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The Particle as a Statistical Ensemble of Events in Stueckelberg-Horwitz-Piron Electrodynamics †

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Abstract: Stueckelberg-Horwitz-Piron (SHP) electrodynamics formalizes the distinction between coordinate time (measured by laboratory clocks) and chronology (temporal ordering) by defining 4D spacetime events x^μ as functions of an external evolution parameter τ. Classical spacetime events $x^\mu(\tau)$ evolve as τ grows monotonically, tracing out particle worldlines dynamically and inducing the *five* U(1) gauge potentials through which events interact. Since Lorentz invariance imposes time reversal symmetry on x^0 but not τ, the formalism resolves grandfather paradoxes and related problems of irreversibility. The action involves standard first order field derivatives but includes a higher order τ derivative that while preserving gauge and Lorentz invariance removes certain singularities and makes the related QFT super-renormalizable. The resulting field equations are Maxwell-like but τ-dependent and sourced by a current that represents a statistical ensemble of instantaneous events distributed along the worldline. The width λ of this distribution defines a correlation time for the interactions and a mass spectrum for the photons that carry the interaction. As λ becomes very large, the photon mass goes to zero and the field equations become τ-independent Maxwell's equations. Maxwell theory thus emerges as an equilibrium limit of SHP, in which λ is larger than any other relevant time scale. Particles and fields are not constrained to mass shells in SHP theory, and by exchanging mass may produce pair creation/annihilation processes at the classical level. On-shell evolution with fixed particle masses is restored through a self-interaction associated with the 5D wave equation.

Keywords: classical gauge theory; pair creation/annihilation; temporal paradoxes

1. Introduction

In developing his interpretation of antiparticles as particles traveling backward in time, Stueckelberg [1,2] hoped to demonstrate that pair creation/annihilation processes would appear naturally in a thoroughly deterministic and classical relativistic Hamiltonian mechanics, once such a formalism could be properly constructed. In this picture, a particle is the worldline traced out by a spacetime event $x^\mu(\tau)$ that evolves dynamically as the Poincaré invariant parameter proceeds monotonically from $\tau = -\infty$ to $\tau = \infty$. Stueckelberg argued that pair annihilation is observed when the coordinate time $x^0(\tau)$ reverses direction, because for some values $x_2^0 > x_1^0$ on the laboratory clock there will be two solutions to $x^0(\tau) = x_1^0$ but no solution to $x^0(\tau) = x_2^0$. He proposed a covariant evolution equation of the form

$$\frac{D}{D\tau}\dot{x}^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{e}{M}\left[F^{\mu\nu}(x)g_{\nu\rho}\frac{dx^\rho}{d\tau} + G^\mu(x)\right] \tag{1}$$

in which the vector field $G^\mu(x)$ is required to overcome the mass-shell constraint

$$\frac{D}{D\tau}\left(\frac{1}{2}M\dot{x}^2\right) = M\dot{x}_\mu \frac{D\dot{x}^\mu}{D\tau} = e\dot{x}_\mu G^\mu(x) \xrightarrow[G^\mu \to 0]{} 0 \tag{2}$$

of standard relativity. By keeping $\dot{x}^2$ from changing sign, this constraint prevents the event from crossing the spacelike region that separates future-oriented trajectories from past-oriented trajectories. However, Stueckelberg was not satisfied that he could justify a Hamiltonian K that produces his evolution equation in flat spacetime from the unconstrained symplectic equations

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \frac{dp^\mu}{d\tau} = \dot{p}^\mu = -\frac{\partial K}{\partial x_\mu} \tag{3}$$

and instead continued his program in quantum mechanics, where, as in Feynman's spacetime diagrams, the event may tunnel probabilistically across the spacelike region.

Horwitz and Piron [3] returned to some of these questions in constructing a canonical relativistic mechanics for the two-body problem. Introducing an invariant scalar interaction of the type

$$K = \frac{p_1^2}{2M_1} + \frac{p_2^2}{2M_2} + V\left(|x_1 - x_2|\right) \tag{4}$$

and replacing

$$R = \sqrt{(\mathbf{x_1} - \mathbf{x_2})^2} \qquad \longrightarrow \qquad \rho = \sqrt{(\mathbf{x_1} - \mathbf{x_2})^2 - (t_1 - t_2)^2} \tag{5}$$

in the argument of nonrelativistic scalar potentials, Horwitz et. al. found solutions for relativistic generalizations of the standard central force problems, including quantum mechanical potential scattering and bound states [4–8]. Examination of radiative transitions [9–11] associated with these bound states suggests that the scalar interaction V is required along with the four-vector potential A^μ in order to account for known phenomenology. Although the electromagnetic tensor interaction produced by $F^{\mu\nu}$ leaves individual particle masses invariant, the scalar interaction, which by way of (3) leads to the vector field $G^\mu = -\partial K/\partial x_\mu$ proposed by Stueckelberg, permits mass exchange in such transitions.

A related issue arises in defining the interaction picture in quantum field theory, required for the Dyson time-ordered perturbation expansion. On the one hand, the Haag theorem proves that any field obtained by unitary transformation of a free field must itself be a free field. On the other hand, the interacting fields of perturbation theory are defined by acting on free fields with a unitary transformation generated by an interaction Hamiltonian. To resolve this apparent contradiction, Seidewitz has observed [12] that Haag's proof relies on transformations generated by a Hamiltonian which is the 0-component of a four-vector and parameterized by x^0. He demonstrates that the construction of standard QFT in the Stueckelberg framework leads to an interaction picture obtained by acting on free fields with a unitary transformation generated by a scalar Stueckelberg Hamiltonian, as for example in equation (11) below, and parameterized by τ. Because this generator is invariant under Lorentz transformations, a crucial step in the Haag theorem is inapplicable, and the no-go result is averted. It is worth noting that even if the Stueckelberg Hamiltonian is τ-independent, the interaction picture Hamiltonian will depend on τ and so permit mass exchange among particles and fields (just as a t-dependent nonrelativistic Hamiltonian permits nonconservation of energy).

