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A zeroth law compatible model to Kerr black hole thermodynamics

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Abstract: We consider the thermodynamic and stability problem of Kerr black holes arising from the nonextensive/nonadditive nature of the Bekenstein-Hawking entropy formula. Nonadditive thermodynamics is often criticized by asserting that the zeroth law cannot be compatible with nonadditive composition rules, so in this work we follow the so-called formal logarithm method to derive an additive entropy function for Kerr black holes satisfying also the zeroth law's requirement. Starting from the most general, equilibrium compatible, nonadditive entropy composition rule of Abe, we consider the simplest, non-parametric approach that is generated by the explicit nonadditive form of the Bekenstein-Hawking formula. This analysis extends our previous results on the Schwarzschild case and shows that the zeroth law compatible temperature function in the model is independent of the mass-energy parameter of the black hole. By applying the Poincaré turning point method we also study the thermodynamic stability problem in the system.

Keywords: black holes; entropy; nonextensive thermodynamics; stability

1. Introduction

After the finding of the Kerr solution [1], the mathematical and physical properties of the corresponding black hole spacetimes have been widely investigated. As a culmination of the work done in the first decade, the four laws of black hole mechanics were formulated by Bardeen, Carter and Hawking [2]. These four laws show a striking similarity to the laws of thermodynamics and can be summarized as the following:

- The zeroth law of black hole mechanics states that the surface gravity κ of a stationary black hole is constant over the horizon, which is essentially the requirement of transitivity of the equilibrium state.
- The first law manifests a relation between variations in the mass M, horizon area A and angular momentum J if the black hole is perturbed,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_{\rm H}\delta J,$$

  where $\Omega_{\rm H}$ is the angular velocity of the horizon.
- The second law of black hole mechanics is Hawking's area theorem which states that the surface area of the event horizon never decreases with time,

$$\delta A \geq 0.$$

- The third law is formulated by stating that it is impossible to achieve $\kappa = 0$ in a finite series of physical processes.

The close analogy between the four laws of black hole mechanics and the laws of ordinary thermodynamics is intriguing. In this analogy, the mass of the black hole mathematically correspond to the energy of a thermodynamic system, the area of the horizon to the entropy and the surface gravity to the temperature. While the correspondence between mass and energy is a physical identity, the other two correspondences are only analogy in classical general relativity. Classical black holes have zero temperature and the area of the horizon has a dimension of length squared. With the physical penetration of the black hole entropy by Bekenstein [3] and the semi-classical derivation of black hole radiation by Hawking [4], we can treat black holes as thermodynamic systems with the Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{1}{4} A,$$

at the Hawking temperature

$$T_{\text{H}} = \frac{\kappa}{2\pi}.$$

Investigations in gravitating systems have pointed out that entropy functions with nonextensive nature tend to appear in systems with long-range interactions like gravity (see e.g [5] and references therein). The Bekenstein-Hawking black hole entropy seems also to be nonextensive because it is proportional to the area of the horizon instead of being proportional to the black hole volume (although the problem of how to compute the volume of a general black hole and how it is related to the horizon area is also a nontrivial question [6]). The nonextensive nature of black hole thermodynamics has been studied with many approaches and various methods [7–16], and in the present paper we also consider a simple model to the Kerr problem that we have already investigated for the Schwarzschild case [17]. The main idea behind this approach is to study the system in equilibrium while also requiring to satisfy the zeroth law of thermodynamics.

As a fundamental formula, we start with the most general functional form of nonadditive entropy composition rules (derived by Abe [18]) requiring only the existence of equilibrium in the system. Assuming that the total entropy $S(A, B)$ is given by the entropies of the subsystems $S(A)$ and $S(B)$ as $S(A, B) = f(S(A), S(B))$ where f is a C^2 class symmetric function, Abe derived the most general pseudoadditivity rule for the entropy,

$$H_\lambda(S(A, B)) = \frac{H_\lambda(S(A)) + H_\lambda(S(B)) + \lambda H_\lambda(S(A)) H_\lambda(S(B)) - H_\lambda(0)}{1 + \lambda H_\lambda(0)}, \tag{1}$$

where $H_\lambda(S)$ is a differential function of S and $\lambda \in \mathbb{R}$ is a constant parameter. By setting $H_\lambda(0)$ to zero with the physical requirement of zero joint entropy for zero subsystem entropies, Abe's equation becomes

$$H_\lambda(S(A, B)) = H_\lambda(S(A)) + H_\lambda(S(B)) + \lambda H_\lambda(S(A)) H_\lambda(S(B)). \tag{2}$$

