing the coordinate operator $\pmb{x}$ and the momentum operator $\pmb{p}$ and leaving invariant the
151 commutator $[\pmb{x},\pmb{p}]_- = i$. As $\pmb{x}$ and $\pmb{p}$ are linked with the operators $\pmb{z}$ and $\pmb{p}$ through the linear
152 relations (12), we may also take a definition of linear canonical transformation as linear
153 transformation mixing $\pmb{z}$ and $\pmb{p}$

154

155
$$\begin{cases} \pmb{p}' = \Pi\pmb{p} + \Theta\pmb{z} \\ \pmb{z}' = \Xi\pmb{p} + \Lambda\pmb{z} \end{cases} \Leftrightarrow (\pmb{p}' \quad \pmb{z}') = (\pmb{p} \quad \pmb{z})\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \tag{34}$$

156 in which

157
$$\begin{cases} \varkappa' = \dfrac{(x'-X')}{\sqrt{2}a'} = \dfrac{(x'-X')}{\sqrt{2}\mathcal{A}'} = \sqrt{2}\mathcal{b}'(x'-X') = \sqrt{2\mathcal{B}'}(x'-X') \\ \varrho' = \dfrac{(p'-P')}{\sqrt{2}\mathcal{b}'} = \dfrac{(p'-P')}{\sqrt{2\mathcal{B}'}} = \sqrt{2}a'(p'-P') = \sqrt{2\mathcal{A}'}(p'-P') \end{cases}$$
(35)

158
$$[\varkappa',\varrho']_- = \frac{1}{2a'\mathcal{b}'}[(x'-X'),(p'-P')]_- = [x',p']_-$$
(36)

159 where x' and p' are the new coordinate and momentum operators resulting from the
160 transformation. If we have a linear canonical transformation, we must have

161
$$[x',p']_- = [x,p]_- = i$$
(37)

162 So taking into account the relation (36), we must have

163
$$[\varkappa',\varrho']_- = [x',p']_- = [x,p]_- = i = [\varkappa,\varrho]_-$$
(38)

164 Then in our case the full definition of the linear canonical transformation is

165
$$\begin{cases} \varrho' = \Pi\varrho + \Theta\varkappa \\ \varkappa' = \Xi\varrho + \Lambda\varkappa \\ [\varkappa',\varrho'] = [\varkappa,\varrho] = i \end{cases}$$
(39)

166 The last condition $[\varkappa',\varrho']_- = [\varkappa,\varrho]_- = i$ leads to the relation

167
$$\Pi\Lambda - \Theta\Xi = 1 \iff \begin{vmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{vmatrix} = 1$$
(40)

168 If we consider real linear canonical transformation (the parameters Π, Λ, Ξ and Θ are real), the
169 relation (40) means that the matrix $\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}$ is an element of the special linear group $SL(2,\mathbb{R})$. We
170 may write it in the form

171
$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\mathcal{M}} = e^{\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & \mathcal{M}_4 \end{pmatrix}}$$
(41)

172 with $\mathcal{M}$ an element of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of the Lie group $SL(2,\mathbb{R})$, we have

173
$$\Pi\Lambda - \Theta\Xi = 1 \iff \mathcal{M}_4 = -\mathcal{M}_1 \iff \mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & -\mathcal{M}_1 \end{pmatrix}$$
(42)

174 Then, for an infinitesimal linear canonical transformation, we have

175
$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = 1 + \mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & -\mathcal{M}_1 \end{pmatrix} = \begin{pmatrix} 1+\mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & 1-\mathcal{M}_1 \end{pmatrix}$$
(43)

176 So

177
$$\begin{cases} \varrho' = \Pi\varrho + \Theta\varkappa = \varrho + \mathcal{M}_1\varrho + \mathcal{M}_2\varkappa \\ \varkappa' = \Xi\varrho + \Lambda\varkappa = \varkappa + \mathcal{M}_3\varrho - \mathcal{M}_1\varkappa \end{cases}$$
(44)

178 *3.2. Unitary representation and relation with the dispersion operators algebra*

179 As the linear canonical transformation is a transformation which affects quantum operators, we may
180 represent it by an unitary transformation

181
$$\begin{cases} \varrho' = \Pi\varrho + \Theta\varkappa = U\varrho U^\dagger \\ \varkappa' = \Xi\varrho + \Lambda\varkappa = U\varkappa U^\dagger \end{cases}$$
(45)

182 where U is a unitary operator which can be considered as acting in the state space $\mathcal{E}$ of a particle. It
183 may be verified that the commutator $[\varkappa,\varrho] = i$ is invariant under the unitary representation defined
184 in (45) as expected for a linear canonical transformation

185
$$[\varkappa',\varrho']_- = [U\varkappa U^\dagger, U\varrho U^\dagger]_- = U\varkappa U^\dagger U\varrho U^\dagger - U\varrho U^\dagger U\varkappa U^\dagger = U[\varkappa,\varrho]U^\dagger = iUU^\dagger = i$$
(46)

186   It can be shown that U may be written in the form

187    $$U = e^{i \beth} = e^{i(\theta_+ \beth^+ + \theta_- \beth^- + \theta_\times \beth^\times)} \tag{47}$$

188   in which $\beth = \theta_+ \beth^+ + \theta_- \beth^- + \theta_\times \beth^\times$ is an element of the dispersion operators algebra and

189    $\theta_+, \theta_-, \theta_\times$ three real numbers. In fact, for an infinitesimal transformation, we have

190    $\quad U = 1 + i(\theta_+ \beth^+ + \theta_- \beth^- + \theta_\times \beth^\times) \qquad\qquad U^\dagger = 1 - i(\theta_+ \beth^+ + \theta_- \beth^- + \theta_\times \beth^\times)$