Sa'ad, Horwitz, and Arshansky [13] found a more fundamental justification for the scalar field by studying the gauge invariance associated with (4). It has been shown [14] that the most general classical interaction consistent with the unconstrained quantum commutation relations

$$[x^\mu, x^\nu] = 0 \qquad m[x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu}(x) \tag{6}$$

is given by Stueckelberg's evolution equation (1) with the substitutions

$$F^{\mu\nu}(x) \to f^{\mu\nu}(x,\tau) \qquad\qquad G^\mu(x) \to f^{5\mu}(x,\tau), \tag{7}$$

and that in flat Minkowski space this system is equivalent to the Lagrangian

$$L = \dot{x}^\mu p_\mu - K = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} \phi(x,\tau) \tag{8}$$

where

$$f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu \qquad\qquad f^{5\mu} = \partial_\tau a^\mu - \partial^\mu \phi. \tag{9}$$

This classical Lagrangian is unique up to the τ-dependent gauge transformations

$$a_\mu(x,\tau) \to a_\mu(x,\tau) + \partial_\mu \Lambda(x,\tau) \qquad\qquad \phi(x,\tau) \to \phi(x,\tau) + \partial_\tau \Lambda(x,\tau) \tag{10}$$

and the associated quantum mechanics

$$i\hbar \partial_\tau \psi(x,\tau) = K\psi(x,\tau) = \left[\frac{1}{2M} \left(p^\mu - \frac{e}{c} a^\mu \right) \left(p_\mu - \frac{e}{c} a_\mu \right) - \frac{e}{c} \phi \right] \psi(x,\tau) \tag{11}$$

admits the additional invariance

$$\psi(x,\tau) \to \exp\left[\frac{ie}{\hbar c} \Lambda(x,\tau) \right] \psi(x,\tau) \tag{12}$$

when taken together with (10).

The generalization of Stueckelberg's framework to include τ-dependent fields and gauge transformations thus succeeds in implementing his model of pair processes in classical mechanics, but also raises new questions. Perhaps most significantly, while defining the system in an unconstrained 8D phase space relaxes the *a priori* mass shell relation $\dot{x}^2 = 1$ and thus permits classical trajectories that reverse the direction of their time evolution, it also eliminates reparametrization invariance[1]. Moreover, in Stueckelberg-Horwitz-Piron (SHP) electrodynamics the evolution parameter τ cannot be identified as the proper time of the motion, but is a dynamical quantity proportional to it through

$$c^2 \, ds^2(\tau) = -g_{\mu\nu} dx^\mu dx^\nu = -\dot{x}^2(\tau) \, d\tau^2 \qquad\qquad g_{\mu\nu} = \text{diag}(-1,1,1,1). \tag{13}$$

Therefore, the parameter τ plays the role of an irreducible chronological time, independent of the spacetime coordinates and similar to the external time t in nonrelativistic Newtonian mechanics. It determines the temporal ordering of events, the order of their physical occurrence, which may be different from the order of observed coordinate times x^0 registered by laboratory clocks as the events appear in measuring apparatus. For example, in a classical, continuous version of a Feynman spacetime diagram, the event trajectory

$$x(\tau) = \left(c \left(t_0 + \dot{t}_0 \tau - \tfrac{1}{2} \ddot{t}_0 \tau^2 \right), \dot{\mathbf{x}}(\tau) \right) \tag{14}$$

where t_0, $\dot{t}_0$ and $\ddot{t}_0$ are positive constants, reverses time direction at $\tau_0 = \dot{t}_0 / \ddot{t}_0$, and represents a pair annihilation process. Singling out three events in their τ-chronological order of occurrence,

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[1]   The mass shell constraint and reparametrization invariance are related features of a Lagrangian that is homogeneous of first degree in the velocities, which is not the case for (8.

$$
\begin{array}{lll}
\tau = 0 & x^0 = ct_0 & \dot{x}^0 = c\dot{t}_0 \\
\tau = \tau_0 & x^0 = c\left(t_0 + \dot{t}_0^2/2\ddot{t}_0\right) & \dot{x}^0 = 0 \\
\tau = 2\tau_0 & x^0 = ct_0 & \dot{x}^0 = -c\dot{t}_0
\end{array}
\tag{15}
$$

the laboratory apparatus will first record both particle and antiparticle trajectories at $x^0 = ct_0$, then the annihilation event at $x^0 = c\left(t_0 + \dot{t}_0^2/2\ddot{t}_0\right)$ and no particles for subsequent values of x^0. The classical antiparticle is identified here by its negative energy and no charge conjugation operation is necessary[2]. Although the laboratory apparatus observes the two events at $x^0 = ct_0$ as simultaneous, they may be distinguished in SHP theory by other τ-dependent interactions.

As Horwitz has observed, grandfather paradoxes may be resolved by noticing that the return trip to a past coordinate time x^0 must take place while the chronological time τ continues to increase. The *occurrence* of event $x^\mu(\tau_1)$ at τ_1 is understood to be an irreversible process that cannot be changed by a subsequent event occurring at the same spacetime location, $x^\mu(\tau_2) = x^\mu(\tau_1)$ with $\tau_2 > \tau_1$. This absence of closed timelike curves similarly applies in SHP quantum electrodynamics [15] where the particle propagator $G(x_2 - x_1, \tau_2 - \tau_1)$ vanishes unless $\tau_2 > \tau_1$, thus preventing divergent matter loops, when $x_2 = x_1$.[3] In the microscopic event dynamics described by this explicit distinction between chronological and coordinate time [18] a covariant Hamiltonian generates evolution of a 4D block universe defined at τ to an infinitesimally close 4D block universe defined at $\tau + d\tau$. Standard Maxwell electrodynamics emerges as an equilibrium limit in which the system becomes τ-independent, and the 4D block universe remains static. The details of this τ-dependence in the interacting fields and currents can be studied by reconciling classical SHP with classical Maxwell electromagnetic phenomenology.

