Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0067.v1

On the stability of a laminated beam with structural damping and Gurtin-Pipkin thermal law

Wenjun Liu and Weifan Zhao
College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China. E-mail: wjliu@nuist.edu.cn.

In this paper, we investigate the stabilization of a one-dimensional thermoelastic laminated beam with structural damping, coupled to a heat equation modeling an expectedly dissipative effect through heat conduction governed by Gurtin-Pipkin thermal law. Under some assumptions on the relaxation function g, we establish the well-posedness for the problem. Furthermore, we prove the exponential stability and lack of exponential stability for the problem. To achieve our goals, we make use of the semigroup method, the perturbed energy method and Gearhart-Herbst-Prüss-Huang theorem.

Keywords: laminated beam, Gurtin-Pipkin thermal law, well-posedness, exponential stability, lack of exponential stability.

AMS Subject Classification (2010): 35B40, 74F05, 93D20.

1 Introduction

The aim of this paper is to study the well-posedness and asymptotic stability of a thermoelastic laminated beam with structural damping and Gurtin-Pipkin thermal law, i.e.,

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi-\varphi_x) + \delta\theta_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\alpha w_t = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ k\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\mathrm{d}s + \delta(3w-\psi)_{tx} = 0, & (x,t) \in (0,1)\times(0,+\infty), \end{cases} \tag{1.1}$$

with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \psi(x,0)=\psi_0(x), w(x,0)=w_0(x), \theta(x,0)=\theta_0(x), & x\in[0,1], \\ \varphi_t(x,0)=\varphi_1(x), \psi_t(x,0)=\psi_1(x), w_t(x,0)=w_1(x), \theta(-s)|_{s>0}=\theta_0(s), & x\in[0,1], \\ \varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta_x(0,t)=0, & t\in[0,+\infty), \\ \varphi(1,t)=\psi_x(1,t)=w_x(1,t)=\theta(1,t)=0, & t\in[0,+\infty), \end{cases} \tag{1.2}$$

where the functions $\varphi(x,t), \psi(x,t), 3w(x,t)-\psi(x,t), \theta(x,t), g(s)$ denote the transverse displacement of the beam which departs from its equilibrium position, rotation angle, effective rotation angle, relative temperature, and the memory kernel, respectively; $w(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x; $g(s)$ is the heat conductivity relaxation kernel, whose properties will be specified later; $(1.1)_3$ describes the dynamics of the slip; $\rho, G, I_\rho, D, \gamma, \beta$ are the density of the beams, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness of the beams, and adhesive damping parameter, respectively; Moreover, $\rho, G, I_\rho, D, \delta, \gamma, \alpha, k, \beta$ are positive constant coefficients.

Problem likes (1.1) is called laminated beam, which was first introduced by Hansen and Spies in [12]. In that paper, the authors derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, namely,

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi - \varphi_x) = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\alpha w_t = 0, & (x,t) \in (0,1) \times (0,+\infty). \end{cases} \tag{1.3}$$

Later on, Wang et al. [26] considered system (1.3) with the cantilever boundary conditions and two different wave speeds ($\sqrt{G/\rho}$ and $\sqrt{D/I_\rho}$), they pointed out that system (1.3) can reach the asymptotic stability but it does not reach the exponential stability due to the action of the slip w. To achieve the exponential decay result, the authors in [26] added an additional boundary control such that the boundary conditions become

$$\varphi(0,t) = \xi(0,t) = w(0,t) = 0, \;\; w_x(1,t) = 0,$$

$$3w(1,t) - \xi(1,t) - \varphi_x(1,t) = u_1(t) := k_1\varphi_t(1,t),$$

$$\xi_x(1,t) = u_2(t) := -k_2\xi_t(1,t),$$

where $\xi = 3w - \psi$ and k_1 and k_2 are positive constant feedback gains. Furthermore, Cao et al. [6] proved the exponential stability for system (1.3) with following boundary conditions

$$\psi(0,t) - \varphi_x(0,t) = u_1(t) := -k_1\varphi_t(0,t) - \varphi(0,t),$$

$$3w_x(1,t) - \psi_x(1,t) = u_2(t) := -k_2\xi_t(1,t) - \xi(1,t),$$

provided $k_1 \neq \sqrt{\rho/G}$ and $k_2 \neq \sqrt{I_\rho/D}$. More importantly, the authors proved that the dominant part of the system is itself exponentially stable.

The general thermoelastic laminated beam model reads

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi - \varphi_x) + \delta\theta_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\alpha w_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ k\theta_t + q_x + \delta(3w-\psi)_{tx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.4}$$

where $\theta(x,t)$ is the relative temperature and $q(x,t)$ is the heat flux vector. If we assume Cattaneo law of heat conduction

$$\tau q_t + \kappa q + \theta_x = 0, \tag{1.5}$$

 doi:10.20944/preprints201801.0067.v1

where $\kappa > 0$ is a fixed constant and $\tau > 0$ is small, then we can get a laminated beam with second sound. The stabilization of system (1.4)-(1.5) has been analyzed in [1]. There, Apalara obtained the well-posedness and uniform stability results depending on the following stability number

$$\chi_\tau = \left(1 - \frac{\tau k G}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau G \delta^2}{\rho I_\rho}.$$

If we assume Gurtin-Pipkin thermal law of heat conduction

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)\mathrm{d}s = 0, \tag{1.6}$$

where g is called the memory kernel, then we can get the desired laminated beam with Gurtin-Pipkin thermal law and structural damping, i.e., (1.1)-(1.2). In fact, Cattaneo law (1.5) can be reduced as a particular instance of (1.6), which have been proved in [9]. For other asymptotic behavior results to laminated beams, we refer the reader to [6, 12, 19, 20, 21, 25, 26] and the references therein.

For the case of the beams with Gurtin-Pipkin thermal law, a large number of interesting decay results depending on the stability number have been established. Recently, Dell'Oro and Pata [9] considered Timoshenko system with Gurtin-Pipkin thermal law, i.e.,

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\mathrm{d}s + \delta\psi_{tx} = 0, & (x,t) \in (0,L)\times(0,+\infty), \end{cases}$$

where $\rho_1, \kappa, \rho_2, b, \delta, \rho_3, \beta$ are positive constants. The authors obtained the exponential stability depending on the stability number

$$\xi_g = \left(\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right)\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right) - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b}.$$

Later on, Dell'Oro [10] considered the thermoelastic Bresse-Gurtin-Pipkin system, i.e.,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi+lw) + \delta\theta_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x+\psi+lw) = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_3\theta_t - k_1\displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\mathrm{d}s + \delta\psi_{tx} = 0, & (x,t) \in (0,L)\times(0,+\infty), \end{cases}$$

and obtained that the system is exponentially stable if and only if

$$\alpha_g := \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k_1}\frac{\rho_1\delta^2}{\rho_3 b k} = 0 \textit{ and } k = k_0.$$

For other related results, we refer the reader to [2, 3, 5, 7, 8, 14, 15, 17, 18, 22, 27, 28].

In this paper, we first prove the well-posedness by using Lumer-Philips theorem. And then, by using the perturbed energy method, we establish an exponential stability result depending on the stability number

$$\chi_g = \left(1 - \frac{\beta}{g(0)}\frac{kG}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\beta}{g(0)}\frac{G\delta^2}{\rho I_\rho}.$$

To overcome the difficulty brought by Gurtin-Pipkin thermal law, we use some appropriated multipliers to construct a Lyapunov functional. For the case $\chi_g \neq 0$, we prove the lack of exponential stability by using Gearhart-Herbst-Prüss-Huang theorem.

The remaining part of this paper is organized as follows. In Section 2, we introduce some hypotheses and present our main results. In Section 3, we prove the well-posedness for problem (1.1)-(1.2). In Section 4, we establish an exponential decay result to problem (1.1)-(1.2). In Section 5, we prove the lack of exponential stability for problem (1.1)-(1.2). Section 6 is devoted to the conclusion and open problem. Throughout this paper, we use c to denote a generic positive constant.

2 Preliminaries and main results

In this section, we first introduce some notation and present our hypotheses. Then we give some lemmas which will be used in the proof of main results.

