The results on vertex domination in Fuzzy graphs

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Abstract

We do fuzzification the concept of domination in crisp graph by using membership values of nodes, α-strong and arcs. In this paper, we introduce a new variation on the domination theme which we call vertex domination. We determine the vertex domination number γ_v for several classes of fuzzy graphs, specially complete fuzzy graph and complete bipartite fuzzy graphs. The bounds is obtained for the vertex domination number of fuzzy graphs. Also the relationship between M-strong arcs and α-strong is obtained. In fuzzy graphs, monotone decreasing property and monotone increasing property is introduced. We prove the vizing's conjecture is monotone decreasing fuzzy graph property for vertex domination. we prove also the Grarier-Khelladi's conjecture is monotone decreasing fuzzy graph property for it. We obtain Nordhaus-Gaddum (NG) type results for these parameters. The relationship between several classes of operations on fuzzy graphs with the vertex domination number of them is studied.

Keywords: Fuzzy graph, α-strong arcs, Weight of nodes, vertex domination
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1   Introduction

L.A. Zadeh introduced the concept of a fuzzy subset of a set as a way for representing uncertainty. Zadeh's ideas stirred the interest of researchers worldwide. His ideas have been applied to a wide range of scientific areas. Theoretical mathematics has also been touched by the notion of a fuzzy subset. In 1965, Zadeh published his seminal paper "fuzzy sets" [77] which described fuzzy set theory and consequently fuzzy logic. The purpose of Zadeh's paper was to develop a theory which could deal with ambiguity and imprecision of certain classes or sets in Human thinking, particularly in the domains of pattern recognition, communication of information, and observation. This theory proposed making the grade of membership of an element in a subset of a universal set a value in the closed interval $[0, 1]$ of real numbers. Zadeh's idea have found applications in computer science, artificial intelligence, decision analysis, information science, system science, control engineering, expert systems, pattern recognition, management science, operations research, and robotics. Theoretical mathematics has also been touched by fuzzy set theory. In the classical set theory introduced by Cantor, values of elements in a set are either 0 or 1. That is for any element, there are only two possibilities: the element is the set or it is not. Therefore, Cantor set theory cannot handle data with ambiguity and uncertainty. The ideas of fuzzy set theory have been introduced into topology, abstract algebra, geometry, graph theory, and analysis. Analytical

representation of physical phenomena can be fruitful as models of reality, but are ²⁰
sometimes difficult to understand because they do not explain much by themselves, and ²¹
may remain unclear to the non-specialist. In other words, Zadeh proposed fuzzy theory ²²
and introduced fuzzy set theory which can be considered as the phenomenon of ²³
ambiguity across all systems displaying this property and its consequences. ²⁴

Graph theory is one of the branches of modern mathematics having experienced a ²⁵
most impressive development in recent years. The origin of graph theory can be traced ²⁶
back to Euler's work on the Konigsberg bridge problem (1735) which subsequently led ²⁷
to the concept of an Eulerian graph. The first text book on graph theory was written by ²⁸
D'enesKonig and published in 1936. A later text book by Frank Harary published in ²⁹
1968, was enormously popular and enabled mathematicians, chemists, electrical ³⁰
engineers and social scientists to have common platform to dialogue with each other. ³¹
Graphs are represented graphically by taking a set of points on the plane and it is ³²
desired to find some structure among the points in the form of edges containing a subset ³³
of the pair of points. Graph theory plays a vital role as far as application side is ³⁴
concerned. Graph theory is intimately related to many branches of mathematics ³⁵
including group theory, matrix theory, numerical analysis, probability, topology and ³⁶
combinatorics because of its diagrammatic representation and its intuitive and aesthetic ³⁷
appeal. ³⁸

One of the most interesting problems in graph theory is that of Domination Theory. ³⁹
The earliest ideas of dominating sets are found in the classical problems of covering chess ⁴⁰
board with minimum number of chess pieces. Nowadays domination theory ranks top ⁴¹
among the most prominent areas of research in graph theory and combinatorics. The ⁴²
concept of domination in graphs, with its many variations, is now well studied in graph ⁴³
theory. The book by Chartrand and Lesniak [16] includes a chapter on domination. For ⁴⁴
a more thorough study of domination in graphs, see Haynes et al. [24]. The current list ⁴⁵
of papers on domination in [24] has over 1200 entries.The theory of domination is ⁴⁶
formalized by Clauge Berge in his book "Theory of graphs and its application" (1962). ⁴⁷
Berge mentions the strategies of keeping a number of locations under surveillance, by a ⁴⁸
set of radar station. Oystein Ore was a first person to use the term domination number ⁴⁹
in his book on Graph Theory. The theory of domination has been the nucleus of ⁵⁰
research activity in graph theory in recent times. The fastest growing area within graph ⁵¹
theory is a study of domination and related subset problems such independence, ⁵²
covering, matching, decomposition and labelling. Domination boasts a host of ⁵³
applications to social network theory, land surveying, game theory, interconnection ⁵⁴
network, parallel computing and image processing and so on. Today, this theory gained ⁵⁵
popularity and remains as a major area of research due to the contributions of ⁵⁶
O.Ore [50], C.Berge [6], E.J.Cockayne [19], S.T.Hedetniemi [24], T.W.Haynes [24], ⁵⁷
R.C.Laskar [25], P.J.Slater [66], V.R.Kulli [32], E.Sampathkumar [62], S.Arumugam [4]. ⁵⁸

Fuzzy graph theory has numerous applications in various fields like clustering ⁵⁹
analysis, database theory, network analysis, information theory, etc. [44]. Fuzzy models ⁶⁰
can be used in problems handling uncertainty to get more accurate and precise ⁶¹
solutions [74–76]. As in graphs, connectivity concepts play a key role in applications ⁶²
related with fuzzy graphs [44,70]. The fuzzy definition of fuzzy graphs was proposed by ⁶³
Kaufmann [28], from the fuzzy relations introduced by Zadeh. Although Rosenfeld ⁶⁴
introduced another elaborated definition, including fuzzy vertex and fuzzy edges. Fuzzy ⁶⁵
graphs were introduced by Rosenfeld [56] and Yeh and Bang [73] independently in 1975. ⁶⁶
Rosenfeld in his paper "Fuzzy Graphs" presented the basic structural and connectivity ⁶⁷
concepts while Yeh and Bang introduced different connectivity parameters of a fuzzy ⁶⁸
graph and discussed their applications in the paper titled "Fuzzy relations, Fuzzy ⁶⁹
graphs and their applications to clustering analysis" [73]. Rosenfeld considered fuzzy ⁷⁰
relations on fuzzy sets and developed the structure of fuzzy graphs, obtaining analogues ⁷¹

of several graph theoretical concepts. He introduced and examined such concepts as paths, connectedness and clusters, bridges, cut vertices, forests and trees. Fuzzy graphs introduced by Rosenfeld are finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the difference between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems and AI. After the pioneering work of Rosenfeld and Yeh and Bang in 1975, when some basic fuzzy graph theoretic concepts and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts. This include fuzzy trees [21,67], fuzzy line graphs [42], operations on fuzzy graphs [43], automorphism of fuzzy graphs [9,11], fuzzy interval graphs [20], cycles and cocycles of fuzzy graphs [45], and metric aspects in fuzzy graphs [65]. Bhutani and Rosenfeld have introduced the concept of strong arcs [12]. Different parameters like sum distance in fuzzy graphs and chromatic number of fuzzy graphs were discussed in [29,69]. The work on fuzzy graphs was also done by Akram, Samanta, Nayeem, Pramanik, Rashmanlou and Pal [1–3, 48, 51–55, 57–61]. P.Bhattacharya [10] discussed some properties of fuzzy graphs and introduced the notion of eccentricity and centre in fuzzy graphs. K.R.Bhutani [13] introduced the concept of complete fuzzy graphs and concluded that a complete fuzzy graph has no cut nodes. Xu [72] applied connectivity parameters of fuzzy graphs to problems in chemical structures.

The concept of domination in fuzzy graphs was investigated by A.Somasundaram and S.Somasundaram. A.Somasundaram presented the concepts of independent domination, total domination, connected domination and domination in cartesian products and composition of fuzzy graphs [63]. Somasundaram and Somasundaram discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graph [63,64]. Nagoorgani and Chandrasekharan defined domination in fuzzy graphs using strong arcs [47]. Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs [36,37]. A. Selvam Avadayappan, G. Mahadevan, A. Mydeenbibi, T.A. Subramanian, A. Nagarajan, A. Rajeswari have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs.

Motivated by the notion of dominating sets and their applicability, we focused on introducing some dominating parameters in fuzzy graph theory. For fuzzification of the following problems, types of nodes (based on advantages) and types of connection with nodes can be assigned by different values. So the question is based on based on values on nodes and ratio of total of values of adjacent α-strong connections to total of values of adjacent connections?

Chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. Harary et al. [5] explained an interesting application in voting situations using the concept of domination. A number of strategic locations are to be kept under observations. One of the important areas of applications of domination is communication network, where a dominating set represents a set of cities which, acting as transmitting stations, can transmit messages to every city in the network. Another area of application of domination is voting situations. Suppose the commander of the Army Postal services plans to set up a few post offices in an important region with minimum number of post offices to control the whole region. Now-a-day almost all schools operate school buses for transporting children to and from schools. Among many points, three important points to be noted are 1. The running time of a bus between school and its terminus. 2. Maximum number of students in a

bus at any one time and 3. The maximum distance a student has to walk to board a   124
school bus. Consider a computer network modeled by a 4-cube. The vertices of the   125
4-cube represents computers and edges represent direct communication link between two   126
computers. So, in this model we have 16 computers or processers to which it is directly   127
connected. The problem is to collect information from all processors and we like to do it   128
relatively often and relatively fast. So we identify a small set of processors called   129
collecting processors and ask each processor to send its information to one of a small set   130
of collecting processors. We assume that at most a one-unit delay between the time a   131
processor sends its information and time it arrives at a nearest collector is allowed. So,   132
we have to find an dominating set among the set of a processors. Consider the problem   133
of locating a single fire station, police station or a similar such service facility to serve   134
the communities. Also, we would like to locate such a service facility in one of these   135
communities and not at an arbitrary point along the road, due to some reasons. Let P_n 136
be a set of points in general position on the plane. The unit distance graph $UDG(P_n)$ 137
associated to P_n is a graph whose vertex set consists of the elements of P_n, two of which   138
are connected if they are at distance at most one. Unit distance graphs are used to   139
model various types of wireless networks, including cellular networks, sensor networks,   140
ad-hoc networks and others in which the nodes represent broadcast stations with a   141
uniform broadcast range we shall refer to networks that can be modeled using unit   142
distance graphs as unit distance wireless networks, abbreviated as UDW networks.   143

We first briefly illustrate our opinion. The rest of this paper is organized as follows.   144
In Section 2, we lay down the preliminary results which recall some basic concepts of   145
fuzzy graph, path, cycle, connectedness, complete fuzzy graph, order, size, complement,   146
types of arcs consists of α-strong, β-strong,δ-strong and M-strong, bipartite fuzzy   147
graph, complete bipartite fuzzy graph, star fuzzy graph, be isolated, domatic partition,   148
Vizing's conjecture, Gravier and Khelladi's conjecture, some operations on fuzzy graphs   149
consists of cartesian product, join and union, Nordhaus-Gaddum (NG) results and   150
finally we conclude this section with Remark (2.1) and In Section 3, The α-strong   151
domination number of a fuzzy graph is defined in a classic way, Definition (3.1), (3.3),   152
(3.4). We determine vertex domination number for several classes of fuzzy graphs   153
consists of complete fuzzy graph, Proposition (3.10), empty fuzzy graph, Proposition   154
(3.11), star fuzzy graph, Proposition (3.13), complete bipartite fuzzy graph, Proposition   155
(3.14). We give an upper bound for the vertex domination number of fuzzy graphs,   156
Proposition (3.15). For any fuzzy graph the Nordhaus-Gaddum(NG)'result holds,   157
Theorem (3.16). Finding domatic partition of size two in fuzzy graph G of order $n \geq 2$ 158
is studied, Theorem (3.19). We improve upper bound for the vertex domination number   159
of fuzzy graphs without isolated nodes, Theorem (3.20). We also improve   160
Nordhaus-Gaddum(NG)'result for fuzzy graphs without isolated nodes, Corollary (3.21).   161
We give the relationship between M-strong arcs and α-strong arcs, Corollary (3.24). We   162
give a necessary and sufficient condition for vertex domination which is half of order, In   163
fact fuzzy graphs with vertex domination which is half of order is characterized in the   164
special case, Theorem (3.26). The vertex domination of union of two fuzzy graphs is   165
studied, Proposition (3.27). Also the vertex domination of union of fuzzy graphs Family   166
is discussed, Corollary (3.28). The concepts of both monotone increasing fuzzy graph   167
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(3.31), are introduced. The result in relation with vizing's conjecture by using α-strong   169
arc and monotone decreasing fuzzy graph property is determined, Theorem (3.34).   170
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Family is discussed, Corollary (3.37). The result in relation with Gravier and Khelladi's   174
conjecture by using α-strong arc and monotone decreasing fuzzy graph property is   175

determined, Theorem (3.38). We conclude this section with Some result in relation with Gravier and Khelladi's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.39). In Section 4, We give 9 practical applications in relation with these concepts.

2　Preliminary

We provide some basic background for the paper in this section.

Some of the books discussing these various themes are Bezdek and Pal [7], Lootsma [35], Morderson and Malik [40], Comelius . T. Leondes [34] and Klir and Bo Yuan [31]. We shall now list below some basic definitions and results from [41], [56]. Also Background on fuzzy graphs and the following definitions can be found in them.

we lay down the preliminary results which recall some basic concepts of fuzzy graph, path, cycle, connectedness, complete fuzzy graph, order, size, complement, types of arcs consists of α-strong, β-strong,δ-strong and M-strong, bipartite fuzzy graph, complete bipartite fuzzy graph, star fuzzy graph, be isolated, domatic partition, Vizing's conjecture, Gravier and Khelladi's conjecture, some operations on fuzzy graphs consists of cartesian product, join and union, Nordhaus-Gaddum (NG) results and finally we conclude this section with Remark (2.1)

We recall that a fuzzy subset of a set S is a function of S into the closed interval [0, 1], [77]. A fuzzy graph is denoted by $G = (V, \sigma, \mu)$ such that $\mu(\{x, y\}) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$ where V is a vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on V. We call σ the fuzzy node set (or fuzzy vertex set) of G and μ the fuzzy arc set (or fuzzy edge set) of G, respectively. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(\{x, x\}) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(\{x, y\}) = \mu(\{y, x\})$, for all $x, y \in V$). In all the examples σ and μ is chosen suitably. In any fuzzy graph, the underlying crisp graph is denoted by $G^* = (V, E)$ where V and E are domain of σ and μ, respectively. This definition of fuzzy graph is essentially the same as the one appearing in [56]. The fuzzy graph $H = (\tau, \nu)$ is called a partial fuzzy subgraph of $G = (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. Similarly, the fuzzy graph $H = (\tau, \nu)$ is called a fuzzy subgraph of $G = (V, \sigma, \mu)$ induced by P if $P \subseteq V, \tau(x) = \sigma(x)$ for all $x \in P$ and $\nu(\{x, y\}) = \mu(\{x, y\})$ for all $x, y \in P$. For the sake of simplicity, we sometimes call H a fuzzy subgraph of G. We say that the partial fuzzy subgraph (τ, ν) spans the fuzzy graph (σ, μ) if $\sigma = \tau$. In this case, we call (τ, ν) a spanning fuzzy subgraph of (σ, μ).

For the sake of simplicity, we sometimes write xy instead of $\{x, y\}$

A path P of length n is a sequence of distinct nodes $u_0, u_1, \cdots, u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \cdots, n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

A fuzzy graph $G = (V, \sigma, \mu)$ is connected if for every x, y in V, $CONN_G(x, y) > 0$.

An arc uv of a fuzzy graph is called an M-strong arc if $\mu(uv) = \sigma(u) \wedge \sigma(v)$. In order to avoid confusion with the notion of strong arcs introduced by Bhutani and Rosenfeld [15], we shall call strong in the sense of Mordeson as M-strong [46].

A fuzzy graph G is said complete if $\mu(uv) = \sigma(x) \wedge \sigma(y)$. for all $u, v \in V$.

The order p and size q of a fuzzy graph $G = (V, \sigma, \mu)$ are defined $p = \Sigma_{x \in V} \sigma(x)$ and $q = \Sigma_{x, y \in V} \mu(xy)$.

The complement of a fuzzy graph G, denoted by $\bar{G}$ is defined to $\bar{G} = (V, \sigma, \bar{\mu})$ where $\bar{\mu}(xy) = \sigma(x) \wedge \sigma(y) - \mu(xy)$ for all $x, y \in V$.

