The Geometrization of Maxwell's Homogenous Equation and the Emergence of Gravity

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ABSTRACT

A recently proposed classical field theory in which the Maxwell tensor is coupled to the Riemann-Christoffel curvature tensor in a fundamentally new way is reviewed and extended. This proposed geometrization of the Maxwell tensor leaves the classical equations of electromagnetism unchanged, but also leads to the emergence of gravity as all solutions of the proposed field equations are shown to be solutions of Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and/or dark energy. Using specific solutions to the proposed theory, the unification brought to electromagnetic and gravitational phenomena as well as the consistency of those solutions with those of the classical Maxwell and Einstein field equations are emphasized throughout. Unique to the four fundamental field equations that comprise the proposed theory, and based on specific solutions to them are: the emergence of antimatter and its behavior in gravitational fields, the emergence of dark matter and dark energy mimicking terms in the context of General Relativity, an underlying relationship between electromagnetic and gravitational radiation, the impossibility of negative mass solutions that would generate repulsive gravitational fields or antigravity, and a method for quantizing the charge and mass of particle-like solutions.

Keywords: Maxwell's equations, General Relativity, unification of electromagnetism and gravitation, dark matter and dark energy, electromagnetic and gravitational radiation, antimatter, antigravity, quantization

1. INTRODUCTION

Electromagnetic and gravitational fields have long range interactions characterized by speed of light propagation; similarities that suggest these fields should be coupled together at the classical physics level. Although this coupling or unification is a well-worn problem with many potential solutions having been proposed, it is fair to say that there is still no generally accepted classical field theory that can explain both electromagnetism and gravitation in a coupled or unified framework.[1] Today, the existence of electromagnetic and gravitational fields are generally understood to be distinct and independent, with electromagnetic fields described by Maxwell's equations and gravitational fields described by Einstein's General Relativity. The purpose of this manuscript is to assess a recently proposed set of four field equations that introduce a geometric description of electromagnetism and from which gravitation then emerges.

Assuming the geometry of nature is Riemannian with four dimensions, the four field equations in Table I are proposed to provide a description of classical physics at the level of the Maxwell and Einstein Field Equations (M&EFEs)[2] but to then to go further by geometrically unifying electromagnetism and gravitation.

Table I. Fundamental field equations of proposed theory

Equations	Equation number
$F_{\mu\nu;\kappa} = a^{\lambda} R_{\lambda\kappa\mu\nu}$	(1)
$F^{\mu\nu}{}_{;\mu} = -\rho_c u^{\nu}$	(2)
$u^{\lambda} u_{\lambda} = -1$	(3)
$\left(\rho_m u^{\mu} u^{\nu} + F^{\mu}{}_{\lambda} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)_{;\nu} = 0$	(4)

In equations (1) through (4), $F_{\mu\nu}$ is the Maxwell tensor, $g_{\mu\nu}$ is the metric tensor, $R_{\lambda\mu\nu\kappa}$ is the Riemann-Christoffel (R-C) curvature tensor, u^{λ} is the four-velocity, and ρ_c and ρ_m are the charge and mass density, respectively. Of note are equations (2) through (4), all of which are already well accepted foundations of classical physics, with (2) being Maxwell's inhomogeneous equation, (3) being the normalization of the four-velocity u^{λ}, and (4) the conservation of energy and momentum for matter and electromagnetic fields. Only equation (1), which couples the derivatives of the Maxwell tensor to the components of the Riemann-Christoffel (R-C) curvature tensor through the vector field a^{λ} is new.

Collectively, the equations of Table I are very close to the foundational equations of classical electromagnetism. In fact, if equation (1) in Table I were to be replaced by Maxwell's homogenous equation, then the equations of Table I would correspond exactly to the foundational equations of classical electromagnetism. In section 2.1 it will be shown that Maxwell's homogeneous equation is a direct consequence of equation (1) and the properties of the R-C tensor, a result that ensures the field theory based on equations (1) through (4) encompasses classical electromagnetism in its entirety. It is in this sense that equation (1) can be thought of as a geometricized version of Maxwell's homogenous equation, and the equations in Table I collectively thought of as a geometricized version of classical electromagnetism.

The goal of this manuscript is to show through an axiomatic development that the continuous field theory based on fundamental field equations (1) through (4) not only cover classical electromagnetism but also gravitational phenomena which emerge due to the coupling of the Maxwell tensor to the R-C tensor in equation (1). In the view of the proposed theory, both electromagnetic and gravitational phenomena are put on an equal footing with both being intimately tied to the curvature of space-time. Additionally, it will be shown that the description of gravity that emerges is consistent with Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and/or dark energy.

Listed in Table II are the continuous field variables that the proposed field theory's fundamental equations (1) through (4) solve for. Except for a^{λ}, all the field variables listed in Table II are familiar to classical physics.

Table II. Dynamic fields

Field	Description
$g_{\mu\nu}$	Metric tensor
$F_{\mu\nu}$	Maxwell tensor
u^{λ}	Four-velocity vector field
a^{λ}	Four-vector coupling electromagnetism to gravitation
ρ_c	Proper charge density scalar field

	(Rest value in a locally inertial frame)
ρ_m	Proper mass density scalar field (Rest value in a locally inertial frame)

Just as it does in classical continuum physics, the four-velocity vector field u^λ describes the motion of the charge density field ρ_c and the mass density field ρ_m, which are assumed to be comoving. However, unlike the situation in classical physics where ρ_c and ρ_m are treated as external fields, in the proposed theory they are treated as dynamic fields that are solved for as part of any solution to fundamental equations (1) through (4). For a classical field theory, this treatment of ρ_c and ρ_m as dynamic fields results in several interesting predictions: the emergence of antimatter and its behavior in gravitational fields, the emergence of dark matter and dark energy mimicking terms in the context of General Relativity, an underlying relationship between electromagnetic and gravitational radiation, the impossibility of negative mass solutions that would generate repulsive gravitational fields or antigravity, and a method for quantizing the charge and mass of particle-like solutions. All of these points are brought out through specific solutions that are investigated in this manuscript.

Throughout the manuscript, geometric units are used with a metric tensor having signature [+, +, +, -]. Spatial indices run from 1 to 3 with 4 the time index. The notation within uses commas before tensor indices to indicate ordinary derivatives and semicolons before tensor indices to indicate covariant derivatives. Finally, for the definitions of the R-C curvature tensor and the Ricci tensor, the conventions used by Weinberg are followed. [3]

2. CONSEQUENCES OF FUNDAMENTAL FIELD EQUATIONS

2.1 The equations of electromagnetism

In theoretical derivations of the classical equations of electromagnetism, Maxwell's homogeneous equation, which is well verified experimentally must be taken as an axiom so that the vector potential A_μ can be introduced. The introduction of A_μ is necessary so that a Lagrangian consisting of a matter term, a free electromagnetic field term, and a matter-field interaction term can be defined and written in terms of A_μ. The classical equations of electromagnetism are then derived using a stationary-action calculation in which A_μ and the spatial positions of masses and charges are treated as the dynamic variables.[4] As shown in the following, all these equations of classical electromagnetism are also a direct result of fundamental field equations (1) through (4). The only difference between the derivation of the classical equations of electromagnetism using the proposed theory and their derivation using the traditional approach is that Maxwell's homogenous equation is no longer taken as an axiom in the proposed theory but rather is replaced by fundamental equation (1) as an axiom, with Maxwell's homogeneous equation then becoming a derived consequence of (1) and the properties of the R-C tensor.

Here I give a short description of the derivation of the classical equations of electromagnetism in the framework of the proposed theory. The point in going through this purely formal development is to show that the classical equations of electromagnetism are derivative only to the fundamental field equations (1) through (4) and the algebraic properties of the R-C tensor.

To begin, I establish the antisymmetry of $F_{\mu\nu}$ in the proposed theory using the algebraic property of the R-C tensor,

$$R_{\lambda\kappa\mu\nu} = -R_{\lambda\kappa\nu\mu} \ , \tag{5}$$

which on contracting with a^λ gives,

$$a^\lambda R_{\lambda\kappa\mu\nu} = -a^\lambda R_{\lambda\kappa\nu\mu} \ . \tag{6}$$

Comparing (6) and (1) then gives,

$$F_{\mu\nu} = -F_{\nu\mu} \ , \tag{7}$$

establishing that $F_{\mu\nu}$ is antisymmetric.

Next, I derive Maxwell's homogeneous equation using the algebraic property of R-C tensor,

$$R_{\lambda\kappa\mu\nu} + R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} = 0 \ , \tag{8}$$

which on contracting with a^λ gives,

$$a^\lambda R_{\lambda\kappa\mu\nu} + a^\lambda R_{\lambda\mu\nu\kappa} + a^\lambda R_{\lambda\nu\kappa\mu} = 0 \ . \tag{9}$$

Comparing (9) and (1) then gives,

$$F_{\mu\nu;\kappa} + F_{\nu\kappa;\mu} + F_{\kappa\mu;\nu} = 0 \quad or \quad F_{\mu\nu,\kappa} + F_{\nu\kappa,\mu} + F_{\kappa\mu,\nu} = 0 \tag{10}$$

which is the desired result and where the switch from the covariant to ordinary derivatives is justified by the antisymmetry of $F_{\mu\nu}$.

Having established the antisymmetry of $F_{\mu\nu}$ in (7), and then the vanishing of its anti-symmetrized derivative in (10), the converse of Poincaré's lemma establishes that $F_{\mu\nu}$ can itself be expressed as the anti-symmetrized derivative of a vector function,

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \ , \tag{11}$$

where A_μ is the classical electromagnetic vector potential. Because $F_{\mu\nu}$ can be expressed as the anti-symmetrized derivative of the vector potential A_μ, its value will be unaffected by a gauge transformation in which a gradient field is added to A_μ,

$$A_\mu \rightarrow A_\mu + \partial_\mu \varphi \ . \tag{12}$$

It is based on the forgoing development that I refer to equation (1) as the geometricized version of Maxwell's homogenous equation.

Next, I derive the conservation of charge. Returning to the antisymmetry of $F_{\mu\nu}$, it follows that,

$$F^{\mu\nu}{}_{;\mu;\nu} \equiv 0 , \tag{13}$$

which is an identity for all antisymmetric tensors. Comparing (13) to the contracted derivative of Maxwell's inhomogeneous equation (2),

$$F^{\mu\nu}{}_{;\mu;\nu} = -\left(\rho_c u^\nu\right)_{;\nu} \ , \tag{14}$$

then gives,

$$\left(\rho_c u^\nu\right)_{;\nu} = 0 \ , \tag{15}$$

which is the conservation of charge equation.