To deal with the memory, we introduce a new auxiliary variable $\eta = \eta^t(x,s)$ by (see [11, 9])

$$\eta = \eta^t(x,s) = \int_0^s \theta(x, t-\sigma)\mathrm{d}\sigma,\ (x,t,s) \in [0,1] \times [0,\infty) \times \mathbb{R}^+,$$

which satisfies the following boundary conditions

$$\eta^t(1,s) = 0,\ \eta^t_x(0,s) = 0.$$

Then θ satisfies

$$\eta_t + \eta_s = \theta(t),$$

where

$$\eta^t(x,0) = 0,\ t \in [0,\infty)$$

and

$$\eta^0(x,s) = \eta_0(s) = \int_0^s \theta_0(\sigma)\mathrm{d}\sigma,\ s \in \mathbb{R}^+.$$

Assume $g(\infty) = 0$, a change of variable and a formal integration by parts yield

$$\int_0^\infty g(s)\theta_{xx}(t-s)\mathrm{d}s = -\int_0^\infty g'(s)\eta_{xx}(s)\mathrm{d}s.$$

Now, we denote

$$\mu(s) = -g'(s),$$

then

$$\int_0^\infty g(s)\theta_{xx}(t-s)\mathrm{d}s = \int_0^\infty \mu(s)\eta_{xx}(s)\mathrm{d}s.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0067.v1

Hence system (1.1)-(1.2) can be written as

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi-\varphi_x) + \delta\theta_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\alpha w_t = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ k\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty \mu(s)\eta_{xx}(s)\mathrm{d}s + \delta(3w-\psi)_{tx} = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \eta_t + \eta_s = \theta, & (x,t) \in (0,1)\times(0,+\infty). \end{cases} \tag{2.1}$$

with initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \psi(x,0) = \psi_0(x), w(x,0) = w_0(x), \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \varphi_t(x,0) = \varphi_1(x), \psi_t(x,0) = \psi_1(x), w_t(x,0) = w_1(x), & x \in [0,1], \\ \eta(x,0) = 0, \eta_0(x,s) = \int_0^s \theta_0(x,\sigma)\mathrm{d}\sigma, & x \in [0,1], \\ \varphi_x(0,t) = \psi(0,t) = w(0,t) = \theta_x(0,t) = \eta_x^t(0,s) = 0, & t \in [0,+\infty), \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = \theta(1,t) = \eta^t(1,s) = 0, & t \in [0,+\infty). \end{cases} \tag{2.2}$$

For the memory kernel g, we assume that

(G1) g is a bounded convex summable function on $[0,\infty)$.

(G2) g has a total mass

$$\int_0^\infty g(s)\mathrm{d}s = 1.$$

(G3) g' is an absolutely continuous function on $\mathbb{R}^+$ so that

$$g'(s) \le 0,\ \ g''(s) \ge 0,\ \ g'(0) = \lim_{s\to 0} g'(s) \in (-\infty, 0).$$

(G4) There exists a positive constant ξ so that, for almost every $s > 0$,

$$g''(s) + \xi g'(s) \ge 0.$$

Remark 1 *In particular, μ is summable on $\mathbb{R}^+$ with*

$$\int_0^\infty \mu(s)\mathrm{d}s = g(0).$$

Furthermore, noting that $g(s)$ has total mass 1, we have

$$\int_0^\infty s\mu(s)\mathrm{d}s = 1.$$

Next, we introduce the vector function

$$U = (\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta, \eta)^T.$$

Then system (2.1)-(2.2) can be written as

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, \eta_0)^T, \end{cases} \tag{2.3}$$

 doi:10.20944/preprints201801.0067.v1

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi_t \\ -\frac{G}{\rho}(\psi - \varphi_x)_x \\ (3w-\psi)_t \\ \frac{D}{I_\rho}(3w-\psi)_{xx} + \frac{G}{I_\rho}(\psi-\varphi_x) - \frac{\delta}{I_\rho}\theta_x \\ w_t \\ \frac{D}{I_\rho}w_{xx} - \frac{G}{I_\rho}(\psi_-\varphi_x) - \frac{4\gamma}{3I_\rho}w - \frac{4\alpha}{3I_\rho}w_t \\ \frac{1}{k\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\mathrm{d}s - \frac{\delta}{k}(3w-\psi)_{tx} \\ -\eta_s + \theta \end{pmatrix}.$$

We consider the following spaces

$$H_*^1(0,1) = \left\{\eta \,\middle|\, \eta \in H^1(0,1) : \eta(0) = 0\right\},\ \tilde{H}_*^1(0,1) = \left\{\eta \,\middle|\, \eta \in H^1(0,1) : \eta(1) = 0\right\},$$

$$H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1),\ \tilde{H}_*^2(0,1) = H^2(0,1) \cap \tilde{H}_*^1(0,1),$$

and the energy space

$$\mathcal{H} = \tilde{H}_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathcal{M}, \tag{2.4}$$

where

$$\mathcal{M} = L_\mu^2\left(\mathbb{R}^+, \tilde{H}_*^1(0,1)\right) = \left\{\eta : \mathbb{R}^+ \to \tilde{H}_*^1(0,1) \,\middle|\, \int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s < \infty\right\}$$

equipped with the norm

$$\|\varphi\|_{\mathcal{M}}^2 = \int_0^\infty \mu(s)\|\varphi_x(s)\|_2^2\mathrm{d}s$$

and inner product

$$\langle\varphi,\psi\rangle_{\mathcal{M}} = \int_0^\infty \mu(s)\int_0^1 \varphi_x(s)\psi_x(s)\mathrm{d}x\mathrm{d}s.$$

In particular,

$$\langle -\eta_s, \eta\rangle_{\mathcal{M}} = \frac{1}{2}\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s.$$

Moreover, in light of $(G4)$ on μ, we deduce

$$\xi\int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s \le -\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s. \tag{2.5}$$

Besides, $\mathcal{H}$ is the Hilbert space equipped with the norm

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 =& \|(\varphi,\varphi_t,3w-\psi,(3w-\psi)_t,w,w_t,\theta,\eta)\|_{\mathcal{H}}^2 \\ =& \rho\|\varphi_t\|_2^2 + I_\rho\|(3w-\psi)_t\|_2^2 + 3I_\rho\|w_t\|_2^2 + G\|(\psi-\varphi_x)\|_2^2 + D\|(3w-\psi)_x\|_2^2\end{aligned}$$

$$+4\gamma\|w\|_2^2+3D\|w_x\|_2^2+k\|\theta\|_2^2+\frac{1}{\beta}\|\eta\|_{\mathcal{M}}^2,$$

and the inner product

$$\begin{aligned}\left(U,\tilde{U}\right)_{\mathcal{H}}=&\rho\int_0^1\varphi_t\tilde{\varphi}_t\mathrm{d}x+I_\rho\int_0^1(3w-\psi)_t(3\tilde{w}-\tilde{\psi})_t\mathrm{d}x+3I_\rho\int_0^1 w_t\tilde{w}_t\mathrm{d}x+k\int_0^1\theta\tilde{\theta}\mathrm{d}x\\&+G\int_0^1(\psi-\varphi_x)(\tilde{\psi}-\tilde{\varphi}_x)\mathrm{d}x+D\int_0^1(3w-\psi)_x(3\tilde{w}-\tilde{\psi})_x\mathrm{d}x+4\gamma\int_0^1 w\tilde{w}\mathrm{d}x\\&+3D\int_0^1 w_x\tilde{w}_x\mathrm{d}x+\frac{1}{\beta}\int_0^\infty\mu(s)\int_0^1\eta_x\tilde{\eta}_x\mathrm{d}x\mathrm{d}s,\end{aligned}$$

for $U=(\varphi,\varphi_t,3w-\psi,(3w-\psi)_t,w,w_t,\theta,\eta)^T$ and $\tilde{U}=(\tilde{\varphi},\tilde{\varphi}_t,3\tilde{w}-\tilde{\psi},(3\tilde{w}-\tilde{\psi})_t,\tilde{w},\tilde{w}_t,\tilde{\theta},\tilde{\eta})^T$.

