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Energy and Entropy Measures of Fuzzy Relations for Data Analysis

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Abstract: We present a new method for assessing the strength of fuzzy rules with respect to a dataset, based on the measures of the greatest energy and smallest entropy of a fuzzy relation. Considering a fuzzy automaton (relation) in which A is the input fuzzy set and B the output fuzzy set, the fuzzy relation R1 with greatest energy provides information about the greatest strength of the input-output and the fuzzy relation R2 with the smallest entropy provides information about uncertainty of the relationship input-output. We consider a new index of the fuzziness of the input-output based over R1 and R2. In our method this index is calculated for each pair of input and output fuzzy sets in a fuzzy rule. A threshold value is set for choosing the most relevant fuzzy rules with respect to the data.

Keywords: fuzzy entropy, fuzzy energy, fuzzy rules, fuzzy sets, fuzzy relations

1. Introduction

Let X = {x1,…,xm} be a finite set and A be a fuzzy set of X. In [2, 3] two categories of fuzziness measures are defined as energy and entropy (see, e.g., also [10]). The energy measure of fuzziness of A is given by

$$E(A)=\sum_{i=1}^{m} e\left(A(x_i)\right) \quad (1)$$

where e: [0,1] → [0,1] is a continuous monotonically increasing function, with e(0) = 0 and e(1) = 1. A particular energy function is given by the identity e(u) = u for any u ∊ [0,1]. In this case the minimum value of the fuzzy energy is 0 and the maximum value is given by E(A) = Card(X) = m. The entropy measure of fuzziness of the fuzzy set A is given as

$$H(A)=\sum_{i=1}^{m} h\left(A(x_i)\right) \quad (2)$$

where h: [0,1] → [0,1] is a continuous monotonically increasing function in [0, ½] and monotonically decreasing in [½, 1], with h(0) = h(1) = 0 and h(u) = h(1-u). A simple entropy function is given by h(u) = u if u ≤ ½ and h(u)=1–u if u > ½.

Now we consider another finite set Y = {y1,…,yn} and a fuzzy relation R defined on X×Y. We have that

$$E(R)=\sum_{i=1}^{m}\sum_{j=1}^{n} e\left(R(x_i,y_j)\right) \quad (3)$$

And

$$H(R)=\sum_{i=1}^{m}\sum_{j=1}^{n} h\left(R(x_i,y_j)\right) \quad (4)$$

Take a continuous t-norm t and a max-t fuzzy relation equation, that is of the following type:

$$\vee_{i=1}^{m}\left(R(x_i,y_j)\,t\,A(x_i)\right)=B(y_j) \qquad j=1,\dots,n \quad (5)$$

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where A (resp., B) is a known input (resp., output) fuzzy set and R is a n unknown fuzzy automaton (relation) connecting the inputs-output via fuzzy rules.

Solutions for fuzzy relation equation (5) were proposed in [4, 5, 8] (see, e.g., [9] if t=min). In particular, if we consider the t-norm of Yager [11], the unique greatest fuzzy relation R_1 is defined as where, following [4], we have:

$$a\tau b = \begin{cases} \left((1-a)^p - (1-b)^p\right)^{1/p} & \text{if } a \ge b \\ 1 & \text{if } a < b \end{cases} \qquad a,b \in [0,1],\quad p \ge 1 \tag{6}$$

R_1 is the fuzzy relation having the maximum energy E. Furthermore in [4,5] the authors propose an algorithm for finding the relation R_2, solution of (5) not unique, having the minimum entropy H.

Many works in data and decision analysis present methods to minimize the fuzzy entropy for obtaining the solution with the smallest ambiguity. Some research works as [1, 6, 7, 10, 12, 13] present fuzzy decision algorithms for classification analysis using minimum fuzzy entropy.

We propose a new method for measuring the strength of fuzzy rules with respect to a set of input-output data based on the maximum energy and minimum entropy measures.