An arc of a fuzzy graph is called α-strong if its weights is greater than strength of   226
connectedness of its end nodes when it is deleted. Depending on $CONN_G(x,y)$ of an   227
arc xy in a fuzzy graph G, Mathew and Sunitha [68] defined three types of arcs. Note   228
that $CONN_{G-xy}(x,y)$ is the strength of connectedness between x and y in the fuzzy   229
graph obtained from G by deleting the arc xy. An arc xy in G is α-strong if   230
$\mu(xy) > CONN_{G-xy}(x,y)$. An arc xy in G is β-strong if $\mu(xy) = CONN_{G-xy}(x,y)$.   231
An arc xy in G is δ-arc if $\mu(xy) < CONN_{G-xy}(x,y)$.   232

A fuzzy graph G is said bipartite if the vertex set V can be partitioned into two   233
nonempty sets V_1 and V_2 such that $\mu(v_1 v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Moreover,   234
if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a complete bipartite   235
graph and is denoted by $K\sigma_1, \sigma_2$, where σ_1 and σ_2 are respectively the restrictions of σ 236
to V_1 and V_2. In this case, If $|V_1| = 1$ or $|V_2| = 1$ then the complete bipartite graph is   237
said a star fuzzy graph which is denoted by $K_{1,\sigma}$.   238

A node u is said isolated if $\mu(uv) = 0$ for all $v \neq u$.   239

A domatic partition is a partition of the vertices of a graph into disjoint dominating   240
sets. The maximum number of disjoint dominating sets in a domatic partition of a   241
graph is called its domatic number.   242

In graph theory, Vizing's conjecture [17] concerns a relation between the domination
number and the cartesian product of graphs. This conjecture was first stated by Vadim
G. Vizing (1968), and states that, if $\gamma(G)$ denotes the minimum number of vertices in a
dominating set for G, then

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

Vizing's conjecture from 1968 asserts that the domination number of the. Cartesian   243
product of two graphs is at least as large as the product of their domination numbers.   244

Gravier and Khelladi (1995) conjectured a similar bound for the domination number   245
of the tensor product of graphs; however, a counterexample was found by Klavžar   246
Zmazek (1996) [30]. Since Vizing proposed his conjecture, many mathematicians have   247
worked on it, with partial results described below. For a more detailed overview of these   248
results, see Brešar et al. (2012) [8]   249

The cartesian product $G = G_1 \times G_2$ [39] of two fuzzy graphs   250
$G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is defined as a fuzzy graph $G = (V \times V, \sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ 251
where $E = \{\{uu_2, uv_2\} | u \in V_1, u_2 v_2 \in E_2\} \cup \{\{u_1 w, v_1 w\} | u_1 v_1 \in E_1, w \in V_2\}$. Fuzzy   252
sets $\sigma_1 \times \sigma_2$ and $\mu_1 \times \mu_2$ are defined as $(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$ and   253
$\forall u \in V_1, \forall u_2 v_2 \in E_2, (\mu_1 \times \mu_2)(\{uu_2, uv_2\}) = \sigma_1(u) \wedge \mu_2(u_2 v_2)$ and   254
$\forall u_1 v_1 \in E_1, \forall w \in V_2, (\mu_1 \times \mu_2)(\{u_1 w, v_w\}) = \mu_1(u_1 v_1) \wedge \sigma_2(w)$.   255

The union $G = G_1 \cup G_2$ [39] of two fuzzy graphs $G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is defined   256
as a fuzzy graph $G = (V_1 \cup V_2, \sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ where $E = E_1 \cup E_2$. Fuzzy sets $\sigma_1 \cup \sigma_2$ 257
and $\mu_1 \cup \mu_2$ are defined as $(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u)$ if $u \in V_1 - V_2, (\sigma_1 \cup \sigma_2)(u) = \sigma_2(u)$ if   258
$u \in V_2 - V_1$, and $(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u) \vee \sigma_2(u)$ if $u \in V_1 \cap V_2$. Also   259
$(\mu_1 \cup \mu_2)(uv) = \mu_1(uv)$ if $uv \in E_1 - E_2$ and $(\mu_1 \cup \mu_2)(uv) = \mu_2(uv)$ if $uv \in E_2 - E_1$,   260
and $(\mu_1 \cup \mu_2)(uv) = \mu_1(uv) \vee \mu_2(uv)$ if $uv \in E_1 \cap E_2$.   261

Let $G = G_1 + G_2$ denote the join [39] of two fuzzy graphs $G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is   262
defined as a fuzzy graph $G = (V_1 \cup V_2, \sigma_1 + \sigma_2, \mu_1 + \mu_2)$ where $E = E_1 \cup E_2 \cup E^{'}$ and   263
$E^{'}$ is the set of all edges joining vertices of V_1 with the vertices of V_2, and we assume   264
that $V_1 \cap V_2 = \emptyset$. Fuzzy sets $\sigma_1 + \sigma_2$ and $\mu_1 + \mu_2$ are defined as   265
$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u)$ and $\forall u \in V_1 \cup V_2; (\mu_1 + \mu_2)(uv) = (\mu_1 \cup \mu_2)(uv)$ if   266
$uv \in E_1 \cup E_2$ and $(\mu_1 + \mu_2)(uv) = \sigma_1(u) \wedge \sigma_2(v)$ if $uv \in E^{'}$.   267

The classical paper [49] of Nordhaus and Gaddum established the inequalities for the   268
chromatic numbers of a graph $G = (V, E)$ and its complement $\bar{G}$. We are concerned   269
with analogous inequalities involving domination parameters in graphs. We begin with a   270
brief overview of Nordhaus-Gaddum (NG) inequalities for several domination-related   271
parameters. For each generic invariant μ of a graph G, let $\mu = \mu(G)$ and $\bar{\mu} = \mu(\bar{G})$.   272

Inequalities on $\mu + \bar{\mu}$ and $\mu.\bar{\mu}$ exist in the literature for only a few of the many domination-related parameters and most of these results are of the additive form. In 1972 Jaeger and Payan [26] published the first NG results involving domination. Cockayne and Hedetniemi [18] sharpened the upper bound for the sum. Laskar and Peters [33] improved this bound for the case when both G and $\bar{G}$ are connected. A much improved bound was established for the case when neither G nor $\bar{G}$ has isolated nodes by Bollob*á*s and Cockayne [14] and by Joseph and Arumugam [27] independently.

Remark 2.1. For the sake of simplicity, we do sometimes

- writing xy instead of $\{x, y\}$.

- calling x both vertex and node.

- calling xy both edge and arc.

- writing Cartesian product both $\square$ and $\times$.

- saying $\sigma(x)$ and $\mu(xy)$ with different literature, e.g. value, weight, membership value and etc.

3   Main Results

In this section, we provide the main results. We first briefly illustrate our opinion.

The terms "dominating set", and "domination number" of a graph G = (V, E) were first defined by O. Ore in 1962. A subset $A \subseteq V$ is a dominating set for G if each element of V is either in A, or is adjacent to an element of A. The domination number $\gamma(G)$, which is the most commonly used domination number, is the minimum cardinality among all dominating sets of G.

More than 1200 papers already published on domination in graphs. Without a doubt, the literature on this subject is growing rapidly, and a considerable amount of work has been dedicated to find different bounds for the domination numbers of graphs. In Analogous to them in fuzzy graphs are existed many variations of concepts of dominations as mentioned some results on introduction. a few researchers defined, sometimes redefined, and studied other domination variations: such as connected domination, strong domination, total domination, (1,2)-vertex domination, 2-domination, domination and etc. However, from practical point of view, it was necessary to define other types of dominations. Most of these new variations required the dominating set to have additional properties such as: being as independent set, inducing a connected subgraph, or inducing a clique. These properties were reflected in their names as an adjective: independent domination, connected domination, and clique domination, respectively.

The α-strong domination number of a fuzzy graph is defined in a classic way, Definition (3.1), (3.3), (3.4).

Definition 3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph on V. Let $x, y \in V$. We say that x **dominates** y in G as α-strong if the arc $\{x, y\}$ is α-strong.

Example 3.2. By attention to fuzzy graph In Figure (1), the arcs v_2v_5, v_2v_4, v_3v_4 and v_1v_3 are α-strong and the arcs v_1v_4, v_1v_2 and v_4v_5 are not α-strong.

Definition 3.3. A subset S of V is called a α-**strong dominating set** in G if for every $v \notin S$, there exists $u \in S$ such that u dominates v.

Definition 3.4. Let S be the set of all α-strong dominating sets in G, the **vertex domination number** of G is defined as $\min_{D \in S}[\Sigma_{u \in D}(\sigma(u) + \frac{d_s(u)}{d(u)})]$ and it is denoted

by $\gamma_{\mathbf{v}}(\mathbf{G})$. If $d(u) = 0$, then we consider $\frac{d_s(u)}{d(u)}$ equal with 0. The α-strong dominating set that is correspond to $\gamma_v(G)$ is called by **vertex dominating set**. We also say $\Sigma_{u \in D}(\sigma(u) + \frac{d_s(u)}{d(u)})$, **vertex weight** of D, for every $D \in S$ and it is denoted by $\mathbf{w_v(D)}$.

Example 3.5. By attention to fuzzy graph In Figure (1), the set $\{v_2, v_3\}$ is the α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35$. So $\gamma_v(G) = 3.35$.

