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Four Particular Cases of the Fourier Transform

Jens V. Fischer

German Aerospace Center (DLR), Microwaves and Radar Institute, 82234 Wessling, Germany;
jens.fischer@dlr.de, Tel.: +49-8153-28-3057

Abstract: In previous studies we used Laurent Schwartz' theory of distributions to rigorously introduce discretizations and periodizations on tempered distributions. These results are now used in this study to derive a validity statement for four interlinking formulas. They are variants of Poisson's Summation Formula and connect four commonly defined Fourier transforms to one another, the integral Fourier transform, the Discrete-Time Fourier Transform (DTFT), the Discrete Fourier Transform (DFT) and the Integral Fourier transform for periodic functions – used to analyze Fourier series. We prove that under certain conditions, these four Fourier transforms become particular cases of the Fourier transform in the tempered distributions sense. We first derive four interlinking formulas from four definitions of the Fourier transform pure symbolically. Then, using our previous results, we specify three conditions for the validity of these formulas in the tempered distributions sense.

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1. Introduction

Poisson's Summation Formula plays a very central role in mathematics. Its generalization tells us that discretizations and periodizations are dual operations and due to their reciprocity one of the two sums converges faster than the other. It can e.g. be used to speed up summations [1]. In this study, we will see that there are basically two variants of these formulas. We refer to the ones given in Gasquet [2], equations (37.1) and (37.2), as

$$\sum_{k=-\infty}^{+\infty} f(t-kT) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} \hat{f}(\frac{k}{T}) \, e^{2\pi t \frac{k}{T}} \tag{1}$$

and

$$\sum_{k=-\infty}^{+\infty} \hat{g}(\sigma - \frac{k}{T}) = T \sum_{k=-\infty}^{+\infty} g(kT) \, e^{-2\pi \sigma kT} \tag{2}$$

where $T > 0$ is real and $\hat{f}$ and $\hat{g}$ denote the Fourier transforms of f and g, respectively. As can easily be seen, these two reduce to special variants of the Poisson Summation Formula if $t = 0$ or $\sigma = 0$ or if $T = 1$. Choosing $t = 0$ and $T = 2\pi$ in (1), yields the formula originally found by Poisson [1,3–6]. Another variant arises if f is the Dirac impulse δ, i.e., a tempered distribution [5], or if g is the function that is constantly 1, i.e., a non-integrable function. Choosing, vice versa, $g = \delta$ or $f = 1$, both equations fail. Equivalently, if one chooses $g = sinc$ and $f = rect$, let $sinc(t) := sin(\pi t)/(\pi t)$ and $rect$ be its Fourier transform, they hold. But choosing $f = sinc$ or $g = rect$, they fail. In this paper, we use two

simple criterions found in a previous study [6], which are dual to one another, for deciding whether a Poisson Summation Formula of type (1) or (2) will hold or fail in the tempered distributions sense. Our results will then be used in order to decide under what conditions four usually defined Fourier transforms (Section 7.1) will hold in the tempered distributions sense and reduce to only one Fourier transform, the Fourier transform on tempered distributions (Section 7.2).

Section 2 prepares the reader to the particularities of the space of tempered distributions. Readers who are familiar with Laurent Schwartz' theory of distributions may skip this section. Section 3 introduces to the notations and the terminology used and Section 4 prepares the theorem given in Section 5. In Section 6, we discuss these findings and conclude this study.

2. Preliminaries

2.1. The Fourier Transform and the Theory of Infinitely Differentiable Functions

Whenever we use the Fourier transform, we actually develop a function f, no matter whether it is differentiable or not, into a superposition (integral)

$$f(t) \ = \ \int_{-\infty}^{\infty} c(\sigma) \, e^{2\pi i t\sigma} d\sigma$$

of infinitely differentiable functions $e^{2\pi i t\sigma}$ where $c(\sigma)$ are the coefficients. Hence, functions that can be Fourier transformed must unconditionally be infinitely differentiable in some sense. The theory that made this notion rigorous is the theory of distributions (generalized functions theory) summarized in Laurent Schwartz' encompassing two-volume work [7,8]. It has already become a standard setting in Fourier analysis [2,5,9–14], wavelet theory [15] and beside mathematics [16–29], also in quantum physics [30–33] where its origin [34], the Dirac delta [35] can be found, and in electrical engineering [36,37] where the Dirac delta is used to formally describe sampling [38]. Laurent Schwartz' theory of distributions is also part of wider theories, such as those on pseudo-differential operators [39–41] or modulation spaces [42–44] including Feichtinger's algebra [45]. Compared to these wider theories, the approach in this study requires less precautions. We only rely on three subspaces of the space of tempered distributions in Laurent Schwartz' standard distribution theory [8,20–24]. All functions and generalized functions will be kept infinitely differentiable in this way.

The circumstance that convolutions and, correspondingly, multiplications among distributions cannot arbitrarily be applied, is sometimes considered a major disadvantage of Laurent Schwartz' distribution theory [46,47]. It is, however, not a disadvantage of the theory – it is rather owned to Heisenberg's uncertainty principle [29]. Intuitively it is clear that convolutions fail if they are not summable. Equivalently, the corresponding multiplications must fail because they are coupled via the Fourier transform. Any other outcome would violate the fact that multiplications and convolutions correspond to one another in dual domains. So, when convolutions fail, their corresponding multiplications fail and, vice versa, when multiplications fail, their corresponding convolutions fail. We therefore stay, for convenience only, within the space of tempered distributions $\mathcal{S}'$ where we have a clear understanding of the Fourier transform as an automorphism on tempered distributions, where we will be able to infinitely derive all functions and where we have a clear understanding of allowed multiplications and convolutions among distributions. They are regulated in Laurent Schwartz' theorem on the multiplication-convolution equivalence in $\mathcal{S}'$, presented as Lemma 1 in the next section – for the readers convenience. Another consequence of this central law is the circumstance that discretizations and periodizations on tempered distributions cannot arbitrarily be allowed. One may recall that discretization, in other words, "sampling" of a generalized function can be defined by *multiplying* it with a Dirac comb [1,9,48]

$$\text{III}_T(t) := \sum_{k=-\infty}^{+\infty} \delta_{kT},$$

where $T > 0$ is real-valued and $\delta_{kT} := \tau_{kT}\delta$ extends to $\tau_{kT}f := f(t - kT)$ for ordinary functions f. Let us write III instead of III_T if $T = 1$. Equivalently, the periodization of a generalized function can be defined by *convolving* it with a suitable Dirac comb [1,9,48]. Therefore, in a most rigorous approach, these operations must obey Laurent Schwartz' equivalence of allowed multiplications and convolutions within the space of tempered distributions.

2.2. Convolution vs. Multiplication

Because of the fact that both, convolution and multiplication, may fail among arbitrary tempered distributions, we need to consider three important subspaces of the space $\mathcal{S}'$ of tempered distributions. For a deeper understanding one may refer to [6,18–24,28,29,49–54]. We require the subspace $\mathcal{O}_C'$ of convolution operators in $\mathcal{S}'$, the subspace $\mathcal{O}_M$ of multiplication operators in $\mathcal{S}'$ and the Schwartz space $\mathcal{S}$ which consists of (ordinary) functions which are both, convolution and multiplication operators. All three were introduced by Larent Schwartz [8]. We build on the following theorem which can also be found in Trèves [19], Horváth [20] and Barros-Neto [23] for the reader's convenience. It plays a very central role in Laurent Schwartz' theory of distributions because it extends the commonly known duality between multiplication and convolution to the space of tempered distributions.