The domain of $\mathcal{A}$ is given by

$$\begin{aligned}D(\mathcal{A})=\Bigg\{U\in\mathcal{H}\ \Bigg|\ &\varphi\in\tilde{H}_*^2(0,1),\varphi_t\in\tilde{H}_*^1(0,1),3w-\psi\in H_*^2(0,1),(3w-\psi)_t\in H_*^1(0,1),\\&w\in H_*^2(0,1),w_t\in H_*^1(0,1),\theta\in\tilde{H}_*^1(0,1),\eta\in\mathcal{N},\int_0^\infty\mu(s)\eta(s)\mathrm{d}s\in\tilde{H}_*^2(0,1),\\&\varphi_x(0,t)=\psi_x(1,t)=w_x(1,t)=\theta_x(0,t)=\eta_x(0,s)=0\Bigg\},\end{aligned}$$

where $\mathcal{N}=\mathscr{L}_\mu^2\left(\mathbb{R}^+,\tilde{H}_*^1(0,1)\right)=\left\{\eta\in\mathcal{M}\Big|\eta_s\in\mathcal{M},\eta(0)=0\right\}$. Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The energy associated with problem (2.1)-(2.2) is defined by

$$\begin{aligned}E(t)=&\frac{1}{2}\Bigg(\rho\int_0^1\varphi_t^2\mathrm{d}x+I_\rho\int_0^1(3w_t-\psi_t)^2\mathrm{d}x+4\gamma\int_0^1 w^2\mathrm{d}x+3I_\rho\int_0^1 w_t^2\mathrm{d}x+G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x\\&+D\int_0^1(3w_x-\psi_x)^2\mathrm{d}x+3D\int_0^1 w_x^2\mathrm{d}x+k\int_0^1\theta^2\mathrm{d}x+\frac{1}{\beta}\int_0^\infty\mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s\Bigg).\end{aligned}\tag{2.6}$$

Now, we give our main results in this paper as follows.

Theorem 2.1 *Let $U_0\in\mathcal{H}$, then problem (2.3) exists a unique weak solution $U\in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C(\mathbb{R}^+;D(\mathcal{A}))\cap C^1(\mathbb{R}^+;\mathcal{H}).$$

Theorem 2.2 *Assume that $\chi_g=0$. Let $U_0\in\mathcal{H}$, then there exists positive constants a,b such that the energy $E(t)$ associated with problem (2.1)-(2.2) satisfies*

$$E(t)\le ae^{-bt},\ t\ge 0.\tag{2.7}$$

Theorem 2.3 *Assume that $\chi_g\neq 0$. Let $U_0\in\mathcal{H}$, then problem (2.1)-(2.2) is not exponentially stable.*

Based on two propositions from [9, Proposition 11 and Proposition 12], we give the full equivalence between Cattaneo law and Gurtin-Pipkin thermal law.

Theorem 2.4 *If the laminated beam with structural damping and Cattaneo law is exponentially stable, then so is the laminated beam with structural damping and Gurtin-Pipkin thermal law, and vice versa.*

3 Well-posedness: proof of Theorem 2.1

To obtain the well-posedness, we need to prove that $\mathcal{A}$: $D(\mathcal{A}) \sim \mathcal{H}$ is a maximal monotone operator. To achieve this goal, we need to prove that $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective.

Using the inner product and integration by parts, we can easily obtain

$$\begin{aligned}
(\mathcal{A}U, U)_{\mathcal{H}} =& -\int_0^1 G\left(\psi-\varphi_x\right)_x \varphi_t \mathrm{d}x + \int_0^1 \left[D(3w-\psi)_{xx} + G(\psi-\varphi_x) - \delta\theta_x\right](3w-\psi)_t \mathrm{d}x \\
&+ \int_0^1 \left[3Dw_{xx} - 3G(\psi-\varphi_x) - 4\gamma w - 4\alpha w_t\right] w_t \mathrm{d}x \\
&+ \int_0^1 \left[\frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\mathrm{d}s - \delta(3w-\psi)_{tx}\right]\theta \mathrm{d}x + G\int_0^1 (\psi-\varphi_x)(\psi_t-\varphi_{xt})\mathrm{d}x \\
&+ D\int_0^1 (3w-\psi)_{xt}(3w-\psi)_x \mathrm{d}x + 4\gamma\int_0^1 w_t w \mathrm{d}x + 3D\int_0^1 w_{xt}w_x \mathrm{d}x \\
&+ \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^1 (-\eta_{xs}+\theta_x)\eta_x \mathrm{d}x\mathrm{d}s \\
=& -4\alpha\int_0^1 w_t^2 \mathrm{d}x + \frac{1}{\beta}\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2 \mathrm{d}s \le 0, \qquad (3.1)
\end{aligned}$$

for any $U \in D(\mathcal{A})$. Hence $\mathcal{A}$ is dissipative.

Next, we turn to prove $Id - \mathcal{A}$ is surjective, i.e., for any $F = (f_1, f_2, \cdots, f_8) \in \mathcal{H}$, there exists $V = (v_1, v_2, \cdots, v_8) \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})V = F, \qquad (3.2)$$

that is,

$$\begin{cases}
v_1 - v_2 = f_1, \\
\rho v_2 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho f_2, \\
v_3 - v_4 = f_3, \\
I_\rho v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx}v_3 - 3Gv_5 + \delta\partial_x v_7 = I_\rho f_4, \\
v_5 - v_6 = f_5, \\
\left(I_\rho + \frac{4}{3}\alpha\right)v_6 - G\partial_x v_1 - Gv_3 + \left(3G + \frac{4\gamma}{3}\right)v_5 - D\partial_{xx}v_5 = I_\rho f_6, \\
kv_7 - \frac{1}{\beta}\int_0^\infty \mu(s)\partial_{xx}v_8 \mathrm{d}s + \delta\partial_x v_4 = kf_7, \\
v_8 + \partial_s v_8 - v_7 = f_8.
\end{cases} \qquad (3.3)$$

From $(3.3)_1$, $(3.3)_3$, $(3.3)_5$, $(3.3)_8$ and $v_8(0) = 0$, we have

$$\begin{cases}
v_2 = v_1 - f_1, \\
v_4 = v_3 - f_3, \\
v_6 = v_5 - f_5, \\
v_8 = (1 - e^{-s})v_7 + \int_0^s e^{\tau - s} f_8(\tau)\mathrm{d}\tau.
\end{cases} \qquad (3.4)$$

Inserting (3.4) into $(3.3)_2$, $(3.3)_4$, $(3.3)_6$ and $(3.3)_7$, we obtain

$$\begin{cases} \rho v_1 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho(f_1+f_2), \\ (I_\rho + G)v_3 + G\partial_x v_1 - D\partial_{xx}v_3 - 3Gv_5 + \delta\partial_x v_7 = I_\rho(f_3+f_4), \\ \left(I_\rho + 3G + \frac{4\gamma}{3} + \frac{4\alpha}{3}\right) v_5 - G\partial_x v_1 - Gv_3 - D\partial_{xx}v_5 = I_\rho(f_5+f_6) + \frac{4}{3}\alpha f_5, \\ kv_7 - \frac{1}{\beta}\int_0^\infty (1-e^{-s})\mu(s)\partial_{xx}v_7 \mathrm{d}s + \delta\partial_x v_3 \\ \qquad = kf_7 + \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^s e^{\tau-s}\partial_{xx}f_8(\tau)\mathrm{d}\tau\mathrm{d}s + \delta\partial_x f_3. \end{cases} \tag{3.5}$$