191   Then it can be deduced from (45) that

192    $$\begin{cases} \boldsymbol{p}' = \boldsymbol{p} + i\theta_+ [\beth^+, \boldsymbol{p}]_- + i\theta_- [\beth^-, \boldsymbol{p}]_- + i\theta_\times [\beth^\times, \boldsymbol{p}]_- \\ \boldsymbol{x}' = \boldsymbol{x} + i\theta_+ [\beth^+, \boldsymbol{x}]_- + i\theta_- [\beth^-, \boldsymbol{x}]_- + i\theta_\times [\beth^\times, \boldsymbol{x}]_- \end{cases} \tag{48}$$

193   Taking into account the relations (23) and (24), we obtain

194    $$\begin{cases} \boldsymbol{p}' = \boldsymbol{p} - \dfrac{1}{2}\theta_+ \boldsymbol{x} + \dfrac{1}{2}\theta_- \boldsymbol{x} - \dfrac{1}{2}\theta_\times \boldsymbol{p} = \boldsymbol{p} - \dfrac{1}{2}\theta_\times \boldsymbol{p} + \dfrac{1}{2}(\theta_- - \theta_+)\boldsymbol{x} \\ \boldsymbol{x}' = \boldsymbol{x} + \dfrac{1}{2}\theta_+ \boldsymbol{p} + \dfrac{1}{2}\theta_- \boldsymbol{p} + \dfrac{1}{2}\theta_\times \boldsymbol{x} = \boldsymbol{x} + \dfrac{1}{2}(\theta_- + \theta_+)\boldsymbol{p} + \dfrac{1}{2}\theta_\times \boldsymbol{x} \end{cases} \tag{49}$$

195   identifying the relations (44) and (49) it can be deduced that

196    $$\begin{cases} \mathcal{M}_1 = -\dfrac{1}{2}\theta_\times \\ \mathcal{M}_2 = \dfrac{1}{2}(\theta_- - \theta_+) \\ \mathcal{M}_3 = \dfrac{1}{2}(\theta_- + \theta_+) \end{cases} \tag{50}$$

197   So briefly, we have for a linear canonical transformation

198    $$\begin{cases} \boldsymbol{p}' = \Pi \boldsymbol{p} + \Theta \boldsymbol{x} = U\boldsymbol{p}U^\dagger \\ \boldsymbol{x}' = \Xi \boldsymbol{p} + \Lambda \boldsymbol{x} = U\boldsymbol{x}U^\dagger \end{cases} \Leftrightarrow (\boldsymbol{p}' \quad \boldsymbol{x}') = (\boldsymbol{p} \quad \boldsymbol{x}) \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \tag{51}$$

199   with

200    $$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\mathcal{M}} = e^{\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & \mathcal{M}_4 \end{pmatrix}} = e^{\frac{1}{2}\begin{pmatrix} -\theta_\times & \theta_+ + \theta_- \\ \theta_- - \theta_+ & \theta_\times \end{pmatrix}} \tag{52}$$

201   and

202    $$U = e^{i(\theta_+ \beth^+ + \theta_- \beth^- + \theta_\times \beth^\times)} \tag{53}$$

203   The unitarity of U results from the hermiticity of the operators $\beth^+, \beth^-$ and $\beth^\times$.

204   *3.2. Transformation law of the basis* $\{\beth^+, \beth^-, \beth^\times\}$ *of the dispersion operators algebra*

205   Taking into account the relation (14) and (21), we may define the operators

206    $$\begin{cases} \daleth^{+'} = \mathcal{B}'((\boldsymbol{p}')^2 + (\boldsymbol{x}')^2) \\ \daleth^{-'} = \mathcal{B}'((\boldsymbol{p}')^2 - (\boldsymbol{x}')^2) \\ \daleth^{\times'} = \mathcal{B}'(\boldsymbol{p}'\boldsymbol{x}' + \boldsymbol{x}'\boldsymbol{p}') \end{cases} \tag{54}$$

207    $$\begin{cases} \beth^{+'} = \dfrac{\daleth^{+'}}{4\mathcal{B}'} = \dfrac{1}{4}((\boldsymbol{p}')^2 + (\boldsymbol{x}')^2) \\ \beth^{-'} = \dfrac{\daleth^{-'}}{4\mathcal{B}'} = \dfrac{1}{4}((\boldsymbol{p}')^2 - (\boldsymbol{x}')^2) \\ \beth^{\times'} = \dfrac{\daleth^{\times'}}{4\mathcal{B}'} = \dfrac{1}{4}(\boldsymbol{p}'\boldsymbol{x}' + \boldsymbol{x}'\boldsymbol{p}') \end{cases} \tag{55}$$

208   Taking into account the relations (51), we obtain

209

$$\begin{cases} ℶ^{+'} = \dfrac{1}{2}[(\Pi)^2 + (\Theta)^2]ℶ^+ + \dfrac{1}{2}[(\Xi)^2 - (\Lambda)^2]ℶ^- + (\Pi\Theta + \Xi\Lambda)ℶ^\times = U ℶ^+ U^\dagger \\[2mm] ℶ^{-'} = \dfrac{1}{2}[(\Pi)^2 + (\Theta)^2]ℶ^+ - \dfrac{1}{2}[(\Xi)^2 - (\Lambda)^2]ℶ^- + (\Pi\Theta - \Xi\Lambda)ℶ^\times = U ℶ^- U^\dagger \\[2mm] ℶ^{\times'} = (\Pi\Xi + \Theta\Lambda)ℶ^+ + (\Pi\Xi - \Theta\Lambda)ℶ^- + (\Pi\Lambda + \Theta\Xi)ℶ^\times = U ℶ^\times U^\dagger \end{cases} \tag{56}$$