Our idea is to calculate for any pair of input and output fuzzy sets a normalized index of the strength of the rule with respect to the data, which is function of the maximum energy and minimum entropy. We find the best input-output fuzzy sets pair such that the corresponding index is maximum. If this index is greater or equal to a pre-defined threshold, then we consider that fuzzy rule which is more relevant with respect to the data.

In Section 2 we describe the algorithm presented in [4, 5] for calculating the solutions R_1 and R_2 of the equation (5) with the Yager t-norm. In Section 3 our algorithm is presented for evaluating the strength of fuzzy rules with respect to the data. In Section 4 we present the results of a set of experiments in which we apply our algorithm. The final considerations are shown in Section 5.

2. Algorithm for calculating fuzzy relations having the greatest energy and smallest entropy measures

Let $X = \{x_1,\dots,x_m\}$, $Y = \{y_1,\dots,y_n\}$, A (resp., B) be a fuzzy set on X (resp., Y). Then in [4, 5] the maximum energy measure of a relation in the equation (5) is given from R_1. For the calculus of R_2, the following algorithm is developed in [4, 5]. Let h as defined in Section 1. For each $y_j \in Y$, we consider $\Gamma(y_j) = \{x_i \in X: A(x_i) \ge B(y_j)\}$. If $B(y_j)>0$, the algorithm finds some in $x_c \in \Gamma(y_j)$ (generally not unique) such that $A(x_c)\tau B(y_j)$ is not zero and $h(A(x_c)\tau B(y_j))$ assumes the minimum value. Then $R_2(x_i,y_j) = A(x_i)\tau B(y_j)$ if $x_i = x_c$ and $R_2(x_i, y_j)=0$ if $x_i \neq x_c$. If $B(y_j)=0$, $R_2(x_i, y_j) = 0$ for each $i = 1,\dots,m$. Below we show the related pseudocodes.

Algorithm: **Calculate R_1**

Description:	*Calculate the matrix R_1*
Input:	*X,Y,A,B*
Output:	*R_1*

```
FOR  j = 1 TO n
{
    FOR  i = 1 TO m
    {
        R1(xi,yj):= A(xi) τB(yj) ;
    }
}
END
```