Multiplying $(3.5)_1$-$(3.5)_4$ by $\tilde{v}_1$, $\tilde{v}_3$, $3\tilde{v}_5$ and $\tilde{v}_7$ respectively, and integrating over $(0,1)$, we can obtain

$$\begin{cases} \int_0^1 \rho v_1\tilde{v}_1 \mathrm{d}x - \int_0^1 G\partial_{xx}v_1\tilde{v}_1\mathrm{d}x - \int_0^1 G\partial_x v_3\tilde{v}_1\mathrm{d}x + \int_0^1 3G\partial_x v_5\tilde{v}_1\mathrm{d}x = \int_0^1 \rho(f_1+f_2)\tilde{v}_1\mathrm{d}x, \\ \int_0^1 (I_\rho+G)v_3\tilde{v}_3\mathrm{d}x + \int_0^1 G\partial_x v_1\tilde{v}_3\mathrm{d}x - \int_0^1 D\partial_{xx}v_3\tilde{v}_3\mathrm{d}x - \int_0^1 3Gv_5\tilde{v}_3\mathrm{d}x + \int_0^1 \delta\partial_x v_7\tilde{v}_3\mathrm{d}x \\ \qquad = \int_0^1 I_\rho f_4\tilde{v}_3\mathrm{d}x, \\ \int_0^1 (3I_\rho + 9G + 4\gamma + 4\alpha)\, v_5\tilde{v}_5\mathrm{d}x - \int_0^1 3G\partial_x v_1\tilde{v}_5\mathrm{d}x - \int_0^1 3Gv_3\tilde{v}_5\mathrm{d}x - \int_0^1 3D\partial_{xx}v_5\tilde{v}_5\mathrm{d}x \\ \qquad = \int_0^1 3I_\rho(f_5+f_6)\tilde{v}_5\mathrm{d}x + \int_0^1 4\alpha f_5\tilde{v}_5\mathrm{d}x, \\ \int_0^1 kv_7\tilde{v}_7\mathrm{d}x - \frac{1}{\beta}\int_0^1 \tilde{v}_7\int_0^\infty (1-e^{-s})\mu(s)\partial_{xx}v_7\mathrm{d}s\mathrm{d}x + \int_0^1 \delta\partial_x v_3\tilde{v}_7\mathrm{d}x \\ \qquad = \int_0^1 \delta\partial_x f_3\tilde{v}_7\mathrm{d}x + \frac{1}{\beta}\int_0^1 \tilde{v}_7\int_0^\infty \mu(s)\int_0^s e^{\tau-s}\partial_{xx}f_8(\tau)\mathrm{d}\tau\mathrm{d}s\mathrm{d}x + \int_0^1 kf_7\tilde{v}_7\mathrm{d}x. \end{cases} \tag{3.6}$$

From (3.6), we have the following variational formulation:

$$\begin{aligned} &B\left((v_1,v_3,v_5,v_7)^T, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right) = F\left((\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right), \\ &\forall\, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T \in \tilde{H}^1_*(0,1)\times H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &B\left((v_1,v_3,v_5,v_7)^T, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right) \\ =& \int_0^1 G(-\partial_x v_1 - v_3 + 3v_5)(-\partial_x\tilde{v}_1 - \tilde{v}_3 + 3\tilde{v}_5)\mathrm{d}x + \int_0^1 \rho v_1\tilde{v}_1\mathrm{d}x + \int_0^1 I_\rho v_3\tilde{v}_3\mathrm{d}x \\ &+ \int_0^1 (3I_\rho + 4\gamma + 4\alpha)v_5\tilde{v}_5\mathrm{d}x + \int_0^1 kv_7\tilde{v}_7\mathrm{d}x + \int_0^1 D\partial_x v_3\partial_x\tilde{v}_3\mathrm{d}x + \int_0^1 3D\partial_x v_5\partial_x\tilde{v}_5\mathrm{d}x \\ &+ \int_0^1 \frac{1}{\beta}\left(g(0) - \int_0^\infty e^{-s}\mu(s)\mathrm{d}s\right)\partial_x v_7\partial_x\tilde{v}_7\mathrm{d}x + \delta\int_0^1 (\partial_x v_7)\tilde{v}_3\mathrm{d}x + \delta\int_0^1 (\partial_x v_3)\tilde{v}_7\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned}&F\left((\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right)\\ =&\int_0^1\left[\rho(f_1+f_2)\tilde{v}_1+I_\rho(f_3+f_4)\tilde{v}_3+3I_\rho(f_5+f_6)\tilde{v}_5+4\alpha f_5\tilde{v}_5+\delta\partial_x f_3\tilde{v}_7+kf_7\tilde{v}_7\right]\mathrm{d}x.\\ &+\frac{1}{\beta}\int_0^1\tilde{v}_7\int_0^\infty\mu(s)\int_0^s e^{\tau-s}\partial_{xx}f_8(\tau)\mathrm{d}\tau\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Now, we introduce the Hilbert space $V=\tilde{H}^1_*(0,1)\times H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)$ equipped with the norm

$$\|(v_1,v_3,v_5,v_7)\|_V^2=\|-\partial_x v_1-v_3+3v_5\|_2^2+\|v_1\|_2^2+\|\partial_x v_3\|_2^2+\|\partial_x v_5\|_2^2+\|\partial_x v_7\|_2^2.$$

Then $B(\cdot,\cdot)$ and $F(\cdot)$ are bounded. Furthermore, we obtain that there exists a positive constant c such that

$$\begin{aligned}&B\left((v_1,v_3,v_5,v_7)^T,(v_1,v_3,v_5,v_7)^T\right)\\ =&\int_0^1 G(-\partial_x v_1-v_3+3v_5)^2\mathrm{d}x+\int_0^1\rho v_1^2\mathrm{d}x+\int_0^1 I_\rho v_3^2\mathrm{d}x+\int_0^1(3I_\rho+4\gamma+4\alpha)v_5^2\mathrm{d}x+\int_0^1 kv_7^2\mathrm{d}x\\ &+\int_0^1 D(\partial_x v_3)^2\mathrm{d}x+\int_0^1 3D(\partial_x v_5)^2\mathrm{d}x+\frac{1}{\beta}\left(g(0)-\int_0^\infty e^{-s}\mu(s)\mathrm{d}s\right)\int_0^1(\partial_x v_7)^2\mathrm{d}x\\ \geq&\, c\,\|(v_1,v_3,v_5,v_7)\|_V^2.\end{aligned}$$

Hence $B(\cdot,\cdot)$ is coercive.

As a consequence, by applying Lax-Milgram lemma [23], we can obtain that (3.6) has a unique solution $(v_1,v_3,v_5,v_7)^T\in V$. Then, substituting v_1,v_3,v_5 into $(3.4)_1$-$(3.4)_3$, we obtain

$$v_2\in\tilde{H}^1_*(0,1),v_4\in H^1_*(0,1),v_6\in H^1_*(0,1).$$

Using $(3.4)_4$ and the method in [29, Proposition 2.2], we have

$$\begin{aligned}\int_0^\infty\mu(s)\|\partial_x v_8(s)\|_2^2\mathrm{d}s\leq&2\int_0^\infty\mu(s)\|\left(1-e^{-s}\right)\partial_x v_7\|_2^2\mathrm{d}s+2\int_0^\infty\mu(s)\left\|\int_0^s e^{\tau-s}\partial_x f_8(\tau)\mathrm{d}\tau\right\|_2^2\mathrm{d}s\\ =&2\int_0^\infty\left(1-e^{-s}\right)\mu(s)\mathrm{d}s\|\partial_x v_7\|_2^2+2\left\|\int_0^s e^{\tau-s}f_8(\tau)\mathrm{d}\tau\right\|_{\mathcal{M}}^2\\ \leq&2g(0)\|\partial_x v_7\|_2^2+2\|f_8\|_{\mathcal{M}}^2,\end{aligned}$$

which gives us $v_8\in\mathcal{M}$. Then from $(3.3)_7$, we can obtain

$$\partial_s v_8=v_7-v_8+f_8\in\mathcal{M}.$$

Hence, $v_8\in\mathcal{N}$. Next, we turn to prove that

$$v_1\in\tilde{H}^2_*(0,1),v_3\in H^2_*(0,1),v_5\in H^2_*(0,1),v_7\in\tilde{H}^1_*(0,1),$$

$$\partial_x v_1(0)=\partial_x v_3(1)=\partial_x v_5(1)=\partial_x v_7(0)=0.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0067.v1