Obviously, for standard additive thermodynamic systems $H_\lambda(S)$ is the identity function and the λ parameter is zero. In more general cases however, equilibrium compatibility requires the nonadditive entropy composition rules to satisfy eq. (2) for any thermodynamic system.

Maybe the most well-known example for a nonstandard case was proposed by Tsallis in 1988 [19], where he introduced an entropic function

$$S_q = \frac{1 - \sum p_i^q}{q - 1}, \tag{3}$$

by generalizing the Boltzmann-Gibbs formula $S_{\text{BG}} = -\sum_i p_i \ln p_i$. Based on his nonadditive entropy S_q, Tsallis has also generalized the standard Boltzmann-Gibbs theory to nonextensive statistical mechanics [20]. It is easy to see that the Tsallis entropy recovers the Boltzmann-Gibbs entropy for

$q \to 1$, and when we consider two independent systems A and B, the Tsallis entropy of the composed system satisfies

$$S_q(A, B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B). \tag{4}$$

Therefore the Tsallis composition rule (4) satisfies Abe's formula (2) when $H_\lambda(S)$ is the identity function and $\lambda = 1 - q$, so it is an equilibrium compatible nonadditive composition rule.

One of the crucial criticism of nonextensive thermodynamics is that the transitivity property of thermal equilibrium and the definition of an empirical temperature (i.e. the zeroth law of thermodynamics) do not seem to be compatible with nonadditive entropy composition rules. In standard thermodynamics the empirical temperature is defined as

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{5}$$

This definition is derived from total energy conservation and maximum entropy postulate in the system with the assumption of additivity of these two thermodynamic variables. In a recent paper [21], Biró and Ván generalized this derivation to the case of nonadditive composition laws of the entropy and the energy functions as well. In their analysis, by keeping the maximum entropy principle and the total energy conservation, they found that a new, zeroth law compatible temperature can be obtained from the more general definition

$$\frac{1}{T} = \frac{\partial L(S)}{\partial \hat{L}(E)}, \tag{6}$$

where L and $\hat{L}$ are the so-called *formal logarithm* functions of the nonadditive thermal variables, which map the original (nonadditive) composition laws to additive ones, i.e.

$$L(S(A, B)) = L(S(A)) + L(S(B)), \tag{7}$$
$$\hat{L}(E(A, B)) = \hat{L}(E(A)) + \hat{L}(E(B)). \tag{8}$$

It can be easily shown that the formal logarithm of the Tsallis entropy (3)

$$L(S_q) = \frac{1}{1 - q} \ln[1 + (1 - q)S_q] \equiv S_R, \tag{9}$$

is another well-known parametric generalization of the Boltzmann-Gibbs formula, namely the Rényi entropy function

$$S_R = \frac{1}{1 - q} \ln \sum_i p_i^q, \tag{10}$$

which has been introduced by the Hungarian mathematician, Alfréd Rényi in the middle of the last century [22,23]. According to the properties of the formal logarithm mapping, and as opposed to the Tsallis formula, the Rényi entropy is additive for factorizing probabilities, so the corresponding temperature function (obtained by assuming energy additivity in the system)

$$\frac{1}{T_R} = \frac{\partial L(S_q)}{\partial E} = \frac{\partial S_R}{\partial E}, \tag{11}$$

is a well defined empirical temperature which is compatible with all laws of thermodynamics, including the zeroth's law as well.