210

$$\begin{cases} ℸ^{+'} = \dfrac{1}{2}\dfrac{\mathcal{B}'}{\mathcal{B}}\{[(\Pi)^2 + (\Theta)^2]ℸ^+ + \dfrac{1}{2}[(\Xi)^2 - (\Lambda)^2]ℸ^- + (\Pi\Theta + \Xi\Lambda)ℸ^\times\} = \dfrac{\mathcal{B}'}{\mathcal{B}} U ℸ^+ U^\dagger \\[2mm] ℸ^{-'} = \dfrac{1}{2}\dfrac{\mathcal{B}'}{\mathcal{B}}\{[(\Pi)^2 + (\Theta)^2]ℸ^+ - \dfrac{1}{2}[(\Xi)^2 - (\Lambda)^2]ℸ^- + (\Pi\Theta - \Xi\Lambda)ℸ^\times\} = \dfrac{\mathcal{B}'}{\mathcal{B}} U ℸ^- U^\dagger \\[2mm] ℸ^{\times'} = \dfrac{\mathcal{B}'}{\mathcal{B}}\{[(\Pi\Xi + \Theta\Lambda)ℸ^+ + (\Pi\Xi - \Theta\Lambda)ℸ^- + (\Pi\Lambda + \Theta\Xi)ℸ^\times]\} = \dfrac{\mathcal{B}'}{\mathcal{B}} U ℸ^\times U^\dagger \end{cases} \tag{57}$$

211 ## 4. Multidimensional Generalization

212 *4.1. Dispersion Operators Algebra*

213 We may generalize the operators $ℸ^+, ℸ^-, ℸ^\times$ by the following tensor operators

214
$$\begin{cases} ℸ^+_{\mu\nu} = \dfrac{1}{2}[(p_\mu - P_\mu)(p_\nu - P_\nu) + 4\mathcal{B}_{\mu\alpha}\mathcal{B}_{\nu\beta}(x^\alpha - X^\alpha)(x^\beta - X^\beta)] \\[2mm] ℸ^-_{\mu\nu} = \dfrac{1}{2}[(p_\mu - P_\mu)(p_\nu - P_\nu) - 4\mathcal{B}_{\mu\alpha}\mathcal{B}_{\nu\beta}(x^\alpha - X^\alpha)(x^\beta - X^\beta)] \\[2mm] ℸ^\times_{\mu\nu} = \mathcal{B}_{\mu\alpha}[(p_\nu - P_\nu)(x^\alpha - X^\alpha) + (x^\alpha - X^\alpha)(p_\nu - P_\nu)] \end{cases} \tag{58}$$

215 in which $\mathcal{B}_{\mu\nu}$ are the components of the momentum dispersion-codispersion tensor [1]. Let $\eta_{\mu\nu}$ be
216 the components of the symmetric bilinear form η associated with the considered space. For the case
217 of a general N-dimensional pseudo-Euclidian space, if (N_+, N_-) is the signature of $(N_+ + N_- = N)$,
218 we have

219
$$\eta_{\mu\nu} = \begin{cases} 1 & for \mu = \nu = 0,1,\dots,N_+ - 1 \\ -1 & for \mu = \nu = N_+, N_+ + 1, \dots, N - 1 \\ 0 & for \mu \neq \nu \end{cases} \tag{59}$$

220 for instance, in the case of Minkowski space, the signature of η is $(1,3)$. So we have

221
$$\eta_{\mu\nu} = \begin{cases} 1 & for \mu = \nu = 0 \\ -1 & for \mu = \nu = 1,2,3 \\ 0 & for \mu \neq \nu \end{cases} \tag{60}$$

222 If we introduce the operators $\mathfrak{p}_\mu$ and $\mathfrak{x}^\nu$ defined by the relations

223
$$\begin{cases} p_\mu = \sqrt{2}\,\mathcal{b}^\nu_\mu \mathfrak{p}_\nu + P_\mu \\ x^\nu = \sqrt{2}\,a^\nu_\mu \mathfrak{x}^\mu + X^\mu \end{cases} \tag{61}$$

224 with a^ν_μ and $\mathcal{b}^\nu_\mu$ verifying the relations

225
$$\begin{cases} \mathcal{B}_{\mu\alpha} a^\alpha_\nu = \dfrac{1}{2}\mathcal{b}_{\mu\nu} = \eta_{\mu\rho}\mathcal{b}^\rho_\nu \\[2mm] a^\lambda_\mu \mathcal{b}^\nu_\lambda = \dfrac{1}{2}\delta^\nu_\mu \end{cases} \tag{62}$$

226 (δ^ν_μ being the components of the Kronecker's symbol) we obtain

227
$$\begin{cases} ℸ^+_{\mu\nu} = \mathcal{b}^\rho_\mu \mathcal{b}^\lambda_\nu ℸ^+_{\rho\lambda} \\ ℸ^-_{\mu\nu} = \mathcal{b}^\rho_\mu \mathcal{b}^\lambda_\nu ℸ^-_{\rho\lambda} \\ ℸ^\times_{\mu\nu} = \mathcal{b}^\rho_\mu \mathcal{b}^\lambda_\nu ℸ^\times_{\rho\lambda} \end{cases} \tag{63}$$

228 in which

229
$$\begin{cases} \beth_{\mu\nu}^{+} = \dfrac{1}{4}\left(\boldsymbol{p}_{\mu}\boldsymbol{p}_{\nu} + \boldsymbol{x}_{\mu}\boldsymbol{x}_{\nu}\right) \\[2mm] \beth_{\mu\nu}^{-} = \dfrac{1}{4}\left(\boldsymbol{p}_{\mu}\boldsymbol{p}_{\nu} - \boldsymbol{x}_{\mu}\boldsymbol{x}_{\nu}\right) \\[2mm] \beth_{\mu\nu}^{\times} = \dfrac{1}{4}\left(\boldsymbol{p}_{\mu}\boldsymbol{x}_{\nu} + \boldsymbol{x}_{\nu}\boldsymbol{p}_{\mu}\right) \end{cases} \qquad (64)$$

230   If we define also the operators

231
$$\begin{cases} \boldsymbol{z}_{\mu}^{-} = \dfrac{1}{\sqrt{2}}\left(\boldsymbol{p}_{\mu} - i\boldsymbol{x}_{\mu}\right) \\[2mm] \boldsymbol{z}_{\mu}^{+} = \dfrac{1}{\sqrt{2}}\left(\boldsymbol{p}_{\mu} + i\boldsymbol{x}_{\mu}\right) \end{cases} \qquad (65)$$