Consider next the Lorentz force law and the conservation of mass, both of which are derived from the conserved energy-momentum tensor given in equation (4). To see this, distribute the covariant derivative in (4),

$$\left(\rho_m u^\nu\right)_{;\nu} u^\mu + \rho_m u^\mu{}_{;\nu} u^\nu + F^{\mu\lambda} F^\nu{}_{\lambda;\nu} + F^{\mu\lambda}{}_{;\nu} F^\nu{}_\lambda - \frac{1}{2} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma;\nu} = 0, \qquad (16)$$

which, with some substitutions and rearrangements using Maxwell's homogenous equation (10) and inhomogeneous equation (2), can be re written as,

$$\left(\rho_m u^\nu\right)_{;\nu} u^\mu + \rho_m u^\mu{}_{;\nu} u^\nu - \rho_c F^\mu{}_\lambda u^\lambda = 0. \qquad (17)$$

Contracting (17) with u_μ, the 2nd and 3rd terms on the LHS of (17) are zeroed due to (3) and the antisymmetry of $F_{\mu\nu}$, respectively, leaving the conservation of mass equation,

$$\left(\rho_m u^\nu\right)_{;\nu} = 0\ . \qquad (18)$$

Using (18) to zero out the conservation of mass term in (17) then leaves the Lorentz force law,

$$\rho_m \frac{Du^\mu}{D\tau} = \rho_c F^\mu{}_\lambda u^\lambda\ , \qquad (19)$$

where $\dfrac{Du^\mu}{D\tau} \equiv u^\mu{}_{;\lambda} u^\lambda$.

This completes the derivation of the classical equations of electromagnetism, establishing that classical electromagnetism is a derivable consequence of the fundamental field equations of the proposed theory and the properties of the R-C tensor. The next result I derive goes beyond classical electromagnetism by establishing a relationship between ρ_c, u^ν, a^λ , and the Ricci tensor $R^\lambda{}_\nu$, which will be useful when finding solutions to the full set of fundamental equations (1) through (4).

Beginning with (1) and then contracting the μ and κ indices gives,

$$F^{\mu\nu}{}_{;\mu} = a^\lambda R_{\lambda\mu}{}^{\mu\nu} = -a^\lambda R_\lambda{}^\nu\ . \qquad (20)$$

Comparing (20) to Maxwell's inhomogeneous equation (2), the RHS's of these equations can be equated giving,

$$a^{\lambda} R_{\lambda}{}^{\nu} = \rho_c u^{\nu} \ . \tag{21}$$

Several related expressions that follow directly from (21) and (3) which will be useful when finding specific solutions to equations (1) through (4) are:

$$\rho_c{}^2 = -a^{\lambda} R_{\lambda}{}^{\nu} a^{\sigma} R_{\sigma\nu} \tag{22}$$

and

$$\rho_c = -a^{\lambda} u^{\nu} R_{\lambda\nu} \, . \tag{23}$$

In summary, the theory of electromagnetism being proposed here in no way changes the traditional development of classical electromagnetism except that Maxwell's homogenous equation is no longer taken as an axiom, but rather is a derived consequence of fundamental field equation (1) and the properties of the R-C tensor. However, adopting equation (1) as an axiom does introduce conceptual changes that go beyond the classical Maxwell equations. For example, the charge density ρ_c is no longer an externally inserted field as it is in the classical physics picture. This can already be seen in equation (23) where the scalar charge density is shown to be dependent on the Ricci tensor. In subsequent sections, the proposed theory will be developed further using specific solutions to the fundamental field equations to show that electromagnetic and gravitational phenomena are put on the same footing, with both being intrinsically tied to nonzero curvatures. The cost of this unification is the introduction of the new vector field a^{λ}, a field not known to traditional classical physics and which in the proposed theory couples the derivatives of the Maxwell tensor to the components of the R-C tensor. Additionally, and as will be shown in section 3.3, a^{λ} is directly relatable to the familiar vector potential A_{ν} of classical electromagnetism.

2.2 A theory of gravitation

While evident from the preceding discussion that equations of classical electromagnetism follow from the fundamental field equations (1) through (4), at this point it is not obvious that the same can be said of Einstein's equation of General Relativity,

$$G^{\mu\nu} = -8\pi \, T^{\mu\nu} \, , \tag{24}$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the Einstein tensor. As will be shown, the particle-like solution to be analyzed in section 4.1 demonstrates that the Reissner-Nordström metric is an exact solution of the fundamental field equations (1) through (4), establishing that the new theory and classical General Relativity (24) support the same metric field solutions, at least in the case of spherical symmetry. However, one must go further to determine if Einstein's field equation is a derivable consequence of the fundamental equations (1) through (4).

To investigate this question, I start by considering the conserved energy-momentum tensor (4). An immediate consequence of $G^{\mu\nu}$ and $T^{\mu\nu}$ being both symmetric and independently conserved, independently conserved because $G^{\mu\nu}{}_{;\nu} \equiv 0$ by the Bianchi identity and $T^{\mu\nu}{}_{;\nu} = 0$ by (4), is that for any constant α, one can define a tensor field $\Lambda^{\mu\nu}$ such that,

$$\Lambda^{\mu\nu} \equiv G^{\mu\nu} - \alpha T^{\mu\nu} \ . \tag{25}$$

With this definition for $\Lambda^{\mu\nu}$, it is constrained to be both symmetric,

$$\Lambda^{\mu\nu} = \Lambda^{\nu\mu} , \tag{26}$$

and conserved,

$$\Lambda^{\mu\nu}{}_{;\nu} = 0 \ . \tag{27}$$

The value of the constant α in (25) is completely arbitrary and without physical significance because $\Lambda^{\mu\nu}$ as defined can absorb any change in α such that (25) remains satisfied. Taking advantage of this arbitrariness and setting the value of the constant $\alpha = -8\pi$ gives with a slight rearrangement of (25),

$$G^{\mu\nu} = -8\pi\, T^{\mu\nu} + \Lambda^{\mu\nu} \ , \tag{28}$$

which is recognized as Einstein's equation of General Relativity (24) augmented on its RHS by the term $\Lambda^{\mu\nu}$, a term that from the perspective of classical General Relativity mimics the properties of dark matter and/or dark energy, *viz.*, it is a conserved and symmetric tensor field, it is a source of gravitational fields in addition to $T^{\mu\nu}$, and it has no interaction signature beyond the gravitational fields it sources.

It is important to recognize at this point in the development of the proposed theory that (28) is a trivial result, and any solution of the fundamental field equations (1) through (4) must also necessarily be a solution of (28) for some choice $\Lambda^{\mu\nu}$. In fact, the validity of (28) rests only on the existence of a conserved energy-momentum tensor and the properties of the R-C tensor and so will be true in any physical theory that has a conserved energy-momentum tensor. However, the interesting point in the context of the proposed theory here is that the value of $\Lambda^{\mu\nu}$ can be calculated in terms of the dynamic field variables given in Table II, and without recourse to postulating the existence of exotic forms of matter or energy such as dark matter and dark energy. This feature will be investigated further in subsequent sections in which specific solutions to the fundamental field equations (1) through (4) are developed.

In the view of the proposed theory, gravitation emerges as a manifestation of the geometricized theory of electromagnetism based on field equations (1) through (4). While this theory of electromagnetism encompass all of classical electromagnetism, it does start with a slightly different set of foundational equations, specifically with Maxwell's homogenous equation (10) being replaced by equation (1). It is equation (1) with its coupling of the derivatives of the Maxwell tensor to the R-C tensor that brings gravitation into the picture. While viewing gravitation as a manifestation of electromagnetism is not new,[5] the specific approach being followed here with equation (1) is fundamentally new.

3. MATHEMATICAL STRUCTURE OF FUNDAMENTAL FIELD EQUATIONS

3.1 Symmetries of fundamental field equations

Three important global symmetries of the fundamental field equations (1) through (4) that are shared by all their solutions are reviewed here. The first of these global symmetries corresponds to charge-conjugation,

$$\begin{pmatrix} u^{\lambda} \\ a^{\lambda} \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \rightarrow \begin{pmatrix} u^{\lambda} \\ -a^{\lambda} \\ -F^{\mu\nu} \\ g_{\mu\nu} \\ -\rho_c \\ \rho_m \end{pmatrix}, \tag{29}$$

the second corresponds to a matter-to-antimatter transformation as will be discussed in section 5.2,

$$\begin{pmatrix} u^{\lambda} \\ a^{\lambda} \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \to \begin{pmatrix} -u^{\lambda} \\ -a^{\lambda} \\ -F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix}, \tag{30}$$

and the third to the product of the first two,

$$\begin{pmatrix} u^{\lambda} \\ a^{\lambda} \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \to \begin{pmatrix} -u^{\lambda} \\ a^{\lambda} \\ F^{\mu\nu} \\ g_{\mu\nu} \\ -\rho_c \\ \rho_m \end{pmatrix}. \tag{31}$$

All three transformations (29) through (31) leave the fundamental equations (1) through (4) unchanged. Adding the identity transformation to these symmetries forms the Klein-4 group, with the product of any two of the symmetries giving the remaining symmetry.

Note that among the fundamental fields of the theory, only $g_{\mu\nu}$ and ρ_m are unchanged by all the symmetry transformations, a fact that will be useful in section 5.4 for defining boundary conditions that lead to quantized mass, charge and angular momentum of particle-like solutions as well as for the treatment of antimatter. Finally, in addition to the global symmetries (29) through (31), and general covariance, the proposed theory also exhibits the electromagnetic gauge covariance of classical electromagnetism as detailed in equations (11) and (12).

3.2 Relationship of a^{λ} to the conventional electromagnetic vector potential A^{λ}

The only really new piece of physics in the foregoing development has been the introduction of the vector field a^{λ}, a field that has no counterpart in the conventionally accepted development of classical physics but here serves to couple through (1) the derivatives of the Maxwell tensor with its 20 independent components to the R-C tensor with its 20 independent components (in four dimensions). In some respects, this new vector field a^{λ} plays a role similar to the conventional electromagnetic vector potential

A^λ of classical electromagnetism as defined in (11), and in fact can be directly related to it. To see this, take the covariant derivative of both sides of (11) giving,

$$F_{\mu\nu;\kappa} = A_{\nu;\mu;\kappa} - A_{\mu;\nu;\kappa} \ . \tag{32}$$

Now compare (32) to equation (1) rewritten as,

$$F_{\mu\nu;\kappa} = a^\lambda R_{\lambda\kappa\mu\nu} = -a_{\kappa;\mu;\nu} + a_{\kappa;\nu;\mu} \ , \tag{33}$$

where the RHS of (33) follows from the commutation property of covariant derivatives. Equating the RHS's of equations (32) and (33) gives,

$$a_{\kappa;\nu;\mu} - a_{\kappa;\mu;\nu} = A_{\nu;\mu;\kappa} - A_{\mu;\nu;\kappa} \, , \tag{34}$$

establishing the connection between a^λ and A^λ.