Recently, based on the above formal logarithm approach to the nonextensive problem of black hole thermodynamics, we investigated the thermodynamic [24] and stability [25] properties of the Schwarzschild solution. In the analysis of [24], we regarded the Bekenstein-Hawking entropy of black hole event horizons as a nonextensive Tsallis entropy, which, for small energies and for small

parameter values ($\lambda = 1 - q$), was assumed to follow the nonadditive Tsallis composition law (4). As a result, the corresponding zeroth law compatible entropy function became the Rényi entropy, and the resulting temperature exhibited some interesting features. Most notably, we observed that the relation between the temperature and the mass-energy parameter of the black hole had the same form as the one obtained from a black hole in AdS space by using the original Boltzmann-Gibbs entropy formula. The temperature has a minimum in both cases, and from the temperature – mass-energy relation we could derive a correspondence between the nonextensivity parameter (λ) of the Schwarzschild-Rényi model and the AdS curvature parameter.

In [25], we investigated the thermodynamic stability of the problem. In the standard picture, the Schwarzschild black hole seems to be thermodynamically unstable in the canonical treatment because the heat capacity is always negative in the Bekenstein-Hawking approach. However the standard stability analysis of extensive systems is generally not applicable to black holes since it strongly depends on the additivity of the entropy function [26]. To avoid the inapplicability of the standard Hessian analysis to nonadditive systems, Kaburaki et al. [27] used the Poincaré turning point method [28] to investigate the thermodynamic stability of black holes. This method is a topological approach which does not depend on the additivity of the entropy function, and it has been widely applied to problems in astrophysical and gravitating systems [26,27,29–31] where, as we have already mentioned, entropy functions with nonextensive nature tend to appear frequently due to the long-range interaction property of the gravitational field. By applying the Poincaré approach one can easily separate stable and unstable configurations for cases of a one-parameter series of equilibria. Here we briefly summarize the essence of this method as we will also use it in the next section. For more details however, please consult with the original references.

Suppose $Z(x^i, y)$ is a distribution function whose extrema $\partial Z / \partial x^i = 0$ define stable equilibrium configurations if the extremal value of Z is a maximum. Consider now the equilibrium value $Z(y) = Z[X^i(y), y]$, where $X^i(y)$ is a solution of $\partial Z / \partial x^i = 0$. If the derivative function dZ/dy plotted versus y has the topology of a continuous and differentiable curve, it can be shown that stability changes will occur only at points where the tangents of the curve are vertical. The distribution function Z is called *Massieu function*, y is called *control parameter*, and the points with vertical tangents are called *turning points*. The branch with negative slope near the turning point is always unstable, while the branch with positive slope near the turning point is more stable.

In our Schwarzschild-Rényi approach, we could apply both thermodynamic stability analysis methods since the entropy function given by the formal logarithm approach is an additive function. In [25], first we investigated the question of a pure, isolated black hole in the microcanonical ensemble. We showed that these configurations are stable against spherically symmetric perturbations, just like in the classical picture. Next we also investigated the problem of a black hole surrounded by a bath of thermal radiation in the canonical treatment. We found that in contrast to the standard picture, Schwarzschild black holes can be in stable equilibrium with thermal radiation at a fixed temperature in the Rényi approach. We observed that a stability change occurs at a certain value of the mass-energy parameter which belongs to the minimum temperature solution. Smaller mass black holes are unstable while larger mass black holes become stable in this model. These results are similar to the ones obtained by Hawking and Page in AdS space within the standard Boltzmann entropy approach [32]. Motivated by this similarity, we also investigated the question of a possible phase transition in the canonical picture and found, that a Hawking-Page-like black hole phase transition occurs in a very similar manner as in AdS space. The corresponding critical temperature depends only on the Rényi entropy parameter.

To extend the above studies on the Schwarzschild-Rényi model, we have also examined the thermodynamic and stability properties of Kerr black holes within the Tsallis-Rényi approach [33]. We analyzed the thermodynamic stability of the problem in both the microcanonical and canonical ensemble and found a stability change in the latter case. We showed that a Hawking-Page-like phase transition and a first order small black hole/large black hole transition occur in this system analogous

to the ones of rotating black holes in AdS space [34–36]. By this we confirmed the similarity between the Rényi-asymptotically flat and Boltzmann-AdS approaches to black hole thermodynamics in the rotating case as well.