232   We have

233
$$\begin{cases} \beth_{\mu\nu}^{+} = \dfrac{1}{4}\left(\boldsymbol{z}_{\mu}^{+}\boldsymbol{z}_{\nu}^{-} + \boldsymbol{z}_{\mu}^{-}\boldsymbol{z}_{\nu}^{+}\right) \\[2mm] \beth_{\mu\nu}^{-} = \dfrac{1}{4}\left(\boldsymbol{z}_{\mu}^{+}\boldsymbol{z}_{\nu}^{+} + \boldsymbol{z}_{\mu}^{-}\boldsymbol{z}_{\nu}^{-}\right) \\[2mm] \beth_{\mu\nu}^{\times} = \dfrac{i}{8}\left([\boldsymbol{z}_{\mu}^{+},\boldsymbol{z}_{\nu}^{-}]_{+} - [\boldsymbol{z}_{\mu}^{+},\boldsymbol{z}_{\nu}^{+}]_{+} + [\boldsymbol{z}_{\mu}^{-},\boldsymbol{z}_{\nu}^{-}]_{+} - [\boldsymbol{z}_{\mu}^{-},\boldsymbol{z}_{\nu}^{+}]_{+}\right) \end{cases} \qquad (66)$$

234     $[\boldsymbol{z}_{\mu}^{+},\boldsymbol{z}_{\nu}^{-}]_{+}$ being the anticomutator

235
$$[\boldsymbol{z}_{\mu}^{+},\boldsymbol{z}_{\nu}^{-}]_{+} = \boldsymbol{z}_{\mu}^{+}\boldsymbol{z}_{\nu}^{-} + \boldsymbol{z}_{\nu}^{-}\boldsymbol{z}_{\mu}^{+}$$

236

237   From the commutation relation $[\boldsymbol{p}_{\mu}, x^{\nu}]_{-} = i\delta_{\mu}^{\nu}$ we may deduce the following commutators

238
$$[\boldsymbol{p}_{\mu}, \boldsymbol{x}_{\nu}]_{-} = i\eta_{\mu\nu} \qquad (67)$$

239
$$[\boldsymbol{z}_{\mu}^{+}, \boldsymbol{z}_{\nu}^{-}]_{-} = \eta_{\mu\nu} \qquad (68)$$

240
$$\begin{cases} [\boldsymbol{x}_{\mu}\boldsymbol{x}_{\nu}, \boldsymbol{p}_{\rho}]_{-} = -i\eta_{\nu\rho}\boldsymbol{x}_{\mu} - i\eta_{\mu\rho}\boldsymbol{x}_{\nu} \\[1mm] [\boldsymbol{p}_{\mu}\boldsymbol{x}_{\nu}, \boldsymbol{p}_{\rho}]_{-} = -i\eta_{\nu\rho}\boldsymbol{p}_{\mu} \\[1mm] [\boldsymbol{x}_{\mu}\boldsymbol{p}_{\nu}, \boldsymbol{p}_{\rho}]_{-} = -i\eta_{\mu\rho}\boldsymbol{p}_{\nu} \end{cases} \qquad (69)$$

241
$$\begin{cases} [\boldsymbol{p}_{\mu}\boldsymbol{p}_{\nu}, \boldsymbol{x}_{\rho}]_{-} = i\eta_{\nu\rho}\boldsymbol{p}_{\mu} + i\eta_{\mu\rho}\boldsymbol{p}_{\nu} \\[1mm] [\boldsymbol{p}_{\mu}\boldsymbol{x}_{\nu}, \boldsymbol{x}_{\rho}]_{-} = i\eta_{\mu\rho}\boldsymbol{x}_{\nu} \\[1mm] [\boldsymbol{x}_{\mu}\boldsymbol{p}_{\nu}, \boldsymbol{x}_{\rho}]_{-} = i\eta_{\nu\rho}\boldsymbol{x}_{\mu} \end{cases} \qquad (70)$$

242
$$\begin{cases} [\beth_{\mu\nu}^{+}, \boldsymbol{p}_{\rho}]_{-} = -\dfrac{1}{4}\left(i\eta_{\nu\rho}\boldsymbol{x}_{\mu} + i\eta_{\mu\rho}\boldsymbol{x}_{\nu}\right) \\[2mm] [\beth_{\mu\nu}^{-}, \boldsymbol{p}_{\rho}]_{-} = \dfrac{1}{4}\left(i\eta_{\nu\rho}\boldsymbol{x}_{\mu} + i\eta_{\mu\rho}\boldsymbol{x}_{\nu}\right) \\[2mm] [\beth_{\mu\nu}^{\times}, \boldsymbol{p}_{\rho}]_{-} = -\dfrac{i}{2}\eta_{\nu\rho}\boldsymbol{p}_{\mu} \end{cases} \qquad (71)$$

243
$$\begin{cases} [\beth_{\mu\nu}^{+}, \boldsymbol{x}_{\rho}]_{-} = \dfrac{1}{4}\left(i\eta_{\nu\rho}\boldsymbol{p}_{\mu} + i\eta_{\mu\rho}\boldsymbol{p}_{\nu}\right) \\[2mm] [\beth_{\mu\nu}^{-}, \boldsymbol{x}_{\rho}]_{-} = \dfrac{1}{4}\left(i\eta_{\nu\rho}\boldsymbol{p}_{\mu} + i\eta_{\mu\rho}\boldsymbol{p}_{\nu}\right) \\[2mm] [\beth_{\mu\nu}^{\times}, \boldsymbol{x}_{\rho}]_{-} = \dfrac{i}{2}\eta_{\mu\rho}x_{\nu} \end{cases} \qquad (72)$$