Lemma 1 (Convolution vs. Multiplication). *Let $g \in \mathcal{S}'$, $f \in \mathcal{O}_C'$ and $\alpha \in \mathcal{O}_M$, then*

$$\mathcal{F}(g * f) = \mathcal{F}(g) \cdot \mathcal{F}(f) \qquad \text{and} \tag{3}$$
$$\mathcal{F}(g \cdot \alpha) = \mathcal{F}(g) * \mathcal{F}(\alpha) \tag{4}$$

hold in the tempered distributions sense.

2.3. Discretization vs. Periodization

Using Lemma 1 and the fact that $\text{III} \in \mathcal{S}'$ and $\mathcal{F}(\text{III}) = \text{III}$, see e.g. [5], one easily verifies the following lemma where we may think of "$\text{III} * f$" as the periodization of f with increments of $T = 1$ and of "$\text{III} \cdot \alpha$" as the discretization of α at integers.

Lemma 2 (Discretization vs. Periodization). *Let $f \in \mathcal{O}_C'$ and $\alpha \in \mathcal{O}_M$, then*

$$\mathcal{F}(\text{III} * f) = \mathcal{F}(\text{III}) \cdot \mathcal{F}(f) \qquad \text{and} \tag{5}$$
$$\mathcal{F}(\text{III} \cdot \alpha) = \mathcal{F}(\text{III}) * \mathcal{F}(\alpha) \tag{6}$$

hold in the tempered distributions sense.

The role of III as an ideal sampling and periodization operator can be studied in Kammler [9] or Bracewell [48], for example. Hence, by using $\triangle\!\triangle\!\triangle f := \text{III} * f$ and $\sqcup\!\sqcup\!\sqcup \alpha := \text{III} \cdot \alpha$, where $\triangle\!\triangle\!\triangle$ is periodization and $\sqcup\!\sqcup\!\sqcup$ is discretization (sampling), the latter lemma becomes

$$\mathcal{F}(\triangle\!\triangle\!\triangle f) = \sqcup\!\sqcup\!\sqcup(\mathcal{F}f) \qquad \text{and} \tag{7}$$
$$\mathcal{F}(\sqcup\!\sqcup\!\sqcup \alpha) = \triangle\!\triangle\!\triangle(\mathcal{F}\alpha) \tag{8}$$

in the tempered distributions sense. It tells us that "periodizing f discretizes its Fourier transform" and, vice versa, "discretizing α periodizes its Fourier transform". Without loss of generality, one may think of α as the Fourier transform $\mathcal{F}f = \alpha$ of f. Allowing moreover arbitrary increments $T > 0$, i.e., using the definition $\triangle\!\triangle\!\triangle_T f := \text{III}_T * f$ and $\sqcup\!\sqcup\!\sqcup_T \alpha := \text{III}_T \cdot \alpha$ we obtain

$$\mathcal{F}(\triangle\!\triangle\!\triangle_T f) = \frac{1}{T}\,\sqcup\!\sqcup\!\sqcup_{\frac{1}{T}}(\mathcal{F}f) \qquad \text{and} \tag{9}$$
$$\mathcal{F}(\sqcup\!\sqcup\!\sqcup_{\frac{1}{T}} \alpha) = T\triangle\!\triangle\!\triangle_T(\mathcal{F}\alpha) \tag{10}$$

in the tempered distributions sense [28], due to the reciprocity between time and frequency domain. However, it is worth noting that the opposite, i.e., "discretizing f" and "periodizing α" is not allowed most arbitrarily. It requires that $\alpha, f \in \mathcal{S}$ because $\mathcal{S} = \mathcal{O}_M \cap \mathcal{O}_C{}'$ as shown in Lemma 3 below.

2.4. Poisson's Summation Formula

One may think of the formulas (5), (6) and (7), (8) as well as (9), (10) as generalized versions of the Poisson Summation Formula and its dual [2,4]. This can be seen by inserting the definition of $\triangle\!\triangle\!\triangle_T$ and applying the Fourier transform rule $\mathcal{F}^{-1}\{\delta(\sigma - \frac{k}{T})\} = e^{2\pi t \frac{k}{T}}$ in (1). It becomes

$$(\triangle\!\triangle\!\triangle_T f)(t) \;=\; \frac{1}{T}\, \mathcal{F}^{-1}\Big\{ \sum_{k=-\infty}^{+\infty} \hat{f}(\frac{k}{T})\, \delta(\sigma - \frac{k}{T}) \Big\} \tag{11}$$

and inserting the definition of $\underset{\frac{1}{T}}{\text{⊔⊔}}$ on the right and applying $\mathcal{F}$ on both sides, it yields (9).

Vice versa, inserting the definition of $\triangle\!\triangle\!\triangle_{\frac{1}{T}}$ and applying the Fourier transform rule $\mathcal{F}\{\delta(t - kT)\} = e^{-2\pi \sigma kT}$ in (2) yields

$$(\triangle\!\triangle\!\triangle_{\frac{1}{T}}\hat{g})(t) \;=\; T\,\mathcal{F}\Big\{ \sum_{k=-\infty}^{+\infty} g(kT)\, \delta(t - kT) \Big\} \tag{12}$$

and inserting the definition of ⊔⊔⊔_T on the right and dividing both sides by T, it yields (10).

Let us recall now that (3), (5), (7), (9), (11) and (1) are true if $f \in \mathcal{O}_C{}'$ and (4), (6), (8), (10), (12) and (2) are true if $\alpha \in \mathcal{O}_M$ according to Lemma 1 and Lemma 2. Because $\mathcal{F}f = \alpha$, it is moreover sufficient to either fulfill $f \in \mathcal{O}_C{}'$ or $\alpha \in \mathcal{O}_M$ as $\mathcal{F}(\mathcal{O}_C{}') = \mathcal{O}_M$ and $\mathcal{F}(\mathcal{O}_M) = \mathcal{O}_C{}'$, see e.g. [8,19,20]. These two conditions form, accordingly, an outer framework for validity statements on variants of the Poisson Summation Formula in the tempered distributions sense. They in particular apply to (1) and (2) and therewith to many variants of the Poisson Summation Formula found in the literature.

2.5. Validity Statement

The validity statement $f \in \mathcal{O}_C{}'$ and, equivalently, $\alpha \in \mathcal{O}_M$ is moreover most general in $\mathcal{S}'$ because it cannot be widened up in any way. The space $\mathcal{O}_C{}'$ already includes all convolution operators and $\mathcal{O}_M$ already includes all multiplication operators and beyond these two spaces no other convolution or multiplication will be possible in $\mathcal{S}'$. It is clear that otherwise the corresponding generalized function already belongs to one of these two spaces. This validity statement is furthermore obtained in good agreement with many other publications devoted to this topic, including [4,55–59], except for the fact that, in this as well as in our previous studies, we generally do not move away from the overall principle that all functions must be infinitely differentiable. It is the default, tacit assumption in distribution theory.

One now easily verifies that in order to sample a tempered distribution f, it must be smooth in the ordinary functions sense and should not grow faster than any polynomial. If one of these two conditions is not fulfilled, then f is either no tempered distribution or f is not band-limited. If f is not band-limited (in the sense that $\mathcal{F}f \in \mathcal{O}_C{}'$) then its Fourier transform $\mathcal{F}f$ cannot be periodized, i.e., the periodization of $\mathcal{F}f$ fails to converge if f is not smooth in the ordinary functions sense. One may say, "differentiability in one domain implies summability in the other domain and, vice versa, summability in one domain implies differentiability the other domain". In Table 1 we summarize – for the readers convenience – symbolic calculation rules including their validity scope. They are derived in [6,28,29]. The validities hold on tempered distributions and in particular on Lebesgue square-integrable functions under the same conditions.