Abe's general equilibrium compatible entropy composition formula (2), and the formal logarithm approach is not restricted of course to the Tsallis-Rényi model only. In [21], Biró and Ván has derived the formal logarithm map to all of those nonadditive entropy functions which satisfy Abe's composition law, and showed that the most general form of equilibrium and zeroth law compatible entropy functions reads as

$$L(S) = \frac{1}{\lambda} \ln[1 + \lambda H_\lambda(S)]. \tag{12}$$

The Tsallis-Rényi model in this framework is an interesting approach to the nonextensive problem of black hole thermodynamics, however there is another relevant nonadditive composition rule which we should also consider in connection with the problem, namely the original nonadditive composition rule arising from the Bekenstein-Hawking entropy expression. Indeed, we started with the question of the nonextensivity/nonadditivity of the standard black hole entropy function, but we haven't discussed whether it can satisfy the equilibrium compatibility condition of Abe's formula yet. In a recent paper [17], we have investigated the thermodynamic and stability problem of a Schwarzschild black hole by considering the nonadditive entropy composition rule that is emerging from the Bekenstein-Hawking description. We showed that in this case the entropy composition rule of the Schwarzschild problem takes the form

$$S(A, B) = S(A) + S(B) + 2\sqrt{S(A)}\sqrt{S(B)}, \tag{13}$$

which, as it can be easily verified, corresponds to the case of $H_\lambda(S) = \sqrt{S}$ and $\lambda = 0$ in Abe's formula (2). Starting from this non-parametric composition rule of Schwarzschild black holes and also applying the formal logarithm method of Biró and Ván, we investigated the thermodynamic and stability of the problem. The new additive entropy function for the black hole was obtained as

$$S \equiv L(S_{\mathrm{BH}}) = 2\sqrt{S_{\mathrm{BH}}}, \tag{14}$$

by mapping the composition law (13) to its formal logarithm. Assuming additivity of the energy function $E(A, B) = E(A) + E(B)$ in the system, we obtained the new, zeroth law compatible entropy formula for Schwarzschild black holes as

$$S = 4\sqrt{\pi}E, \tag{15}$$

which is a linear function of the mass-energy parameter. From definition (6), one obtains the corresponding, zeroth law compatible temperature of the black hole as

$$T \equiv \frac{1}{S'(E)} = \frac{1}{4\sqrt{\pi}}, \tag{16}$$

which is an energy independent constant. Therefore, in this model, all black hole has the same temperature which is independent of their size or mass. In this case, the heat capacity of the hole diverges and the Hessian method can not provide a definite result for the thermodynamic stability of the problem. The Poincaré turning point method, on the other hand, can still be applied, and we could show that pure isolated black holes are thermodynamically stable against spherically symmetric perturbations within this approach.

In the present paper, we extend the non-parametric, zeroth law compatible approach to black hole thermodynamics by considering the problem of rotating black holes. We study the function

$H_\lambda(S)$ arising from the Bekenstein-Hawking formula in the the Kerr case without any parameter but satisfying Abe's composition rule $H_\lambda(S_{BH}(A, B)) = H_\lambda(S_{BH}(A)) + H_\lambda(S_{BH}(B))$. By regarding this $H_\lambda(S_{BH})$ as the new, additive entropy function of the Kerr black holes via the formal logarithm approach, we investigate the thermodynamic and stability properties of the rotating problem.

2. Results

2.1. Kerr black holes

The spacetime metric of a rotating black hole is given by the Kerr solution

$$ds^2 = -dt^2 + \frac{2Mr}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin\theta d\phi^2 \,, \tag{17}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr. \tag{18}$$

Here, M is the mass-energy of the black hole and a is its rotational parameter. The thermodynamic variables of the black hole can be expressed by the horizon radius $r_+ = M + \sqrt{M^2 - a^2}$ and the rotational parameter a. The mass-energy parameter can be written as

$$M = \frac{r_+^2 + a^2}{2r_+}, \tag{19}$$

the angular momentum of the black hole is

$$J = \frac{a}{2r_+}(r_+^2 + a^2), \tag{20}$$

and the angular velocity of the horizon is

$$\Omega = \frac{a}{r_+^2 + a^2}. \tag{21}$$

The Bekenstein-Hawking entropy is

$$S_{\mathrm{BH}} = \pi(r_+^2 + a^2), \tag{22}$$

while the Hawking temperature of the black hole horizon is

$$T_{\mathrm{H}} = \frac{1}{2\pi}\left[\frac{r_+}{r_+^2 + a^2} - \frac{1}{2r_+}\right]. \tag{23}$$