3.3 A system of first order partial differential equations – Do solutions exist?

Equation (1) represents a mixed system of first order partial differential equations for $F_{\mu\nu}$ and illustrates one of the mathematical complexities of the fundamental field equations (1) through (4) that must be dealt with when attempting to find solutions. Specifically, mixed systems of first order partial differential equations must satisfy integrability conditions if solutions are to exist.[6] Although there are several ways of stating what these integrability conditions are, perhaps the simplest is given by,

$$F_{\mu\nu;\kappa;\lambda} - F_{\mu\nu;\lambda;\kappa} = -F_{\mu\sigma} R^\sigma{}_{\nu k\lambda} - F_{\sigma\nu} R^\sigma{}_{\mu\kappa\lambda} \ , \tag{35}$$

which can be derived using the commutation relations for covariant derivatives. Using (1) to substitute for $F_{\mu\nu;\kappa}$ in (35) gives,

$$\left(a^\rho R_{\rho\kappa\mu\nu}\right)_{;\lambda} - \left(a^\rho R_{\rho\lambda\mu\nu}\right)_{;k} = -F_{\mu\sigma} R^\sigma{}_{\nu k\lambda} - F_{\sigma\nu} R^\sigma{}_{\mu\kappa\lambda} \ , \tag{36}$$

which can be interpreted as conditions that are automatically satisfied by any solution consisting of expressions for $g_{\mu\nu}$, a^λ and $F_{\mu\nu}$ that satisfy (1). With (36) as integrability conditions that must be satisfied by any solution of (1), the question that naturally arises is this: Are these integrability conditions so restrictive that perhaps no solution exists to the proposed theory? Although this view could be construed as making the proposed field theory based on the fundamental field equations (1) through (4) uninteresting because perhaps no solutions exist, it will be shown that solutions that are consistent with known solutions of the M&EFEs can be found. Finally, to further elucidate this and other questions regarding solutions of the fundamental field equations, an outline showing how they can be solved numerically is given in the appendix where an analysis is presented of fundamental equations (1) through (4) in terms of a Cauchy initial value problem.

4. SOLUTIONS TO FUNDAMENTAL FIELD EQUATIONS

4.1 Spherically symmetric charged particle

In this section a solution representing a non-rotating, spherically symmetric charged mass is investigated. It is demonstrated that the Reissner-Nordström metric with an appropriate choice for the fields $F_{\mu\nu}, a^{\lambda}, u^{\lambda}, \rho_c$ and ρ_m satisfy the fundamental field equations (1) through (4). Although the presentation in this section is purely formal, it is included here for several reasons. First, if the theory could not describe the asymptotic electric and gravitational fields of a charged particle it would be of no interest on physical grounds. Second and as already discussed, the presented theory requires the solution of a mixed system of first order partial differential equations, a system that may be so restrictive that no solutions exist and so an outline of at least one methodology to a solution is warranted. Finally, having an exact solution to field equations (1) through (4) that corresponds to a known solution of the M&EFEs enables a direct comparison and contrast of the two solutions to be made.

To proceed, I draw on a solution for a spherically symmetric charged particle that was previously derived.[7] Starting with the Reissner-Nordström metric[8],

$$g_{\mu\nu} = \begin{pmatrix} \dfrac{1}{1+\dfrac{q^2}{r^2}-\dfrac{2m}{r}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\mathrm{Sin}^2(\theta) & 0 \\ 0 & 0 & 0 & -1-\dfrac{q^2}{r^2}+\dfrac{2m}{r} \end{pmatrix}, \tag{37}$$

and a guess for a^{λ},

$$a^{\lambda} = \left(0,0,0,c_1\right), \tag{38}$$

where c_1 is a yet to be determined constant, ρ_c can be determined from (22) as,

$$\rho_c = \pm \frac{q^2\sqrt{q^2 + r(r-2m)}}{r^5}|c_1|. \tag{39}$$

Using equation (21), u^λ is then found to be,

$$u^\lambda = \left(0, 0, 0, \pm \frac{r}{\sqrt{q^2 + r(r-2m)}}\frac{c_1}{|c_1|}\right). \tag{40}$$

The next step is to satisfy (1) by solving for $F_{\mu\nu}$. Rather than tackle this head on by directly trying to find a solution to the mixed system of first order partial differential equations (1), I instead solve the integrability equations (35) which are linear in $F_{\mu\nu}$ for $F_{\mu\nu}$. Proceeding in this manner I find all the integrability equations are satisfied for $F_{\mu\nu}$ given by,

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_\phi & -B_\theta & E_r \\ -B_\phi & 0 & B_r & E_\theta \\ B_\theta & -B_r & 0 & E_\phi \\ -E_r & -E_\theta & -E_\phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{(mr-q^2)}{r^3}c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(mr-q^2)}{r^3}c_1 & 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

By direct substitution it is easily verified that $F_{\mu\nu}$ as given in (41) is a solution of (1).[9] Choosing the value of the undetermined constant $c_1 = \pm q/m$ then gives an electric field which agrees with the Coulomb field of a point charge to leading order in $1/r$. Finally, the remaining unknown field, the scalar mass density field ρ_m is found using the conserved energy-momentum tensor (4). Substituting the known fields into (4) and then solving for ρ_m gives,

$$\rho_m = \frac{q^4(q^2 - 2mr + r^2)}{m^2 r^6}. \tag{42}$$

To summarize, the following expressions for $g_{\mu\nu}$, $F_{\mu\nu}$, a^λ, u^λ, ρ_c and ρ_m are an exact solution to the fundamental field equations (1) through (4):

$$
\begin{gathered}
g_{\mu\nu} = \begin{pmatrix} \dfrac{1}{1-\dfrac{2m}{r}+\dfrac{q^2}{r^2}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\mathrm{Sin}^2(\theta) & 0 \\ 0 & 0 & 0 & -\left(1-\dfrac{2m}{r}+\dfrac{q^2}{r^2}\right) \end{pmatrix} \\
F_{\mu\nu} = \begin{pmatrix} 0 & B_\phi & -B_\theta & E_r \\ -B_\phi & 0 & B_r & E_\theta \\ B_\theta & -B_r & 0 & E_\phi \\ -E_r & -E_\theta & -E_\phi & 0 \end{pmatrix} = s\begin{pmatrix} 0 & 0 & 0 & \dfrac{q}{r^2}-\dfrac{q^3/m}{r^3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{q}{r^2}+\dfrac{q^3/m}{r^3} & 0 & 0 & 0 \end{pmatrix} \\
u^\lambda = s\left(0,0,0,\frac{1}{\sqrt{1-\dfrac{2m}{r}+\dfrac{q^2}{r^2}}}\right) \\
a^\lambda = s\left(0,0,0,\frac{q}{m}\right) \\
\rho_c = \frac{q^3}{m}\frac{\sqrt{1-\dfrac{2m}{r}+\dfrac{q^2}{r^2}}}{r^4} \\
\rho_m = \frac{q^4}{m^2}\frac{(1-\dfrac{2m}{r}+\dfrac{q^2}{r^2})}{r^4}
\end{gathered}
\tag{43}
$$

In (43) I have introduced a parameter s where $s=\pm1$ corresponding to the global matter-to-antimatter transformation symmetry (30), which will be further discussed in section 5. The matter and antimatter solutions to the proposed theory that emerge here are a direct result of treating ρ_m and ρ_c as dynamic fields to be solved for rather than the traditional classical treatment in which they are treated as external fields inserted into the theory. Except for the possibility of both matter and antimatter solutions, the physical interpretation of solution (43) is almost identical to that of the M&EFEs, *i.e.*, a non-rotating, spherically symmetric particle having charge q and mass m. Of note is the metric tensor which is identical to the Reissner-Nordström metric, establishing that the new theory predicts a gravitational field identical to the prediction of Einstein's General Relativity. However, the solution (43) does differ from the classical picture of a point charge in an important way, that being the mass and charge of the particle in (43) are not localized, with both ρ_m and ρ_c tailing off as $1/r^4$.

Also, of note in the solution (43) is the form of the radial electric field,

$$E_r = \frac{q}{r^2} - \frac{q^3/m}{r^3} = \frac{q}{r^2}\left(1 - \frac{q^2/m}{r}\right), \tag{44}$$

which agrees with the Coulomb field q/r^2 to leading order in $1/r$. Going beyond the leading order term, the next term in the radial electric field depends on both the charge and mass of the particle. Taking an electron as an example, its electric field as given by (44) is,

$$E_r = \frac{q_e}{r^2}\left(1 - \frac{r_e}{r}\right) \approx \frac{q_e}{r^2}\left(1 - \frac{2.82x10^{-15}m}{r}\right), \tag{45}$$

where $r_e = q_e^2/m_e$ is recognized as the classical radius of an electron (~$2.82x10^{-15}$m). Although the correction term to the Coulomb field is small, being only ~53 ppm at a distance of a Bohr radius, it may have interesting consequences in various situations because it depends on both the charge and the mass of the particle.