Algorithm: **Calculate R_2**

Description:	*Calculate the matrix R_2*

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```
Input:                    X,Y,A,B
Output:                   R2
FOR  j = 1 TO n
{
   IF  B(yj)>0
   {
      xc:= 0:
      hmin:= 1;
      FOR each x in Γ(yj)
      {
         IF h(A(x), B(yj)) < hmin THEN
         {
            hmin:= h(A(x), B(yj));
            xc:= x;
         }
      }
      FOR i = 1 TO m
      {
         IF (xi = xc)
            R2(xi,yj):= A(xi) τB(yj) ;
         ELSE
            R2(xi,yj):= 0;
      }
   }
   ELSE
   {
      FOR i = 1 TO m
         R2(xi,yj):= 0;
   }
}
END
```

As example, let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, A = (0.2, 0.3, 0.5, 0.8) and B = (0.4, 0.0, 0.6, 0.7). For $p = 2$ in formula (6), we obtain that

$$R_1 = \begin{vmatrix} 1.00 & 0.40 & 1.00 & 1.00 \\ 1.00 & 0.29 & 1.00 & 1.00 \\ 0.67 & 0.13 & 1.00 & 1.00 \\ 0.43 & 0.02 & 0.65 & 0.78 \end{vmatrix}$$

For R_2, we have $\Gamma(y_1)=\{x_3, x_4\}$, $\Gamma(y_3)=\{x_4\}$, $\Gamma(y_4)=\{x_4\}$ and hence $R_2(x_3, y_1) = 0.67$, $R_2(x_4, y_3) = 0.65$ and $R_2(x_4, y_4) = 0.78$. For $B(y_2) = 0$, we have that $R_2(x_i, y_2) = 0$ for each $i = 1,\dots,4$. Then the minimum entropy fuzzy relation is given by

$$R_2 = \begin{vmatrix} 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \\ 0.67 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.65 & 0.78 \end{vmatrix}$$

3. Evaluating the strength of the fuzzy rules with respect to the data

Our goal is to evaluate the strength of the fuzzy rules considered in a domain's expert, with respect to dataset. Transferring its knowledge of the domain, the expert builds a fuzzy partition of q fuzzy sets $\{A_1,\dots,A_q\}$ of the universe of the discourse U_x of the input variable x, and a fuzzy partition of s fuzzy sets $\{B_1,\dots,B_s\}$ of the universe of the discourse U_y of the output variable y. Subsequently, he defines a set of fuzzy rules relating the input and the output variables in the following form:

$$\textbf{rk : IF } x \text{ is } A_w \textbf{ THEN } y \text{ is } B_z,\ w = 1,\dots,q,\ z = 1,\dots,s \qquad (7)$$