The heat capacity at constant Ω is given by

$$C_\Omega = T_{\mathrm{H}}\left(\frac{\partial S_{\mathrm{BH}}}{\partial T_{\mathrm{H}}}\right)_\Omega = \frac{2\pi r_+^2(a^2 - r_+^2)}{r_+^2 + a^2}, \tag{24}$$

and the heat capacity at constant J is

$$C_J = T_{\mathrm{H}}\left(\frac{\partial S_{\mathrm{BH}}}{\partial T_{\mathrm{H}}}\right)_J = \frac{2\pi(r_+^2 - a^2)(r_+^2 + a^2)^2}{3a^4 + 6r_+^2 a^2 - r_+^4} \,. \tag{25}$$

We plotted the heat capacities C_Ω and C_J as functions of $h \equiv \frac{|a|}{r_+}$ in Fig. 1. C_Ω is negative for all h and C_J diverges at $h = h_c = \sqrt{\frac{2}{3}\sqrt{3} - 1}$. C_J is negative for $h < h_c$ and positive for $h > h_c$. Kaburaki et al. [27] considered the thermodynamic stability of standard Kerr black holes by using the Poincaré turning point method. They concluded that isolated Kerr black holes are thermodynamically stable

with respect to axisymmetric perturbations. They also found that faster rotating black holes ($h > h_c$) are more stable than slower rotating ones ($h < h_c$) in a heat bath.
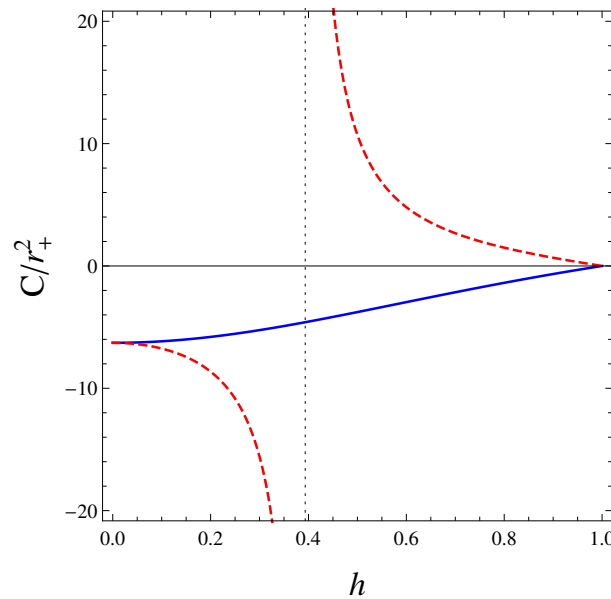

Figure 1. Plots of heat capacities C_Ω (blue solid) and C_J (red dashed) as functions of h.

2.2. The formal logarithm approach

In order to extend the non-parametric results of the Schwarzschild problem obtained by the formal logarithm approach, the following non-trivial equation has to be solved for Kerr black holes,

$$H(S_{BH}(A, B)) = H(S_{BH}(A)) + H(S_{BH}(B)), \tag{26}$$

where, and from now on, we omit the λ index of H for obvious reasons, and S_{BH} is the Bekenstein-Hawking entropy of Kerr black holes. Clearly the simple $H(S) = \sqrt{S}$ Schwarzschild solution is not working in the rotating case, but after some effort one can show that

$$H(S) \equiv L(S_{BH}) = \frac{2S_{BH}}{\sqrt{S_{BH} - \pi a^2}} = 2\sqrt{\pi}\frac{r_+^2 + a^2}{r_+}, \tag{27}$$