As already mentioned, the new theory's solution (43) goes further than the M&EFE's solution by solving for the spatial dependence of both the charge density ρ_c and the mass density ρ_m, solutions which then define the spatial dependence of the energy-momentum tensor,

$$T^{\mu\nu} = \rho_m u^\mu u^\nu + F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}. \tag{46}$$

By direct substitution it can be verified that the resulting spatial dependence of the energy momentum tensor (46) along with the Reissner-Nordström metric (37) do not satisfy Einstein's equation of General Relativity (24). However, Einstein's equation of General Relativity augmented by the $\Lambda^{\mu\nu}$ term on its RHS (28) is trivially satisfied. For completeness the values of $G^{\mu\nu}$, $T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ that go with solution (43) are given here:

$$G^{\mu\nu}=\begin{pmatrix}\frac{q^2\left(q^2+r(-2m+r)\right)}{r^6} & 0 & 0 & 0\\ 0 & -\frac{q^2}{r^6} & 0 & 0\\ 0 & 0 & -\frac{q^2\csc^2(\theta)}{r^6} & 0\\ 0 & 0 & 0 & -\frac{q^2}{r^2\left(q^2+r(-2m+r)\right)}\end{pmatrix}$$

$$T^{\mu\nu}=\begin{pmatrix}-\frac{q^2\left(q^2-mr\right)^2\left(q^2+r(-2m+r)\right)}{2m^2r^8} & 0 & 0 & 0\\ 0 & \frac{q^2\left(q^2-mr\right)^2}{2m^2r^8} & 0 & 0\\ 0 & 0 & \frac{q^2\left(q^2-mr\right)^2\csc^2(\theta)}{2m^2r^8} & 0\\ 0 & 0 & 0 & \frac{3q^6+m^2q^2r^2+2q^4r(-3m+r)}{2m^2r^4\left(q^2+r(-2m+r)\right)}\end{pmatrix}$$

$$\Lambda^{\mu\nu}=\begin{pmatrix}-\frac{q^2\left(q^2+r(-2m+r)\right)\left(-m^2r^2+4\pi\left(q^2-mr\right)^2\right)}{m^2r^8} & 0 & 0 & 0\\ 0 & -\frac{q^2}{r^6}+\frac{4\pi q^2\left(q^2-mr\right)^2}{m^2r^8} & 0 & 0\\ 0 & 0 & \frac{q^2\left(-m^2r^2+4\pi\left(q^2-mr\right)^2\right)\csc^2(\theta)}{m^2r^8} & 0\\ 0 & 0 & 0 & -\frac{q^2\left(m^2r^2-4\pi\left(3q^4+m^2r^2+2q^2r(-3m+r)\right)\right)}{m^2r^4\left(q^2+r(-2m+r)\right)}\end{pmatrix}$$

(47)

In the context of classical General Relativity (24), the interpretation of $\Lambda^{\mu\nu}$ in (47) is that of dark matter and/or dark energy, which serves as a source term for gravitational fields in addition $T^{\mu\nu}$. However, in the context of the proposed theory, $\Lambda^{\mu\nu}$ depends only on the existence of normal matter and normal energy and is a consequence of the fundamental field equations (1) through (4), again emphasizing that theory of gravitation emerging from the proposed theory differs from that of classical General Relativity as given by (24).

Another feature that results from the spatial distributions for both the charge and mass density being specified as part of the solution to the fundamental field equations (1) through (4) is that it gives a mechanism for quantizing both the mass m and charge q parameters of the Reissner-Nordström metric (37). By requiring that both the mass m and charge q be consistent with the spatially integrated mass density and charge density, respectively, self-consistency equations can be derived that put additional requirements on the physically allowable values these parameters can have, a topic that will be picked up in section 5.4.

Finally, an interesting constraint on particle-like solutions with metrics that depend explicitly on a charge parameter q, such as the Reissner-Nordström metric used in (43), follows from the charge-conjugation symmetry (29) of fundamental field equations (1) through (4). The charge conjugation symmetry transformation (29) takes $g_{\mu\nu} \to g_{\mu\nu}$ and $\rho_c \to -\rho_c$, or equivalently $q \to -q$, as will be justified in section 5.4. This forces the conclusion that the sign of q has no impact on the metric, *i.e.*, the metric can only depend on the absolute value of q (or the absolute value of ρ_c) since it is unchanged by the transformation $q \to -q$. This result, which is applicable to all solutions of fundamental field equations (1) through (4) is also in-line with known charge containing solutions of Einstein's field equation such as the Reissner-Nordström and Kerr-Newman metrics, both of which depend on q^2.

4.2 Electromagnetic radiation

Working in the weak field limit, derived here are expressions for a propagating electromagnetic plane waves in terms of the vector field a^λ and the metric tensor $g_{\mu\nu}$. This example is useful because it establishes a fundamental relationship between electromagnetic and gravitational radiation imposed by the field equations (1) through (4), and predicts that electromagnetic and gravitational waves are both manifestations of wave propagation of the underlying metric $g_{\mu\nu}$. To begin, consider an electromagnetic plane wave having frequency ω, propagating in the *+z*-direction and polarized in the *x*-direction. The Maxwell tensor for this field is given by,

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -B_y & E_x \\ 0 & 0 & 0 & 0 \\ B_y & 0 & 0 & 0 \\ -E_x & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \tag{48}$$

where E_x and B_y are the constant field amplitudes of the electromagnetic wave. Assume a near-Minkowski weak field metric given by,

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} e^{i\omega(t-z)} \\ |h_{\mu\nu}| \ll 1 \end{gathered} \tag{49}$$

where $\eta_{\mu\nu} = diag[1,1,1,-1]$, $h_{\mu\nu}$ are complex constants, and the vector field a^λ is also assumed to be constant and given by,

$$a^{\lambda} = \left(a^1, a^2, a^3, a^4\right) . \tag{50}$$

I proceed by substituting for $F_{\mu\nu}$ from (48), $g_{\mu\nu}$ from (49) and a^{λ} from (50) into (1), and then only retain terms to first order in the fields $h_{\mu\nu}$ and $F_{\mu\nu}$, both of which are assumed to be small and of the same order.[10] Doing this leads to a set of 8 independent linear equations for the 16 unknown constants: $h_{\mu\nu}$, a^{λ}, E_x and B_y. Solving these 8 independent equations, the 8 field components E_x, B_y, h_{13}, h_{22}, h_{23}, h_{34}, a^2 and a^3 can be solved for in terms of 8 free constants a^1, a^4, h_{11}, h_{12}, h_{14}, h_{24}, h_{33}, and h_{44}:

$$\begin{aligned} E_x &= i\,\omega \frac{\left(h_{11}{}^2 + h_{12}{}^2\right)}{2\,h_{11}} a^1 , \\ B_y &= E_x \end{aligned} \tag{51}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & -h_{14} & h_{14} \\ h_{12} & -h_{11} & -h_{24} & h_{24} \\ -h_{14} & -h_{24} & h_{33} & -\frac{1}{2}\left(h_{33}+h_{44}\right) \\ h_{14} & h_{24} & -\frac{1}{2}\left(h_{33}+h_{44}\right) & h_{44} \end{pmatrix} e^{i\omega(t-z)} , \tag{52}$$

and

$$a^{\lambda} = \left(a^1, a^1\frac{h_{12}}{h_{11}}, a^4, a^4\right) . \tag{53}$$

This solution illustrates several ways in which the new theory departs from the traditional view of electromagnetic radiation. Of most significance, the undulations in the electromagnetic field are due to underlying undulations in the metric field $g_{\mu\nu}$ (52) via the coupling defined in (1). This result also underscores that the existence of electromagnetic radiation is forbidden in strictly flat space-time. An interesting aspect of this solution is that while electromagnetic radiation necessitates the presence of an underlying gravitational radiation field, the underlying gravitational radiation is not completely defined by the electromagnetic radiation. The supporting gravitational radiation has 6 undetermined constants $\left(h_{11}, h_{12}, h_{14}, h_{24}, h_{33}, h_{44}\right)$ with the only restriction being $\left|h_{\mu\nu}\right| \ll 1$ and $h_{11} \neq 0$ as required by (51).

Further insight into the physical content of the metric (52) is evident after making the infinitesimal coordinate transformation from $x^\mu \to x'^\mu$ given by,

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x + \dfrac{i}{\omega} h_{14}\, e^{i\omega(t-z)} \\ y + \dfrac{i}{\omega} h_{24}\, e^{i\omega(t-z)} \\ z - \dfrac{i}{2\omega} h_{33}\, e^{i\omega(t-z)} \\ t - \dfrac{i}{2\omega} h_{44}\, e^{i\omega(t-z)} \end{pmatrix}, \tag{54}$$

and only retaining terms to first order in the *h's*. Doing this, the metric (52) is transformed to,

$$g'_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & -h_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)}\ , \tag{55}$$

while E'_x and B'_y, the transformed electric and magnetic field amplitudes, respectively, are identical to E_x and B_y given in (51). Note, only the h_{11} and h_{12} components of the metric (55) have an absolute physical significance, and $h_{22} = -h_{11}$ which makes the plane wave solution (55) identical to the gravitational plane wave solution of the classical Einstein field equations.[11], [12]

Because the underlying gravitational wave couples to both charged and uncharged matter, one consequence of the solution here is that there will be an uncertainty when describing the interaction of electromagnetic radiation with matter if the gravitational wave component of the problem is ignored. However, for nonrelativistic matter, this gravitational interaction (55) vanishes to first order in the *h's*. To see this, consider the following expansion of the Lorentz force law,

$$\begin{gathered} \rho_m \frac{Du^\mu}{D\tau} = \rho_c u^\lambda F^\mu{}_\lambda \\ \downarrow \\ \rho_m \frac{du^\mu}{d\tau} = -\rho_m u^\nu u^\lambda \Gamma^\mu{}_{\nu\lambda} + \rho_c u^\lambda F^u{}_\lambda \end{gathered} \quad . \tag{56}$$

The first term on the RHS in the line above represents the gravitational interaction. This gravitational interaction term vanishes for nonrelativistic matter with $u^{\lambda} \approx (0,0,0,1)$ because for the metric (55) all the $\Gamma^{\mu}{}_{44}$ vanish to first order in the *h's*.

4.3 Gravitational radiation

The forgoing analysis demonstrates the necessity of having an underlying gravitational wave to support the presence of an electromagnetic wave, but the converse is not true and gravitational radiation can exist independent of any electromagnetic radiation. The following analysis demonstrates this by solving for the structure of gravitational radiation in the absence of electromagnetic radiation. Following the same weak field formalism for the unknown fields $h_{\mu\nu}$ given in (49), but this time zeroing out E_x and B_y in (48), leads to the following solutions for $g_{\mu\nu}$ and a^{λ},

$$g_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & -h_{14} & h_{14} \\ h_{12} & \dfrac{h_{12}{}^2}{h_{11}} & -h_{24} & h_{24} \\ -h_{14} & -h_{24} & h_{33} & -\dfrac{h_{33}+h_{44}}{2} \\ h_{14} & h_{24} & -\dfrac{h_{33}+h_{44}}{2} & h_{44} \end{pmatrix} e^{i\omega(t-z)} \tag{57}$$

and

$$a^{\lambda} = \left(a^1, -a^1 \frac{h_{11}}{h_{12}}, a^4, a^4 \right) . \tag{58}$$

Both $g_{\mu\nu}$ given by (57) and a^{λ} given by (58) are modified from their solutions in the presence of an electromagnetic wave as given by (52) and (53), respectively. Performing a transformation to the same primed coordinate system as given in (54), here gives the metric field,

$$g'_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & \dfrac{h_{12}{}^2}{h_{11}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} , \tag{59}$$

illustrating again that only the h_{11} and h_{12} components have an absolute physical significance. The interaction of nonrelativistic matter with the gravitational wave (59) vanishes to first order in the *h's* for the same reason that it vanished for the gravitational wave (55) that accompanies electromagnetic radiation. Of particular note is the change in the value of the h_{22} component depending on whether the gravitational wave supports an electromagnetic wave as in (55) or is standalone as in (59).