Table 1. Discrete Functions vs. Periodic Functions

No	Rule	Remark	Requirement
1	$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_T f) = \frac{1}{T}\,\mathrm{III}_{\frac{1}{T}}(\mathcal{F}f)$	Poisson Summation Formula (generalized)	$f \in \mathcal{O}_C{}',\ T > 0$
2	$\mathcal{F}(\mathrm{III}_{\frac{1}{T}} f) = T\,\triangle\!\!\!\triangle\!\!\!\triangle_T(\mathcal{F}f)$	Poisson Summation Formula (its dual)	$f \in \mathcal{O}_M,\ T > 0$
3	$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle f) = \mathrm{III}(\mathcal{F}f)$	Rule 1, special case $T = 1$	$f \in \mathcal{O}_C{}'$
4	$\mathcal{F}(\mathrm{III} f) = \triangle\!\!\!\triangle\!\!\!\triangle(\mathcal{F}f)$	Rule 2, special case $T = 1$	$f \in \mathcal{O}_M$
5	$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle \delta) = \mathrm{III}(\mathcal{F}\delta)$	Rule 3, special case $f \equiv \delta$ and $\mathcal{F}\delta = 1$	–
6	$\mathcal{F}(\mathrm{III} 1) = \triangle\!\!\!\triangle\!\!\!\triangle(\mathcal{F}1)$	Rule 4, special case $f \equiv 1$ and $\mathcal{F}1 = \delta$	–
7	$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle \delta) = \triangle\!\!\!\triangle\!\!\!\triangle \delta$	Rule 5 + 9, Dirac Comb Invariance	–
8	$\mathcal{F}(\mathrm{III} 1) = \mathrm{III} 1$	Rule 6 + 9, Dirac Comb Invariance	–
9	$\triangle\!\!\!\triangle\!\!\!\triangle \delta = \mathrm{III} 1$	Dirac Comb Identity (by definition)	–
10	$\mathrm{III}_T 1 = \triangle\!\!\!\triangle\!\!\!\triangle_T \delta$	Dirac Comb Identity (by definition)	$T > 0$
11	$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_T \delta) = \frac{1}{T}\,\mathrm{III}_{\frac{1}{T}} 1$	Rule 1, $f \equiv \delta$, Dirac comb reciprocity	$T > 0$
12	$\mathcal{F}(\mathrm{III}_{\frac{1}{T}} 1) = T\,\triangle\!\!\!\triangle\!\!\!\triangle_T \delta$	Rule 2, $f \equiv 1$, Dirac comb reciprocity	$T > 0$
13	$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g$	Convolution vs. Multiplication	$f \in \mathcal{O}_C{}'$
14	$\mathcal{F}(f \cdot g) = \mathcal{F}f * \mathcal{F}g$	Multiplication vs. Convolution	$f \in \mathcal{O}_M$
15	$\triangle\!\!\!\triangle\!\!\!\triangle g * f = \triangle\!\!\!\triangle\!\!\!\triangle(g * f) = g * \triangle\!\!\!\triangle\!\!\!\triangle f$	Periodization Rule	$f, g \in \mathcal{O}_C{}'$
16	$\mathrm{III} g \cdot f = \mathrm{III}(g \cdot f) = g \cdot \mathrm{III} f$	Discretization Rule	$f, g \in \mathcal{O}_M$
17	$\triangle\!\!\!\triangle\!\!\!\triangle \delta * f = \triangle\!\!\!\triangle\!\!\!\triangle(\delta * f) = \triangle\!\!\!\triangle\!\!\!\triangle f$	Rule 15, $g \equiv \delta$, Periodization of f	$f \in \mathcal{O}_C{}'$
18	$\mathrm{III} 1 \cdot f = \mathrm{III}(1 \cdot f) = \mathrm{III} f$	Rule 16, $g \equiv 1$, Discretization of f	$f \in \mathcal{O}_M$
19	$\triangle\!\!\!\triangle\!\!\!\triangle(rect) = 1$	Rule 17, $f \equiv rect$, Periodization of $rect$	–
20	$\mathrm{III}(sinc) = \delta$	Rule 18, $f \equiv sinc$, Discretization of $sinc$	–

3. Notation

3.1. Generalized Functions

Although generalized functions cannot be treated pointwise and must always be applied as a whole, we nevertheless denote them as $f(t)$ instead of f to indicate that they would depend on $t \in \mathbb{R}^n$ or $\sigma \in \mathbb{R}^n$ or $k \in \mathbb{Z}^n$ or $m \in \mathbb{Z}^n$ if they were ordinary functions. We denote Fourier transformed functions as $(\mathcal{F}f)(\sigma) = \hat{f}(\sigma)$ or $\mathcal{F}(f) = \hat{f}$ or just $\mathcal{F}f$. For simplicity, we do not assume vector-valued functions in this study, i.e., we let $n = 1$ in $t, \sigma \in \mathbb{R}^n$ or $k, m \in \mathbb{Z}^n$ although the n-dimensional case looks very similar. For cases of $n > 1$, one may refer to our previous studies [6,28,29].

3.2. Definitions

As explained above, discretizations and periodizations, may fail on tempered distributions. In order to avoid this and to make our definitions rigorous, we require according to Lemma 2, that only $f \in \mathcal{O}_M$ can be discretized and only $f \in \mathcal{O}_C{}'$ can be periodized. For any $f \in \mathcal{O}_M$, any real-valued $T > 0$, $k \in \mathbb{Z}$ and the Dirac comb $\mathrm{III} \in \mathcal{S}'$, one may therefore define

$$(\mathrm{III}_T f)(t) := (\mathrm{III}_T \cdot f)(t) = \sum_{k=-\infty}^{\infty} f(kT)\,\delta(t - kT) \tag{13}$$

and for any $f \in \mathcal{O}_C{}'$ one may define

$$(\triangle\!\!\!\triangle\!\!\!\triangle_T f)(t) := (\mathrm{III}_T * f)(t) = \sum_{k=-\infty}^{\infty} f(t - kT) \tag{14}$$

where (13) is called the "discretization of f" and (14) is called the "periodization of f" with increments T. The result $\perp\!\!\!\perp_T f$ of discretization is again a tempered distribution. Equivalently, the result $\triangle\!\!\triangle\!\!\triangle_T f$ of periodization is again a tempered distribution. We call $\perp\!\!\!\perp_T f$ "discrete function" and $\triangle\!\!\triangle\!\!\triangle_T f$ "periodic function". It is either a periodic function in the usual sense or a generalized periodic function, such as the Dirac comb III_T which is T-periodic. Vice versa, every ordinary or generalized T-periodic function g can be written as $g = \triangle\!\!\triangle\!\!\triangle_T f$ for some $f \in \mathcal{O}_C{}'$. Equivalently, for every discrete function $g = \perp\!\!\!\perp \alpha$ there is some $\alpha \in \mathcal{O}_M$. It is interesting to note in this context that both these operations have already been defined in 1953 in Woodward [1] and later in Brandwood [13] with symbols *comb* and *rep* instead of $\perp\!\!\!\perp$ and $\triangle\!\!\triangle\!\!\triangle$, but pure symbolically, i.e., without statement on the actually permitted domain and the resulting image of these operations in the tempered distributions sense.

4. Preparations

4.1. Schwartz Functions

The following lemma may appear obvious but cannot be found in the literature to the best of this author's knowledge. It explains the special role of Schwartz functions.