can describe the Kerr solution, where we have fixed the overall factor of S to reproduce the Schwarzschild result (14) in the $a \to 0$ limit. Comparing this result with the mass-energy parameter of the black hole (19), we can conclude that the new, equilibrium and zeroth law compatible entropy function of Kerr black holes obtained by the formal logarithm method based on the nonadditive and non-parametric approach arising from the Bekenstein-Hawking model is the same linear function of the energy,

$$S = 4\sqrt{\pi}E, \tag{28}$$

as in the Schwarzschild problem. By assuming energy additivity in the system as before, the corresponding zeroth law compatible temperature function from (6) becomes

$$T = \frac{1}{4\sqrt{\pi}}, \tag{29}$$

which is again the same energy independent constant that we obtained in the Schwarzschild case. Furthermore, both heat capacities, C_Ω and C_J diverge, and the entropy and temperature functions versus the energy of the hole are plotted in Figs. 2 and 3 respectively.
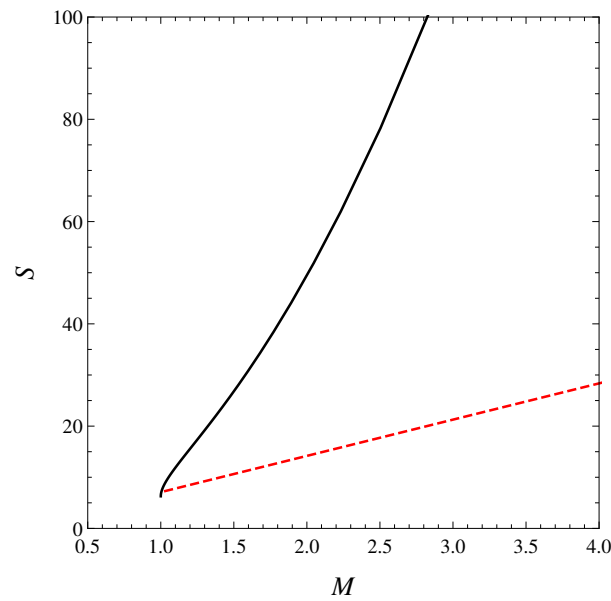
Figure 2. Entropy vs. mass-energy, standard (black solid) and formal logarithm (red dashed) approaches.
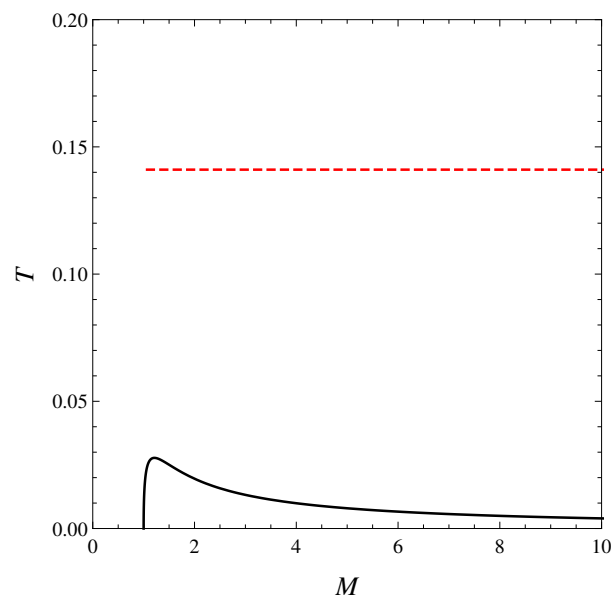

Figure 3. Temperature vs. mass-energy, standard (black solid) and formal logarithm (red dashed) approaches.

2.3. Stability analysis

Although the new, formal logarithmic entropy function (27) of Kerr black holes is additive for composition, the Hessian method for stability analysis can not be used because, just like in the Schwarzschild problem, the heat capacities are divergent. Nevertheless, the Poincaré method can still provide information about the stability of the system. In cases when the black holes are isolated from their surroundings, the thermodynamic problem can be described in the microcanonical ensemble. Here the Massieu function of the system is the entropy function, and the corresponding control parameters are the mass-energy, M and the angular momentum, J. The derivative functions are $\beta = \frac{1}{T} = \frac{\partial S}{\partial M}$ and $-\alpha = \frac{\Omega}{T} = \frac{\partial S}{\partial J}$, and we plotted the $\beta(M)$ stability curves at constant J and the

$-\alpha(J)$ stability curves at constant M on Figs. 4 and 5, respectively. One can see on the plots that in the standard thermodynamic treatment there are no turning points on the curves, therefore, Kaburaki et al. [27] concluded that isolated Kerr black holes are stable against axisymmetric perturbations.