One of the successes of fundamental field equations (1) through (4) is the existence of solutions describing both electromagnetic and gravitational radiation, unifying both phenomena as undulations of the underlying metric field $g_{\mu\nu}$. In some respects, this is not too surprising. Equation (1) with $F_{\mu\nu}=0$ is a system of second order partial differential equations, $a^{\lambda}R_{\lambda\kappa\mu\nu}=0$, in the metric field components $g_{\mu\nu}$ just as Einstein's field equations are, so the fact that both sets of field equations give similar solutions for gravitational waves is not to be completely unexpected. Finally, because both gravitational and electromagnetic radiation are due to undulations of the metric field $g_{\mu\nu}$ in the new theory, the speed of propagation of these waves is predicted to be identical, a result that has recently been refined experimentally with observations made during the binary neutron star merger in NGC 4993, 130 million light years from Earth.[13] The nearly simultaneous detection, within 2 seconds of each other, of gravity waves[14] and a burst of gamma rays[15] from this event constrain the propagation speed of electromagnetic and gravitational radiation to be the same to better than 1 part in 10^{15}.

4.4 Isotropic and homogenous universe

As shown in a section 4.1, the M&EFEs and fundamental field equations (1) through (4) share particle-like solutions having similar character. However, when considering non-static metrics, differences between the predictions of the two theories start to emerge. To illustrate some of these differences, here I investigate the Friedmann–Lemaître–Robertson–Walker (FLRW) metric,

$$g_{\mu\nu}=\begin{pmatrix} \dfrac{R_{cs}{}^{2}(t)}{1-kr^{2}} & 0 & 0 & 0 \\ 0 & R_{cs}{}^{2}(t)r^{2} & 0 & 0 \\ 0 & 0 & R_{cs}{}^{2}(t)r^{2}\,\mathrm{Sin}^{2}(\theta) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{60}$$

where *k* equals +1, 0 or -1 depending on whether the spatial curvature is positive, zero or negative, respectively, and $R_{cs}(t)$ is the cosmic scale factor at time *t*. Just as in the case of Einstein's equation of General Relativity where the FLRW metric is a cosmological solution representing a homogenous and isotropic universe, it is the same for fundamental field equations (1) through (4) with an appropriate choice for the time development of the cosmic scale parameter $R_{cs}(t)$. To derive the time dependence of this cosmic scale factor I start by noting that the 3-dimensional spatial subspace of (60) is maximally symmetric and so any tensor fields that inhabit that subspace must also be maximally symmetric.[16] Specifically, this restricts the form of a^{μ} to be,

$$a^{\mu} = \left(0,0,0,a^{4}(t)\right), \tag{61}$$

and forces the antisymmetric Maxwell tensor to vanish,

$$F_{\mu\nu} = 0 \ . \tag{62}$$

Because $F_{\mu\nu}$ vanishes so must $F_{\mu\nu;\kappa}$,

$$F_{\mu\nu;\kappa} = 0 \ , \tag{63}$$

which on substitution in (1) forces,

$$a^{\lambda} R_{\lambda\kappa\mu\nu} = 0 \, . \tag{64}$$

This in turn forces,

$$a^{\lambda} R_{\lambda}^{\ \nu} = 0 \, , \tag{65}$$

which is just equation (21) with $\rho_c = 0$. Substituting a^{μ} given by (61), and the FLRW metric given by (60) into (64) then leads to the following set of equations to be satisfied,

$$\begin{aligned}
a^{4}(t) R_{4114} &= a^{4}(t)\left(\frac{R_{cs}(t)}{k\,r^{2}-1}\,\frac{d^{2}R_{cs}(t)}{dt^{2}}\right) = 0 \\
a^{4}(t) R_{4224} &= a^{4}(t)\left(-r^{2}\,R_{cs}(t)\,\frac{d^{2}R_{cs}(t)}{dt^{2}}\right) = 0 \\
a^{4}(t) R_{4334} &= a^{4}(t)\left(-r^{2}\,R_{cs}(t)\,\mathrm{Sin}^{2}(\theta)\,\frac{d^{2}R_{cs}(t)}{dt^{2}}\right) = 0
\end{aligned} \tag{66}$$

with all other components of (64) not listed in (66) being trivially satisfied, *i.e.*, $0=0$. The nontrivial component equations (66) are all satisfied if,

$$\frac{d^2 R_{cs}(t)}{dt^2} = 0 , \tag{67}$$

which gives,

$$R_{cs}(t) = R_{cs\,0} + v_{cs}\, t \;, \tag{68}$$

where $R_{cs\,0}$ is the cosmic scale factor at time *t=0* and v_{cs} is the rate of change of the cosmic scale factor. The solution for $R_{cs}(t)$ given in (68) ensures that the metric (60) satisfies both (64) and (65) for all values of *k*. Based on this solution, the predictions of the new theory for a homogenous and isotropic universe are:

1. It must be charge neutral, *i.e.*, $\rho_c = 0$.
2. The cosmic scale factor changes linearly with cosmic time.
3. The spatial curvature of the universe can be positive, negative or 0.

The second prediction above runs counter to results of the Friedmann models of classical General Relativity in which the growth of the cosmic scale factor is divided into three regimes: the radiation dominated regime with the scale factor growing as $t^{1/2}$, the matter dominated regime with the scale factor growing as $t^{2/3}$, and the dark energy dominated regime with the scale factor growing exponentially with time. This emphasizes one of the challenges facing the proposed theory based on fundamental equations (1) through (4), that of finding additional solutions that are in agreement with the interpretation of recent observations and analyses indicating an accelerating universe.

Just as in the case of the spherically symmetric particle-like solution analyzed in section 4.1, the cosmological solution of fundamental field equations (1) through (4) analyzed here must satisfy Einstein's equation of General Relativity augmented by a $\Lambda^{\mu\nu}$ term on its RHS (28). For the cosmological solution considered here it is straight forward to calculate the $G^{\mu\nu}, T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ by taking the energy-momentum tensor to be,

$$T^{\mu\nu} = (p+\rho_m) u^{\mu} u^{\nu} + p\, g^{\mu\nu} \;, \tag{69}$$

and the four-velocity vector field to be,

$$u^{\mu} = (0,0,0,1) \ , \tag{70}$$

where the form of the energy-momentum tensor (69) as a perfect fluid with pressure p and mass-density ρ_m, and the form of the four-velocity u^{μ} in (70) are both dictated by the requirement that they be maximally symmetric in the 3-dimensional spatial subspace of (60). For completeness the values of $G^{\mu\nu}, T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ that go with (68), (69) and (70) are given here:

$$G^{\mu\nu} = \begin{pmatrix} \frac{k-k^2r^2+v_{CS}{}^2-kr^2v_{CS}{}^2}{\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 & 0 & 0 \\ 0 & \frac{k+v_{CS}{}^2}{r^2\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 & 0 \\ 0 & 0 & \frac{\left(k+v_{CS}{}^2\right)\csc^2(\theta)}{r^2\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 \\ 0 & 0 & 0 & -\frac{3\left(k+v_{CS}{}^2\right)}{\left(R_{CS0}+t\,v_{CS}\right)^2} \end{pmatrix}$$

$$T^{\mu\nu} = \begin{pmatrix} \frac{\left(1-kr^2\right)p(t)}{\left(R_{CS0}+t\,v_{CS}\right)^2} & 0 & 0 & 0 \\ 0 & \frac{p(t)}{r^2\left(R_{CS0}+t\,v_{CS}\right)^2} & 0 & 0 \\ 0 & 0 & \frac{\csc^2(\theta)\,p(t)}{r^2\left(R_{CS0}+t\,v_{CS}\right)^2} & 0 \\ 0 & 0 & 0 & \rho_m(t) \end{pmatrix}$$

$$\Lambda^{\mu\nu} = \begin{pmatrix} -\frac{\left(-1+kr^2\right)\left(k+v_{CS}{}^2+8\pi\left(R_{CS0}+t\,v_{CS}\right)^2 p(t)\right)}{\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 & 0 & 0 \\ 0 & \frac{k+v_{CS}{}^2+8\pi\left(R_{CS0}+t\,v_{CS}\right)^2 p(t)}{r^2\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 & 0 \\ 0 & 0 & \frac{\csc^2(\theta)\left(k+v_{CS}{}^2+8\pi\left(R_{CS0}+t\,v_{CS}\right)^2 p(t)\right)}{r^2\left(R_{CS0}+t\,v_{CS}\right)^4} & 0 \\ 0 & 0 & 0 & -\frac{3\left(k+v_{CS}{}^2\right)}{\left(R_{CS0}+t\,v_{CS}\right)^2}+8\pi\rho_m(t) \end{pmatrix}$$

(71)

Just as was the case for the spherically symmetric particle-like solution studied in section 4.1, in the context of classical General Relativity (24) the interpretation of $\Lambda^{\mu\nu}$ in (71) is that of dark matter and/or dark energy, a source of gravitational fields in addition to those generated by $T^{\mu\nu}$. However, in the context of the new theory, $\Lambda^{\mu\nu}$ depends only on the existence of normal matter and normal energy as determined by fundamental field equations (1) through (4).

5. DISCUSSION

5.1 Dark matter and dark energy

With General Relativity as the foundation of observational gravitational physics today, dark matter and dark energy have been postulated to exist because of the many galactic and cosmological scale observations that cannot be understood using General Relativity with normal matter and normal energy alone. Specifically, some of the large-scale gravitational features of galaxies and galactic clusters dating back to Zwicky's observations in the 1930's have been explained using dark matter[17], and the acceleration of the universe discovered in the 1990's has been explained using dark energy[18]. Another example of modifications made to the original General Relativity field equations (24) to satisfy a perceived need, was the cosmological constant term $\lambda g_{\mu\nu}$ that Einstein added to their RHS,

$$G_{\mu\nu} = -8\pi T_{\mu\nu} + \lambda g_{\mu\nu} . \tag{72}$$

Specifically, this was done to enable a static universe solution but then subsequently dropped after expansion was discovered. Today this term is again in vogue as a possible representation of dark energy.