Now, if $(\tilde{v}_3, \tilde{v}_5, \tilde{v}_7) \equiv (0,0,0) \in H^1_*(0,1) \times H^1_*(0,1) \times L^2(0,1)$, then (3.7) reduces to

$$\int_0^1 G(\partial_x v_1 - v_3 + 3v_5)\partial_x \tilde{v}_1 \mathrm{d}x = \int_0^1 \rho v_1 \tilde{v}_1 \mathrm{d}x - \int_0^1 \rho(f_1 + f_2)\tilde{v}_1 \mathrm{d}x, \tag{3.8}$$

for all $\tilde{v}_1 \in \tilde{H}^1_*(0,1)$, which implies

$$G\partial_{xx} v_1 = \rho v_1 - G\partial_x v_3 + 3G\partial_x v_5 - \rho(f_1 + f_2) \in L^2(0,1). \tag{3.9}$$

From the regularity theory for the linear elliptic equations, we obtain

$$v_1 \in \tilde{H}^2_*(0,1).$$

Moreover, (3.8) is also true for any $\phi \in C^1([0,1]) \subset H^1_*(0,1)$ $(\phi(1) = 0)$. Thus, we get

$$\int_0^1 G\partial_x v_1 \partial_x \phi \mathrm{d}x + \int_0^1 \rho v_1 \phi \mathrm{d}x - \int_0^1 G(\partial_x v_3)\phi \mathrm{d}x + \int_0^1 3G(\partial_x v_5)\phi \mathrm{d}x = \int_0^1 \rho(f_1 + f_2)\phi \mathrm{d}x,$$

for $\forall\ \phi \in C^1([0,1]), \phi(1) = 0$. Using (3.9) and the integration by parts, we have

$$\partial_x v_1(0)\phi(0) = 0,\ \forall \phi \in C^1([0,1]),\ \phi(1) = 0.$$

Hence,

$$\partial_x v_1(0) = 0.$$

In the same way, we get

$$v_3 \in H^2_*(0,1), v_5 \in H^2_*(0,1), v_7 \in \tilde{H}^1_*(0,1), \partial_x v_3(1) = \partial_x v_5(1) = \partial_x v_7(0) = 0.$$

From the classical regularity theory for the linear elliptic equations, we know that there exists a unique solution $U \in D(\mathcal{A})$ such that (3.2) is satisfied. So the operator $Id - \mathcal{A}$ is surjective.

As a consequence, $\mathcal{A}$ is a maximal monotone operator. Therefore, we established the well-posedness result stated in Theorem 2.1 by using Lumer-Philips theorem (see [4, 16]).

4 Exponential decay: proof of Theorem 2.2

In this section, we prove the exponential stability for system (2.1)-(2.2) when $\chi_g = 0$. It will be achieved by using the perturbed energy method. Before we prove our result, we need some useful lemmas.

Lemma 4.1 *Let $(\varphi, \psi, w, \theta)$ be the solution of problem (2.1)-(2.2). Then the energy function $E(t)$ satisfies*

$$\frac{d}{dt}E(t) = -4\alpha \int_0^1 w_t^2 \mathrm{d}x + \frac{1}{\beta}\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2 \mathrm{d}s \le 0,\ \forall\ t \ge 0. \tag{4.1}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0067.v1

Proof. Multiplying $(2.1)_1$ by φ_t, $(2.1)_2$ by $(3w-\psi)_t$, $(2.1)_3$ by $3w_t$, $(2.1)_4$ by θ and integrating over $(0,1)$, using integration by parts the boundary conditions in (2.2), we can obtain

$$\frac{d}{dt}E(t) = -4\alpha\int_0^1 w_t^2\mathrm{d}x + \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^1 \theta\eta_{xx}(s)\mathrm{d}x\mathrm{d}s + \frac{1}{2\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s. \tag{4.2}$$

From $(2.1)_5$ we know that

$$\frac{1}{2\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s = \frac{1}{\beta}\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s + \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^1 \theta_x\eta_x(s)\mathrm{d}x\mathrm{d}s. \tag{4.3}$$

Combining (4.2) and (4.3), we could obtain (4.1). This completes the proof.

Lemma 4.2 *Let* (φ,ψ,w,θ) *be the solution of* (2.1)-(2.2). *Then the functional*

$$F_1(t) = -\frac{k}{g(0)}\int_0^\infty \mu(s)\int_0^1 \theta\eta(s)\mathrm{d}x\mathrm{d}s$$

satisfies the estimate

$$F_1'(t) \le -\frac{k}{2}\int_0^1 \theta^2\mathrm{d}x - c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s + \varepsilon_1\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x, \tag{4.4}$$

for any $\varepsilon_1 > 0$.

Proof. Taking the derivative of $F_1(t)$ with respect to t, using $(2.1)_4$, $(2.1)_5$ and integrating by parts, we get

$$\begin{aligned} F_1'(t) = &-k\int_0^1 \theta^2\mathrm{d}x + \frac{k}{g(0)}\int_0^\infty \mu(s)\int_0^1 \theta\eta_s(s)\mathrm{d}x\mathrm{d}s + \frac{1}{\beta g(0)}\left\|\int_0^\infty \mu(s)\eta_x(s)\mathrm{d}s\right\|_2^2 \\ &- \frac{\delta}{g(0)}\int_0^\infty \mu(s)\int_0^1 (3w-\psi)_t\eta_x(s)\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{4.5}$$

Using integrate by parts and Young's inequality with $\varepsilon_1 > 0$, we infer that

$$\begin{aligned} \frac{k}{g(0)}\int_0^\infty \mu(s)\int_0^1 \theta\eta_s(s)\mathrm{d}x\mathrm{d}s &= -\frac{k}{g(0)}\int_0^\infty \mu'(s)\int_0^1 \theta\eta(s)\mathrm{d}x\mathrm{d}s \\ &\le \varepsilon\int_0^1 \theta^2\mathrm{d}x - \frac{c}{\varepsilon}\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s, \end{aligned}$$

$$\frac{1}{\beta g(0)}\left\|\int_0^\infty \mu(s)\eta_x(s)\mathrm{d}s\right\|_2^2 \le c\int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s,$$

$$-\frac{\delta}{g(0)}\int_0^\infty \mu(s)\int_0^1 (3w-\psi)_t\eta_x(s)\mathrm{d}x\mathrm{d}s \le \varepsilon_1\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + \frac{c}{\varepsilon_1}\int_0^\infty \mu(s)\|\eta_x(s)\|_2^2\mathrm{d}s.$$

Then we can get (4.4) by using above inequalities and (2.5). This completes the proof.

Lemma 4.3 *Let* (φ,ψ,w,θ) *be the solution of* (2.1)-(2.2). *Then the functional*

$$F_2(t) = \frac{kI_\rho}{\delta}\int_0^1 (3w-\psi)_t\int_0^x \theta(y)\mathrm{d}y\mathrm{d}x$$

satisfies the estimate

$$\begin{aligned}F_2'(t) \leq & -\frac{I_\rho}{2}\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + \varepsilon_2\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + \varepsilon_2\int_0^1(3w_x-\psi_x)^2\mathrm{d}x \\ & + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1\theta^2\mathrm{d}x - c\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s, \end{aligned} \tag{4.6}$$

for any $\varepsilon_2>0$.

Proof. Taking the derivative of $F_2(t)$ with respect to t, using $(2.1)_2$, $(2.1)_4$ and integrating by parts, we get

$$\begin{aligned}F_2'(t) = & -I_\rho\int_0^1(3w_t-\psi_t)^2\mathrm{d}x + \frac{kG}{\delta}\int_0^1(\psi-\varphi_x)\int_0^x\theta(y)\mathrm{d}y\mathrm{d}x - \frac{kD}{\delta}\int_0^1(3w-\psi)_x\theta\mathrm{d}x \\ & + k\int_0^1\theta^2\mathrm{d}x + \frac{I_\rho}{\beta\delta}\int_0^\infty\mu(s)\int_0^1(3w-\psi)_t\eta_x(s)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Using (2.5), Young's and Cauchy-Schwarz inequalities with $\varepsilon_2>0$, we establish (4.6).