In the case of the formal logarithm approach, the stability curve $\beta(M)$ is simply a horizontal line while $-\alpha(J)$ is a monotonically decreasing convex curve. There are no turning points (i.e. points with vertical tangents) on these curves either, so based on these results we can conclude that isolated Kerr black holes are thermodynamically stable against axisymmetric perturbations in the formal logarithm approach as well.
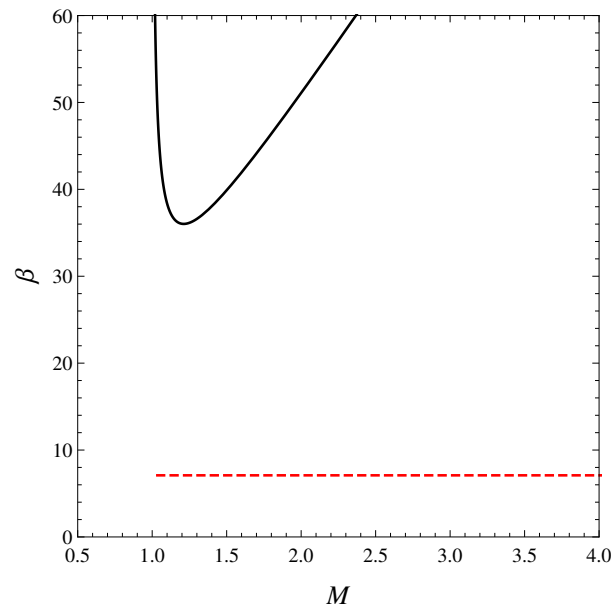

Figure 4. Plots of stability curves $\beta(M)$ at fixed J of standard (black solid) and formal logarithm (red dashed) approaches in the microcanonical treatment. By rotating the figure clockwise with $\frac{\pi}{2}$, the stability curves of the canonical treatment can be obtained, i.e. $-M(\beta)$ at fixed J.
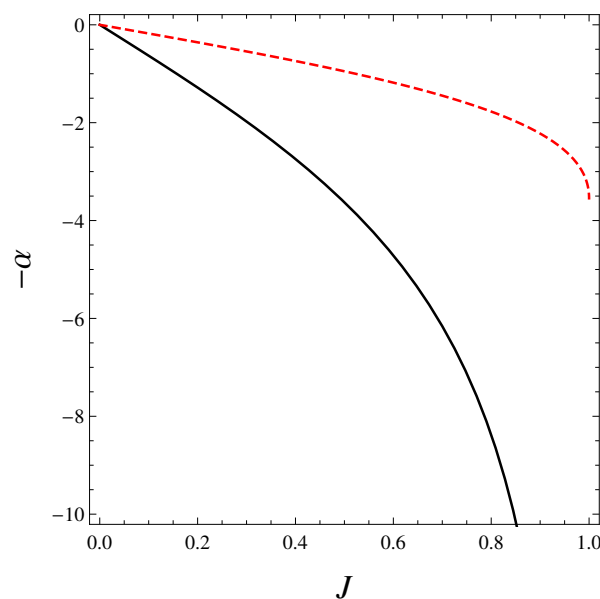

Figure 5. Plots of stability curves $-\alpha(J)$ at fixed M of standard (black solid) and formal logarithm (red dashed) approaches in the microcanonical treatment.