Theorem 3.6. *[38] If G is a complete fuzzy graph, then all arcs are strong.*

Theorem 3.7. *[38] If G is a complete bipartite fuzzy graph, then all arcs are strong.*

Remark 3.8. If G is a complete fuzzy graph, then all arcs areα-strong.

Remark 3.9. If G is a complete bipartite fuzzy graph, then all arcs are α-strong.

It is well known and generally accepted that the problem of determining the domination number of an arbitrary graph is a difficult one. Because of this, researchers have turned their attention to the study of classes of graphs for which the domination problem can be solved in polynomial time.

We determine vertex domination number for several classes of fuzzy graphs consists of complete fuzzy graph, Proposition (3.10), empty fuzzy graph, Proposition (3.11), star fuzzy graph, Proposition (3.13), complete bipartite fuzzy graph, Proposition (3.14).

Proposition 3.10 (Complete fuzzy graph)**.** *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then $\gamma_v(G) = \min_{u \in V}(\sigma(u)) + 1$.*

Proof. Since G is a complete fuzzy graph, all arcs are α-strong by Remark (3.8) and each node is incident to all other nodes. Hence $D = \{u\}$ is a α-strong dominating set and $d_s(u) = d(u)$ for each $u \in V$ Hence the result follows. $\square$

Proposition 3.11 (Empty fuzzy graph)**.** *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then $\gamma_v(G) = p$, if G be edgeless, i.e.$G = \bar{K}_n$.*

Proof. Since G is edgeless, Hence V is only α-strong dominating set in G and all arcs are not α-strong. so we have $\gamma_v(G) = \min_{D \in S}[\Sigma_{u \in D}\sigma(u)] = \Sigma_{u \in v}\sigma(u) = p$ by Definition (3.4). so we can write $\gamma_v(\bar{K}_n) = p$ by our notations. $\square$

It is interesting to note the converse of Proposition (3.11) that does not hold.

Example 3.12. We show the converse of Proposition (3.11) does not hold. For this purpose, Let $V = \{v_1, v_2, v_3, v_4, v_5\}$. We define σ on V by $\sigma : V \to [0, 1]$ such that
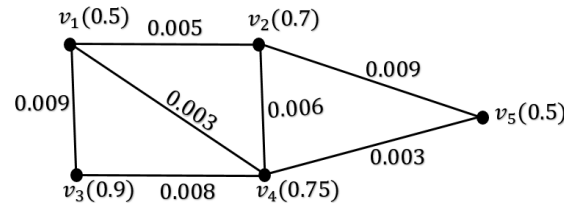

Figure 1. **Vertex domination**

$$\sigma(v_1) = 0.5, \sigma(v_2) = 0.7, \sigma(v_3) = 0.9, \sigma(v_4) = 0.75, \sigma(v_5) = 0.5$$

Now, The function $\mu : V \times V \to [0, 1]$ is defined by

$$\mu(v_1v_2) = 0.005, \mu(v_1v_4) = 0.003, \mu(v_1v_3) = 0.009, \mu(v_2v_4) = 0.006, \mu(v_2v_5) = 0.009,$$

$\mu(v_3v_4) = 0.008, \mu(v_4v_5) = 0.003$ such that $\forall u, v \in V, \mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. Finally, Let V, σ, and μ be the vertices, value of vertices and value of edges respectively. In other words, By attention to fuzzy graph In Figure (1), the arcs v_2v_5, v_2v_4, v_3v_4 and v_1v_3 are α-strong and the arcs v_1v_4, v_1v_2 and v_4v_5 are not α-strong. So the set $\{v_2, v_3\}$ is the α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35 = \Sigma_{u \in v}\sigma(u) = p$. So $G \neq \bar{K}_5$ but $\gamma_v(G) = p$.

Proposition 3.13 (Star fuzzy graph)**.** *Let G be a star fuzzy graph. Then $G = K_{1,\sigma}$ and $\gamma_v(K_{1,\sigma}) = \sigma(u) + 1$ where u is center of G.*   353
                                                                                          354

Proof. Let G be the star fuzzy graph with $V = \{u, v_1, v_2, \cdots, v_n\}$ such that u and v_i 355
are center and leaves of G, for $1 \leq i \leq n$ respectively. So $G^* = K_{1,n}^*$ is underlying crisp   356
graph of G. $\{u\}$ is vertex dominating set in G and all arcs are α-strong by Remark (3.9)   357
and due to G is bipartite fuzzy graph. Hence the result follows.   □   358

Proposition 3.14 (Bipartite fuzzy graph)**.** *Let G be the bipartite fuzzy graph which is*   359
not star fuzzy graph. Then $G = K_{\sigma_1, \sigma_2}$ and $\gamma_v(K_{\sigma_1, \sigma_2}) = \min_{u \in V_1, v \in V_2}(\sigma(u) + \sigma(v)) + 2$.   360

Proof. Let $G \neq K_{1,\sigma}$ be bipartite fuzzy graph. Then both of V_1 and V_2 include more   361
than one vertex. In K_{σ_1, σ_2} , all arcs are α-strong by Remark (3.9). Also each node in V_1 362
is dominated as α-strong with all nodes in V_2 and conversely. Hence in K_{σ_1, σ_2}, the   363
α-strong dominating sets are V_1 and V_2 and any set containing 2 nodes, one in V_1 and   364
other in V_2. Hence $\gamma_v(K_{\sigma_1, \sigma_2}) = \min_{u \in V_1, v \in V_2}(\sigma(u) + \sigma(v)) + 2$. So the theorem is   365
proved.   □   366

We give an upper bound for the vertex domination number of fuzzy graphs,   367
Proposition (3.15).   368

Proposition 3.15. *For any fuzzy graph $G = (V, \sigma, \mu)$, We have $\gamma_v \leq p$.*   369

Proof. $\gamma_v(\bar{K}_n) = p$ by Theorem (3.11). So the result follows.   □   370

For the vertex domination number γ_v the following theorem gives a   371
Nordhaus-Gaddum type result.   372
For any fuzzy graph the Nordhaus-Gaddum(NG)'result holds, Theorem (3.16).   373

Theorem 3.16. *For any fuzzy graph $G = (V, \sigma, \mu)$, The Nordhaus-Gaddum result holds.*   374
In other words, we have $\gamma_v + \bar{\gamma}_v \leq 2p$.   375

Proof. G is fuzzy graph. So $\bar{G}$ is also fuzzy graph. We implement Theorem (3.15) on G 376
and $\bar{G}$. Then $\gamma_v \leq p$ and $\bar{\gamma}_v \leq p$. Hence $\gamma_v + \bar{\gamma}_v \leq 2p$. So the theorem is proved.   □   377

The following theorems on dominating sets in graphs are the first results about   378
domination and were presented by Ore in his book Theory of Graphs [69].   379

Definition 3.17 ([47])**.** A α-strong dominating set D is called a minimal α-strong   380
dominating set if no proper subset of D is a α-strong dominating set.   381

Theorem 3.18 ([47])**.** *Let G be a fuzzy graph without isolated nodes. If D is a*   382
minimal α-strong dominating set then $V - D$ is a α-strong dominating set.   383

Finding a domatic partition of size 1 is trivial and finding a domatic partition of size   384
2 (or establishing that none exists) is easy but finding a maximum-size domatic   385
partition (i.e., the domatic number), is computationally hard. Finding domatic   386
partition of size two in fuzzy graph G of order $n \geq 2$ is easy by the following.   387

Theorem 3.19 ([47])**.** *Every connected graph G of order $n \geq 2$ has a α-strong*   388
dominating set D whose complement $V - D$ is also a α-strong dominating set.   389

We improve upper bound for the vertex domination number of fuzzy graphs without   390
isolated nodes, Theorem (3.20).   391

Theorem 3.20. *For any fuzzy graph $G = (V, \sigma, \mu)$ without isolated nodes, We have*   392
$\gamma_v \leq \frac{p}{2}$.   393

Proof. Let D be a minimal dominating set of G. Then by Theorem (3.19), V-D is a   394
α-strong dominating set of G. Then $\gamma_v(G) \leq w_v(D)$ and $\gamma_v(G) \leq w_v(V - D)$.   395

Therefore $2\gamma_v(G) \leq w_v(D) + w_v(V - D) \leq p$ which implies $\gamma_v \leq \frac{p}{2}$. Hence the proof   396
is completed.   □   397

We also improve Nordhaus-Gaddum(NG)'result for fuzzy graphs without isolated   398
nodes, Corollary (3.21).   399

Corollary 3.21. *Let G be a fuzzy graph such that both G and $\bar{G}$ have no isolated*   400
nodes. Then $\gamma_v + \bar{\gamma}_v \leq p$, where $\bar{\gamma}_v$ is the vertex domination number of $\bar{G}$. Moreover,   401
equality holds if and only if $\gamma_v = \bar{\gamma}_v = \frac{p}{2}$.   402