244

$$
\begin{cases}
\left[\boldsymbol{p}_\mu \boldsymbol{p}_\nu, \boldsymbol{x}_\rho \boldsymbol{x}_\lambda\right]_- = i\eta_{\lambda\nu}\boldsymbol{p}_\mu \boldsymbol{x}_\rho + i\eta_{\rho\nu}\boldsymbol{p}_\mu \boldsymbol{x}_\lambda + i\eta_{\lambda\mu}\boldsymbol{x}_\rho \boldsymbol{p}_\nu + i\eta_{\rho\mu}\boldsymbol{x}_\lambda \boldsymbol{p}_\nu \\
\left[\boldsymbol{p}_\mu \boldsymbol{p}_\nu, \boldsymbol{p}_\rho \boldsymbol{x}_\lambda\right]_- = i\eta_{\lambda\nu}\boldsymbol{p}_\mu \boldsymbol{p}_\rho + i\eta_{\lambda\mu}\boldsymbol{p}_\rho \boldsymbol{p}_\nu \\
\left[\boldsymbol{p}_\mu \boldsymbol{p}_\nu, \boldsymbol{x}_\rho \boldsymbol{p}_\lambda\right]_- = i\eta_{\rho\nu}\boldsymbol{p}_\mu \boldsymbol{p}_\lambda + i\eta_{\rho\mu}\boldsymbol{p}_\lambda \boldsymbol{p}_\nu \\
\left[\boldsymbol{x}_\mu \boldsymbol{x}_\nu, \boldsymbol{p}_\rho \boldsymbol{x}_\lambda\right]_- = -i\eta_{\rho\nu}\boldsymbol{x}_\mu \boldsymbol{x}_\lambda - i\eta_{\rho\mu}\boldsymbol{x}_\lambda \boldsymbol{x}_\nu \\
\left[\boldsymbol{x}_\mu \boldsymbol{x}_\nu, \boldsymbol{x}_\rho \boldsymbol{p}_\lambda\right]_- = -i\eta_{\lambda\nu}\boldsymbol{x}_\mu \boldsymbol{x}_\rho - i\eta_{\lambda\mu}\boldsymbol{x}_\rho \boldsymbol{x}_\nu \\
\left[\boldsymbol{p}_\mu \boldsymbol{x}_\nu, \boldsymbol{p}_\rho \boldsymbol{x}_\lambda\right]_- = -i\eta_{\rho\nu}\boldsymbol{p}_\mu \boldsymbol{x}_\lambda + i\eta_{\lambda\mu}\boldsymbol{p}_\rho \boldsymbol{x}_\nu \\
\left[\boldsymbol{p}_\mu \boldsymbol{x}_\nu, \boldsymbol{x}_\rho \boldsymbol{p}_\lambda\right]_- = -i\eta_{\lambda\nu}\boldsymbol{p}_\mu \boldsymbol{x}_\rho + i\eta_{\rho\mu}\boldsymbol{p}_\lambda \boldsymbol{x}_\nu \\
\left[\boldsymbol{x}_\mu \boldsymbol{p}_\nu, \boldsymbol{x}_\rho \boldsymbol{p}_\lambda\right]_- = i\eta_{\rho\nu}\boldsymbol{x}_\mu \boldsymbol{p}_\lambda - i\eta_{\lambda\mu}\boldsymbol{x}_\rho \boldsymbol{p}_\nu
\end{cases}
\tag{73}
$$

245

$$
\begin{cases}
\left[\beth_{\mu\nu}^+, \beth_{\rho\lambda}^+\right]_- = \dfrac{i}{8}\left[\eta_{\nu\lambda}\left(\beth_{\mu\rho}^\times - \beth_{\rho\mu}^\times\right) + \eta_{\nu\rho}\left(\beth_{\mu\lambda}^\times - \beth_{\lambda\mu}^\times\right) + \eta_{\mu\lambda}\left(\beth_{\nu\rho}^\times - \beth_{\rho\nu}^\times\right) + \eta_{\mu\rho}\left(\beth_{\nu\lambda}^\times - \beth_{\lambda\nu}^\times\right)\right] \\[2mm]
\left[\beth_{\mu\nu}^-, \beth_{\rho\lambda}^-\right]_- = -\dfrac{i}{8}\left[\eta_{\nu\lambda}\left(\beth_{\mu\rho}^\times - \beth_{\rho\mu}^\times\right) + \eta_{\nu\rho}\left(\beth_{\mu\lambda}^\times - \beth_{\lambda\mu}^\times\right) + \eta_{\mu\lambda}\left(\beth_{\nu\rho}^\times - \beth_{\rho\nu}^\times\right) + \eta_{\mu\rho}\left(\beth_{\nu\lambda}^\times - \beth_{\lambda\nu}^\times\right)\right] \\[2mm]
\left[\beth_{\mu\nu}^\times, \beth_{\rho\lambda}^\times\right]_- = \dfrac{i}{2}\left(\eta_{\nu\mu}\beth_{\rho\lambda}^\times - \eta_{\rho\lambda}\beth_{\mu\nu}^\times\right)
\end{cases}
\tag{74}
$$