Lemma 3 (Schwartzfunctions). *Schwartz functions belong to $\mathcal{O}_M$ and $\mathcal{O}_C{}'$ and, vice versa, any distribution in $\mathcal{O}_M \cap \mathcal{O}_C{}'$ is a Schwartz function. Briefly,*

$$\mathcal{S} = \mathcal{O}_M \cap \mathcal{O}_C{}'.$$

Proof. Let $\varphi \in \mathcal{S}$. Then, because $\mathcal{S} \subset \mathcal{O}_M$ and $\mathcal{S} \subset \mathcal{O}_C{}'$ (see e.g. [8,28]), it follows that $\varphi \in \mathcal{O}_M \cap \mathcal{O}_C{}'$. Let now, vice versa, $\varphi \in \mathcal{O}_M \cap \mathcal{O}_C{}'$. Then $\varphi \in \mathcal{O}_M$ means it is a regular function (infinitely differentiable in the ordinary functions sense) in $\mathcal{S}'$ and $\varphi \in \mathcal{O}_C{}'$ means it is local (decays faster to zero than any polynomial). The latter is a common property of elements in $\mathcal{S}$ and in $\mathcal{O}_C{}'$ but elements in $\mathcal{O}_C{}'$ are not necessarily regular (infinitely differentiable in the ordinary functions sense). It remains to show that there is no space X of regular functions in $\mathcal{S}'$ satisfying $\mathcal{S} \subset X \subset \mathcal{O}_C{}'$ except for $\mathcal{S} \equiv X$. This, however, is clear because $\mathcal{S}$ is dense in $\mathcal{O}_C{}'$. Hence, $\varphi \in \mathcal{S}$. $\square$

4.2. Discrete Periodic Functions

In contrast to Schwartz functions $\varphi \in \mathcal{S}$ which are fully smooth (i.e., f and $\mathcal{F}f$ are both ordinarily smooth), let us now denote

$$G := \complement\,\mathcal{O}_M \cap \complement\,\mathcal{O}_C{}'$$

the subspace of fully generalized functions (i.e., f and $\mathcal{F}f$ are both *not* ordinarily smooth) in $\mathcal{S}'$. Then, the following lemma holds in $\mathcal{S}'$.

Lemma 4 (Discrete Periodic Functions). *Let $\varphi \in \mathcal{S}$ and $T, B > 0$ be real-valued, $N = TB$ an integer, then*

$$\triangle\!\!\triangle\!\!\triangle_T \perp\!\!\!\perp_{\frac{1}{B}} \varphi = \perp\!\!\!\perp_{\frac{1}{B}} \triangle\!\!\triangle\!\!\triangle_T \varphi$$

holds in the tempered distributions sense. It is a fully generalized function, i.e., it belongs to G.

Proof. Because $\varphi \in \mathcal{S} = \mathcal{O}_M \cap \mathcal{O}_C{}'$ we are allowed to discretize and periodize φ. Discretization $\perp\!\!\!\perp$ is an injection from $\mathcal{O}_M$ into $\complement\,\mathcal{O}_M$ and periodization $\triangle\!\!\triangle\!\!\triangle$ is an injection from $\mathcal{O}_C{}'$ into $\complement\,\mathcal{O}_C{}'$ such that the result is in G. The fact that both operations commute on Schwartz functions is a result of $T = N/B$ which implies that first discretizing one period of length T and then periodizing this function with increments of T is the same as first periodizing the function with increments of T and then discretizing all periods at once. $\square$

4.3. Finite Summations

Links between Poisson's summation formula and the Discrete Fourier Transform (DFT) have already occasionally been investigated, for example in [60,61]. The following lemma helps to further understand this connection.

Lemma 5 (Discrete Fourier Transform). *Nesting two dual infinite-sum PSF variants*

$$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_T f) = \tfrac{1}{T}\, Ш_{\frac{1}{T}}\, \mathcal{F}(f) \tag{15}$$

$$\mathcal{F}(Ш_{\frac{1}{B}}\alpha) = B\,\triangle\!\!\!\triangle\!\!\!\triangle_B\, \mathcal{F}(\alpha) \tag{16}$$

into one another, in two different ways, yields

$$\mathcal{F}(Ш\,\triangle\!\!\!\triangle\!\!\!\triangle_N\varphi) = \tfrac{1}{N}\,\triangle\!\!\!\triangle\!\!\!\triangle\,Ш_{\frac{1}{N}}\, \mathcal{F}(\varphi) \tag{17}$$

$$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle\,Ш_{\frac{1}{N}}\varphi) = N\,Ш\,\triangle\!\!\!\triangle\!\!\!\triangle_N\, \mathcal{F}(\varphi), \tag{18}$$

i.e., two dual finite-sum PSF variants. The equations hold for integer-valued time-bandwidth products $N = TB$, where $T > 0$ and $B > 0$ are real-valued and $f \in \mathcal{O}_C{}'$, $\alpha \in \mathcal{O}_M$, $\varphi \in \mathcal{S}$ in the tempered distributions sense. The equations (17), (18) represent the Discrete Fourier Transform (DFT) and its inverse (37), (38).

Proof. Let $\varphi \in \mathcal{S}$. Then, since $\mathcal{S} = \mathcal{O}_C{}' \cap \mathcal{O}_M$, φ can be discretized in $\mathcal{S}'$. Inserting $f = Ш_{\frac{1}{B}}\varphi \in \mathcal{O}_C{}' \cap \complement\,\mathcal{O}_M$ in (15) and applying Rule 1 and Rule 2 (Table 1), we obtain

$$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_T (Ш_{\frac{1}{B}}\varphi)) = \frac{1}{T}\,Ш_{\frac{1}{T}}(\mathcal{F}(Ш_{\frac{1}{B}}\varphi)) = \frac{B}{T}\,Ш_{\frac{1}{T}}(\triangle\!\!\!\triangle\!\!\!\triangle_B\,\mathcal{F}(\varphi)) \tag{19}$$

Accordingly, since $\mathcal{S} = \mathcal{O}_C{}' \cap \mathcal{O}_M$, φ can be periodized in $\mathcal{S}'$. Inserting $\alpha = \triangle\!\!\!\triangle\!\!\!\triangle_T\varphi \in \complement\,\mathcal{O}_C{}' \cap \mathcal{O}_M$ in (16) and applying Rule 2 and Rule 1 (Table 1), we obtain

$$\mathcal{F}(Ш_{\frac{1}{B}}(\triangle\!\!\!\triangle\!\!\!\triangle_T\varphi)) = B\,\triangle\!\!\!\triangle\!\!\!\triangle_B(\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_T\varphi)) = \frac{B}{T}\,\triangle\!\!\!\triangle\!\!\!\triangle_B(Ш_{\frac{1}{T}}(\mathcal{F}\varphi)). \tag{20}$$

Now, an integer $N = TB$ arises because (19) satisfies $B = N(1/T)$ where $(1/T)$ is the discretization step. Equivalently, N arises in (20) as $T = N(1/B)$ where $(1/B)$ is the discretization step. The assertion now follows because, without loss of generality, one either has $B = 1$ in (19), (20) such that

$$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_N\,Ш\,\varphi) = \frac{1}{N}\,Ш_{\frac{1}{N}}(\mathcal{F}(Ш\varphi)) = \frac{1}{N}\,Ш_{\frac{1}{N}}\,\triangle\!\!\!\triangle\!\!\!\triangle\,\mathcal{F}(\varphi) \tag{21}$$

$$\mathcal{F}(Ш\,\triangle\!\!\!\triangle\!\!\!\triangle_N\varphi) = \triangle\!\!\!\triangle\!\!\!\triangle(\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle_N\varphi)) = \frac{1}{N}\,\triangle\!\!\!\triangle\!\!\!\triangle\,Ш_{\frac{1}{N}}\,\mathcal{F}(\varphi) \tag{22}$$

or $T = 1$ in (19), (20) yielding

$$\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle\,Ш_{\frac{1}{N}}\varphi) = N\,Ш(\mathcal{F}(Ш_{\frac{1}{N}}\varphi)) = N\,Ш\,\triangle\!\!\!\triangle\!\!\!\triangle_N\,\mathcal{F}(\varphi) \tag{23}$$

$$\mathcal{F}(Ш_{\frac{1}{N}}\,\triangle\!\!\!\triangle\!\!\!\triangle\,\varphi) = N\,\triangle\!\!\!\triangle\!\!\!\triangle_N(\mathcal{F}(\triangle\!\!\!\triangle\!\!\!\triangle\varphi)) = N\,\triangle\!\!\!\triangle\!\!\!\triangle_N\,Ш(\mathcal{F}\varphi). \tag{24}$$

where (21) and (22) as well as (23) and (24) represent the same formula, respectively, according to Lemma 5. Hence, (22) and (23) remain, as claimed. The fact that (17) and (18) are the Discrete Fourier Transform and its inverse is shown in Sections 8.3 and 8.4. □