One of the vexing problems facing dark matter and dark energy-based explanations of various observational phenomena today is an ongoing inability to directly detect these forms of matter and energy, a situation which only adds to their ad hoc character. However, fundamental field equations (1) through (4) offer the prospect that dark matter and dark energy effects can be explained in terms of normal matter and normal energy alone, *i.e.*, the $\Lambda^{\mu\nu}$ term in (28) representing dark matter and dark energy in the context of General Relativity is provided with a mechanism for directly calculating its structure using fundamental field equations (1) through (4) with only normal matter and normal energy. The already investigated spherically symmetric particle-like solution which assumed a Reissner-Nordström metric and the cosmological solution which assumed an FLRW metric are two accessible examples that outline such a direct calculation of $\Lambda^{\mu\nu}$. With questions today regarding the validity of classical General Relativity beyond the confines of our own solar system[19] and the inability to directly detect dark matter and dark energy, the possible interpretation of the $\Lambda^{\mu\nu}$ term in (28) using only normal matter and normal energy is an enticing feature of fundamental field equations (1) through (4) . However, it must be acknowledged that one of the challenging tasks facing the proposed field theory, and one well beyond the analysis presented in this manuscript, is that of finding additional solutions that could be interpreted as agreeing with the rapidly developing observational understanding of galactic and cosmological structures.

5.2 Emergence of antimatter from solutions of fundamental field equations and it's behavior in electromagnetic and gravitational fields

One of the unique features of fundamental field equations (1) through (4) as a classical field theory is that the properties of matter and antimatter emerge naturally in solutions. This feature of solutions to the fundamental field equations is intimately tied to the treatment of ρ_c and ρ_m as dynamic fields that are solved for rather than being externally inserted as done in the traditional classical physics picture. For example, every matter containing solution to equations (1) through (4) has a corresponding antimatter solution generated by the symmetry transformation (30). This is evident in the spherically symmetric particle-like solution (43) where the multiplicative factor s in the expressions for $F_{\mu\nu}$, a^{λ} and u^{λ} is defined by,

$$s = \begin{cases} +1 \ \mathit{for\ matter} \\ -1 \ \mathit{for\ antimatter} \end{cases} \tag{73}$$

and accounts for the matter-antimatter symmetry expressed in (30). The physical interpretation of the $s=-1$ solution is that it represents a particle having the same mass but opposite charge and four-velocity to that of the $s=+1$ solution. This is equivalent to the view that a particle's antiparticle is the particle moving backwards through time.[20] Said another way, the time-like component of the four-velocity is positive for matter and negative for antimatter,

$$u^4 \begin{cases} >0 \ \mathit{for\ matter} \\ <0 \ \mathit{for\ antimatter} \end{cases} . \tag{74}$$

With these definitions for the four-velocity of matter and antimatter, charged mass density can annihilate similarly charged anti-mass density and satisfy both the local conservation of charge (15) and local conservation of mass (18). Additionally, such annihilation reactions must conserve energy by (4).

Building on the distinction between matter and antimatter, their behavior in external electromagnetic and gravitational fields in the context of the theory based on fundamental field equations (1) through (4) is briefly reviewed here. As already mentioned, antimatter can be viewed as matter moving backwards through time. To see this more rigorously consider the four-velocity associated with a fixed quantity of charge and mass density,

$$u^{\lambda} = \frac{dx^{\lambda}}{d\tau} \; . \tag{75}$$

Under the matter-antimatter transformation (30), $u^{\lambda} \to -u^{\lambda}$, or equivalently $d\tau \to -d\tau$. This motivates the following expression for the four-velocity in terms of the coordinate time in locally inertial coordinate systems,

$$u^{\lambda} = \frac{dx^{\lambda}}{d\tau} = s\gamma \frac{dx^{\lambda}}{dt} = s\gamma \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} , \tag{76}$$

where s is the matter-antimatter parameter defined in (73), $\vec{v} = \left(v_x, v_y, v_z\right)$ is the ordinary 3-space velocity of the charge and mass density, and $\gamma = 1/\sqrt{1-\vec{v}^2}$. Equation (76) establishes that corresponding matter and antimatter solutions travel in opposite time directions relative to each other. One of the unusual aspects of the matter-antimatter transformation (30) is that ρ_c does not change sign under the transformation. To see that this is consistent with the usual view in which antiparticles have the opposite charge of their corresponding particles, I use (76) to illustrate the behavior of a charged matter and antimatter density in an external electromagnetic field. Consider a region with an externally defined electromagnetic field,

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{77}$$

in a locally inertial coordinate system. Starting with the Lorentz force law (19), and then expanding and rearranging slightly leads to the following development,

$$
\begin{gathered}
\rho_p \frac{Du^\mu}{D\tau} = \rho_c F^\mu{}_\lambda u^\lambda \\
\downarrow \\
\rho_p s\gamma \frac{du^\mu}{dt} = \rho_c F^\mu{}_\lambda u^\lambda \\
\downarrow \\
\rho_p s\gamma \frac{d}{dt}\begin{pmatrix} s\gamma\vec{v} \\ s\gamma \end{pmatrix} = \rho_c \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} \begin{pmatrix} s\gamma v_x \\ s\gamma v_y \\ s\gamma v_z \\ s\gamma \end{pmatrix} \\
\downarrow \\
\rho_p \frac{d}{dt}\begin{pmatrix} \gamma\vec{v} \\ \gamma \end{pmatrix} = s\,\rho_c \begin{pmatrix} \vec{E} + \vec{v}\times\vec{B} \\ \vec{v}\cdot\vec{E} \end{pmatrix}
\end{gathered} \tag{78}
$$

which on the last line above ends up at the conventional form of the Lorentz force law except for the extra factor of s on the RHS. This factor of s in (78) gives the product $s\rho_c$ the appearance that antimatter charge density has the opposite sign to that of matter charge density when interacting with an external electromagnetic field.

Next, I investigate the behavior of antimatter in an external gravitational field. There is no question about the gravitational fields generated by matter and antimatter, they are identical under the matter-antimatter symmetry (30) as $g_{\mu\nu}$ is unchanged by that transformation. To understand whether antimatter is attracted or repelled by an external gravitational field, I again go to the Lorentz force law (19) but this time assume there is no electromagnetic field present, just a gravitational field given by a Schwarzschild metric generated by a central mass $m > 0$ that is composed of either matter or antimatter. I explicitly call out $m > 0$ because I am endeavoring to develop a physical theory that axiomatically flows from fundamental field equations (1) through (4) and at this point in the development there is nothing to preclude the existence of negative mass density $\rho_m < 0$, a consideration I will return to in section 5.3. Placing a test particle having mass m_{test} composed of either matter or antimatter a distance r from the center of the gravitational field and assuming it to be initially at rest, the trajectory of the test particle is that of a geodesic given by the following development,

$$
\begin{gathered}
m_{test}\frac{Du^{\mu}}{D\tau}=0 \\
\downarrow \\
s\gamma\frac{du^{\mu}}{dt}=-\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho} \\
\downarrow \\
s\gamma\frac{d}{dt}\left(s\gamma\frac{d}{dt}\begin{pmatrix} r \\ \theta \\ \phi \\ t \end{pmatrix}\right)=-\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}\approx-\Gamma^{\mu}{}_{44}u^{4}u^{4}=-\begin{pmatrix} \left(1-\frac{2m}{r}\right)\frac{m}{r^{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}\left(\frac{s}{\sqrt{1-\frac{2m}{r}}}\right)^{2}=-\begin{pmatrix} \frac{m}{r^{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}s^{2}
\end{gathered} \tag{79}
$$

where $s=\pm 1$ references whether the test particle is composed of matter or antimatter (73). In the last line of (79) I have approximated the RHS using the initial at rest value of the test particle's four-velocity $u^{\mu}=\left(0,0,0,s/\sqrt{1-2m/r}\right)$, and additionally used the fact that the only nonzero $\Gamma^{\mu}{}_{44}$ in a Schwarzschild metric is $\Gamma^{1}{}_{44}=\left(1-\frac{2m}{r}\right)m/r^{2}$. Simplifying the LHS of the last line in (79) by noting that initially $\gamma=1$ then gives,

$$
\frac{d^{2}r}{dt^{2}}\approx-\frac{m}{r^{2}}, \tag{80}
$$

which is independent of s, and so demonstrates that the proposed theory predicts both matter and antimatter test particles will be attracted by the source of the gravitational field, and this regardless of whether the source of the gravitational field is matter or antimatter. The result that the test particle is attracted toward the source of the gravitational field is also independent of whether the test particle's mass, m_{test} is positive or negative, this because the geodesic trajectory (80) is independent of m_{test}.

5.3 Possibility of negative mass solutions and antigravity

As already noted, there appears to be nothing in the fundamental equations (1) through (4) that precludes the possibility of negative mass density $\rho_m<0$. The existence of negative mass density is equivalent to the existence of antigravity because negative mass density generates gravitational fields that are repulsive, *viz*., equation (80) with $m<0$. However, logical inconsistencies are introduced if negative mass density were to exist. As just shown, equation (80) with $m>0$ predicts a test particle at some distance from the origin will feel an attractive gravitational force regardless of whether the test particle is comprised of matter or antimatter and regardless of whether its mass is positive or negative. Now consider equation (80) with the central mass $m < 0$. Using the same argument as in the previous section, the test particle in this case will feel a repulsive gravitational force regardless of whether it is comprised of matter or antimatter and regardless of whether its mass is positive or negative. These two situations directly contradict each other. For example, in the first case the negative mass test particle is gravitationally attracted toward the positive mass particle located at the origin, but in the second case the positive mass test particle is gravitationally repelled by the negative mass particle located at the origin. This contradiction makes fundamental equations (1) through (4) logically inconsistent if negative mass density were allowed exist. The only way to avoid this logical contradiction is to require that mass density be non-negative always. This condition that mass density ρ_m be non-negative always is also consistent with the global symmetry transformations (29) through (31) where it was noted that the field ρ_m does not change sign under any of the symmetry transformations.

It is interesting to note that the existence of negative mass in the context of classical General Relativity has been studied extensively[21], [22] and invoked particularly when trying to find stable particle-like solutions using the conventional Einstein field equations.[23], [24], [25] However, in the context of the present theory the existence of negative mass density leads to a logical contradiction that can only be resolved by requiring mass density be non-negative always, *i.e.*, $\rho_m \geq 0$.