246

$$
\begin{cases}
\left[\beth_{\mu\nu}^+, \beth_{\rho\lambda}^-\right]_- = \dfrac{i}{8}\left[\eta_{\nu\lambda}\left(\beth_{\mu\rho}^\times + \beth_{\rho\mu}^\times\right) + \eta_{\nu\rho}\left(\beth_{\mu\lambda}^\times + \beth_{\lambda\mu}^\times\right) + \eta_{\mu\lambda}\left(\beth_{\nu\rho}^\times + \beth_{\rho\nu}^\times\right) + \eta_{\mu\rho}\left(\beth_{\nu\lambda}^\times + \beth_{\lambda\nu}^\times\right)\right] \\[2mm]
\left[\beth_{\mu\nu}^-, \beth_{\rho\lambda}^\times\right]_- = \dfrac{i}{4}\left[\eta_{\lambda\nu}\left(\beth_{\mu\rho}^+ + \beth_{\mu\rho}^-\right) + \eta_{\lambda\mu}\left(\beth_{\rho\nu}^+ + \beth_{\rho\nu}^-\right) + \eta_{\rho\nu}\left(\beth_{\mu\lambda}^+ - \beth_{\mu\lambda}^-\right) + \eta_{\rho\mu}\left(\beth_{\lambda\nu}^+ - \beth_{\lambda\nu}^-\right)\right] \\[2mm]
\left[\beth_{\mu\nu}^\times, \beth_{\rho\lambda}^+\right]_- = -\dfrac{i}{4}\left[\eta_{\nu\lambda}\left(\beth_{\mu\rho}^+ + \beth_{\mu\rho}^-\right) + \eta_{\nu\rho}\left(\beth_{\mu\lambda}^+ + \beth_{\mu\lambda}^-\right) - \eta_{\mu\lambda}\left(\beth_{\rho\nu}^+ - \beth_{\rho\nu}^-\right) - \eta_{\mu\rho}\left(\beth_{\nu\lambda}^+ - \beth_{\nu\lambda}^-\right)\right]
\end{cases}
\tag{75}
$$

247 It can be deduced easily from the relations (74) and (75) that the set $\{\beth_{\mu\nu}^+, \beth_{\mu\nu}^-, \beth_{\mu\nu}^\times\}$ is a basis of a Lie

248 algebra which is the dispersion operators algebra for the multidimensional case.

249 As the indices μ, ν, ρ, λ run from 0 to $N-1$, the dimension D of this dispersion operators algebra

250 which is equal to the numbers of the elements of the basis $\{\beth_{\mu\nu}^+, \beth_{\mu\nu}^-, \beth_{\mu\nu}^\times\}$ is

251
$$
D = \frac{N(N+1)}{2} + \frac{N(N+1)}{2} + N^2 = N(2N+1)
\tag{76}
$$

252 In fact, from the relation (64) we can deduce that the number of operators $\beth_{\mu\nu}^+$ is equal to $\frac{N(N+1)}{2}$,

253 the number of operators $\beth_{\mu\nu}^-$ equal to $\frac{N(N+1)}{2}$ and the number of operators $\beth_{\mu\nu}^\times$ equal to N^2.

254 *4.2. Dispersion Operators Algebra*

255 We may define the linear canonical transformation as the linear transformation given by the

256 relation

257
$$
\begin{cases}
\boldsymbol{p}_\mu{}' = \Pi_\mu^\nu \boldsymbol{p}_\nu + \Theta_\mu^\nu \boldsymbol{x}_\nu \\
\boldsymbol{x}_\mu{}' = \Xi_\mu^\nu \boldsymbol{p}_\nu + \Lambda_\mu^\nu \boldsymbol{x}_\nu
\end{cases}
\Leftrightarrow
\begin{pmatrix} \boldsymbol{p}' & \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} & \boldsymbol{x} \end{pmatrix}
\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}
\tag{77}
$$

258 and which leave invariant the canonical commutation relations

259

260
$$
\begin{cases}
\left[\boldsymbol{p}_\mu{}', \boldsymbol{p}_\nu{}'\right]_- = 0 \\
\left[\boldsymbol{x}_\mu{}', \boldsymbol{x}_\nu{}'\right]_- = 0 \\
\left[\boldsymbol{p}_\mu{}', \boldsymbol{x}_\nu{}'\right] = i\eta_{\mu\nu}
\end{cases}
\tag{78}
$$

261 we obtain the following conditions

262

263
$$\begin{cases} \Pi_\mu^\rho \eta_{\rho\lambda} \Theta_\nu^\lambda - \Theta_\mu^\rho \eta_{\rho\lambda} \Pi_\nu^\lambda = 0 \\ \Xi_\mu^\rho \eta_{\rho\lambda} \Lambda_\nu^\lambda - \Lambda_\mu^\rho \eta_{\rho\lambda} \Xi_\nu^\lambda = 0 \\ \Pi_\mu^\rho \eta_{\rho\lambda} \Lambda_\nu^\lambda - \Xi_\nu^\lambda \eta_{\rho\lambda} \Theta_\mu^\rho = \eta_{\mu\nu} \end{cases} \Leftrightarrow \begin{cases} \Pi^t \eta \Theta - \Theta^t \eta \Pi = 0 \\ \Xi^t \eta \Lambda - \Lambda^t \eta \Xi = 0 \\ \Pi^t \eta \Lambda - \Theta^t \eta \Xi = \eta \end{cases} \Leftrightarrow \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}^t \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \quad (79)$$

264 If the signature of η is $(N, 0)$, it is equal to the $N \times N$ identity matrix $\eta = I_N$ (case of Euclidian
265 space), the relation (79) becomes

266
$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}^t \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \quad (80)$$

267 according to this relation the $2N \times 2N$ matrix $\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}$ is in this case an element of the symplectic

268 group $Sp(2N)$. We may generalize this result for the general case of pseudo-Euclidian space i.e

269 with η having a signature (N_+, N_-) : in that case, we may call a matrix $\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}$ verifying the

270 relation (79) a pseudo-symplectic matrix and their set the pseudo-symplectic group. We may denote

271 this Lie group $Sp(2N_+, 2N_-)$ and its Lie algebra $\mathfrak{sp}(2N_+, 2N_-)$. The matrix $\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}$ can be written

272 in the form

273
$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\mathcal{M}} = e^{\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & \mathcal{M}_4 \end{pmatrix}} \quad (81)$$

274 in which $\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & \mathcal{M}_4 \end{pmatrix}$ is an element of the Lie algebra $\mathfrak{sp}(2N_+, 2N_-)$, we have

275
$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix}^t \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \Leftrightarrow \begin{cases} \mathcal{M}_2^t = \eta \mathcal{M}_2 \eta \\ \mathcal{M}_4 = -\eta \mathcal{M}_1^t \eta \\ \mathcal{M}_4 = -\eta \mathcal{M}_1^t \eta \\ \mathcal{M}_3^t = \eta \mathcal{M}_3 \eta \end{cases} \quad (82)$$