Another explanation for seeing that both, "time domain" and "frequency domain" of a function, are coupled via the same $N = TB$ can be found for example in [61,62]. The fact that $Ш_T$ and $\triangle\!\!\!\triangle\!\!\!\triangle_B$ "commute if the sampling distance (sampling grid) is a refinement of the periodization net" is for example described in [62] for Feichtinger's algebra S_0 [45] which includes the Schwartz space $\mathcal{S}$. Using

the commutator $[A, B] := AB - BA$ notation which is customary practice in quantum mechanics [32] one may write

$$[\triangle\!\triangle\!\triangle, \text{Ш}_{\frac{1}{N}}] \;=\; 0 \;=\; [\triangle\!\triangle\!\triangle_N, \text{Ш}\text{Ш}] \tag{25}$$

with respect to Schwartz functions. We actually have two, dual equations in (25) and they hold because every integer N is divisible by 1 (left-hand side) and N (right-hand side). Schwartz functions play an important role in quantum mechanics [63]. They are simultaneously used as test functions in both, distribution theory and quantum physics. In distribution theory, they test tempered distributions [7,8] and in quantum physics they test Hamiltonians and other relevant operators [33]. This double meaning is well demonstrated in (25) where both these meanings coincide. Let us now summarize these rules in Table 2.

Table 2. Discrete Periodic Functions

No	Rule	Remark	Requirement
i	$\mathcal{F}\big(\text{Ш}_{\frac{1}{T}}\triangle\!\triangle\!\triangle_B f\big) \;=\; \frac{T}{B}\,\triangle\!\triangle\!\triangle_T\text{Ш}_{\frac{1}{B}}(\mathcal{F}f)$	Rule 1 + 2	$f \in \mathcal{S}$, real $T > 0$, real $B > 0$
ii	$\mathcal{F}\big(\triangle\!\triangle\!\triangle_B\,\text{Ш}_{\frac{1}{T}}f\big) \;=\; \frac{T}{B}\,\text{Ш}_{\frac{1}{B}}\triangle\!\triangle\!\triangle_T(\mathcal{F}f)$	Rule 2 + 1	$f \in \mathcal{S}$, real $T > 0$, real $B > 0$
iii	$\triangle\!\triangle\!\triangle\,\text{Ш}_{\frac{1}{N}}f \;=\; \text{Ш}_{\frac{1}{N}}\triangle\!\triangle\!\triangle f$	identity	$f \in \mathcal{S}$, integer $N > 0$
iv	$\text{Ш}\text{Ш}\,\triangle\!\triangle\!\triangle_N f \;=\; \triangle\!\triangle\!\triangle_N\,\text{Ш}\text{Ш}f$	identity	$f \in \mathcal{S}$, integer $N > 0$
v	$\mathcal{F}\big(\text{Ш}\text{Ш}\,\triangle\!\triangle\!\triangle_N f\big) \;=\; \frac{1}{N}\,\triangle\!\triangle\!\triangle\,\text{Ш}_{\frac{1}{N}}(\mathcal{F}f)$	Rule 3 + 2	$f \in \mathcal{S}$, integer $N > 0$
vi	$\mathcal{F}\big(\triangle\!\triangle\!\triangle\,\text{Ш}_{\frac{1}{N}}f\big) \;=\; N\,\text{Ш}\text{Ш}\,\triangle\!\triangle\!\triangle_N(\mathcal{F}f)$	Rule 4 + 1	$f \in \mathcal{S}$, integer $N > 0$

It is interesting to observe that two infinite sums nested into one another yield finite sums. Hence, the summation becomes fast. Simplifying nested infinite sums actually falls into a recent research topic [64]. This simplification could be done because Schwartz functions fade to zero for large arguments combined with the (this time positive) effect of "aliasing" [65] that trickily folds the infinite sum. In this way, it allows us to be ready in a finite number of summation steps.

5. Four Fourier Transforms

It is known today that Poisson's Summation Formula is true if $f \in \mathcal{S}$, see e.g. [11]. However, the next theorem extends our understanding of the Poisson Summation Formula to a validity on tempered distributions. Now, even generalized functions $f \in \mathcal{O}_C{}'$ and non-integrable functions $f \in \mathcal{O}_M$ are allowed including the Dirac impulse $\delta \in \mathcal{O}_C{}'$ in (27) and the function that is constantly $1 \in \mathcal{O}_M$ in (28) as their most prominent members.

Theorem 1 (Four Fourier Transforms). *Let four Fourier transforms be defined as (31),(33),(35),(37) and let us denote the integral Fourier transform (31) as*

$$\mathcal{F}(f) = \hat{f}. \tag{26}$$

It then follows that the other three Fourier transform definitions (33),(35),(37) reduce to the known rules

$$\mathcal{F}\big(\triangle\!\triangle\!\triangle_T f\big) \;=\; \frac{1}{T}\,\text{Ш}_{\frac{1}{T}}\mathcal{F}(f) \tag{27}$$

$$\mathcal{F}\big(\text{Ш}_{\frac{1}{T}}f\big) \;=\; T\,\triangle\!\triangle\!\triangle_T\mathcal{F}(f) \tag{28}$$

$$\mathcal{F}\big(\text{Ш}\text{Ш}\,\triangle\!\triangle\!\triangle_N f\big) \;=\; \frac{1}{N}\,\triangle\!\triangle\!\triangle\,\text{Ш}_{\frac{1}{N}}\mathcal{F}(f) \tag{29}$$

$$\mathcal{F}\big(\triangle\!\triangle\!\triangle\,\text{Ш}_{\frac{1}{N}}f\big) \;=\; N\,\text{Ш}\text{Ш}\,\triangle\!\triangle\!\triangle_N\mathcal{F}(f) \tag{30}$$

where $\text{Ш}\text{Ш}$ *is discretization,* $\triangle\!\triangle\!\triangle$ *is periodization, defined as in (13), (14), $T > 0$ is real, $N > 0$ an integer and*

(26) is the Integral Fourier Transform $\mathcal{F}$ (for non-discrete non-periodic functions),

(27) *is the Integral Fourier Transform $\mathcal{F}_{per}$ for (non-discrete) periodic functions,*
(28) *is its inverse, the Discrete-Time Fourier Transform (DTFT) for discrete (non-periodic) functions,*
(29) *is the Discrete Fourier Transform (DFT) for discrete periodic functions,*
(30) *is its inverse, respectively,*

and the equations (26),(27),(28),(29) hold if

$f \in \mathcal{S}' = \mathcal{F}(\mathcal{S}')$ *in (26),*
$f \in \mathcal{O}_C' = \mathcal{F}(\mathcal{O}_M)$ *in (27),*
$f \in \mathcal{O}_M = \mathcal{F}(\mathcal{O}_C')$ *in (28) and*
$f \in \mathcal{S} = \mathcal{O}_C' \cap \mathcal{O}_M = \mathcal{F}(\mathcal{S})$ *in (29) and (30).*

Furthermore,

(27) *and* (28) *are dual generalizations of the Poisson Summation Formula (PSF) and*
(29) *and* (30) *are* (27) *and* (28) *nested into one another in two different ways.*

Finally, for any $f \in \mathcal{S}$, it follows that (26),(27),(28),(29),(30) hold simultaneously in $\mathcal{S}'$.