Lemma 4.4 *Let* (φ,ψ,w,θ) *be the solution of* (2.1)-(2.2). *Then the functional*

$$\begin{aligned}F_3(t) = & \rho D\int_0^1\varphi_t(3w-\psi)_x\mathrm{d}x - I_\rho G\int_0^1(3w-\psi)_t(\psi-\varphi_x)\mathrm{d}x \\ & + \frac{\rho k I_\rho}{\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^1\theta\varphi_t\mathrm{d}x - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty\mu(s)\int_0^1(\psi-\varphi_x)\eta_x(s)\mathrm{d}x\mathrm{d}s\end{aligned}$$

satisfies the estimate

$$F_3'(t) \leq -\frac{G}{2}\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + c\int_0^1\left[(3w_t-\psi_t)^2+w_t^2\right]\mathrm{d}x - c\int_0^\infty\mu'(s)\|\eta_x(s)\|_2^2\mathrm{d}s. \tag{4.7}$$

Proof. By $(2.1)_1$, $(2.1)_2$, $(2.1)_4$ and integrating by parts, we get

$$\begin{aligned}F_3'(t) = & -G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - I_\rho G\int_0^1(3w-\psi)_t\psi_t\mathrm{d}x - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty\mu(s)\int_0^1\psi_t\eta_x(s)\mathrm{d}x\mathrm{d}s \\ & - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty\mu'(s)\int_0^1(\psi-\varphi_x)\eta_x(s)\mathrm{d}x\mathrm{d}s - \frac{\rho I_\rho}{\delta}\frac{g(0)}{\beta}\chi_g\int_0^1\theta_x(\psi-\varphi_x)\mathrm{d}x.\end{aligned} \tag{4.8}$$

Using (2.5), Young's inequality and $\chi_g=0$, we get (4.7).

Lemma 4.5 *Let* (φ,ψ,w,θ) *be the solution of* (2.1)-(2.2). *Then the functional*

$$F_4(t) = -\rho\int_0^1\varphi\varphi_t\mathrm{d}x$$

satisfies the estimate

$$F_4'(t) \leq -\rho\int_0^1\varphi_t^2\mathrm{d}x + \varepsilon_4\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + \varepsilon_4\int_0^1 w_x^2\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1(\psi-\varphi_x)^2\mathrm{d}x, \tag{4.9}$$

for any $\varepsilon_4>0$.

Proof. By differentiating $F_4(t)$ with respect to t, using $(2.1)_1$ and integrating by parts, we obtain

$$F_4'(t) = -\rho\int_0^1 \varphi_t^2\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x - G\int_0^1 \psi(\psi-\varphi_x)\mathrm{d}x.$$

Using Young's inequality, we obtain

$$F_4'(t) \le -\rho\int_0^1 \varphi_t^2\mathrm{d} + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + \varepsilon_4\int_0^1 \psi_x^2\mathrm{d}x,$$

for $\varepsilon_4 > 0$. Note that

$$\int_0^1 \psi_x^2\mathrm{d}x = \int_0^1 (\psi_x - 3w_x + 3w_x)^2\mathrm{d}x \le 2\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + 18\int_0^1 w_x^2\mathrm{d}x.$$

Then estimate (4.9) is obtained.

Lemma 4.6 *Let (φ,ψ,w,θ) be the solution of* (2.1)-(2.2). *Then the functional*

$$F_5(t) = I_\rho\int_0^1 (3w-\psi)(3w-\psi)_t\mathrm{d}x$$

satisfies the estimate

$$F_5'(t) \le -\frac{D}{2}\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + c\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + c\int_0^1 \theta^2\mathrm{d}x. \tag{4.10}$$

Proof. Taking the derivative of $F_5(t)$ with respect to t, using $(2.1)_2$ and integrating by parts, we get

$$\begin{aligned} F_5'(t) = &- D\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x \\ &+ G\int_0^1 (\psi-\varphi_x)(3w-\psi)\mathrm{d}x + \delta\int_0^1 (3w-\psi)_x\theta\mathrm{d}x. \end{aligned}$$

Then, using Young's inequality, we arrive at (4.10).

Lemma 4.7 *Let (φ,ψ,w,θ) be the solution of* (2.1)-(2.2). *Then the functional*

$$F_6(t) = I_\rho\int_0^1 ww_t\mathrm{d}x$$

satisfies the estimate

$$F_6'(t) \le -\frac{2\gamma}{3}\int_0^1 w^2\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x + c\int_0^1 w_t^2\mathrm{d}x + c\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x. \tag{4.11}$$

Proof. By differentiating $F_6(t)$ with respect to t, using $(2.1)_3$ and integrating by parts, we obtain

$$F_6'(t) = I_\rho\int_0^1 w_t^2\mathrm{d}x - G\int_0^1 w(\psi-\varphi_x)\mathrm{d}x - \frac{4\gamma}{3}\int_0^1 w^2\mathrm{d}x - \frac{4\alpha}{3}\int_0^1 ww_t\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x.$$

We then use Young's inequality to obtain (4.11). This completes the proof.

Now we define the following Lyapunov functional

$$\mathscr{L}(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + F_5(t) + F_6(t). \tag{4.12}$$

where N, N_1, N_2, N_3 are positive constants to be selected later. Then we have the lemma as follows.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0067.v1

Lemma 4.8 *Let $(\varphi, \psi, w, \theta)$ be the solution of (2.1)-(2.2). For N large enough, there exists a positive c depending on N and ε such that*

$$(N-c)E(t) \le \mathscr{L}(t) \le (N+c)E(t), \tag{4.13}$$

for any $t \ge 0$.

Proof. Using Young's and Cauchy-Schwarz inequalities, we can easily obtain that

$$\begin{aligned} |\mathscr{L}(t) - NE(t)| \le{}& \alpha_1 \int_0^1 \varphi_t^2 \mathrm{d}x + \alpha_2 \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \alpha_3 \int_0^1 w_t^2 \mathrm{d}x + \alpha_4 \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ &+ \alpha_5 \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \alpha_6 \int_0^1 w^2 \mathrm{d}x + \alpha_7 \int_0^1 w_x^2 \mathrm{d}x + \alpha_8 \int_0^1 \theta^2 \mathrm{d}x \\ &+ \alpha_9 \int_0^\infty \mu(s) \|\eta_x(s)\|_2^2 \mathrm{d}s, \end{aligned} \tag{4.14}$$

where $\alpha_i (i = 1, 2, \cdots, 9)$ are positive constants. It from (2.6) and (4.14) that there exists a positive constant c such that

$$|\mathscr{L}(t) - NE(t)| \le cE(t),$$

which completes the proof.

Now, we are ready to prove the main result in this section.

Proof of Theorem 2.2. From (4.4), (4.6), (4.7), (4.9), (4.10) and (4.11), we can obtain

$$\begin{aligned} \mathscr{L}'(t) \le{}& -\rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho}{2} N_2 - N_1 \varepsilon_1 - cN_3 - I_\rho\right] \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &- (4\alpha N - cN_3 - c) \int_0^1 w_t^2 \mathrm{d}x - \left[\frac{G}{2} N_3 - N_2 \varepsilon_2 - c\left(1 + \frac{1}{\varepsilon_4}\right) - 2c\right] \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ &- \left(\frac{D}{2} - N_2 \varepsilon_2 - \varepsilon_4\right) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \mathrm{d}x \\ &- (D - \varepsilon_4) \int_0^1 w_x^2 \mathrm{d}x - \left[\frac{k}{2} N_1 - cN_2 \left(1 + \frac{1}{\varepsilon_2}\right) - c\right] \int_0^1 \theta^2 \mathrm{d}x \\ &+ \left[\frac{1}{\beta} N - cN_1 \left(1 + \frac{1}{\varepsilon_1}\right) - cN_2 - cN_3\right] \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2 \mathrm{d}s. \end{aligned} \tag{4.15}$$