2. Classical SHP electrodynamics

In analogy with the notation $x^0 = ct$ we adopt the formal designations

$$
x^5 = c_5\tau \qquad \partial_5 = \frac{1}{c_5}\partial_\tau \qquad j^5 = c_5\rho \qquad a_5 = \frac{1}{c_5}\phi
\tag{16}
$$

and the conventions

$$
\mu, \nu = 0, 1, 2, 3 \qquad \alpha, \beta, \gamma = 0, 1, 2, 3, 5 \qquad g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1, \pm 1)
\tag{17}
$$

so that c_5/c characterizes the relative rate of evolution in τ, and as shown in section 4, SHP becomes Maxwell theory in the limit $c_5 \to 0$. Writing (8) and (9) as

$$
L = \frac{1}{2}M\dot{x}^\mu \dot{x}_\mu + \frac{e}{c}\dot{x}^\alpha a_\alpha \qquad f^\alpha{}_\beta = \partial^\alpha a_\beta - \partial_\beta a^\alpha
\tag{18}
$$

the event dynamics are given by the Lorentz force

$$
M\ddot{x}^\mu = \frac{e}{c}f^\mu{}_\alpha(x, \tau)\dot{x}^\alpha \qquad \frac{d}{d\tau}\left(-\tfrac{1}{2}M\dot{x}^2\right) = g_{55}\frac{ec_5}{c}f^{5\mu}\dot{x}_\mu
\tag{19}
$$

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2   Looking ahead to the current defined in (20) one sees that the electric charge $Q = \int d^3x\, j^0(x, \tau)$ will similarly reverse sign on this section of the trajectory.

3   The equivalence of τ-retarded causality to the Feynman contour in evaluating propagators was first observed by Feynman in connection with the path integral for the Klein-Gordon equation [16,17]. In SHP QED it also emerges from the vacuum expectation value of τ-ordered operator products.

equivalent to substituting the τ-dependent fields (7) into Stueckelberg's evolution equation (1) in flat spacetime. For the moment we understand the factor g_{55} as a choice of sign for the second of (19) and not a hint at some 5D metric structure.

To complete the dynamical picture we re-express the velocity-potential interaction as a current-potential integral

$$\dot{X}^{\alpha}a_{\alpha} \to \int d^4x\, \dot{X}^{\alpha}(\tau)\delta^4\left(x - X(\tau)\right)a_{\alpha}(x,\tau) = \frac{1}{c}\int d^4x\, j^{\alpha}(x,\tau)a_{\alpha}(x,\tau)$$

$$j^{\alpha}(x,\tau) = c\dot{X}^{\alpha}(\tau)\delta^4\left(x - X(\tau)\right) \tag{20}$$

and choose some kinetic action term for the fields, the most obvious candidate being

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{4c}f^{\alpha\beta}(x,\tau)f_{\alpha\beta}(x,\tau). \tag{21}$$

Combining (20) and (21) the electromagnetic action

$$S_{\text{em}} = \int d^4x d\tau \left\{ \frac{e}{c^2}\, j^{\alpha}(x,\tau)a_{\alpha}(x,\tau) - \frac{1}{4c}f^{\alpha\beta}(x,\tau)f_{\alpha\beta}(x,\tau) \right\} \tag{22}$$

can be varied with respect to a_{α} to produce Maxwell-like field equations, admitting a wave equation with associated Green's function whose solutions describe the fields induced by specific event trajectories. However, one is immediately confronted by conceptual difficulties in attempting to describe even the simple case of low energy Coulomb scattering. First, although the current $j^{\alpha}(x,\tau)$ and the potential $a_{\alpha}(x,\tau)$ are individually constructed to be vector + scalar representations of O(3,1) on physical grounds, the 5D scalar structure of the electromagnetic action (22) places all five components on the same algebraic footing. This suggests an underlying formal symmetry larger than O(3,1) but containing it as a subgroup: O(4,1) for the choice $g_{55} = 1$ or O(3,2) symmetry for $g_{55} = -1$. The formal 5D scalar structure survives in the wave equation and these symmetry considerations cannot be entirely ignored. Second, because x^{μ} and τ are introduced to play very different roles, the rest frame of an event is τ-dependent. Thus, a 'static' particle — an event evolving uniformly along the x^0 axis in its rest frame — is described by $x_{\text{rest}} = (c\tau, \mathbf{0})$. It turns out that the potential at an observation point $x = (ct, \mathbf{x})$ induced by this particle is of the form $\delta\left((\tau - t)^2 - |\mathbf{x}|^2/c^2\right)$ with support sharply focused on the lightcone of the source event's immediate location. A test event evolving uniformly as $x_{\text{test}} = (c(\tau + t_0), \mathbf{R})$ will experience the potential $\delta\left(t_0^2 - |\mathbf{R}|^2/c^2\right)$ for all τ, rendering comparison with experiment nearly impossible. It is worth noting that the problem of clock synchronization between the source and test events (characterized here by $x_{\text{test}}^0(0) = ct_0$) is not a problem for the quantized theory where the use of asymptotic states with sharp mass implies maximum uncertainty in the location x^{μ} of the event in τ. By introducing a degree of uncertainty into the definition of the classical electromagnetic action, a reasonable theory may be constructed.