Proof. The symbolic derivation of (27),(28),(29),(30) from (31),(33),(35),(37) is shown in Appendix 8 and their validities in $\mathcal{S}'$ follow from Lemmas 1-5. The circumstance that (27),(28) describe generalized variants of the Poisson summation formula is shown in Section 2.4 and nesting them into one another yields (29),(30) according to Lemma 5. The last statement, finally, is already known [11]. □

Roughly, the theorem states that "instead of four Fourier transforms there is only one and instead of one Poission Summation Formula there are actually four". Let us furthermore recall that requiring $f \in \mathcal{O}_M$ in (28), which means that it is infinitely differentiable and not growing faster than polynomials, is not too strong for a function to be sampled. In all practical cases, we have that f is band-limited, in the sense that $\mathcal{F}f \in \mathcal{O}_C'$, and this already fulfills the requirement. Vice versa, the requirement $f \in \mathcal{O}_M$ means that only band-limited functions can be sampled.
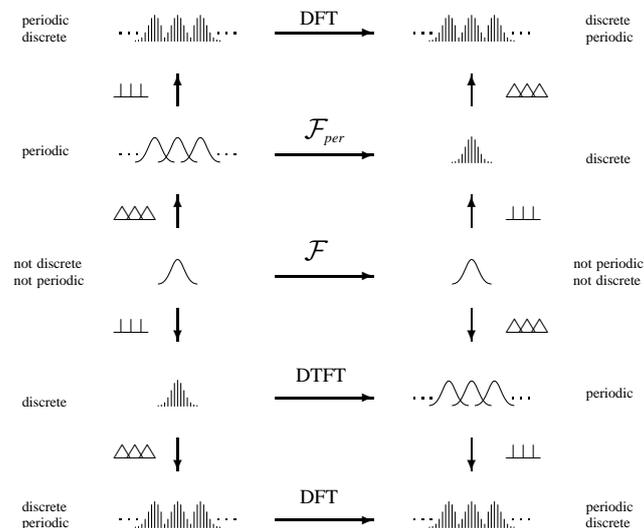
Figure 1. Four Fourier transforms and the links ⊥⊥⊥ and △△△ between them. In the tempered distributions sense, all four transforms $\mathcal{F}, \mathcal{F}_{per}$, DTFT and DFT reduce to only one Fourier transform, the Fourier transform on tempered distributions.

Equations (27), (28), (29), (30) can be visualized in a simple diagram (Figure 1). For simplicity, we set $T = 1$ in all discretizations and periodizations and use a Gaussian, i.e., a so-called self-reciprocal

function [66]. The diagram therefore withholds an existing reciprocity $t^{+1} \mapsto t^{-1}$ or $t^{-1} \mapsto t^{+1}$ between time and frequency (left and right or vice versa). More generally, wide Gaussians would become narrow Gaussians and narrow Gaussians would become wide Gaussians. One may notice, furthermore, that starting from the integral Fourier transform (in the middle) only Schwartz functions can reach the domain of the DFT (above and below). That is what we proved in the theorem. It is also clear that "above" and "below" coincide. Hence, the diagram is double-cyclic. Identifying the upper left side with the lower right side, we may also think of it as a Möbius band. It tells us that there is actually no left and no right-hand side, just opposite sides.

6. Discussion

The Poisson summation formula in its four variants is the actual link between four usually defined Fourier transforms. The naming of these four Fourier transforms $\mathcal{F}$, $\mathcal{F}_{per}$, DTFT and DFT in the literature is often not very appropriate and sometimes confusing. They should be called "Fourier transform", "Fourier transform for periodic functions", "Fourier transform for discrete functions" and "Fourier transform for discrete periodic functions" more appropriately. All four transforms moreover describe special cases of the Fourier transform on tempered distributions; they only differ in the kind of functions they are applied to, i.e., to f, $\triangle\!\!\!\triangle\!\!\!\triangle f$, $\text{Ш} f$ or $\triangle\!\!\!\triangle\!\!\!\triangle\,\text{Ш} f$ and stretched or compressed versions thereof. An introduction of two transforms $\mathcal{F}_{per}$ and DTFT is generally not advisable because they are, apart from having an inverse sign (which indicates that they are inverse to one another), the same transform. Most appropriately, it could be called the "Fourier Series Transform" as it switches between two Fourier series representations, i.e., between discrete functions (Fourier series coefficients) and periodic functions (Fourier series). Its two formulas should be called "Fourier Series Analysis" and "Fourier Series Synthesis" formula, as already done in many textbooks.

We furthermore showed that, in the most rigorous sense, for a function to be sampled, it must be smooth and should not grow faster than polynomials. If one of these two properties is not given, the function cannot be band-limited. Hence, the periodization of its Fourier transform will fail to converge. These two conditions moreover represent a validity statement for the generalization of Poisson's summation formula and its dual on tempered distributions. The Fourier transform pair *rect*, *sinc* mentioned in the introduction is a striking example of what has been shown in this study. The equations (1) and (2) hold if $g = sinc$ and $f = rect$ but they fail if $f = sinc$ and $g = rect$ and the reason is that

$$rect \in \mathcal{O}_C{}' \quad \text{but} \quad rect \notin \mathcal{O}_M \qquad \text{and}$$
$$sinc \in \mathcal{O}_M \quad \text{but} \quad sinc \notin \mathcal{O}_C{}'.$$

which can easily be seen because *rect* is of compact support, in particular it is rapidly decreasing, but it is not smooth in the ordinary functions sense. In contrast, *sinc* is smooth in the ordinary functions sense but slowly decreasing, it goes with $1/t$ towards infinity which is a polynomial decrease. However, since $rect \in \mathcal{O}_C{}'$ it can be periodized and since $sinc \in \mathcal{O}_M$, it can be discretized. A particular case arising from these facts is $\triangle\!\!\!\triangle\!\!\!\triangle(rect) = 1$ and $\text{Ш}(sinc) = \delta$, respectively. Let us add them as Rule 19 and Rule 20 in Table 1. Another consequence of this slow decrease of *sinc* is its failure to serve as universal building block in Shannon's and Kotelnikov's reconstruction formula for arbitrary tempered distributions [2]. A much better alternative are Lighthill's unitary functions [17,67].

7. APPENDIX: Fourier Transforms

There are mainly three ways of how to deal with the factor 2π in Fourier transform definitions. Here, we use the so-called "unitary, ordinary frequency" Fourier transform as it is given for example in [2,9,11]. It is also called the "normalization" of the Fourier transform [11] because it uses 1-periodic exponential functions $e^{2\pi i\sigma}$ rather than 2π-periodic ones $e^{i\sigma}$ and thereby yields the most symmetric

results in Fourier transform pairs, e.g. $\mathcal{F}\delta = 1$ and $\mathcal{F}1 = \delta$. The normalized Fourier transform actually links $\mathbb{Z}$ to $\mathbb{R}$ via 1-periodic functions. As a result of this, "time domain" and "frequency domain" become fully equivalent.

7.1. Conventional Fourier Transforms

Let $f(t)$ be a suitable function such that it can be Fourier transformed and let $\hat{f}(\sigma)$ be its Fourier transform. Then the following four Fourier transform variants are usually defined.

7.1.1. Fourier Transform

The Integral Fourier Transform (for non-discrete non-periodic functions) is defined by

$$\hat{f}(\sigma) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t \sigma} dt \qquad \text{Analysis} \tag{31}$$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\sigma)\, e^{2\pi i \sigma t} d\sigma \qquad \text{Synthesis} \tag{32}$$

for suitable $f(t)$.