At this point, we need to choose our constants very carefully. First, we choose

$$\varepsilon_1 = \frac{I_\rho N_2}{4N_1}, \ \varepsilon_2 = \min\left\{\frac{GN_3}{4N_2}, \frac{D}{8N_2}\right\}, \ \varepsilon_4 = \frac{D}{8},$$

so that

$$\begin{aligned} \mathscr{L}'(t) \le{}& -\rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho}{4} N_2 - cN_3 - I_\rho\right] \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x - (4\alpha N - cN_3 - c) \int_0^1 w_t^2 \mathrm{d}x \\ &- \left[\frac{G}{4} N_3 - \frac{8}{D} c - 3c\right] \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{D}{4} \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \mathrm{d}x \\ &- \frac{7D}{8} \int_0^1 w_x^2 \mathrm{d}x - \left[\frac{k}{2} N_1 - cN_2 \left(1 + \frac{1}{\varepsilon_2}\right) - c\right] \int_0^1 \theta^2 \mathrm{d}x \end{aligned}$$

$$+\left[\frac{1}{\beta}N - cN_1\left(1+\frac{N_1}{N_2}\right) - cN_2 - cN_3\right]\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2 \mathrm{d}s. \tag{4.16}$$

Then, we select N_3 large enough so that

$$\frac{G}{4}N_3 - \frac{8}{D}c - 3c > 0.$$

Next, we select N_2 large enough so that

$$\frac{I_\rho}{4}N_2 - cN_3 - I_\rho > 0.$$

Furthermore, we select N_1 large enough so that

$$\frac{k}{2}N_1 - cN_2\left(1+\frac{1}{\varepsilon_2}\right) - c > 0.$$

Finally, we select N large enough so that

$$4\alpha N - cN_3 - c > 0,\ \frac{1}{\beta}N - cN_1\left(1+\frac{N_1}{N_2}\right) - cN_2 - cN_3 > 0.$$

Using (2.6), we obtain that there exist positive constants M_1 and M_2 such that (4.16) becomes

$$\begin{aligned}\mathscr{L}'(t) &\leq -M_1E(t) + M_2\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2 \mathrm{d}s\\ &\leq -M_1E(t),\ \forall\, t \geq 0.\end{aligned}$$

From Lemma 4.8, we obtain

$$\mathscr{L}'(t) \leq -b\mathscr{L}(t),\ \forall\, t \geq 0, \tag{4.17}$$

where $b = \frac{M_1}{N+c}$. Then, a simple integration of (4.17) over $(0,t)$ yields

$$\mathscr{L}(t) \leq \mathscr{L}(0)e^{-bt},\ \forall\, t \geq 0. \tag{4.18}$$

At last, estimate (4.18) gives exponential stability result (2.7) when be combined with Lemma 4.8. This completes the proof.

5 Lack of exponential stability: proof of Theorem 2.3

Our result is achieved by using Gearhart-Herbst-Prüss-Huang theorem to dissipative systems (see Prüss [24] and Huang [13]).

Lemma 5.1 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supset \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

 doi:10.20944/preprints201801.0067.v1

Proof of Theorem 2.3. We will prove that there exists a sequence of imaginary number λ_μ and function $F_\mu \in \mathcal{H}$ with $\|F_\mu\|_\mathcal{H} \leq 1$ such that $\|(\lambda_\mu I - \mathcal{A})^{-1} F_\mu\|_\mathcal{H} = \|U_\mu\|_\mathcal{H} \to \infty$, where

$$\lambda_\mu U_\mu - \mathcal{A} U_\mu = F_\mu, \tag{5.1}$$

with $U_\mu = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^T$ not bounded. Rewriting spectral equation (5.1) in term of its components, we have for $\lambda_\mu = \lambda$

$$\begin{cases} \lambda v_1 - v_2 = g_1, \\ \rho\lambda v_2 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ \lambda v_3 - v_4 = g_3, \\ I_\rho \lambda v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx} v_3 - 3Gv_5 + \delta\partial_x v_7 = I_\rho g_4, \\ \lambda v_5 - v_6 = g_5, \\ I_\rho \lambda v_6 - G\partial_x v_1 - Gv_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 + \dfrac{4\alpha}{3} v_6 - D\partial_{xx} v_5 = I_\rho g_6, \\ k\lambda v_7 - \dfrac{1}{\beta} \displaystyle\int_0^\infty \mu(s)\partial_{xx} v_8(s) \mathrm{d}s + \delta\partial_x v_4 = k g_7, \\ \lambda v_8 + \partial_s v_8 - v_7 = g_8, \end{cases} \tag{5.2}$$

where $\lambda \in \mathbb{R}$ and $F = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)^T \in \mathcal{H}$. Take $g_1 = g_3 = g_5 = 0$, then the above system becomes

$$\begin{cases} \rho\lambda^2 v_1 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ I_\rho \lambda^2 v_3 + G\partial_x v_1 + Gv_3 - D\partial_{xx} v_3 - 3Gv_5 + \delta\partial_x v_7 = I_\rho g_4, \\ I_\rho \lambda^2 v_5 - G\partial_x v_1 - Gv_3 + \left(3G + \dfrac{4\gamma}{3} + \dfrac{4\alpha}{3}\lambda\right) v_5 - D\partial_{xx} v_5 = I_\rho g_6, \\ k\lambda v_7 - \dfrac{1}{\beta} \displaystyle\int_0^\infty \mu(s)\partial_{xx} v_8(s) \mathrm{d}s + \lambda\delta\partial_x v_3 = k g_7, \\ \lambda v_8 + \partial_s v_8 - v_7 = g_8. \end{cases} \tag{5.3}$$

Due to the boundary conditions in (2.2), we can suppose that

$$v_1 = A\cos\left(\frac{\mu\pi}{2}x\right), v_3 = B\sin\left(\frac{\mu\pi}{2}x\right), v_5 = C\sin\left(\frac{\mu\pi}{2}x\right), v_7 = E\cos\left(\frac{\mu\pi}{2}x\right), v_8 = \phi(s)\cos\left(\frac{\mu\pi}{2}x\right).$$

Choosing

$$g_2 = \frac{1}{\rho}\cos\left(\frac{\mu\pi}{2}x\right), \; g_4 = g_6 = g_7 = g_8 = 0,$$

 doi:10.20944/preprints201801.0067.v1

then we can obtain

$$\begin{cases} \left[\rho\lambda^2 + G\left(\frac{\mu\pi}{2}\right)^2\right] A - G\left(\frac{\mu\pi}{2}\right) B + 3G\left(\frac{\mu\pi}{2}\right) C = 1, \\ -G\left(\frac{\mu\pi}{2}\right) A + \left[I_\rho\lambda^2 + G + D\left(\frac{\mu\pi}{2}\right)^2\right] B - 3GC - \delta\left(\frac{\mu\pi}{2}\right) E = 0, \\ G\left(\frac{\mu\pi}{2}\right) A - GB + \left[I_\rho\lambda^2 + 3G + \frac{4\gamma}{3} + \frac{4\alpha}{3}\lambda + D\left(\frac{\mu\pi}{2}\right)^2\right] C = 0, \\ \lambda\delta\left(\frac{\mu\pi}{2}\right) B + k\lambda E + \frac{1}{\beta}\left(\frac{\mu\pi}{2}\right)^2 \int_0^\infty \mu(s)\phi(s)\mathrm{d}s = 0, \\ \phi'(s) + \lambda\phi(s) - E = 0. \end{cases} \tag{5.4}$$

In the above equaiton, we take $\lambda = \lambda_\mu := i\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)$ such that

$$\rho\lambda^2 + G\left(\frac{\mu\pi}{2}\right)^2 = 0.$$

Solving $(5.4)_5$, we get

$$\phi(s) = \frac{E}{\lambda}\left(1 - e^{-\lambda s}\right). \tag{5.5}$$

Then substituting (5.5) into $(5.4)_4$, we can get

$$E = \frac{\frac{G\delta}{\rho}\left(\frac{\mu\pi}{2}\right)}{\frac{g(0)}{\beta}\left[1 - \frac{kG}{\rho}\frac{\beta}{g(0)}\right] - \frac{1}{\beta}\int_0^\infty \mu(s)e^{-\lambda s}\mathrm{d}s} B.$$

The combination of $(5.4)_2$ and $(5.4)_3$ gives

$$I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 B + \left[\frac{4\gamma}{3} + \frac{4\alpha}{3}\lambda + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right] C - \delta\left(\frac{\mu\pi}{2}\right) E = 0. \tag{5.6}$$

Substituting E into (5.6), we get

$$C = -\frac{\Lambda_\mu}{\Gamma_\mu} B,$$

where

$$\Lambda_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 - \frac{\frac{G\delta^2}{\rho}\left(\frac{\mu\pi}{2}\right)^2}{\frac{g(0)}{\beta}\left[1 - \frac{kG}{\rho}\frac{\beta}{g(0)}\right] - \frac{1}{\beta}\int_0^\infty \mu(s)e^{-\lambda s}\mathrm{d}s},$$

$$\Gamma_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 + \frac{4\alpha}{3}\lambda + \frac{4\gamma}{3}.$$

Substituting C into $(5.4)_1$, we get

$$B = -\frac{\Gamma_\mu}{G\left(\frac{\mu\pi}{2}\right)(\Gamma_\mu + 3\Lambda_\mu)},$$

Similarly, substituting C into $(5.4)_3$, we get

$$A = \frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G\left(\frac{\mu\pi}{2}\right)\Gamma_\mu} B$$

$$= -\frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G^2(\Gamma_\mu + 3\Lambda_\mu)\left(\frac{\mu\pi}{2}\right)^2}.$$

At this point, we introduce the number

$$\gamma_g = 1 - \frac{kG}{\rho}\frac{\beta}{g(0)}$$

and consider separately two cases.