The canonical ensemble depicts the system of a black hole in equilibrium with an infinite reservoir of thermal radiation at constant temperature. The Massieu function in this case is $\Psi(\beta, J) = S - \beta M$ and the control parameters are β and J. The pair of conjugate variables are the functions $-M(\beta)$ at fixed J and $-\alpha(J)$ at fixed β. The stability curve $-M(\beta)$ at constant J for the standard case can be obtained by clockwise rotating Fig. 4 with $\pi/2$. The stability curve $-\alpha(J)$ at fixed β in the standard treatment is plotted in Fig. 6. In this case, the stability curve exhibits a turning point where a stability change occurs in the system. Based on these observations, Kaburaki et al. [27] concluded that fast rotation can stabilize black holes in a heat bath.
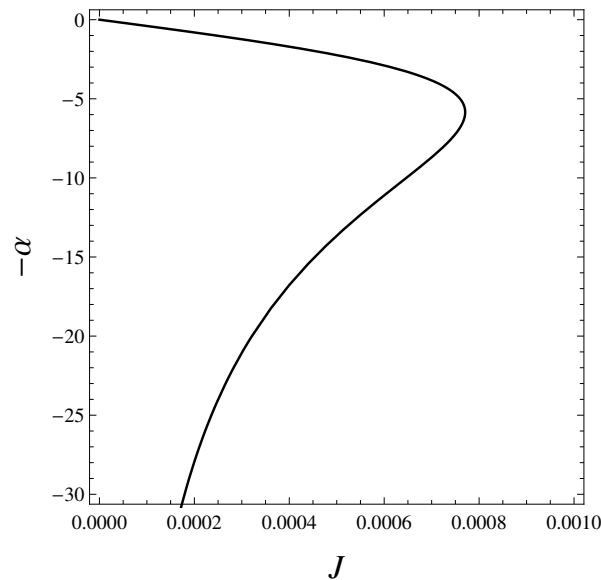

Figure 6. Stability curve $-\alpha(J)$ at fixed β of the standard case in the canonical treatment.

For the formal logarithm approach, the Massieu function Ψ turns out to be identically zero in the canonical case, so we can conclude that all black hole configurations are equally probable or improbable in this model.

3. Discussion

In this paper we considered an equilibrium and zeroth law compatible approach to the thermodynamic and stability problem of rotating black holes in standard 4 dimensions. The model we adopted was based on the nonadditive entropy composition law of Kerr black holes arising from the standard Bekenstein-Hawking entropy function. By solving Abe's formula (2) for $H_\lambda(S)$ with zero λ parameter and applying the formal logarithm method of Biró and Ván [21], we derived the following zeroth law compatible entropy function for Kerr black holes

$$L(S_{\mathrm{BH}}) = H_{\lambda=0}(S_{\mathrm{BH}}) = \frac{2S_{\mathrm{BH}}}{\sqrt{S_{\mathrm{BH}} - \pi a^2}}. \tag{30}$$

We showed that the above entropy expression is a linear function of the black hole's energy, and also that the corresponding temperature function must be a constant, i.e. independent of the mass-energy parameter of the black hole. These results are exactly the same what we had previously found in studying the Schwarzschild solution by using the same approach [17].

In the analysis of the thermodynamic stability problem, we have found some strange and presumably unphysical results. These are e.g. diverging heat capacities, a horizontal stability curve and a Massieu function which is always zero. Indeed, after some careful analysis of the present model, one can show that the $H_{\lambda=0}(S_{\mathrm{BH}})$ function of the Bekenstein-Hawking entropy in the system

is always the mass-energy parameter of the black hole up to some constant factor, so the new entropy function can only result a constant temperature, no matter what complicated black hole spacetime is considered. Comparing these findings with our previous results on the problem by using the parametric Tsallis-Rényi approach [24,25,33], we conclude that the present non-parametric approach to black hole thermodynamics based on the Bekenstein-Hawking entropy nonadditivity is most likely an unrealistic one. Based on this conclusion it is more reasonable to further study the parametric, $\lambda \neq 0$ case, where the zeroth law compatible temperature function of the system is not independent of the black hole mass.

In the Tsallis-Rényi approach, the formal logarithm entropy function $L(S)$ satisfies the triviality condition $L(0) = 0$ and the leading order additivity $L'(0) = 1$ requirement. These conditions are discussed in more detail by Biró and Ván in [21]. In the black hole problem however, the leading order additivity condition $L'(0) = 1$ is not necessarily a natural choice when one considers the classical, not the close to quantum limit of black holes. We intend to study this direction in our future investigations by relaxing the leading order additivity condition.

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