5.4 Conjecture for quantizing charge and mass of particle-like solutions

Consider particle-like solutions such as (43), because the mass density and charge density are specified as part of the solution of fundamental field equations (1) through (4), a self-consistency constraint exists on physically allowed solutions that provides a mechanism for quantizing the charge and mass of such solutions. For example, in solution (43) the particle's total charge q and total mass m, both parameters in the Reissner-Nordström metric, must agree with the spatially integrated charge and mass density, respectively, if the solution is to be self-consistent. For the charge, this amounts to requiring the asymptotic value of the electric field be consistent with the spatially integrated charge density,

$$\lim_{r\to\infty} r^2 F_{14} = \int \rho_c u^4 \sqrt{\gamma_{sp}}\, d^3x = q \;, \tag{81}$$

where q is the total charge of the particle and given by the asymptotic value of $r^2 F_{14}$ per the solution given in (43), and γ_{sp} is the determinant of the spatial metric defined by,[26]

$$\gamma_{sp\ i\,j} = g_{i\,j} - \frac{g_{i4}\, g_{j4}}{g_{44}}, \tag{82}$$

where *i* and *j* run over the spatial dimensions 1, 2 and 3. An analogous quantizing boundary condition for the mass of the particle is arrived at by requiring the asymptotic value of its gravitational field be consistent with the spatially integrated mass density of the solution,

$$\lim_{r\to\infty} r\frac{1+g_{44}}{2} = \int \rho_m \left|u^4\right| \sqrt{\gamma_{sp}}\, d^3x = m \ . \tag{83}$$

The reason for the absolute value of u^4 in the mass boundary condition (83) but not in the charge boundary condition (81) are the global symmetries (29) through (31) exhibited by the theory's fundamental field equations (1) through (4) and the requirement that the boundary conditions exhibit those same symmetries. The conjecture being put forth here is that boundary conditions (81) and (83) represent self-consistency constraints on the charge and the mass, respectively, that any particle-like solution to the fundamental field equations (1) through (4) must satisfy if the solutions are to be physically realizable.

For the spherically-symmetric solution investigated in (43), the RHS of both (81) and (83) diverge leaving no hope for satisfying these quantization/boundary conditions. The upshot of this observation is that while (43) represents a solution that describes the gravitational and electrical fields of a particle-like solution that formally satisfy the fundamental field equations (1) through (4), (43) cannot represent a physically allowed solution. The possibility of finding solutions that satisfy both the fundamental field equations (1) through (4) and the charge and mass boundary conditions (81) and (83) remains an open question at this point. However, interesting possibilities exist beyond the spherically symmetric solution based on the Reissner-Nordström metric investigated within (43). For example, the modified Reissner-Nordström and modified Kerr-Newman metrics developed by S.M. Blinder[27] give finite values for the LHS of both (81) and (83). Finally, when considering metrics that include nonzero angular momentum, as for example would be required for particles having an intrinsic magnetic field, the same methodology used here to quantize the particle's mass and charge can be used to quantize its angular momentum.

6. CONCLUSION

The proposed theory based on equations (1) through (4) is very similar to the classical theory of electromagnetism at the level of Maxwell's equations and the interaction of electromagnetic fields with matter. In fact, replacing the proposed theory's equation (1) with Maxwell's homogenous equation gives exactly the foundational equations of classical electromagnetism. Because Maxwell's homogeneous equation is a direct consequence of (1) and the properties of the R-C tensor, one way to look at the proposed theory is that it is a geometricized theory of electromagnetism that encompasses classical electromagnetism in its entirety. Additionally, due to the coupling of the derivatives of the Maxwell tensor to the R-C tensor in equation (1), the proposed theory puts electromagnetic and gravitational phenomena on an equal footing, with both being intimately tied to the curvature of space-time.

The specific solutions investigated within demonstrate that the same metric solutions of classical General Relativity, *e.g.*, the Reissner-Nordström metric and the FLRW metric, are also solutions of the fundamental equations (1) through (4) with an appropriate choice for the mass and charge density fields, ρ_m and ρ_c, respectively. This is one way in which the proposed theory differs from the traditional classical treatments in which ρ_m and ρ_c are inserted as external fields. Importantly, full solutions of the proposed theory including the solutions for ρ_m and ρ_c do not satisfy the General Relativity field equations (24), but do satisfy those equations augmented by the $\Lambda^{\mu\nu}$ term as in equation (28), a term that from the perspective of classical General Relativity mimics the properties of dark matter and dark energy. This result emphasizes that the classical General Relativity field equations are not a derivable consequence of the proposed theory's field equations (1) through (4). However, because any solution of (1) through (4) will necessarily satisfy the General Relativity field equations augmented by the $\Lambda^{\mu\nu}$ as in equation (28), a result that is forced by the conserved energy-momentum tensor given in equation (4), the proposed theory does enable the determination of the $\Lambda^{\mu\nu}$ using only ordinary matter and energy and so possibly avoids the necessity of postulating the existence of both dark matter and dark energy.

Another consequence of the proposed theory that pushes beyond classical electromagnetism is the underlying relationship between electromagnetic and gravitational radiation that was established by the radiative solutions investigated within. These solutions demonstrate that in the proposed theory, both electromagnetic and gravitational radiation are due to undulations of the underlying metric field $g_{\mu\nu}$, while simultaneously preserving many of their characteristics as traditionally described using Maxwell's equations for electromagnetic radiation and the General Relativity field equations for gravitational radiation.

One of the unique features of solutions to the proposed theory's field equations as illustrated by the particle-like solution investigated within, is the natural emergence of antimatter. Traditionally, the properties of antimatter emerge in quantum mechanical treatments but here emerge in the context of a classical continuous field theory due to the global symmetry (30) of the fundamental equations (1) through (4). Using such solutions, the properties of matter and antimatter in the presence of external electromagnetic and gravitational fields can be unambiguously described. Additionally, the impossibility of negative mass, or equivalently, the impossibility of repulsive gravitational fields can be concluded within the confines of the proposed theory.

Although not yet based on solutions to the proposed theory, a methodology for quantizing the charge and mass of particle like solutions is conjectured based on the treatment of ρ_m and ρ_c as dynamic fields that are solved for as part of full solutions to the fundamental equations. The conjecture is that physically allowed solutions to the fundamental equations (1) through (4) must also simultaneously satisfy the charge and mass self-consistency equations (81) and (83), respectively. With this point of view, the exact particle-like solution presented within cannot represent a physically allowed solution because it does not satisfy (81) and (83). However, exact solutions such as the one presented within are still important in that they offer an existence proof that exact solutions to the proposed theory do exist, albeit ones that are only an approximate description of physical reality. Finally, using the same approach as in the derivation of the self-consistency equations (81) and (83), it is further conjectured that such a condition can be used to quantize the angular momentum of particle like solutions. Traditionally the quantization of mass, charge and angular momentum are introduced in quantum mechanical treatments but here are conjectured within the framework of a classical continuous field-theoretic description of nature and are another example of how the proposed theory differs from the classical M&EFEs.

The genesis of the work presented within was reported in a preliminary form in reference [2]. The same fundamental field equations proposed here were first reported there, although in a somewhat modified form. Also, the quantizing boundary conditions reviewed here were first reported there. New to this manuscript is the discussion of the global symmetries of the fundamental field equations (1) through (4) , and based on those global symmetries the interpretation of the particle-like solution has been advanced. The derivation of the Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and/or dark energy is also new to this manuscript as is the discussion of the cosmological solution based on the FLRW metric. The present manuscript also corrects an error in the weak field analysis of reference [2], leading to the expanded discussion of electromagnetic radiation and its underlying gravitational radiation. The discussion of the impossibility of negative mass solutions and antigravity is new to this manuscript. Finally, the appendix containing the analysis of the Cauchy initial value problem as it relates to the theory's fundamental field equations (1) through (4) is new and included to replace an incorrect discussion of the logical consistency of the fundamental field equations that was given in reference [2].

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8. APPENDIX - The Cauchy problem applied to the fundamental field equations

One of the unusual features of fundamental field equations (1) through (4) is the lack of any explicit derivatives of the vector field a^{λ}, a situation which raises questions about the time dependent development of a^{λ}. To further elucidate this and other questions regarding solutions of the fundamental field equations, and to outline how they could be solved numerically, they are here analyzed in terms of a Cauchy initial value problem.

Given a set of initial conditions comprising the values of the fundamental fields in Table I at all spatial locations, a procedure is outlined that propagates those fields to any other time. To begin, assume $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ are known at all spatial coordinates at some initial coordinate time t_0. Note that the initial values for the field a^{λ} are not required, rather they will be solved for using equation (1) as described below. Also note that in addition to $g_{\mu\nu}$ the initial values of $\frac{\partial g_{\mu\nu}}{\partial t}$ must be specified because the fundamental field equations are second order in the time derivatives of $g_{\mu\nu}$, a situation analogous to classical General Relativity. The goal of the Cauchy method as it applies here is to start with specified initial conditions for $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ at t_0, and then using the fundamental field equations (1) through (4) solve for $a^{\lambda}, R_{\lambda\kappa\mu\nu}, \frac{\partial F_{\mu\nu}}{\partial t}, \frac{\partial u^{\lambda}}{\partial t}, \frac{\partial \rho_m}{\partial t}, \frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at t_0. Armed with these values at t_0, it is straight forward to propagate the fields $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ from their initial conditions at t_0 to $t_0 + dt$ and then solve for $a^{\lambda}, R_{\lambda\kappa\mu\nu}, \frac{\partial F_{\mu\nu}}{\partial t}, \frac{\partial u^{\lambda}}{\partial t}, \frac{\partial \rho_m}{\partial t}, \frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at $t_0 + dt$ using the same procedure that was used to find them at t_0. Repeating this procedure, values for the fundamental fields of the theory can then be found at all times. One additional requirement on the field values specified by initial conditions is that they must be self-consistent with the fundamental field equations (1) through (4), *i.e.*, the specified initial conditions must be consistent with a solution to the fundamental field equations (1) through (4).