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where **rk** is the k*th* fuzzy rule of the fuzzy rule set. For instance, let a dataset be composed by m measures of the input variable x, $X = \{x_1,\ldots,x_m\}$, and a dataset composed by n measures of the output variable y, $Y = \{y_1,\ldots,y_n\}$. For each rule we extract the pair (A_w,B_z) formed by the input and the output fuzzy sets in (7), and we calculate a normalized index based on the fuzzy maximum energy and minimum entropy. The index represents the strength of the k*th* fuzzy rule with respect to the data. Let R be a the fuzzy automaton relation connecting A_w and B_z by means of equation (5) with the Yager t-norm. Let R_{1wz} and R_{2wz} the fuzzy solutions of (5) with maximum energy and minimum entropy calculated via (3) and (4), respectively. The index of fuzziness strength for the pair (A_w,B_z) is defined [4] as

$$I_{wz} = \frac{E(R_{1wz}) - H(R_{2wz})}{m \cdot n} \tag{8}$$

For $I_{wz} = 1$, we obtain $E(R_{1wz}) = n{\cdot}m$ and $H(R_{2wz})=0$. If I_{wz} is greater or equal to a pre-defined threshold, then the fuzzy rule is confirmed by the data. In Fig. 1 this process is schematized as well.

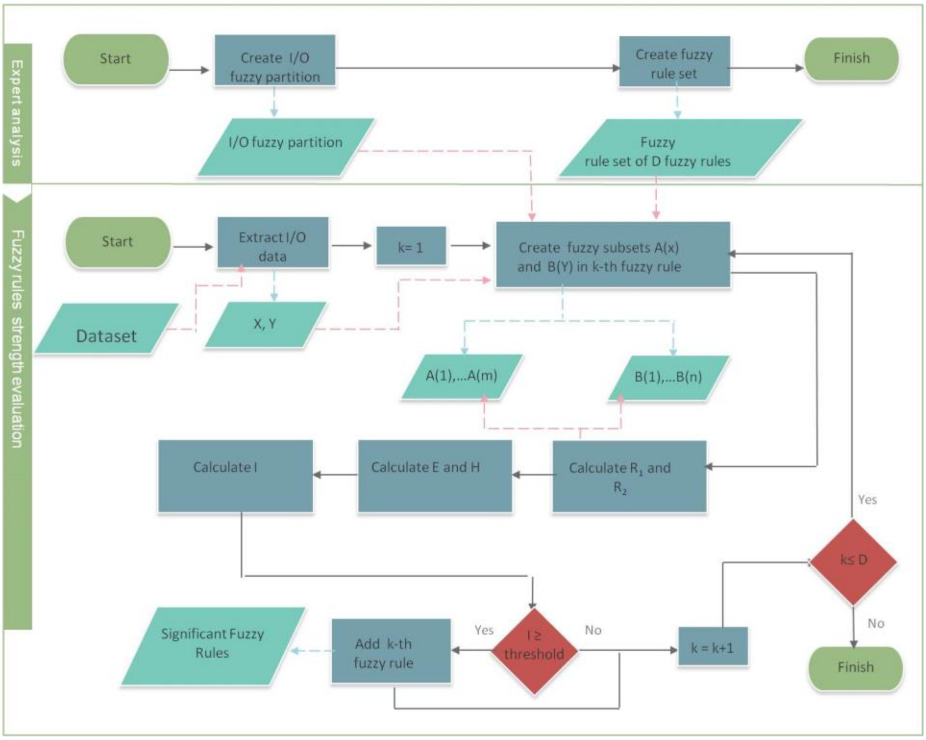

Fig. 1. Schema of the process

The continuous black arrows are related to the two process, the red arrows symbolize the use of data in input and the black arrows symbolize the use of data in output.

In the first phase the expert creates the fuzzy partition for U_x and U_y and creates the fuzzy rule set. Then the expert analyzes each fuzzy rule in the fuzzy rule set with respect to a set of data. For the input-output fuzzy set couple (A_w,B_z), $A_w(x_1),\ldots, A_w(x_m)$, $B_z(y_1),\ldots, B_z(y_n)$, the fuzzy relations R_1 and R_2, the Energy E, the Entropy H and the index I are calculated. If the index I is greater of equal to a prefixed threshold, then the rule is added as significant fuzzy rule set with respect to the input/output data. We can generalize this model to the case of two or more input variables are considered. The generalized form of a fuzzy rule is given by the form:

$$\mathbf{rk:IF}\ (x_1 \text{ is } A^{(1)}_{w1}) \text{ AND } (x_2 \text{ is } A^{(2)}_{w2}) \text{ AND } \ldots \text{ AND } (x_v \text{ is } A^{(v)}_{wv}) \text{ THEN } y \text{ is } B_z \tag{9}$$

where $A^{(1)}_{wv}$ is a fuzzy set of the fuzzy partition of the universe of the discourse of the input variable.

For each pair $(A^{(1)}_{w1},B_z),\dots,(A^{(v)}_{wv},B_z)$, we calculate the corresponding indices $I^{(l)}_{w_l z}$ for l=1,…,v and assign a measure of strength of the fuzzy rule with respect to the data given by

$$I_k = \min_{l=1,\dots,v} I^{(l)}_{w_l z} \tag{10}$$

Below we show the pseudocode of the algorithm.

Algorithm: **Energy-Entropy fuzzy rules evaluation**

Description: *Calculate the matrix R_2*
Input: *X,Y,A,B*
Output: *R_2*