276 It can be deduced easily from the relation (82) that the matrix $\mathcal{M}$ and his transpose $\mathcal{M}^t$ are of the
277 form

278
$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & -\eta \mathcal{M}_1^t \eta \end{pmatrix} \qquad \mathcal{M}^t = \begin{pmatrix} \mathcal{M}_1^t & \eta \mathcal{M}_2 \eta \\ \eta \mathcal{M}_3 \eta & -\eta \mathcal{M}_1 \eta \end{pmatrix} \quad (83)$$

279 If we introduce the parametrization

280
$$\begin{cases} \mathcal{M}_1 = \eta \mathcal{X} \\ \mathcal{M}_2 = \eta \mathcal{Y} \\ \mathcal{M}_3 = \eta \mathcal{Z} \end{cases} \quad (84)$$

281 we obtain for $\mathcal{M}$ and $\mathcal{M}^t$

282
$$\mathcal{M} = \begin{pmatrix} \eta \mathcal{X} & \eta \mathcal{Z} \\ \eta \mathcal{Y} & -\eta \mathcal{X}^t \end{pmatrix} \qquad \mathcal{M}^t = \begin{pmatrix} \mathcal{X}^t \eta & \mathcal{Y} \eta \\ \mathcal{Z} \eta & -\mathcal{X} \eta \end{pmatrix} \quad (85)$$

283 Then for an infinitesimal linear canonical transformation, we have

284
$$(\boldsymbol{p}' \quad \boldsymbol{\varkappa}') = (\boldsymbol{p} \quad \boldsymbol{\varkappa})(1 + \mathcal{M}) = (\boldsymbol{p} \quad \boldsymbol{\varkappa}) \begin{pmatrix} 1 + \eta \mathcal{X} & \eta \mathcal{Z} \\ \eta \mathcal{Y} & 1 - \eta \mathcal{X}^t \end{pmatrix}$$

285
$$\begin{cases} \boldsymbol{p}' = \boldsymbol{p} + \eta \mathcal{X} \boldsymbol{p} + \eta \mathcal{Y} \boldsymbol{\varkappa} \\ \boldsymbol{\varkappa}' = \boldsymbol{\varkappa} + \eta \mathcal{Z} \boldsymbol{p} - \eta \mathcal{X}^t \boldsymbol{\varkappa} \end{cases} \Leftrightarrow \begin{cases} p'_\mu = \boldsymbol{p}_\mu + [\eta \mathcal{X}]_\mu^\nu \boldsymbol{p}_\nu + [\eta \mathcal{Y}]_\mu^\nu \boldsymbol{\varkappa}_\nu \\ x'_\mu = \boldsymbol{\varkappa}_\mu + [\eta \mathcal{Z}]_\mu^\nu \boldsymbol{p}_\nu - [\eta \mathcal{X}^t]_\nu^\mu \boldsymbol{\varkappa}_\nu \end{cases}$$

286
$$\begin{cases} p'_\mu = \boldsymbol{p}_\mu + \eta_{\mu\rho} \mathcal{X}^{\rho\nu} \boldsymbol{p}_\nu + \eta_{\mu\rho} \mathcal{Y}^{\rho\nu} \boldsymbol{\varkappa}_\nu \\ x'_\mu = \boldsymbol{\varkappa}_\mu + \eta_{\mu\rho} \mathcal{Z}^{\rho\nu} \boldsymbol{p}_\nu - \eta_{\mu\rho} \mathcal{X}^{\nu\rho} \boldsymbol{\varkappa}_\nu \end{cases} \quad (86)$$

287 We may introduce a unitary representation of the linear canonical transformation

288

$$\begin{cases} \boldsymbol{p}_\mu^{'} = U\boldsymbol{p}_\mu U^\dagger \\ \boldsymbol{x}_\mu^{'} = U\boldsymbol{x}_\mu U^\dagger \end{cases} \tag{87}$$

289   and we can verify that U and $U^\dagger$ can be   given by the relations

290

$$\begin{cases} U = e^{i(\theta_+^{\rho\lambda}\beth_{\rho\lambda}^+ + \theta_-^{\rho\lambda}\beth_{\rho\lambda}^- + \theta_\times^{\rho\lambda}\beth_{\rho\lambda}^\times)} \\ U^\dagger = e^{-i(\theta_+^{\epsilon\sigma}\beth_{\epsilon\sigma}^+ + \theta_-^{\epsilon\sigma}\beth_{\epsilon\sigma}^- + \theta_\times^{\epsilon\sigma}\beth_{\epsilon\sigma}^\times)} \end{cases} \tag{88}$$

291   In fact, for an infinitesimal transformation

292

$$\begin{cases} U = 1 + i(\theta_+^{\rho\lambda}\beth_{\rho\lambda}^+ + \theta_-^{\rho\lambda}\beth_{\rho\lambda}^- + \theta_\times^{\rho\lambda}\beth_{\rho\lambda}^\times) \\ U^\dagger = 1 - i(\theta_+^{\epsilon\sigma}\beth_{\epsilon\sigma}^+ + \theta_-^{\epsilon\sigma}\beth_{\epsilon\sigma}^- + \theta_\times^{\epsilon\sigma}\beth_{\epsilon\sigma}^\times) \end{cases} \tag{89}$$

293   So (87) become

294

$$\begin{cases} \boldsymbol{p}_\mu^{'} = \boldsymbol{p}_\mu + i\theta_+^{\rho\lambda}[\beth_{\rho\lambda}^+, \boldsymbol{p}_\mu]_- + i\theta_-^{\rho\lambda}[\beth_{\rho\lambda}^-, \boldsymbol{p}_\mu]_- + i\theta_\times^{\rho\lambda}[\beth_{\rho\lambda}^\times, \boldsymbol{p}_\mu]_- \\ \boldsymbol{x}_\mu^{'} = \boldsymbol{x}_\mu + i\theta_+^{\rho\lambda}[\beth_{\rho\lambda}^+, \boldsymbol{x}_\mu]_- + i\theta_-^{\rho\lambda}[\beth_{\rho\lambda}^-, \boldsymbol{x}_\mu]_- + i\theta_\times^{\rho\lambda}[\beth_{\rho\lambda}^\times, \boldsymbol{x}_\mu]_- \end{cases} \tag{90}$$