3. Non-local field kinetics $\leftrightarrow$ ensemble of events

The problem of potentials with δ-function support can be repaired by writing the action with a slightly less obvious candidate for the field kinetic term, the non-local form

$$S_{\text{em}} = \int d^4x d\tau \left\{ \frac{e}{c^2}\, j^{\alpha}(x,\tau)a_{\alpha}(x,\tau) - \int \frac{ds}{\lambda}\frac{1}{4c}\left[f^{\alpha\beta}(x,\tau)\Phi(\tau - s)f_{\alpha\beta}(x,s)\right] \right\} \tag{23}$$

where λ is a parameter with dimensions of time. The field interaction kernel is

$$\Phi(\tau) = \delta(\tau) - (\xi\lambda)^2\delta''(\tau) = \int \frac{d\kappa}{2\pi}\left[1 + (\xi\lambda\kappa)^2\right]e^{-i\kappa\tau} \tag{24}$$

where

$$\xi = \frac{1}{2}\left[1 + \left(\frac{c_5}{c}\right)^2\right] \tag{25}$$

is chosen so that the low energy Lorentz force agrees with Coulomb's law. The kinetic term now includes $\left(\partial_\tau f^{\alpha\beta}(x,\tau)\right)\left(\partial_\tau f_{\alpha\beta}(x,\tau)\right)$ which explicitly breaks the 5D symmetry to O(3,1) while maintaining gauge symmetry. We write the inverse function of the interaction kernel as

$$\varphi(\tau) = \lambda\Phi^{-1}(\tau) = \lambda\int\frac{d\kappa}{2\pi}\frac{e^{-i\kappa\tau}}{1+(\xi\lambda\kappa)^2} = \frac{1}{2\xi}e^{-|\tau|/\xi\lambda} \tag{26}$$

which satisfies

$$\int\frac{ds}{\lambda}\,\varphi(\tau-s)\,\Phi(s) = \delta(\tau) \qquad \int\frac{d\tau}{\lambda}\,\varphi(\tau) = 1. \tag{27}$$

Varying the action (23) with respect to the potentials, leads to field equations

$$\partial_\beta f_\Phi^{\alpha\beta}(x,\tau) = \partial_\beta\int\frac{ds}{\lambda}\Phi(\tau-s)f^{\alpha\beta}(x,s) = \frac{e}{c}\,j^\alpha(x,\tau) \tag{28}$$

describing the non-local superposition of fields $f_\Phi^{\alpha\beta}$ sourced by the instantaneous event current $j^\alpha(x,\tau)$. Using (27) to remove $\Phi(\tau)$ from the LHS and writing the Bianchi identity, we obtain equations for the local field sourced by a non-local superposition of event currents,

$$\partial_\beta f^{\alpha\beta}(x,\tau) = \frac{e}{c}\int ds\,\varphi(\tau-s)\,j^\alpha(x,s) = \frac{e}{c}\,j_\varphi^\alpha(x,\tau) \tag{29}$$

$$\partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0 \tag{30}$$

which are formally similar to Maxwell's equations in 5D and are called pre-Maxwell equations. Rewriting the field equations in 4D tensor, vector and scalar components, they take the form

$$\partial_\nu f^{\mu\nu} - \frac{1}{c_5}\frac{\partial}{\partial\tau}f^{5\mu} = \frac{e}{c}\,j_\varphi^\mu \qquad\qquad \partial_\mu f^{5\mu} = \frac{e}{c}\,j_\varphi^5 = \frac{c_5}{c}\,e\rho_\varphi$$

$$\partial_\mu f_{\nu\sigma} + \partial_\nu f_{\sigma\mu} + \partial_\sigma f_{\mu\nu} = 0 \qquad\qquad \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5}\frac{\partial}{\partial\tau}f_{\mu\nu} = 0 \tag{31}$$

which may be compared with the 3-vector form of Maxwell's equations

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = \frac{e}{c}\,\mathbf{J} \qquad\qquad \nabla\cdot\mathbf{E} = \frac{e}{c}\,J^0$$

$$\nabla\cdot\mathbf{B} = 0 \qquad\qquad \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0 \tag{32}$$

showing that $f^{5\mu}$ plays the role of the vector electric field and $f^{\mu\nu}$ plays the role of the magnetic field. It follows from (29) that $\partial_\alpha j^\alpha = 0$ so that $j^5(x,\tau) = c_5\rho(x,\tau)$ plays the role of an event density, and

$$\frac{d}{d\tau}\int d^4x\,\rho(x,\tau) = -\int d^4x\,\partial_\mu j^\mu(x,\tau) = 0 \tag{33}$$

expresses the conservation of total event number.

Rewriting the source of the inhomogeneous equation (29) as

$$j_\varphi^\alpha(x,\tau) = \int ds\,\varphi(\tau-s)\,j^\alpha(x,s) = \frac{1}{2\xi}\int ds\,e^{-|s|/\xi\lambda}\,j^\alpha(x,\tau-s) \tag{34}$$

we recognize $j_\varphi^\alpha(x, \tau)$ as a weighted superposition of currents, each originating at an event $X^\mu(\tau - s)$ displaced from $X^\mu(\tau)$ by an amount s along the worldline. It is useful to regard this superposition as the current produced by an ensemble of events in the neighborhood of $X^\mu(\tau)$, a view encouraged by the particular weight function $\varphi(s)$. Given a Poisson distribution describing the occurrence of independent random events with a constant average rate of $1/\lambda\xi$ events per second, the average time between events is $\lambda\xi$ and the probability at τ that the next event will occur following a time interval $s > 0$ is just $\varphi(s)/\lambda = e^{-s/\xi\lambda}/\xi\lambda$. Extending the displacement to positive and negative values, the ensemble is constructed by assembling a set of instantaneous event currents $j^\alpha(x, \tau - s)$ along the worldline, each weighted by $\varphi(s)$, the probability that the occurrence of this event is delayed from τ by an interval of at least $|s|$. We will see that the leading term of the Green's function manifestly breaks 5D symmetry to O(3,1) and the causality relations embedded in this term select the one event from this ensemble for which the interacting events are at lightlike separation, depending on their relative τ-synchronization.