7.1.2. Fourier Transform (for periodic functions)

The Integral Fourier Transform for (non-discrete) periodic functions, used for *Fourier Series* analysis, is defined by

$$\hat{f}[m] = \frac{1}{T} \int_{0}^{T} f(t)\, e^{-2\pi i \frac{t}{T} m} dt \qquad \text{Analysis} \tag{33}$$

$$f(t) = \sum_{m=-\infty}^{\infty} \hat{f}[m]\, e^{2\pi i \frac{m}{T} t} \qquad \text{Synthesis} \tag{34}$$

where (34) is the *Fourier Series* of $f(t)$ and (33) determines its coefficients. The coefficients are *discrete*.

7.1.3. Fourier Transform (for discrete functions)

The Fourier Transform for discrete (non-periodic) functions, also called *Discrete-Time Fourier Transform* (DTFT), used for *Fourier Series* synthesis, is defined by

$$\hat{f}(\sigma) = \frac{1}{T} \sum_{k=-\infty}^{\infty} f[k]\, e^{-2\pi i \frac{k}{T} \sigma} \qquad \text{Analysis} \tag{35}$$

$$f[k] = \int_{0}^{T} \hat{f}(\sigma)\, e^{2\pi i \frac{\sigma}{T} k} d\sigma \qquad \text{Synthesis} \tag{36}$$

where (35) is a *Fourier Series*. Hence, it is *periodic* but $f(t)$ itself is *discrete*, its samples are determined by (36).

7.1.4. Fourier Transform (for discrete periodic functions)

The Fourier Transform for discrete periodic functions, called *Discrete Fourier Transform* (DFT) is defined by

$$\hat{f}[m] = \frac{1}{N} \sum_{k=0}^{N-1} f[k]\, e^{-2\pi i \frac{k}{N} m} \qquad \text{Analysis} \tag{37}$$

$$f[k] = \sum_{m=0}^{N-1} \hat{f}[m]\, e^{2\pi i \frac{m}{N} k} \qquad \text{Synthesis} \tag{38}$$

where both, (37) and (38), are (finite-sum) *Fourier Series*. Thus, they are *periodic* and *discrete*, simultaneously.

7.2. Distributional Fourier Transform

Let $\langle f, \varphi \rangle$ be the application of a tempered distribution $f \in \mathcal{S}'$ to some test function $\varphi \in \mathcal{S}$. The Fourier transform of $f \in \mathcal{S}'$ is then defined as

$$\langle \mathcal{F}f, \varphi \rangle := \langle f, \mathcal{F}\varphi \rangle$$

where $\mathcal{F}$ on the right-hand side is the integral Fourier transform given by (31) and its inverse is (32), respectively. One usually uses this definition to test and verify symbolic calculation rules on tempered distributions. Once the rules are established, they can symbolically be used on tempered distributions as if they were ordinary functions. It is known that Fourier transform rules which apply to Lebesgue-square integrable functions do also apply to tempered distributions. Additionally, new rules can be found such as $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$ which are based on rigorous calculations using the above definition. Rules summarized in Table 1 were established in [6]. They do also hold on ordinary square-integrable functions under the same conditions.

Most convenient is the fact that using the Fourier transform in the distributional sense, all functions become infinitely differentiable. In other words, the Fourier transform rule "multiplying a function k times with $2\pi it$ in one domain means to differentiate its Fourier transform k times in the other domain", e.g. in [2,5,9,25,27], can be exploited unrestrictedly. One may recall that $2\pi it$ is the inner derivative of $e^{2\pi it\sigma}$. It means, the Fourier transform rule originates from

$$\frac{d}{d\sigma} e^{2\pi it\sigma} = (2\pi it) \, e^{2\pi it\sigma}$$

because $e^{2\pi it\sigma}$ is infinitely differentiable in the ordinary functions sense. The special role of $e^{2\pi it\sigma}$ arises from the fact that it is a number and a function, simultaneously.

8. APPENDIX: Derivations

In this appendix, we symbolically derive (27),(28),(29),(30) from four definitions of the Fourier transform given in (31),(33),(35),(37). We prove that

- defining the Integral Fourier Transform for periodic functions via (33),(34) leads to (27),
- defining the Discrete-Time Fourier Transform (DTFT) via (35),(36) leads to (28) and
- defining the Discrete Fourier Transform (DFT) via (37),(38) leads to (29) and (30).

Interesting is the fact that there has obviously been no other choice than defining all four Fourier transforms in exactly this way.

8.1. Fourier Transform for periodic functions ($\mathcal{F}_{per}$)

In this subsection we will see that if a *periodic function* is Fourier transformed via the Fourier transform for periodic functions (33) then the result is a *discrete function*. Indeed, inserting some $p(t) = (\triangle\triangle\triangle_T f)(t)$ into (33) yields

$$
\begin{aligned}
\hat{p}(m) &= \frac{1}{T} \int_0^T \triangle\triangle\triangle_T f(t) \, e^{-2\pi i \frac{m}{T} t} dt \\
&= \frac{1}{T} \int_{-\infty}^{\infty} f(t) \, e^{-2\pi i \frac{m}{T} t} dt \\
&= \frac{1}{T} \, (\mathcal{F}f)(\frac{m}{T})
\end{aligned}
$$

where we used the popular *periodization trick* [12] and Fourier transform definition (31). Inserting these coefficients into (34) we obtain

$$
(\triangle\!\triangle\!\triangle_T f)(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} (\mathcal{F}f)(\frac{m}{T})\, e^{2\pi i t\, \frac{m}{T}}
$$

$$
= \frac{1}{T}\, \mathcal{F}^{-1}\left\{ \sum_{m=-\infty}^{\infty} (\mathcal{F}f)(\frac{m}{T})\, \delta(\sigma - \frac{m}{T}) \right\}
$$

$$
= \frac{1}{T}\, \mathcal{F}^{-1}\left\{ (\text{III}_{\frac{1}{T}}(\mathcal{F}f))(\sigma) \right\}
$$

$$
= \frac{1}{T}\, (\mathcal{F}^{-1}\, \text{III}_{\frac{1}{T}}\, \mathcal{F}f)(t)
$$

$$
(\mathcal{F}(\triangle\!\triangle\!\triangle_T f))(\sigma) = \frac{1}{T}\, (\text{III}_{\frac{1}{T}}(\mathcal{F}f))(\sigma).
$$

Therefore, (33) inserted into (34) reduces to

$$
\mathcal{F}(\triangle\!\triangle\!\triangle_T f) = \frac{1}{T}\, \text{III}_{\frac{1}{T}}(\mathcal{F}f)
$$

as a function of $\sigma \in \mathbb{R}$. This is formula (27).

8.2. Fourier Transform for discrete functions (DTFT)

We show that if a *discrete function* is Fourier transformed via the DTFT then the result is a *periodic function*. As (35) is a Fourier series, it is periodic. The ansatz is therefore to let $\hat{d}(\sigma) = (\triangle\!\triangle\!\triangle_T \hat{f})(\sigma) = (\triangle\!\triangle\!\triangle_T(\mathcal{F}f))(\sigma)$ in (36). It yields

$$
d(k) = \int_0^T \triangle\!\triangle\!\triangle_T \hat{f}(\sigma)\, e^{2\pi i \frac{k}{T}\sigma} d\sigma
$$

$$
= \int_{-\infty}^{\infty} \hat{f}(\sigma)\, e^{2\pi i \frac{k}{T}\sigma} d\sigma
$$

$$
= (\mathcal{F}^{-1}(\mathcal{F}f))(\frac{k}{T}) \;=\; f(\frac{k}{T}).
$$

Inserting these coefficients into (35) we obtain

$$
(\triangle\!\triangle\!\triangle_T(\mathcal{F}f))(\sigma) = \frac{1}{T} \sum_{k=-\infty}^{\infty} f(\frac{k}{T})\, e^{-2\pi i \sigma \frac{k}{T}}
$$

$$
= \frac{1}{T}\, \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} f(\frac{k}{T})\, \delta(t - \frac{k}{T}) \right\}
$$

$$
= \frac{1}{T}\, \mathcal{F}\left\{ (\text{III}_{\frac{1}{T}} f)(t) \right\}
$$

$$
= \frac{1}{T}\, (\mathcal{F}(\text{III}_{\frac{1}{T}} f))(\sigma).
$$

Thus, (36) inserted into (35) reduces to

$$
\mathcal{F}(\text{III}_{\frac{1}{T}} f) = T\, \triangle\!\triangle\!\triangle_T(\mathcal{F}f)
$$

as a function of $\sigma \in \mathbb{R}$. This is formula (28).