Proof. By the Implement of Theorem (3.20) on G and $\bar{G}$, we have $\gamma_v(G) = \gamma_v \leq \frac{p}{2}$, and   403
$\gamma_v(\bar{G}) = \bar{\gamma}_v(G) = \bar{\gamma}_v \leq \frac{p}{2}$. So $\gamma_v + \bar{\gamma}_v \leq \frac{p}{2} + \frac{p}{2} = p$. Hence $\gamma_v + \bar{\gamma}_v \leq p$.   404

Suppose $\gamma_v = \bar{\gamma}_v = \frac{p}{2}$, then obviously $\gamma_v + \bar{\gamma}_v = p$. Conversely, suppose $\gamma_v + \bar{\gamma}_v \leq p$.   405
Then we have $\gamma_v \leq \frac{p}{2}$ and $\bar{\gamma}_v \leq \frac{p}{2}$. If either $\gamma_v < \frac{p}{2}$ or $\bar{\gamma}_v < \frac{p}{2}$, then $\gamma_v + \bar{\gamma}_v < p$, which   406
is a contradiction. Hence the only possibility case is $\gamma_v = \bar{\gamma}_v = \frac{p}{2}$.   □   407

Remark 3.22. Note that when we use the definition of domination number in [13,14,15],   408
Theorem (3.20) and Corollary (3.21) are hold.   409

Proposition 3.23. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. If all arcs have equal value, the*   410
G has no α-strong edge.   411

Proof. Obviously the result is hold by using Definition (3.1).   □   412

We give the relationship between M-strong arcs and α-strong arcs, Corollary (3.24).   413

Corollary 3.24. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. If all arcs are M-strong, the G*   414
has no α-strong edge.   415

Proof. Obviously the result is hold by using Proposition (3.23).   □   416

The following   417
example illustrates this concept.   418

Example 3.25.   419
In Figure (2) , all arcs are M-strong   420
but there is no α-strong arcs in this   421
fuzzy graph. Obviously this result   422
is hold by using Definition (3.3).   423

Figure 2. M-strong arcs and
α-strong arcs

We give a necessary   424
and sufficient condition for vertex   425
domination which is half of order,   426
In fact fuzzy graphs with vertex   427
domination which is half of order is characterized in the special case, Theorem (3.26).   428

Theorem 3.26. *In any fuzzy graph $G = (V, \sigma, \mu)$ such that values of nodes are equal*   429
and all arcs have same value, i.e. for $\forall u_i, u_j \in V$ and $\forall e_i, e_j \in E$, we have   430
$\sigma(u_i) = \sigma(u_j)$ and $\mu(e_i) = \mu(e_j)$. $\gamma_v = \frac{p}{2}$ if and only if For any vertex dominating set   431
D in G, we have $|D| = \frac{n}{2}$.   432

Proof. Suppose D has the conditions. $d_s(D) = 0$ by Proposition (3.23). So   433
$\gamma_v(G) = \Sigma_{u \in D}\sigma(u)$ by using Definition (3.4). Since values of nodes are equal and   434
$|D| = \frac{n}{2}$, we have $\gamma_v(G) = \Sigma_{u \in D}\sigma(u) = \frac{n}{2}\sigma(u) = \frac{1}{2}(n\sigma(u)) = \frac{1}{2}(\Sigma_{u \in V}\sigma(u)) = \frac{1}{2}(p) = \frac{p}{2}$.   435
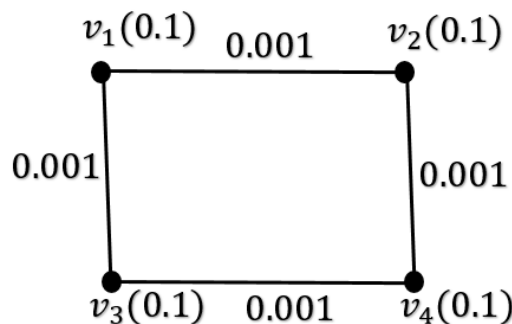Hence the result is hold in this case.   436

Conversely, Suppose $\gamma_v = \frac{p}{2}$. Let $D = \{u_1, u_2, \cdots, u_n\}$ be a vertex dominating set. $d_s(D) = 0$ by Proposition (3.23). So $\gamma_v(G) = \Sigma_{u \in D}\sigma(u)$ by using Definition (3.4). Since $\gamma_v(G) = W_v(D)$, we have $\gamma_v = \frac{p}{2} = \frac{1}{2}(\Sigma_{u \in V}\sigma(u)) = \Sigma_{u \in D}\sigma(u)$. Suppose $n' \neq \frac{n}{2}$. so $\Sigma_{i=1}^{n''}\sigma(v_i)) = 0$ which is a contradiction with $\forall u_i \in V, \sigma(u_i) > 0$. Hence $n' = \frac{n}{2}$, i.e. $|D| = n' = \frac{n}{2}$. So the result is hold in this case. □

The vertex domination of union of two fuzzy graphs is studied, Proposition (3.27).

Proposition 3.27. *Let G_1 and G_2 be fuzzy graphs. The vertex dominating set of $G_1 \cup G_2$ is $D = D_1 \cup D_2$ such that D_1 and D_2 are the vertex dominating set of G_1 and G_2 respectively. Moreover, $\gamma_v(G_1 \cup G_2) = \gamma_v(G_1) + \gamma_v(G_2)$.*

Proof. Obviously the result is hold by using Definition of union of two fuzzy graphs. □

Also the vertex domination of union of fuzzy graphs Family is discussed, Corollary (3.28).

Corollary 3.28. *Let $G_1, G_2, \cdots, G_n$ be fuzzy graphs. The vertex dominating set of $\cup_{i=1}^{n}G_i$ is $D = \cup_{i=1}^{n}D_i$ such that D_i is the vertex dominating set of G_i. Moreover, $\gamma_v(\cup_{i=1}^{n}G_i) = \Sigma_{i=1}^{n}\gamma_v(G_i)$.*

Proof. Obviously the result is hold by using proposition (3.27). □

The concepts of both monotone increasing fuzzy graph property, Definition (3.29), and monotone decreasing fuzzy graph property, Definition (3.31), are introduced.

Definition 3.29. We call a fuzzy graph property P monotone increasing if $G \in P$ implies $G + e \in P$, i.e., adding an edge e to a fuzzy graph G does not destroy the property.

Example 3.30. Connectivity and Hamiltonicity are monotone increasing properties. A monotone increasing property is nontrivial if the empty graph $\bar{K}_n \notin P$ and the complete graph $K_n \in P$.

Definition 3.31. A fuzzy graph property is monotone decreasing if $G \in P$ implies $G - e \in P$, i.e., removing an edge from a graph does not destroy the property.

Example 3.32. Properties of a fuzzy graph not being connected or being planar are examples of monotone decreasing fuzzy graph properties.

Remark 3.33. Obviously, a fuzzy graph property P is monotone increasing if and only if its complement is monotone decreasing. Clearly not all fuzzy graph properties are monotone. For example having at least half of the vertices having a given fixed degree d is not monotone.

Let $\gamma(G)$ denote the domination number of a simple graph G. Then Vizing (1963) [17]conjectured that $\gamma(G)\gamma(H) \leq \gamma(G \times H)$, where $G \times H$ is the graph product. While the full conjecture remains open, Clark and Suen (2000) [23] have proved the looser result $\gamma(G)\gamma(H) \leq 2\gamma(G \times H)$.

Vizing stated the still open conjecture:

Conjecture (Vizing [17]). For all graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G \times H)$. The result in relation with vizing's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.34).

Theorem 3.34. *The vizing's conjecture is monotone decreasing property in fuzzy graph G, if the edge e be α-strong and $\gamma_v(G - e) = \gamma_v(G)$.*

Proof. The fuzzy graph $(G - e) \times H$ is the spanning fuzzy subgraph of $G \times H$, for all fuzzy graph H. So $\gamma_v((G - e) \times H) \geq \gamma_v(G \times H) \geq \gamma_v(G)\gamma_v(H) = \gamma_v(G - e)\gamma_v(H)$. Hence vizing's conjecture is also hold for $G - e$. Then the result follows. □

Some results in relation with vizing's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.35).

Corollary 3.35. *Suppose the vizing's conjecture is hold for G. Let K be the spanning fuzzy subgraph of G such that $\gamma_v(K) = \gamma_v(G)$. Then the vizing's conjecture is hold for K.*

Proof. The fuzzy graph $K \times H$ is the spanning fuzzy subgraph of $G \times H$, for all fuzzy graph H. So $\gamma_v(K \times H) \geq \gamma_v(G \times H) \geq \gamma_v(G)\gamma_v(H) = \gamma_v(K)\gamma_v(H)$. Hence the vizing's conjecture is also hold for K. So the result follows. □

The vertex domination of join of two fuzzy graphs is studied, Proposition (3.36).