The ensemble can also be understood by thinking of $j_\varphi^\alpha(x, \tau)$ as a random variable describing the probability of finding a current density at x at a given τ. Then we may consider a correlation function for the event density of the type

$$\langle \rho(\tau)\rho(s) \rangle = \frac{1}{\mathcal{N}} \int d^4x \, \rho(x, \tau)\rho(x, s) \tag{35}$$

where $\mathcal{N}$ is a normalization. For a uniformly moving event with $X^\mu(\tau) = u^\mu\tau$ the raw event current (20) leads to

$$\langle \rho(\tau)\rho(s) \rangle = \frac{c^2}{\mathcal{N}} \int d^4x \, \delta^4(x - u\tau)\delta^4(x - us) = \frac{c^2\delta^3(\mathbf{0})}{|u^0|\mathcal{N}} \delta(\tau - s) \tag{36}$$

showing that the currents at differing times $\tau \neq s$ are uncorrelated. For the ensemble current defined in (29) the correlation becomes

$$\langle \rho_\varphi(\tau)\rho_\varphi(s) \rangle = \frac{c^2}{\mathcal{N}} \int d\tau' ds' d^4x \, \varphi(\tau - \tau')\varphi(s - s')\delta^4(x - u\tau')\delta^4(x - us')$$

$$= \frac{c^2\delta^3(\mathbf{0})}{|u^0|\mathcal{N}} \int d\tau' \, \varphi(\tau - \tau')\varphi(\tau' - s)$$

$$= \frac{c^2\delta^3(\mathbf{0})}{4\xi^2|u^0|\mathcal{N}} \int d\tau' \, e^{-|\tau-\tau'|/\xi\lambda - |\tau'-s|/\xi\lambda} \tag{37}$$

so that taking $\tau > s$ and separating the integral into three intervals punctuated by s and τ leads to

$$\langle \rho_\varphi(\tau)\rho_\varphi(s) \rangle = \frac{\lambda c^2\delta^3(\mathbf{0})}{4\xi|u^0|\mathcal{N}} \left(1 + \frac{\tau - s}{\xi\lambda}\right) e^{-(\tau-s)/\xi\lambda} \tag{38}$$

with a time dependence characteristic of an Ornstein-Uhlenbeck process. Regarding the smoothed current $j_\varphi^\alpha(x, \tau)$ produced by an event $X^\mu(\tau)$ as the instantaneous current produced by an ensemble of events, this correlation suggests that the ensemble is the result of the Brownian motion found by subjecting $X^\mu(\tau)$ to a random force under viscous drag.

The pre-Maxwell equations in Lorenz gauge lead to the wave equation

$$\partial_\beta\partial^\beta a^\alpha = (\partial_\mu\partial^\mu + \partial_\tau\partial^\tau)a^\alpha = (\partial_\mu\partial^\mu + \frac{g_{55}}{c_5^2} \partial_\tau^2)a^\alpha = -\frac{e}{c} j_\varphi^\alpha(x, \tau) \tag{39}$$

whose solutions may respect 5D symmetries broken by the O(3,1) symmetry of the event dynamics. A Green's function solution to

$$(\partial_\mu\partial^\mu + \frac{g_{55}}{c_5^2} \partial_\tau^2)G(x, \tau) = -\delta^4(x)\delta(\tau) \tag{40}$$

can be used to obtain potentials in the form

$$a^\alpha (x, \tau) = -\frac{e}{c} \int d^4 x' d\tau' \, G(x - x', \tau - \tau') \, j_\varphi^\alpha (x', \tau') \,. \tag{41}$$

The principal part Green's function [19] is

$$G_P(x, \tau) = -\frac{1}{2\pi} \delta(x^2) \delta(\tau) - \frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-g_{55} g_{\alpha\beta} x^\alpha x^\beta) \frac{1}{\sqrt{-g_{55} g_{\alpha\beta} x^\alpha x^\beta}} \tag{42}$$

$$= G_{Maxwell} + G_{Correlation} \tag{43}$$

where $G_{Maxwell}$ breaks a higher symmetry to O(3,1) while the support of $G_{Correlation}$ is

$$-g_{55} g_{\alpha\beta} x^\alpha x^\beta = \begin{cases} -\left(x^2 + c_5^2 \tau^2\right) = c^2 t^2 - \mathbf{x}^2 - c_5^2 \tau^2 > 0 \quad , \quad g_{55} = 1 \\[2mm] \left(x^2 - c_5^2 \tau^2\right) = \mathbf{x}^2 - c^2 t^2 - c_5^2 \tau^2 > 0 \quad , \quad g_{55} = -1 \end{cases} \tag{44}$$

with causality properties dependent on the choice of g_{55}.

The contribution from $G_{Correlation}$ is smaller than that of $G_{Maxwell}$ by c_5/c and drops off as $1/|\mathbf{x}|^2$, so it may be neglected at low energy [20]. The contribution to the potential from $G_{Maxwell}$ is

$$a^\alpha (x, \tau) = -\frac{e}{c} \int d^4 x' d\tau' \, G_{Maxwell}(x - x', \tau - \tau') \, j_\varphi^\alpha (x', \tau')$$

$$= \frac{e}{2\pi c} \int d^4 x' d\tau' ds \, \delta\left[(x - x')^2\right] \delta(\tau - \tau') \, \varphi(\tau' - s) \, j^\alpha(x', s) \tag{45}$$

and because $G_{Maxwell}$ has support at equal-τ this can be written

$$a^\alpha (x, \tau) = \frac{e}{2\pi c} \int ds \, \varphi(\tau - s) \int d^4 x' \, \delta\left[(x - x')^2\right] \, j^\alpha(x', s) \tag{46}$$