295   Then, taking into account the relations (71) and (72), we obtain

296

$$\begin{cases} \boldsymbol{p}_\mu^{'} = \boldsymbol{p}_\mu + \frac{1}{2}\eta_{\rho\mu}(\theta_+^{\rho\nu} - \theta_-^{\rho\nu})\boldsymbol{x}_\nu + \frac{1}{2}\theta_\times^{\nu\rho}\eta_{\rho\mu}\boldsymbol{p}_\nu \\ \boldsymbol{x}_\mu^{'} = \boldsymbol{x}_\mu - \frac{1}{2}\eta_{\rho\mu}(\theta_+^{\rho\nu} + \theta_-^{\nu\rho})\boldsymbol{p}_\nu - \frac{1}{2}\eta_{\rho\mu}\theta_\times^{\rho\nu}x_\nu \end{cases} \tag{91}$$

297   Indefying the relations (86) and (91) gives

298

$$\begin{cases} \mathcal{X}^{\rho\nu} = \frac{1}{2}\theta_\times^{\nu\rho} \\ \mathcal{Y}^{\rho\nu} = \frac{1}{2}(\theta_+^{\rho\nu} - \theta_-^{\rho\nu}) \\ \mathcal{Z}^{\rho\nu} = -\frac{1}{2}(\theta_+^{\rho\nu} + \theta_-^{\rho\nu}) \end{cases} \Leftrightarrow \begin{cases} \mathcal{X} = \frac{1}{2}\theta_\times^t \\ \mathcal{Y} = \frac{1}{2}(\theta_+ - \theta_-) \\ \mathcal{Z} = -\frac{1}{2}(\theta_+ - \theta_-) \end{cases} \tag{92}$$

299   then the relations in (83) and (85) become

300

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_3 \\ \mathcal{M}_2 & -\eta\mathcal{M}_1^t\eta \end{pmatrix} = \begin{pmatrix} \eta\mathcal{X} & \eta\mathcal{Z} \\ \eta\mathcal{Y} & -\eta\mathcal{X}^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \eta\theta_\times^t & -\eta(\theta_+ + \theta_-) \\ \eta(\theta_+ - \theta_-) & -\eta\theta_\times \end{pmatrix} \tag{93}$$

301

$$\mathcal{M}^t = \begin{pmatrix} \mathcal{M}_1^t & \eta\mathcal{M}_2\eta \\ \eta\mathcal{M}_3\eta & -\eta\mathcal{M}_1\eta \end{pmatrix} = \begin{pmatrix} \mathcal{X}^t\eta & \mathcal{Y}\eta \\ \mathcal{Z}\eta & -\mathcal{X}\eta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \theta_\times\eta & (\theta_+ - \theta_-)\eta \\ -(\theta_+ - \theta_-)\eta & -\theta_\times^t\eta \end{pmatrix} \tag{94}$$

302   So briefly, we have for a linear canonical transformation in the case of multidimensional theory
303

304

$$\begin{cases} \boldsymbol{p}_\mu^{'} = \Pi_\mu^\nu\boldsymbol{p}_\nu + \Theta_\mu^\nu\boldsymbol{x}_\nu = U\boldsymbol{p}_\mu U^\dagger \\ \boldsymbol{x}_\mu^{'} = \Xi_\mu^\nu\boldsymbol{p}_\nu + \Lambda_\mu^\nu\boldsymbol{x}_\nu = U\boldsymbol{x}_\mu U^\dagger \end{cases} \Leftrightarrow (\boldsymbol{p}^{'} \quad \boldsymbol{x}^{'}) = (\boldsymbol{p} \quad \boldsymbol{x})\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} \tag{95}$$

305   with

306

$$\begin{pmatrix} \Pi & \Xi \\ \Theta & \Lambda \end{pmatrix} = e^{\mathcal{M}} = e^{\frac{1}{2}\begin{pmatrix} \eta\theta_\times^t & -\eta(\theta_+ + \theta_-) \\ \eta(\theta_+ - \theta_-) & -\eta\theta_\times \end{pmatrix}} \tag{96}$$

307   and

308

$$U = e^{i(\theta_+^{\rho\lambda}\beth_{\rho\lambda}^+ + \theta_-^{\rho\lambda}\beth_{\rho\lambda}^- + \theta_\times^{\rho\lambda}\beth_{\rho\lambda}^\times)} \tag{97}$$

309   **5. Conclusions**

310   The results obtained in this paper show that the introduction of the dispersion operators algebra
311   permits to perform a natural and well description of the link which can be established between the

312 phase space representation of quantum mechanics and linear canonical transformation. This link is a

313 consequence of the existence of relationship between dispersion operators and the phase space

314 representation on one hand and dispersion operator algebra and linear canonical transformation on

315 the other hand. The phase space representation is built with the eigeinstates $|n, X, P, \mathcal{b}\rangle$ of dispersion

316 operators; linear canonical transformation can be represented using the dispersion operators algebra.

317 The relations (32), (33) and (53) allows to conclude that a right way to describe and to represent linear

318 canonical transformation over state space in framework of quantum mechanics is to use the basis

319 $\{|n, X, P, \mathcal{b}\rangle\}$.

320   The calculations performed in the section 4 show that these main results obtained for the case of

321 one dimension quantum mechanics may be generalized in the case of multidimensional theory.

322   The results that we have established in this paper may have many interesting applications in

323 various scientific and technical fields.

324 **Conflicts of Interest:** The authors declare no conflict of interest.

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