Proposition 3.36. *Let G_1 and G_2 be fuzzy graphs. The vertex dominating set of $G_1 \otimes G_2$ is $D = D_1 \cup D_2$ such that D_1 and D_2 are the vertex dominating set of G_1 and G_2 respectively. Moreover, $\gamma_v(G_1 \otimes G_2) = \gamma_v(G_1) + \gamma_v(G_2)$.*

Proof. Obviously the result is hold by using Definition of join of two fuzzy graphs and Corollary (3.24) which state in this case, M-strong arcs between two fuzzy graphs is not α-strong which is weak arc changing strength of connectedness of G. □

Also the vertex domination of join of fuzzy graphs Family is discussed, Corollary (3.37).

Corollary 3.37. *Let $G_1, G_2, \cdots, G_n$ be fuzzy graphs. The vertex dominating set of $\otimes_{i=1}^n G_i$ is $D = \otimes_{i=1}^n D_i$ such that D_i is the vertex dominating set of G_i. Moreover, $\gamma_v(\otimes_{i=1}^n G_i) = \Sigma_{i=1}^n \gamma_v(G_i)$.*

Proof. Obviously the result is hold by using proposition (3.36). □

Gravier and Khelladi [22] conjecture a Vizing-like inequality for the domination number of the cross product of graphs.

Gravier and Khelladi stated the still open conjecture:

Conjecture (Gravier and Khelladi [22]). For all graphs G and H,

$$\gamma(G)\gamma(H) \leq 2\gamma(G \otimes H).$$

The result in relation with Gravier and Khelladi's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.38).

Theorem 3.38. *The Gravier and Khelladi's conjecture is monotone decreasing property in fuzzy graph G, if the edge e be α-strong and $\gamma_v(G - e) = \gamma_v(G)$.*

Proof. The fuzzy graph $(G - e) \times H$ is the spanning fuzzy subgraph of $G \times H$, for all fuzzy graph H. So $\gamma_v((G - e) \times H) \geq \gamma_v(G \times H) \geq \gamma_v(G)\gamma_v(H) = \gamma_v(G - e)\gamma_v(H)$. Hence Gravier and Khelladi's conjecture is also hold for $G - e$. Then the result follows. □

We conclude this section with Some result in relation with Gravier and Khelladi's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.39).

Corollary 3.39. *Suppose the Gravier and Khelladi's conjecture is hold for G. Let K be* ₅₁₇
the spanning fuzzy subgraph of G such that $\gamma_v(K) = \gamma_v(G)$. *Then the Gravier and* ₅₁₈
Khelladi's conjecture is hold for K. ₅₁₉

Proof. The fuzzy graph $K \times H$ is the spanning fuzzy subgraph of $G \times H$, for all fuzzy ₅₂₀
graph H. So $\gamma_v(K \otimes H) \geq \gamma_v(G \otimes H) \geq \gamma_v(G)\gamma_v(H) = \gamma_v(K)\gamma_v(H)$. Hence the Gravier ₅₂₁
and Khelladi's conjecture is also hold for K. So the result follows. □ ₅₂₂

4   Practical applications ₅₂₃

In this Section, We give 9 practical applications in relation with these concepts. ₅₂₄
Domination is a rapidly developing area of research in graph theory. The concept of ₅₂₅
domination ₅₂₆
has existed for a long time and early ₅₂₇
discussions on the topic can be ₅₂₈
found in works of Ore and Berge . ₅₂₉
The summary of the literature ₅₃₀
shows the following ₅₃₁
wide-known problems, which ₅₃₂
are considered among the earliest ₅₃₃
applications for dominating sets. ₅₃₄

Figure 3. Queens dominating the chessboard

Queens Problem ₅₃₅
This problem was mentioned by ₅₃₆
Ore. According to the rules of chess ₅₃₇
a queen can, in one move, advance ₅₃₈
any number of squares horizontally, ₅₃₉
diagonally, or vertically (assuming ₅₄₀
that no other chess figure is on ₅₄₁
its way). How to place a minimum ₅₄₂
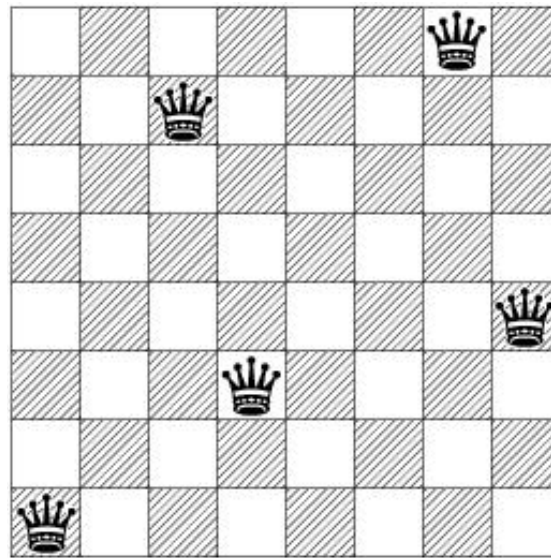number of queens on a chessboard ₅₄₃
so that each square is controlled by ₅₄₄
at least one queen? See one of the solutions in (Fig. 3). For fuzzification of this ₅₄₅
problem, types of square (based on sensitive place in game of chess, chess pieces) and ₅₄₆
types of connection can be assigned by different values. So the question is changed to ₅₄₇
this. How to place a number of queens on a chessboard so that each square is controlled ₅₄₈
by at least one queen based on values on queens and ratio of total of values of adjacent ₅₄₉
α-strong connections to total of values of adjacent connections? ₅₅₀
Locating Radar Stations Problem ₅₅₁
The problem was discussed by Berge. A number of strategic locations are to be kept ₅₅₂
under surveillance. The goal is to locate a radar for the surveillance at as few of these ₅₅₃
locations as possible. How a set of locations in which the radar stations are to be placed ₅₅₄
can be determined? For fuzzification of this problem, types of radar stations (based on ₅₅₅
power of them) and types of connection with locations can be assigned by different ₅₅₆
values. So the question is changed to this. How a set of locations in which the radar ₅₅₇
stations are to be placed can be determined based on values on radar stations and ratio ₅₅₈
of total of values of adjacent α-strong connections to total of values of adjacent ₅₅₉
connections? ₅₆₀
Problem of Communications in a Network ₅₆₁
Suppose that there is a network of cities with communication links. How to set up ₅₆₂
transmitting stations at some of the cities so that every city can receive a message from ₅₆₃
at least one of the transmitting stations? This problem was discussed in detail by Liu. ₅₆₄
For fuzzification of this problem, types of cities (based on population, structure) and ₅₆₅

types of connection with cities can be assigned by different values. So the question is changed to this. How to set up transmitting stations at some of the cities so that every city can receive a message from at least one of the transmitting stations based on values on cities and ratio of total of values of adjacent α-strong connections to total of values of adjacent connections?

Nuclear Power Plants Problem

A similar known problem is a nuclear power plants problem. There are various locations and an arc can be drawn from location x to location y if it is possible for a watchman stationed at x to observe a warning light located at y. How many guards are needed to observe all of the warning lights, and where should they be located? For fuzzification of this problem, types of guards (based on abilities) and types of connection with guards can be assigned by different values. So the question is changed to this. How many guards are needed to observe all of the warning lights, and where should they be located based on values on guards and ratio of total of values of adjacent α-strong connections to total of values of adjacent connections?

At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest. Such applications usually aim to select a subset of nodes that will provide some definite service such that every node in the network is ?close? to some node in the subset. The following examples show when the concept of domination can be applied in modelling real-life problems.

Modelling Biological Networks

Using graph theory as a modelling tool in biological networks allows the utilization of the most graphical invariants in such a way that it is possible to identify secondary RNA (Ribonucleic acid) motifs numerically. Those graphical invariants are variations of the domination number of a graph. The results of the research carried out show that the variations of the domination number can be used for correctly distinguishing among the trees that represent native structures and those that are not likely candidates to represent RNA. For fuzzification of this problem, types of location (based on advantages) and types of connection with locations can be assigned by different values. So the question is based on based on values on locations and ratio of total of values of adjacent α-strong connections to total of values of adjacent connections?

Modelling Social Networks

Dominating sets can be used in modelling social networks and studying the dynamics of relations among numerous individuals in different domains. A social network is a social structure made of individuals (or groups of individuals), which are connected by one or more specific types of interdependency. The choice of initial sets of target individuals is an important problem in the theory of social networks. In the work of Kelleher and Cozzens, social networks are modelled in terms of graph theory and it was shown that some of these sets can be found by using the properties of dominating sets in graphs. For fuzzification of this problem, types of people (based on abilities) and types of connection with people can be assigned by different values. So the question is based on based on values on people and ratio of total of values of adjacent α-strong connections to total of values of adjacent connections?