expressing the potential as an ensemble of single-event potentials. Inserting the current defined in (20)

$$j^\alpha (x, \tau) = c \dot{X}^\alpha(\tau) \delta^4(x - X(\tau))$$

and using the identity

$$\int d\tau f(\tau) \, \delta[g(\tau)] = \frac{f(\tau_R)}{|g'(\tau_R)|}, \tag{47}$$

where τ_R is the retarded time that solves

$$g(\tau) = (x - X(\tau_R))^2 = 0 \qquad \theta^{ret} = \theta\left(x^0 - X^0(\tau_R)\right), \tag{48}$$

we find the potential

$$a^\alpha (x, \tau) = \frac{e}{2\pi} \int ds \, \varphi(\tau - s) \, \dot{X}^\alpha(s) \, \delta\left[(x - X^\alpha(s))^2\right]$$

$$= \frac{e}{4\pi} \varphi(\tau - \tau_R) \frac{\dot{X}^\alpha(\tau_R)}{\left|(x^\mu - X^\mu(\tau_R)) \dot{X}_\mu(\tau_R)\right|} \tag{49}$$

which is the standard Liénard-Wiechert potential multiplied by $\varphi(\tau - \tau_R)$. Thus, while the current that sources the pre-Maxwell field represents an ensemble of events along the worldline, the retarded causality of the Green's function selects the one member of the ensemble that intersects the lightcone of the observation point. The remaining τ-dependence of the fields resides in the finite function φ and expresses the relative time synchronization between the source and a test event experiencing the potential at the spacetime point x at the chronological time τ.

To find the Coulomb potential we specify the event trajectory $X(\tau) = (c\tau, \mathbf{0})$ which produces the instantaneous current

$$j^0(x, \tau) = c^2 \delta(t - \tau) \delta^3(\mathbf{x}) \qquad \mathbf{j}(x, \tau) = 0 \qquad j^5(x, \tau) = cc_5 \delta(t - \tau) \delta^3(\mathbf{x}) \tag{50}$$

and so the potential

$$a^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \varphi\left(\tau - \left(t - \frac{|\mathbf{x}|}{c}\right)\right) \qquad \mathbf{a} = 0 \qquad a^5(x, \tau) = \frac{c_5}{c} a^0(x, \tau) \tag{51}$$

is found from from $G_{Maxwell}$. Consider a test event $x(\tau) = (c(\tau - \tau_0), \mathbf{x})$ evolving along a parallel trajectory at spacial separation $\mathbf{x}$ and time offset τ_0 so that

$$\varphi\left(\tau - \left(t - \frac{|\mathbf{x}|}{c}\right)\right) = \frac{1}{2\xi} e^{-\frac{1}{\xi\lambda}\left|\tau_0 + |\mathbf{x}|/c\right|}. \tag{52}$$

If the test event is precisely on the forward lightcone of the source with $\tau_0 = -|\mathbf{x}|/c$ then $\varphi = 1$ and the interaction is purely Coulomb in form. If $\tau_0 = 0$ so that the events are synchronized at x^0, then

$$a^0(x, \tau) = \frac{1}{2\xi} \frac{e}{4\pi|\mathbf{x}|} e^{-|\mathbf{x}|/\xi\lambda c} \tag{53}$$

which has the form of a Yukawa-type potential with photon mass $m_\gamma \sim \hbar/\xi\lambda c^2$. If the source and test events are slightly desynchronized, with $\tau_0 > 0$, then the interaction is weakened by a factor $e^{-\tau_0/\xi\lambda c}$.

Thus the factor λ that characterizes the width of the ensemble and represents the average event inter-occurrence time, also determines the mass spectrum of the photons mediating the interaction between events. As proposed by Stueckelberg and seen in the second of (19) the masses of the particles and fields are not separately conserved, although Noether's theorem for affine τ-displacement symmetry guarantees that the total mass is a constant of the motion [21]. If λ is small (so that φ/λ approaches a delta function and the current narrows to a small neighborhood around the event) the mass spectrum becomes wide and the interaction range and cross-section decrease. If λ is large, the support of the current spreads along the worldline, the potential becomes Coulomb-like and the photon mass spectrum is small.

A similar role is seen for λ in SHP quantum field theory. From the Fourier expansion for the electromagnetic Green's function (42)

$$G(x - x', \tau - \tau') = \int \frac{d^4k\, d\kappa}{(2\pi)^5} \frac{e^{i[k\cdot(x-x') + g_{55}\kappa c_5(\tau - \tau')]}}{k^2 + g_{55}\kappa^2 - i\epsilon} \tag{54}$$

it appears that photon loops in the 5D theory would render it non-renormalizable. However, quantization of the higher order field kinetic term leads to the photon propagator factor

$$\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right] \frac{-i}{k^2 + g_{55}\kappa^2 - i\epsilon} \frac{1}{1 + (\lambda\xi)^2 \kappa^2} \tag{55}$$

and the theory is super-renormalizable at second order. Again we notice that if λ is large, then large values of photon mass κ are suppressed.

But while mass exchange must be present in any classical theory of pair processes and must also be small to account for standard electromagnetic phenomenology, such a compromise cannot explain the fixed masses of elementary particles. However, it has been shown that under certain circumstances [22] a self-interaction induced through $G_{Correlation}$ has the effect of restoring on-shell evolution in event trajectories and thus returning the particle worldline to the observed fixed mass. As seen in (44) when $g_{55} = 1$ the Green's function has timelike support, permitting the event to interact with the field produced earlier along its worldline. The net effect of this self-interaction is a damping (or anti-damping) force that accelerates the event evolution to its asymptotic mass shell. A more

general approach is found in the statistical mechanics of the many-event system. While the model presented here describes a particle as a weighted ensemble of events $\varphi(s)X^\mu(\tau - s)$ along a single worldline, Horwitz has modeled [23] a particle as an ensemble of n independent spacetime events $X_i^\mu(\tau)$, $i = 1, 2, \ldots, n$ defined at a given τ. He has shown that the total particle mass is determined by a chemical potential. Following collisions governed by a general class of interactions that includes pair processes, particles return to their equilibrium mass values. These developments indicate that the statistical mechanics of event ensembles in the construction of classical particles will be a fruitful way to understand mass and perhaps derive masses from first principles.