```
SET  Ith                                   // set the threshold value
FOR  k = 1 TO D                            // for all the D fuzzy rules in the dataset
{
   Imin := 2;                              // Imin is initialized to a value greater than 1
   Create the fuzzy subsets Bz(y1),…, Bz(yn);
   FOR l = 1 to v
   {
       Create the fuzzy subsets A(l)wl(x1),…, A(l)wl(xm);
       Calculate R1and R2;
       Calculate E and H;
       Calculate I;
       IF ( I < Imin )
              Imin = I;
   }
   IF ( Imin ≥ Ith )
       Annotate the k-th fuzzy rule as significant;
}
END
```

The threshold value I_{th} can be set by the expert by using an opportune calibration. This calibration can be obtained by testing the algorithm applied on a sample dataset for which the expert can evaluate the strength of fuzzy rules with respect to the data. In the next section we present some results obtained by using various datasets. The first experiment is used for calibrating the threshold value I_{th}.

4. Test reults

Here we use e(u)=u for u∈[0,1] and, in accordance to [2, 3], the following fuzzy entropy measure:

$$h(u) = -u \cdot \log_2(u) - (1\text{-u}) \cdot \log_2(1-u) \qquad u \in [0,1] \tag{11}$$

and the equation (5) with the Yager t-norm.

Our tests are applied to datasets extracted from the open data of the city of Naples (Italy) (www.opendata.comune.napoli.it/) and from database of the 15° census population performed during 2011 on the Italian territory by the ISTAT (Italian Statistical National Institute), available at the web site http://dati-censimentopopolazione.istat.it. For brevity we show the results obtained in two experiments.

The city of Naples is partitioned in 10 municipalities. In turn each municipality includes a set of districts; the district of each municipality are listed in Table 1

Table 1. Municipalities of the city of Naples and their districts

Municipality number	Districts

1	Chiaia, Posillipo, S.Ferdinando
2	Avvocata, Montecalvario, Porto, S.Giuseppe, Pendino, Mercato
3	Stella, S.Carlo all'Arena
4	Vicaria, S.Lorenzo, Poggioreale
5	Vomero, Arenella
6	Ponticelli, Barra, S.Giovanni aTeduccio
7	Miano, Secondigliano, S.Pietro a Patierno
8	Chiaiano, Piscinola-Marianella, Scampia
9	Pianura, Soccavo
10	Bagnoli, Fuorigrotta

In the first experiment we consider the input x= Number of inhabitants with less than 5 years of age for each 100 inhabitants and the output y = Number of public kindergartens. The data extracted are shown in Table 2.

Table 2. The I/O data extracted for the 10 municipalities

Municipality	x	y
1	4.26	5
2	4.77	6
3	5.05	6
4	4.93	3
5	3.80	3
6	5.61	9
7	5.40	5
8	5.35	8
9	5.29	6
10	4.11	5

The fuzzy partitions are composed by fuzzy numbers given by semi-trapezoidal or triangular fuzzy sets. The first and last fuzzy sets are semi-trapezoidal fuzzy sets, the intermediate fuzzy sets are triangular fuzzy sets. The triangular fuzzy numbers can be represented with three number as A = (a_1,a_2,a_3) and B = (b_1,b_2,b_3). In Table 3 we show the four fuzzy sets forming the fuzzy partition of the domain U_x.

Table 3. The fuzzy partition for Ux

Label	a_1	a_2	a_3
low	0	2	4
adequate	2	4	5