Case $\gamma_g = 0$. Let $\mu \to \infty$, we get

$$A \to -\frac{\beta\delta^2}{\rho G \int_0^\infty \mu(s)e^{-\lambda s}\mathrm{d}s},\ B \to 0,\ C \to 0.$$

Case $\gamma_g \neq 0$. Let $\mu \to \infty$, we get

$$A \to -\frac{I_\rho \chi_g \left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)}{G^2\left[\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\gamma_g + 3\chi_g\right]},\ B \to 0,\ C \to 0.$$

Thus,

$$\begin{aligned}\|U_\mu\|_{\mathcal{H}}^2 \geq G\int_0^1 (\psi - \varphi_x)^2\mathrm{d}x &= G\left[3C - B + \left(\frac{\mu\pi}{2}\right)A\right]^2 \int_0^1 \sin^2\left(\frac{\mu\pi}{2}x\right)\mathrm{d}x \\ &= \frac{1}{2}G\left[3C - B + \left(\frac{\mu\pi}{2}\right)A\right]^2 \to \infty,\ as\ \mu \to \infty.\end{aligned}$$

This implies that

$$\|U_\mu\|_{\mathcal{H}} \to \infty,\ as\ \mu \to \infty.$$

Therefore, there is no exponential stability. This completes the proof.

6 Conclusion and open problem

In this paper, we first prove the well-posedness for a laminated beam with Gurtin-Pipkin thermal law and structural damping, and then prove that the system is exponentially stable if and only if that stability number is equal to zero ($\chi_g = 0$). When the stability number is not zero ($\chi_g \neq 0$), the problem of whether it is possible to get the polynomial stability for system (2.1)-(2.2) is still an interesting **open problem**.

Acknowledgements

This work was supported by the National Natural Science Foundation of China [grant number 11771216], the Natural Science Foundation of Jiangsu Province [grant number BK20151523], the Six Talent Peaks Project in Jiangsu Province [grant number 2015-XCL-020] and the Qing Lan Project of Jiangsu Province.

References

[1] T. A. Apalara, Uniform stability of a laminated beam with structural damping and second sound, Z. Angew. Math. Phys. **68** (2017), no. 2, 68:41.

[2] J. Bajkowski et al., A thermoviscoelastic beam model for brakes, European J. Appl. Math. **15** (2004), no. 2, 181–202.

[3] F. Boulanouar and S. Drabla, General boundary stabilization result of memory-type thermoelasticity with second sound, Electron. J. Differential Equations **2014** (2014), no. 202, 18 pp.

[4] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011.

[5] A. D. S. Campelo, D. S. Almeida Júnior and M. L. Santos, Stability to the dissipative Reissner-Mindlin-Timoshenko acting on displacement equation, European J. Appl. Math. **27** (2016), no. 2, 157–193.

[6] X.-G. Cao, D.-Y. Liu and G.-Q. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, J. Dyn. Control Syst. **13** (2007), no. 3, 313–336.

[7] M. M. Cavalcanti et al., Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping, Z. Angew. Math. Phys. **65** (2014), no. 6, 1189–1206.

[8] M. M. Chen, W. J. Liu and W. C. Zhou, Existence and general stabilization of the Timoshenko system of thermo-viscoelasticity of type III with frictional damping and delay terms, Adv. Nonlinear Anal., in press. doi:10.1515/anona-2016-0085

[9] F. Dell'Oro and V. Pata, On the stability of Timoshenko systems with Gurtin-Pipkin thermal law, J. Differential Equations **257** (2014), no. 2, 523–548.

[10] F. Dell'Oro, Asymptotic stability of thermoelastic systems of Bresse type, J. Differential Equations **258** (2015), no. 11, 3902–3927.

[11] C. Giorgi, V. Pata and A. Marzocchi, Asymptotic behavior of a semilinear problem in heat conduction with memory, NoDEA Nonlinear Differential Equations Appl. **5** (1998), no. 3, 333–354.

[12] S. W. Hansen and R. Spies, Structural damping in a laminated beams due to interfacial slip, J. Sound Vibration. **204** (1997), no. 2, 183–202.

[13] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations **1** (1985), no. 1, 43–56.

[14] A. A. Keddi, T. A. Apalaras and S. A. Messaoudi, Exponential and polynomial decay in a thermoelastic-Bresse system with second sound, Appl. Math. Optim., in press. DOI: 10.1007/s00245-016-9376-y

[15] M. Khader and B. Said-Houari, On the decay rate of solutions of the Bresse system with Gurtin-Pipkin thermal law, Asymptot. Anal. **103** (2017), no. 1-2, 1–32.

[16] V. Komornik, Exact controllability and stabilization, RAM: Research in Applied Mathematics, Masson, Paris, 1994.

[17] W. J. Liu, K. W. Chen and J. Yu, Existence and general decay for the full von Kármán beam with a thermo-viscoelastic damping, frictional dampings and a delay term, IMA J. Math. Control Inform. **34** (2017), no. 2, 521–542.

[18] W. J. Liu, K. W. Chen and J. Yu, Asymptotic stability for a non-autonomous full von Kármán beam with thermo-viscoelastic damping, Appl. Anal., in press. doi: 10.1080/00036811.2016.1268688

[19] W. J. Liu and W. F. Zhao, Stabilization of a thermoelastic laminated beam with past history, Appl. Math. Optim., in press. doi: 10.1007/s00245-017-9460-y

[20] A. Lo and N. Tatar, Stabilization of laminated beams with interfacial slip, Electron. J. Differential Equations **2015**, No. 129, 14 pp.

[21] A. Lo and N. Tatar, Uniform stability of a laminated beam with structural memory, Qual. Theory Dyn. Syst. **15** (2016), no. 2, 517–540.

[22] S. A. Messaoudi and A. Fareh, Energy decay in a Timoshenko-type system of thermoelasticity of type III with different wave-propagation speeds, Arab. J. Math. (Springer) **2** (2013), no. 2, 199–207.

[23] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, 44, Springer, New York, 1983.

[24] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. **284** (1984), no. 2, 847–857.

[25] N.-E. Tatar, Stabilization of a laminated beam with interfacial slip by boundary controls, Bound. Value Probl. **2015**, 2015:169, 11 pp.

[26] J.-M. Wang, G.-Q. Xu and S.-P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, SIAM J. Control Optim. **44** (2005), no. 5, 1575–1597.

[27] Y. Qin, X.-G. Yang and Z. Ma, Global existence of solutions for the thermoelastic Bresse system, Commun. Pure Appl. Anal. **13** (2014), no. 4, 1395–1406.

[28] F. Tahamtani and A. Peyravi, Asymptotic behavior and blow-up of solution for a nonlinear viscoelastic wave equation with boundary dissipation, Taiwanese J. Math. **17** (2013), no. 6, 1921–1943.

[29] Q. Zhang, Stability analysis of an interactive system of wave equation and heat equation with memory, Z. Angew. Math. Phys. **65** (2014), no. 5, 905–923.