In what follows, Greek indices ($\mu, \nu, \kappa,$ …) take on the usual space-time coordinates 1-4 but Latin indices ($i, j, k,$ …) are restricted to spatial coordinates, 1-3 only. Since the values of $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ are known at all spatial coordinates at time t_0, the values of $\frac{\partial g_{\mu\nu}}{\partial x^i}, \frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial t}$ can be calculated at all spatial coordinates at time t_0. This leaves the ten quantities $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ as the only second derivatives of $g_{\mu\nu}$ not known at t_0. To find the values of $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at t_0 proceed as follows. First find the values of the six $\frac{\partial^2 g_{ij}}{\partial t^2}$ at t_0 using a subset of equations from (1), the subset containing only those equations having spatial derivatives of $F_{\mu\nu}$ on the LHS and at most one time-index in each occurrence of the R-C tensor on the RHS. These equations will be used to solve for the values of a^{λ} at time t_0. In all there are 12 such equations out of the 24 that comprise (1), as listed here:

$$\begin{aligned}
F_{12;1} &= a^{\lambda} R_{\lambda 112} \\
F_{13;1} &= a^{\lambda} R_{\lambda 113} \\
F_{23;1} &= a^{\lambda} R_{\lambda 123} \\
F_{12;2} &= a^{\lambda} R_{\lambda 212} \\
F_{13;2} &= a^{\lambda} R_{\lambda 213} \\
F_{23;2} &= a^{\lambda} R_{\lambda 223} \\
F_{12;3} &= a^{\lambda} R_{\lambda 312} \\
F_{13;3} &= a^{\lambda} R_{\lambda 313} \\
F_{23;3} &= a^{\lambda} R_{\lambda 323} \\
F_{12;4} &= -F_{24;1} - F_{41;2} = a^{\lambda} R_{\lambda 412} \\
F_{13;4} &= -F_{34;1} - F_{41;3} = a^{\lambda} R_{\lambda 413} \\
F_{23;4} &= -F_{34;2} - F_{42;3} = a^{\lambda} R_{\lambda 423}
\end{aligned} \quad . \tag{84}$$

The last three equations in (84) use (10), Maxwell's homogenous equation to express the time derivative of a Maxwell tensor component on the LHS as the sum of the spatial derivatives of two Maxwell tensor components. The importance of having only spatial derivatives of the Maxwell tensor components on the LHS of (84) is that they are all known quantities at time t_0, *i.e.*, since all the $F_{\mu\nu}$ are known at time t_0, all $\frac{\partial F_{\mu\nu}}{\partial x^i}$ and $F_{\mu\nu;i}$ can be calculated at time t_0. Equally important is that the RHS of the 12 equations that comprise (84) contain at most a single time index in each occurrence of their R-C tensor and so are also known at time t_0. To see that this is so I examine the general form of the R-C tensor in a locally inertial coordinate system where all first derivatives of $g_{\mu\nu}$ vanish, *i.e.*,

$$R_{\lambda\kappa\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\lambda}}{\partial x^{\nu} \partial x^{\kappa}} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^{\mu} \partial x^{\kappa}} + \frac{\partial^2 g_{\kappa\nu}}{\partial x^{\mu} \partial x^{\lambda}} \right) . \tag{85}$$

Note, having at most a single time index on the RHS of (85) means that the R-C tensor is made up entirely of terms from $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial t}$, all of which are known at time t_0. Examining the set of equations (84) there are 12 equations for 4 unknowns, the unknowns being the components of a^{λ}. These 12 equations can be solved for a^{λ} at time t_0 if the initial conditions were chosen self-consistently with the fundamental field equations (1) through (4), *i.e.*, chosen such that a solution to the field equations is indeed possible.

Knowing the R-C tensor components with at most one time-index at t_0, I now proceed to determine the R-C tensor components with two time indices. Going back to the 24 equations that comprise the set of equations (1), here I collect the subset of those equations in which the LHS is known at time t_0, *i.e.*, contains only spatial derivatives of the Maxwell tensor, and the RHS has an R-C tensor component that contain two time indices:

$$\begin{aligned} F_{14;1} &= a^{\lambda} R_{\lambda 114} \\ F_{24;1} &= a^{\lambda} R_{\lambda 124} \\ F_{34;1} &= a^{\lambda} R_{\lambda 134} \\ F_{14;2} &= a^{\lambda} R_{\lambda 214} \\ F_{24;2} &= a^{\lambda} R_{\lambda 224} \\ F_{34;2} &= a^{\lambda} R_{\lambda 234} \\ F_{14;3} &= a^{\lambda} R_{\lambda 314} \\ F_{24;3} &= a^{\lambda} R_{\lambda 324} \\ F_{34;3} &= a^{\lambda} R_{\lambda 334} \end{aligned} \quad . \tag{86}$$

Each of the equations in (86) contains only one unknown, the R-C component having two time indices. In total, there are six such independent R-C tensor components:

$$\begin{gathered} R_{1414} \\ R_{1424} \\ R_{1434} \\ R_{2424} \\ R_{2434} \\ R_{3434} \end{gathered} \tag{87}$$

so the system of nine equations (86) can be algebraically solved for the these six unknown R-C components at time t_0. With this I now know the value of all components of the R-C tensor at time t_0. From the t_0 values of the R-C tensor components listed in (87), the values of the six unknown $\dfrac{\partial^2 g_{ij}}{\partial t^2}$ at t_0 can be found.

There are three remaining equations from the set of equations (1) that have not yet been addressed:

$$\begin{aligned} F_{14;4} &= a^{\lambda} R_{\lambda 414} \\ F_{24;4} &= a^{\lambda} R_{\lambda 424} \\ F_{34;4} &= a^{\lambda} R_{\lambda 434} \end{aligned} \qquad . \tag{88}$$

These are the equations for which the temporal derivatives of the Maxwell tensor components are not yet known. Because all values of the R-C tensor and a^{λ} are now known at t_0, these three remaining time-differentiated components of the Maxwell tensor can now be solved for directly using (88), giving complete knowledge of $\frac{\partial F_{\mu\nu}}{\partial t}$ at time t_0.

If the values of the four $\frac{\partial^2 g_{\mu 4}}{\partial t^2}$ could be calculated then all $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ would be known and all $\frac{\partial g_{\mu\nu}}{\partial t}$ could be propagated from t_0 to $t_0 + dt$. Just as is the case with classical General Relativity, the four $\frac{\partial^2 g_{\mu 4}}{\partial t^2}$ can be determined from the four coordinate conditions that are fixed by the choice of coordinate system.[29] Recapping, at t_0 the following quantities are now known: $g_{\mu\nu}$, $F_{\mu\nu}$, u^{λ}, ρ_c, ρ_m and $\frac{\partial g_{\mu\nu}}{\partial t}$ are defined by initial conditions, and a^{λ}, $\frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}}$, $R_{\lambda\kappa\mu\nu}$, and $\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}$ are solved for using those initial conditions, the fundamental field equations, and the four coordinate conditions that are fixed by the choice of coordinate system. Still needed to propagate the initial conditions in time from t_0 to $t_0 + dt$ are $\frac{\partial u^{\mu}}{\partial t}$, $\frac{\partial \rho_m}{\partial t}$ and $\frac{\partial \rho_c}{\partial t}$. Using the Lorentz force law (19), the following development,

$$\begin{gathered} \rho_m \frac{Du^{\mu}}{D\tau} = \rho_c u^{\lambda} F^{\mu}{}_{\lambda} \\ \downarrow \\ \rho_m u^{\mu}{}_{;\nu} u^{\nu} = \rho_c u^{\lambda} F^{\mu}{}_{\lambda} \\ \downarrow \\ \rho_m u^{\mu}{}_{;4} u^4 = -\rho_m u^{\mu}{}_{;i} u^i + \rho_c u^{\lambda} F^{\mu}{}_{\lambda} \\ \downarrow \\ \rho_m \left(\frac{\partial u^{\mu}}{\partial t} + \Gamma^{\mu}{}_{4\sigma} u^{\sigma} \right) u^4 = -\rho_m u^{\mu}{}_{;i} u^i + \rho_c u^{\lambda} F^{\mu}{}_{\lambda} \end{gathered} \tag{89}$$

shows on the last line above that $\frac{\partial u^{\mu}}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. Using the conservation of mass (18) and knowing $\frac{\partial u^{\mu}}{\partial t}$ at t_0, the following development,

$$\begin{gathered}\left(\rho_m u^{\nu}\right)_{;\nu}=0\\ \downarrow\\ \left(\rho_m u^4\right)_{;4}=-\left(\rho_m u^i\right)_{;i}\\ \downarrow\\ \frac{\partial \rho_m}{\partial t}u^4=-\rho_m u^4{}_{;4}-\left(\rho_m u^i\right)_{;i}\end{gathered} \tag{90}$$

shows on the last line above that $\frac{\partial \rho_m}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. Following an analogous development for ρ_c using the charge conservation equation (15), $\frac{\partial \rho_c}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. With these, the values of a^{λ}, $R_{\lambda\kappa\mu\nu}$, $\frac{\partial F_{\mu\nu}}{\partial t}$, $\frac{\partial u^{\lambda}}{\partial t}$, $\frac{\partial \rho_m}{\partial t}$, $\frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ are all known at t_0 and can be used to propagate the initial conditions $g_{\mu\nu}$, $F_{\mu\nu}$, u^{λ}, ρ_c, ρ_m and $\frac{\partial g_{\mu\nu}}{\partial t}$ at t_0 to time t_0+dt. Iterating the process, the values of the fundamental fields can be determined at all times.

9. REFERENCES AND END NOTES

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[3] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, NY 1972. The definition of the Riemann-Christoffel curvature tensor is $R^{\lambda}{}_{\mu\nu\kappa} \equiv \partial_{\kappa}\Gamma^{\lambda}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\nu\eta}$, and the definition of the Ricci tensor is $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$.

[4] See for example: D. E. Soper, *Classical Field Theory*, Ch 8 The Electromagnetic Field, Dover Edition, 2008, and S. Weinberg, *Gravitation and Cosmology*, Ch 12 The Action Principle, John Wiley & Sons, New York, NY 1972.

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[6] L.P. Eisenhart, *An Introduction to Differential Geometry*, Chapter 23, "Systems of Partial Differential Equations of the First Order, Mixed Systems," Princeton University Press 1947.

[7] The interpretation of the solution here is different than in the derivation in reference [2]. Specifically, in reference [2] the charge density was restricted to be positive, a restriction that is lifted here.

[8] C. Misner, K. Thorne, and J. Wheeler, *Gravitation*, p. 840, W.H. Freeman and Company, San Francisco, CA 1970

[9] Calculations within were performed in Mathematica: Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).

[10] This calculation was presented in reference [2] but contained an error that is corrected here. In [2], the electric and magnetic fields where not restricted to the same weak field approximation as the *h's*.

[11] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, NY 1972, section 10.2.

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