Facility Location Problems

The dominating sets in graphs are natural models for facility location problems in operational research. Facility location problems are concerned with the location of one or more facilities in a way that optimizes a certain objective such as minimizing transportation cost, providing equitable service to customers and capturing the largest market share. For fuzzification of this problem, types of location (based on advantages) and types of connection with locations can be assigned by different values. So the

question is based on based on values on locations and ratio of total of values of adjacent   618
α-strong connections to total of values of adjacent connections?   619

Coding Theory   620

The concept of domination is also applied in coding theory as discussed by   621
Kalbfleisch, Stanton and Horton and Cockayne and Hedetniemi. If one defines a graph,   622
the vertices of which are the n-dimensional vectors with coordinates chosen from   623
$\{1, \cdots, p\}, p > 1$, and two vertices are adjacent if they differ in one coordinate, then the   624
sets of vectors which are (n, p)-covering sets, single error correcting codes, or perfect   625
covering sets are all dominating sets of the graph with determined additional properties.   626
For fuzzification of this problem, types of codes (based on types of words, different   627
words, same words) and types of connection with codes can be assigned by different   628
values. So the question is based on based on values on codes and ratio of total of values   629
of adjacent α-strong connections to total of values of adjacent connections?   630

Multiple Domination Problems   631

An important role is played by multiple domination. Multiple domination can be   632
used to construct hierarchical overlay networks in peer-to-peer applications for more   633
efficient index searching. The hierarchical overlay networks usually serve as distributed   634
databases for index searching, e.g. in modern file sharing and instant messaging   635
computer network applications. Dominating sets of several kinds are used for balancing   636
efficiency and fault tolerance as well as in the distributed construction of minimum   637
spanning trees. Another good example of direct, important and quickly developing   638
applications of multiple domination in modern computer networks is a wireless sensor   639
network. A wireless sensor net- work (WSN) usually consists of up to several hundred   640
small autonomous devices to measure some physical parameters. Each device contains a   641
processing unit and a limited memory as well as a radio transmitter and a receiver to be   642
able to communicate with its neighbors. Also, it contains a limited power battery and is   643
constrained in energy consumption. There is a base station, which is a special sensor   644
node used as a sink to collect information gathered by other sensor nodes and to   645
provide a connection between the WSN and a usual network. A routing algorithm   646
allows the sensor nodes to self-organize into a WSN. As stated, an important goal in   647
WSN design is to maximize the functional lifetime of a sensor network by using energy   648
efficient distributed algorithms, networking and routing techniques. To maximize the   649
functional lifetime, it is important to select some sensor nodes to behave as a backbone   650
set to support routing communications. The backbone set can be considered as a   651
dominating set in the corresponding graph. Dominating sets of several different kinds   652
have proved to be useful and effective for modelling backbone sets. In the recent   653
literature, particular attention has been paid to construction of k-connected k-   654
dominating sets in WSNs, and several probabilistic and deterministic approaches have   655
been proposed and analyzed. The backbone set of sensor nodes should be selected as   656
small as possible and, on the other hand, it should guarantee high efficiency and   657
reliability of networking and communications. This trade-off requires construction of   658
multiple dominating sets providing energy efficient and reliable data dissemination and   659
communication protocols. For fuzzification of this problem, types of sensor nodes (based   660
on advantages) and types of connection with sensor nodes can be assigned by different   661
values. So the question is based on based on values on sensor nodes and ratio of total of   662
values of adjacent α-strong connections to total of values of adjacent connections?   663

A homogeneous WSN consists of wireless sensor devices of the same kind. All the   664
devices have the same set of limited resources and, originally, no hierarchy is imposed   665
on the network structure and communications. In a network of this kind, the only   666
special sensor node is a base station. For all the other nodes, it is necessary to construct   667
and switch the backbone sets and communications efficiently so that all the network   668
nodes stay in operation as long as possible. Therefore, in this case, it is important to be   669

able to construct and switch dominating sets and route communications uniformly and   670
efficiently with respect to the energy consumption of each particular sensor node. This   671
has to be done to optimize the functional lifetime of the whole network.   672

Usually, a WSN is mathematically modelled as a unit or quasi-unit disk graph. These   673
are the most natural and general graph models for a WSN. In a unit disk graph model,   674
nodes correspond to sensor locations in the Euclidean plane and are assumed to have   675
identical (unit) transmission ranges. An edge between two nodes means that they can   676
communicate directly, i.e. the distance between them is at most one. A survey of known   677
results on unit disk graphs, including algorithms for constructing dominating sets, can   678
be found. A quasi-unit disk graph model takes into consideration possible transmission   679
obstacles and is much closer to reality: we are sure to have an edge between two nodes   680
if the distance between them is at most a parameter $d, 0 < d < 1$. If the distance   681
between two nodes is in the range from d to 1, the existence of an edge is not specified.   682
A description of several more restricted geometric graph models for WSN design, e.g.   683
the related neighborhood graph, Gabriel graph, Yao graph etc., can be found.   684

Domination is an area in graph theory with an extensive research activity. A book   685
by Haynes, Hedetniemi and Slater on domination published in 1998 lists 1222 articles in   686
this area.   687

5   Conclusion   688

Graph theory is one of the branches of modern mathematics having experienced a most   689
impressive development in recent years. One of the most interesting problems in graph   690
theory is that of Domination Theory. Nowadays domination theory ranks top among   691
the most prominent areas of research in graph theory and combinatorics. The theory of   692
domination has been the nucleus of research activity in graph theory in recent times.   693
The fastest growing area within graph theory is a study of domination and related   694
subset problems such independence, covering, matching, decomposition and labelling.   695
Domination boasts a host of applications to social network theory, land surveying, game   696
theory, interconnection network, parallel computing and image processing and so on.   697
Today, this theory gained popularity and remains as a major area of research. At   698
present, domination is considered to be one of the fundamental concepts in graph theory   699
and its various applications to ad hoc networks, biological networks, distributed   700
computing, social networks and web graphs partly explain the increased interest. More   701
than 1200 papers already published on domination in graphs. Without a doubt, the   702
literature on this subject is growing rapidly, and a considerable amount of work has   703
been dedicated to find different bounds for the domination numbers of graphs. However,   704
from practical point of view, it was necessary to define other types of dominations.   705
Most of these new variations required the dominating set to have additional properties.   706

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References

1. M. Akram, *Bipolar fuzzy graphs*, Information Sciences, 181 (2011) 5548-5564.

2. M. Akram, *Interval-valued fuzzy line graphs*, Neural Computing and
   Applications, 21(2012) 145-150.

3. M. Akram, Sheng-Gang Li and K.P. Shum, *Antipodal bipolar fuzzy graphs*, Italian Journal of Pure and Applied Mathematics, 31 (2013) 425-438.

4. S. Arumugam and R. Kala, *Domination parameters of star graphs*, Ars Combi. 44 (1996) 93-96.

5. D.B. Auer, F. Harary, J. Nieminen, and C.L. Suffel, *Domination Alteration Sets in Graphs*, Discrete Math. 47 (1983) 53-161.

6. C. Berge, *Graphs and Hypergraphs*, North Holland, Amsteddam, 1973.

7. C. Bezdek and S.K. Pal, Fuzzy Models for Pattern Recognition, IEEE Press, (1992).

8. Brešar, Boštjan; Dorbec, Paul; Goddard, Wayne; Hartnell, Bert L.; Henning, Michael A.; Klavžar, Sandi; Rall, Douglas F. (2012), *Vizing's conjecture: a survey and recent results*, Journal of Graph Theory 69 (1) 46-76.

9. P. Bhattacharya, Pattern Recognition Letter, 6 (1987) 297-302.

10. P. Bhattacharya, *Some Remarks on fuzzy graphs*, Pattern Recognition Letters 6 (1987) 297-302.

11. K.R. Bhutani, Pattern Recognition Letter, 9 (1989) 159-162.

12. K.R. Bhutani, A. Rosenfeld, *Strong arcs in fuzzy graphs*, Information Sciences, 152 (2003) 319-322.

13. K.R. Bhutani, J. Mordeson, and A. Rosenfeld, *On degrees of end nodes and cut nodes in fuzzy graphs*, Iranian Journal of Fuzzy Systems 1 1 (2004) 53-60.

14. B. Bollob*á*s and E.J. Cockayne, unpublished.

15. K.R.Bhutani, A.Rosenfeld, *Strong arcs in fuzzy graphs*, Information Sciences 152 (2003) 319-322.

16. G. Chartrand, L. Lesniak, *Graphs and Digraphs*, 3rd ed., Wadsworth and Brooks Cole, Monterey, 1996.

17. W.E. Clark and S. Suen, *An Inequality Related to Vizing's Conjecture*, Electronic J. Combinatorics 7 1 (2000) 1-3, 2000.