4. Maxwell theory as an equilibrium state of SHP

Following an argument by Stueckelberg, Saad et. al. noticed [13] that under the boundary conditions

$$j_\varphi^5(x,\tau) \xrightarrow[\tau\to\pm\infty]{} 0 \qquad f^{5\mu}(x,\tau) \xrightarrow[\tau\to\pm\infty]{} 0 \tag{56}$$

integration of the pre-Maxwell equations provide

$$
\left.
\begin{aligned}
\partial_\beta f^{\alpha\beta}(x,\tau) &= \frac{e}{c} j_\varphi^\alpha(x,\tau) \\[4pt]
\partial_{[\alpha} f_{\beta\gamma]} &= 0 \\[4pt]
\partial_\alpha j^\alpha &= 0
\end{aligned}
\right\}
\xrightarrow{\int \frac{d\tau}{\lambda}}
\left\{
\begin{aligned}
\partial_\nu F^{\mu\nu}(x) &= \frac{e}{c} J^\mu(x) \\[4pt]
\partial_\nu F^{5\nu}(x) &= \frac{e}{c} J^5(x) \\[4pt]
\partial_{[\mu} F_{\nu\rho]} &= 0 \\[4pt]
\partial_\mu J^\mu(x) &= 0
\end{aligned}
\right.
\tag{57}
$$

where

$$A^\alpha(x) = \int \frac{d\tau}{\lambda} \, a^\alpha(x,\tau) \qquad F^{\alpha\nu}(x) = \int \frac{d\tau}{\lambda} \, f^{\alpha\nu}(x,\tau). \tag{58}$$

This integration has been called concatenation and is understood as aggregation of all events that occur at a spacetime point x over all τ. The decoupling of $F^{5\mu}$ from $F^{\mu\nu}$, which now satisfies Maxwell's equations, suggests that the SHP can be seen as an underlying microscopic dynamics for which Maxwell theory is an equilibrium or expectation state. In particular, we see that integration of (53) provides the Coulomb potential and using (27) to integrate the 4-vector current yields the Maxwell current $J^\mu(x)$ in the standard form as

$$
\begin{aligned}
\int d\tau \, j_\varphi^\mu(x,\tau) &= c \int d\tau \, ds \, \varphi(\tau - s) \dot{X}^\mu(s) \delta^4(x - X(s)) \\[4pt]
&= c \int d\tau \, \dot{X}^\mu(\tau) \delta^4(x - X(\tau)).
\end{aligned}
\tag{59}
$$

Similarly, concatenation of (42)

$$\int d\tau \, G_{Maxwell} = D(x) = -\frac{1}{2\pi}\delta(x^2) \qquad \int d\tau \, G_{Correlation} = 0 \tag{60}$$

recovers the 4D Maxwell Green's function.

Another approach [20] to retrieving Maxwell theory from SHP is to slow the τ-evolution to zero by taking $c_5/c \to 0$, thus freezing the microscopic system into a static equilibrium. Under this condition the homogeneous equation (30) imposes the condition

$$c_5\left(\partial_\nu f_{5\mu} - \partial_\mu f_{5\nu}\right) + \partial_\tau f_{\mu\nu} = 0 \xrightarrow[c_5\to 0]{} \partial_\tau f_{\mu\nu} = 0 \tag{61}$$

requiring that the field strength $f^{\mu\nu}$ be τ-independent in this limit. As seen in the Liénard-Wiechert potential, the τ-dependence resides in $\varphi(\tau - \tau_R)$ and can only be suppressed by taking $\lambda \to \infty$. We recall that λ was seen to the be the correlation time for the current j^α, that is the measure of information about the interaction at time τ available from observation of the interaction at time $s < \tau$. Thus, the action-at-a-distance Coulomb law can be seen as the effect of a long-term correlation for the instantaneous potential. Applying the combined limits on c_5/c and λ to (25) and (26), we find $\varphi(\tau) \to 1/2\xi = 1$ and so all field components must be τ-independent. This requirement effectively assigns equal weight to all event currents $j^\alpha(x, \tau)$ in the ensemble $j_\varphi^\alpha(x, \tau)$ defined along the worldline, recovering the standard particle current through

$$j_\varphi^\mu(x, \tau) = \int ds\, \varphi(\tau - s)\, j^\mu(x, s) \quad \longrightarrow \quad \int ds\, 1 \cdot j^\mu(x, s) = J^\mu(x)$$

$$j_\varphi^5(x, \tau) = \int ds\, \varphi(\tau - s)\, j^5(x, s) \quad \longrightarrow \quad \int ds\, j^5(x, s) \tag{62}$$

$$\partial_\mu j_\varphi^\mu(x, \tau) + \frac{1}{c_5}\partial_\tau j_\varphi^5(x, \tau) = 0 \quad \longrightarrow \quad \partial_\mu J^\mu(x) = 0$$

of Maxwell theory[4]. As expected, $f^{\mu\nu}$ decouples from $f^{5\mu}$ and satisfies Maxwell's equations, while the photon mass $m_\gamma \sim \hbar/\xi\lambda c^2$ vanishes.