8.3. Fourier Transform for discrete periodic functions (DFT)

We show that if a *discrete periodic function* is Fourier transformed via the DFT, then the result is again a *discrete periodic function*. First, a simple reasoning tells us that after discretizing a periodic

function $(\triangle\triangle\triangle_T f)(t)$ it can be denoted as $y(t) = (\sqcup\!\sqcup\!\sqcup \triangle\triangle\triangle_N f)(t)$ where N is an integer corresponding to its period T. Inserting now its coefficients $(\triangle\triangle\triangle_N f)(k)$ with $k \in \mathbb{Z}$ into (37) yields

$$
\begin{aligned}
(\mathcal{F}(\triangle\triangle\triangle_N f))(m) &= \frac{1}{N} \sum_{k=0}^{N-1} (\triangle\triangle\triangle_N f)(k)\, e^{-2\pi i \frac{m}{N} k} \\
&= \frac{1}{N} \sum_{k=-\infty}^{\infty} f(k)\, e^{-2\pi i \frac{m}{N} k} \\
&= \frac{1}{N}\, (\mathcal{F}(\sqcup\!\sqcup\!\sqcup f))(\frac{m}{N}) \\
&= \frac{1}{N}\, (\triangle\triangle\triangle(\mathcal{F}f))(\frac{m}{N})
\end{aligned}
$$

using, again, the periodization trick (applied to a discrete function this time), the definition of discretization $\sqcup\!\sqcup\!\sqcup$ and a previous result (Rule 4). Inserting these coefficients into (38) we obtain

$$
\begin{aligned}
(\triangle\triangle\triangle_N f)(k) &= \frac{1}{N} \sum_{m=0}^{N-1} \triangle\triangle\triangle (\mathcal{F}f)(\frac{m}{N})\, e^{2\pi i k \frac{m}{N}} \\
&= \frac{1}{N} \sum_{m=-\infty}^{\infty} (\mathcal{F}f)(\frac{m}{N})\, e^{2\pi i k \frac{m}{N}} \\
&= \frac{1}{N}\, (\mathcal{F}^{-1} \sqcup\!\sqcup\!\sqcup_{\frac{1}{N}} (\mathcal{F}f))(k).
\end{aligned}
$$

again with the periodization trick, discretization $\sqcup\!\sqcup\!\sqcup$ and the definition of $\mathcal{F}^{-1}$ this time. Thus, (37) inserted into (38) reduces to

$$
\begin{aligned}
(\triangle\triangle\triangle_N f)(k) &= \frac{1}{N}\, (\mathcal{F}^{-1} \sqcup\!\sqcup\!\sqcup_{\frac{1}{N}} (\mathcal{F}f))(k), \quad k \in \mathbb{Z} \\
(\sqcup\!\sqcup\!\sqcup \triangle\triangle\triangle_N f)(t) &= \frac{1}{N}\, (\sqcup\!\sqcup\!\sqcup\, \mathcal{F}^{-1} \sqcup\!\sqcup\!\sqcup_{\frac{1}{N}} (\mathcal{F}f))(t), \quad t \in \mathbb{R}
\end{aligned}
$$

and with Rule 4 and Fourier transforming both sides it yields

$$
\mathcal{F}(\sqcup\!\sqcup\!\sqcup \triangle\triangle\triangle_N f) = \frac{1}{N}\, \triangle\triangle\triangle \sqcup\!\sqcup\!\sqcup_{\frac{1}{N}} (\mathcal{F}f)
$$

being a function of $\sigma \in \mathbb{R}$.

8.4. Derivation of the Inverse DFT

As (37) is a Fourier series, it is periodic. Evaluating (38) for $\hat{g}(m) = (\triangle\triangle\triangle_N \hat{f})(m)$ where $\hat{f} = \mathcal{F}f$ yields

$$
\begin{aligned}
g(k) &= \sum_{m=0}^{N-1} (\triangle\triangle\triangle_N (\mathcal{F}f))(m)\, e^{2\pi i \frac{k}{N} m} \\
&= \sum_{k=-\infty}^{\infty} (\mathcal{F}f)(m)\, e^{2\pi i \frac{k}{N} m} \\
&= (\mathcal{F}^{-1} \sqcup\!\sqcup\!\sqcup (\mathcal{F}f))(\frac{k}{N}) \\
&= (\mathcal{F}^{-1} \mathcal{F}(\triangle\triangle\triangle f))(\frac{k}{N}) \\
&= (\triangle\triangle\triangle f)(\frac{k}{N})
\end{aligned}
$$

and inserting this result into (37) we obtain

$$
\begin{aligned}
\left(\triangle\!\triangle\!\triangle_N(\mathcal{F}f)\right)(m) &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\triangle\!\triangle\!\triangle f\right)\!\left(\frac{k}{N}\right)e^{-2\pi i\,m\,\frac{k}{N}}\\
&= \frac{1}{N}\sum_{k=-\infty}^{\infty} f\!\left(\frac{k}{N}\right)e^{-2\pi i\,m\,\frac{k}{N}}\\
&= \frac{1}{N}\left(\mathcal{F}(\text{\cyrille}_{\frac{1}{N}}f)\right)(m).
\end{aligned}
$$

Thus, (38) inserted into (37) reduces to

$$
\begin{aligned}
\left(\mathcal{F}(\text{\cyrille}_{\frac{1}{N}}f)\right)(m) &= N\left(\triangle\!\triangle\!\triangle_N(\mathcal{F}f)\right)(m), \qquad m\in\mathbb{Z}\\
\left(\text{\cyrille}\,\mathcal{F}(\text{\cyrille}_{\frac{1}{N}}f)\right)(\sigma) &= N\left(\text{\cyrille}\,\triangle\!\triangle\!\triangle_N(\mathcal{F}f)\right)(\sigma), \;\; \sigma\in\mathbb{R}
\end{aligned}
$$

and with Rule 4 on the left hand side it yields

$$
\mathcal{F}\!\left(\triangle\!\triangle\!\triangle\,\text{\cyrille}_{\frac{1}{N}}f\right) = N\,\text{\cyrille}\,\triangle\!\triangle\!\triangle_N(\mathcal{F}f)
$$

being a function of $\sigma\in\mathbb{R}$.

Altogether, we proved that the Discrete Fourier Transform (DFT) together with its inverse follow these two rules

$$
\mathcal{F}\!\left(\text{\cyrille}\,\triangle\!\triangle\!\triangle_N f\right) = \frac{1}{N}\,\triangle\!\triangle\!\triangle\,\text{\cyrille}_{\frac{1}{N}}(\mathcal{F}f) \qquad \text{and}
$$

$$
\frac{1}{N}\,\mathcal{F}\!\left(\triangle\!\triangle\!\triangle\,\text{\cyrille}_{\frac{1}{N}}f\right) = \text{\cyrille}\,\triangle\!\triangle\!\triangle_N(\mathcal{F}f)
$$

as functions of $\sigma\in\mathbb{R}$, which are (29) and (30), respectively.

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