18. E.J. Cockayne and S.T. Hedetniemi, *Toward a theory of domination in graphs*, Networks 7 (1977) 247-261.

19. E.J. Cockayne, and S.T. Hedetniemi, *Towards the theory of domination in graphs*, Networks 10 (1977) 247-261.

20. W.L. Craine, Fuzzy Sets and Systems, 68 (1994) 181-193.

21. M. Delgado, J.L. Verdegay and M.A. Vila, European J. Operations Research 22 (1985) 243-249.

22. S. Gravier, A. Khelladi, *On the domination number of cross products of graphs*, Discrete Mathematics, 145 (1995) 273-277.

23. Hartnell, B. and Rall, D. F. *Domination in Cartesian Products: Vizing's Conjecture.* In Domination in Graphs–Advanced Topics (Ed. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater). New York: Dekker, (1998) 163-189.

24. .W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker. New York, 1998.

25. Hedetniemi S.T. and Laskar.R.C., *Topics on Domination*, Vol. 48, North Holland, New York, 1990.

26. F. Jaeger and C. Payan, *Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple*, C.R. Acad. Sci. Paris A 274 (1972) 728-730.

27. J.P. Joseph and S. Arumugam, *A note on domination in graphs*, lnternet. J. Management Systems, to appear.

28. Kaufmann.A., (1975), *Introduction to the theory of Fuzzy Subsets*, Academic Press, New york.

29. Anjaly Kishore and M.S.Sunitha, *Strong chromatic number of fuzzy graphs*, Annals of Pure and Applied Mathematics, 7(2) (2014) 52-60.

30. S. Klavžar, B. Zmazek, *On a Vizing-like conjecture for direct product graphs*, Discrete Mathematics, 156 (1996) 243-246.

31. G.J. Klir and Yuan Bo, Fuzzy Sets and Fuzzy Logic: Theory and Applications, PHI, (1997).

32. Kulli V.R., *Theory of Domination in Graphs*,Vishwa International Publications, 2010.

33. R. Laskar and K. Peters, *Vertex and edge domination parameters in graphs*, Congr. Numer. 48 (1985) 291-305.

34. C.T. Leondes, *Fuzzy Theory Systems: Techniques and Applications* Vol. 1-4, Academic Press, (1999).

35. F.A. Lootsrna, *Fuzzy Logic for Planning and Decision Making*, Kluwer, (1997).

36. O. T. Manjusha and M. S. Sunitha, *Notes on domination in fuzzy graphs*, Journal of intelligent and Fuzzy Systems 27 (2014) 3205-3212.

37. O. T. Manjusha and M. S. Sunitha, *Total domination in fuzzy graphs using strong arcs*, Annals of Pure and Applied Mathematics 9 (1) (2014) 23-33.

38. S. Mathew and M.S. Sunitha, *Node connectivity and arc connectivity of a fuzzy graph*, Information Sciences 180 (2010) 519-531.

39. J.N. Mordeson, C.S. Peng, *Operations on fuzzy graphs*, Information Sciences 79 (1994) 159-170.

40. J.N. Mordeson and D.S. Malik, *Fuzzy Commutative Algebra*, World Scientific (1998).

41. J.N. Mordeson and P.S. Nair, *Trees, Cycles and Cocycles of Fuzzy Graphs*, Encyclopedia of Computer Science and Technology 39 Supplement 24 (1998) 301-322.

42. J.N. Mordeson, Pattern Recognition Letter, 14 (1993) 381-384.

43. J.N. Mordeson and C.S. Peng, Information Sciences, 79 (1994) 159-170.

44. J.N. Mordeson, P.S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, 2000.

45. J.N. Mordeson and P.S. Nair, Information Sciences, 90 (1996) 39-49.

46. J.N. Mordeson, *Fuzzy line graphs*, Pattern Recognition Letters 14 (1993) 381?384.

47. A. Nagoorgani and V. T. Chandrasekaran, *Domination in fuzzy graph*, Adv. in Fuzzy Sets and Systems I (1) (2006) 17-26.

48. Sk. Md. Abu Nayeem and M. Pal, *Diameter constrained fuzzy minimum spanning tree problem*, International Journal of Computational Intelligence Systems, 6(6) (2013) 1040-1051.

49. E.A. Nordhaus and J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly 63 (1956) 175-177.

50. O. Ore,*Theory of Graphs*, Amer. Math. Soc., Collog, Publish, 38, Providence,1962.

51. M.Pal and H.Rashmanlou, *Irregular interval- valued fuzzy graphs*, Annals of Pure and Applied Mathematics, 3(1) (2013) 56-66.

52. T. Pramanik, S. Samanta and M. Pal, *Interval-valued fuzzy planar graphs*, Int. J. Mach. Learn. and Cyber., DOI 10.1007/s13042-014-0284-7.

53. H. Rashmanlou and M. Pal, *Antipodal interval-valued fuzzy graphs*, International journal of Applications of Fuzzy Sets and Artificial Intelligence, 3 (2013) 107-130.

54. H. Rashmanlou and M. Pal, *Isometry on interval-valued fuzzy graphs*, International Journal of Fuzzy Mathematical Archive, 3 (2013) 28-35.

55. H. Rashmanlou, S. Samanta, M. Pal and R.A.Borzooei, *A study on bipolar fuzzy graphs*, Journal of Intelligent and Fuzzy Systems, DOI: 10.3233/IFS-141333.

56. A. Rosenfeld, *Fuzzy Graphs, In Fuzzy Sets and their Applications to Cognitive and Decision Processes*, Zadeh. L.A., Fu, K.S., Shimura, M., Eds ; Academic Press, New York (1975) 77-95.

57. S. Samanta and M. Pal, *Fuzzy tolerance graphs*, International Journal of Latest Trends in Mathematics, 1(2) (2011) 57-67.

58. S. Samanta and M. Pal, *Bipolar fuzzy hypergraphs*, International Journal of Fuzzy Logic Systems, 2(1) (2012) 17-28.

59. S. Samanta and M. Pal, *Fuzzy k-competition graphs and p-competition fuzzy graphs*, Fuzzy Inf. Eng., 5(2) (2013) 191-204.

60. S. Samanta and M. Pal, *Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs*, The Journal of Fuzzy Mathematics, 22(2) (2014) 253-262.

61. S. Samanta and M. Pal, Irregular bipolar fuzzy graphs, International Journal of Applications of Fuzzy Sets, 2 (2012) 91-102.

62. E. Sampathkumar, $(1, k)$-*Domination in a Graph*, Jour. Math. Phy. Sci. 22 (5) (1988) 613-619.

63. A. Somasundaram and S. Somasundaram, *Domination in fuzzy graphs-I*, Pattern Recognition Letters 19 (1998) 787-791.

64. A. Somasundaram, *Domination in fuzzy graphs-II*, J. Fuzzy Math. 13 (2) (2005) 281-288.

65. S. Sunitha and A. Vijaya Kumar, *Some metric aspects of fuzzy graphs*, In Proceedings of the Conference on Graph Connections, Allied Publishers (1999) 111-114.

66. P.J. Slater, *Dominating and Reference Sets in graph*, J.Math and Phy.Sci. 22 (1988) 445-455.

67. M.S. Sunitha and A. Vijaya Kumar, Information Sciences, 113 (1999) 293-300.

68. M.S. Sunitha and A. Vijayakumar, *Complement of a fuzzy graph*, Indian Journal of Pure and Applied Mathematics 33(9) (2002) 1451-1464.

69. M. Tom and M.S. Sunitha, *Sum distance in fuzzy graphs*, Annals of Pure and Applied Mathematics 7(2) (2014) 73-89.

70. Z. Tong, D. Zheng, *An algorithm for finding the connectedness matrix of a fuzzy graph*, Congr. Numer. 120 (1996) 189-192.

71. V.G. Vizing, *The Cartesian Product of Graphs* Vyc. Sis. 9 (1963) 30-43.

72. J. Xu, *The use of fuzzy graphs in chemical structure research*, in: D.H. Rouvry (Ed.), Fuzzy Logic in Chemistry, Academic Press, (1997) 249-282.

73. R.T. Yeh, S.Y. Bang, *Fuzzy relations, Fuzzy relations, fuzzy graphs and their applications to clustering analysis*, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, (1975)125-149.

74. L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.

75. L.A. Zadeh, $Toward a generalized theory of uncertainty (GTU) an outline$, Information Sciences 172 (12) (2005) 1-40.

76. L.A. Zadeh, *Is there a need for fuzzy logic?*, Information Sciences 178 (13) (2008) 2751-2779.

77. L.A. Zadeh, *Fuzzy sets*, Inform. Control, 8 (1965) 338-353.