5. Conclusion

Stueckelberg-Horwitz-Piron electrodynamics can be approached as an abstract gauge theory, exploring the consequences of allowing the gauge transformation (12) of the quantum wave function to depend on the evolution parameter in the dynamical framework. But in another sense, just as Maxwell sought to formalize the empirical results of Cavendish and Coulomb, SHP may be seen as accounting for classical Maxwell electrodynamics in light of the pair creation/annihilation phenomena observed by Anderson. For Stueckelberg, pair processes provide empirical evidence that time must be understood as two distinct physical phenomena, chronology and coordinate, and so must be formalized through independent quantities τ and $(x^0, \mathbf{x})$ in a physically reasonable theory. Having become accustomed, during the two hundred years that separate Cavendish from general gauge theory, to characterizing a single physical time by the evolution of specialized machines (clocks) in a coordinate frame, it is unsurprising that introducing such a distinction raises conceptual difficulties.

A new phenomenon in SHP is the absence of a static configuration — a particle may only remain at the origin in its rest frame for all coordinate time x^0 if its underlying microscopic event continually and uniformly evolves along its time axis, as $x = (c(\tau + \tau_0), \mathbf{0})$. The coordinate $x^0 = c\tau_0$ at $\tau = 0$ is not simply an artifact of initializing a system clock, because the field induced by this event trajectory

$$a^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|}\varphi\left(\tau + \tau_0 - \left(t - \frac{|\mathbf{x}|}{c}\right)\right) \qquad a^5(x, \tau) = \frac{c_5}{c}a^0(x, \tau) \tag{63}$$

depends explicitly on the constant τ_0. The irreversible concatenation performed by measuring apparatus recovers the familiar Coulomb potential

$$A^0(x) = \int \frac{d\tau}{\lambda}\, a^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \tag{64}$$

with no dependence on τ_0 or even on the details of the weight function $\varphi(\tau)$. Similarly, τ_0 plays no role in the quantized theory where sharply defined mass-momentum states retain no information

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[4]   The current $\dfrac{1}{c_5}j_\varphi^5(x, \tau)$ remains finite because $j^5(x, \tau)$ includes the factor $\dot{X}^5 = c_5$.

about the initial conditions of coordinates. Nevertheless, in SHP the microscopic event dynamics are determined by the Lorentz force (19) and so a test event at $x(\tau) = (c(\tau + \tau_0'), \mathbf{x})$ will experience the Coulomb force

$$M\ddot{\mathbf{x}} = -e^2 \frac{1 - g_{55}\frac{c_5}{c}}{1 + \left(\frac{c_5}{c}\right)^2} \nabla \left(\frac{e^{-\left|c(\tau_0 - \tau_0') + |\mathbf{x}|\right|/\xi\lambda c}}{4\pi |\mathbf{x}|} \right) \quad (65)$$

depending on the synchronization $\tau_0 - \tau_0'$. Regarding (63) as a Yukawa potential, the limit $\lambda \to 0$ is understood as extending the range of the interaction to a Coulomb form by taking the mass of the photons that carry the interaction to zero. Viewing λ as a correlation time for the microscopic current density $j^\alpha(x, \tau)$ this limit can be understood as smoothing the interaction to a time-independent Coulomb form by extending the correlation between values of the potential at different times along the entire worldline.

The structure of the source current for the pre-Maxwell field equations was seen to be determined by the choice of kinetic term for the fields. Standard field theory texts note that this choice is not imposed by physical foundations, but recommend the simplest form $f^{\alpha\beta} f_{\alpha\beta}$ because it is Lorentz and gauge invariant, contains only first order derivatives, and in the case of Maxwell theory, recovers the known field equations. For SHP, this choice is equivalent to taking $\varphi(\tau) = \lambda\delta(\tau)$, which does recover the concatenated Coulomb force through (64) but renders the Lorentz force

$$M\ddot{\mathbf{x}} = -\lambda e^2 \left(1 - g_{55}\frac{c_5}{c}\right) \nabla \left[\frac{1}{4\pi |\mathbf{x}|} \delta\left(\tau_0 - \tau_0' + |\mathbf{x}|/c\right) \right] \quad (66)$$

difficult, if not impossible, to reconcile with known phenomenology. The smooth (but non-local) current was found by adding the higher-derivative term $\left(\partial_\tau f^{\alpha\beta}\right)\left(\partial_\tau f_{\alpha\beta}\right)$, which does not affect the Lorentz and gauge invariance of the action. A term of this type has also been considered by Pavsic for brane interactions [24].

The smoothing of the single-event current by $\varphi(\tau)$ can be understood as constructing a statistical ensemble of events as the source for the field equations. An event on the trajectory $X^\mu(\tau)$ is associated with an ensemble whose members are of the form $\varphi(s)X^\mu(\tau - s)$, where the weight $\varphi(s)$ is the probability that a process generating independent random events at a constant average rate will produce an event occurring at displacement s from time τ. A single member of the ensemble is selected by the causal properties of the Green's function when determining the potential induced by the event trajectory. In this way, classical statistical mechanics provides the required element of uncertainty about initial conditions that enables the construction of a well-posed relativistic Hamiltonian theory of electromagnetism.

Limits on the values of the parameters in SHP can be found from standard phenomenology. To describe elastic particle-antiparticle scattering, (65) undergoes

$$-e^2 \left(1 - g_{55}\frac{c_5}{c}\right) \to e^2 \left(1 + g_{55}\frac{c_5}{c}\right) \quad (67)$$

so that the experimental error in the asymmetry of scattering cross-sections places a limit on $c_5/c \ll 1$ and allows us to take $\xi \sim 1/2$. The width of the distribution $\varphi(\tau)$ is characterized by the parameter λ, which also determines as $m_\gamma \sim \hbar/\xi\lambda c^2$ the mass spectrum of the photons that mediate the induced force. Taking m_γ to be the experimental error on the mass of the photon $(10^{-18} eV/c^2)$ we may estimate $\lambda > 10^{-2}$ seconds. Thus, only very low energy interactions will produce phenomena that can be distinguished from standard Maxwell electrodynamics.

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