fair	4	5	6
high	5	6	8

In Table 4 we show the five fuzzy sets forming the fuzzy partition of the domain U_y.

Table 4. The fuzzy partition for Uy

Label	a_1	a_2	a_3
very low	0	1	3
low	1	3	4
mean	3	4	7
high	4	7	10

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very high	7	10	12

In Figs. 2 and 3 we show the graphs of the fuzzy sets of the fuzzy partitions for the domains U_x and U_y, respectively.

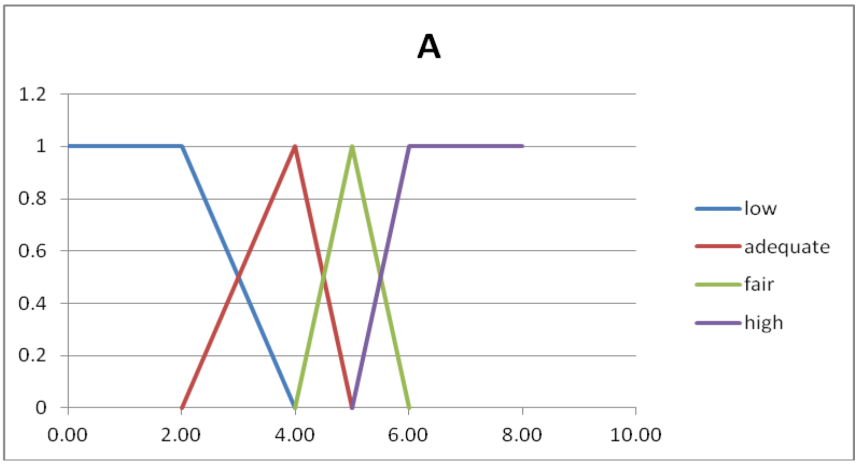

Fig. 2. Graph of the fuzzy sets of the fuzzy partition for U_x

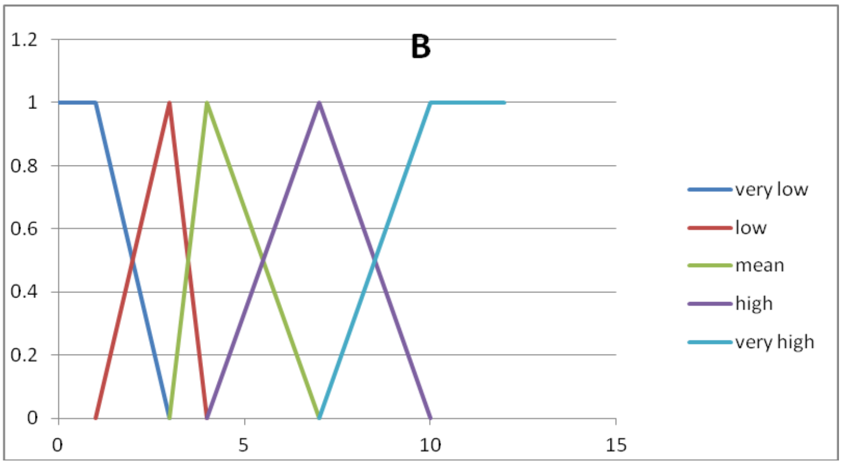

Fig. 3. Graph of the fuzzy sets of the fuzzy partition for U_y

The expert considers significant the following rules:
Rule 1 → IF A= low THEN B = very low
Rule 2 → IF A = adequate THEN B = mean
Rule 3 → IF A = fair THEN B = high

Then, the index of strength of each fuzzy rule is calculated as well. Table 5 (resp., 6) shows E, H, I, corresponding to the three rules for p=1 (resp.,p = 2).

Table 5. E, H, I value obtained by setting p = 1

rule	**p = 1**		
	E	**H**	**I**
Rule1	99.00	0.00	0.99
Rule2	82.50	3.68	0.79

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Rule3	75.78	5.76	0.70

Table 6. E, H, I value obtained by setting p = 2

rule	**p = 1**		
	E	**H**	**I**
Rule1	95.60	0.00	0.95
Rule2	75.85	4.36	0.71
Rule3	64.66	6.87	0.58

For calibrating the threshold value for the index I, after extracting the data x and y, the expert analyzes how each fuzzy rule appears consistent with respect to the data, that is to which degree the fuzzy rule is confirmed from the data. He considers *Rule 1* completely consistent with the data and *Rule 2* sufficiently consistent with the data, therefore the *Rule 3* is considered not sufficiently consistent with the data. For this motivation we set the threshold value to a value less or equal to the strength index I calculated for Rule 2. This value is 0.79 for p = 1 and 0.71 for p = 2. Then in all the experiments we set p = 2 and a threshold value I_{th} = 0.7.

Below we present the results of the second experiment in which two input variables are considered. The inputs are the following: x_1 = *Percent of families in residential properties with respect to the total resident families* and x_2 = *Percent of graduates with respect to the total workforce*. The output is y = *Unemployment rate*.

In Table 7 we show the data extracted for the 10 municipalities.

Table 7. The I/O data extracted for the 10 municipalities

Municipality	**x_1**	**x_2**	**y**
1	30.86	60.86	13.46
2	13.62	52.52	26.77
3	11.58	53.47	26.53
4	8.330	48.41	30.34
5	29.94	69.54	13.53
6	4.410	43.85	36.51
7	4.280	36.34	41.52
8	5.640	36.21	40.69
9	6.880	54.69	31.42
10	12.84	62.39	22.76

In Tables 8, 9, 10 we show the fuzzy sets forming the fuzzy partitions of the domain U_{x1}, U_{x2}, U_y, respectively.

Table 8. The fuzzy partition for U_{x1}

Label	**a_1**	**a_2**	**a_3**
very low	0	1	3
low	1	3	4
mean	3	4	7
high	4	7	10
very high	7	10	12

Table 9. The fuzzy partition for U_{x2}

Label	a_1	a_2	a_3
low	0	30	40
adequate	30	40	60
fair	40	60	80
high	60	80	100

Table 10. The fuzzy partition for Uy

Label	a_1	a_2	a_3
very low	0	10	15
low	10	15	30
mean	15	30	50
high	30	50	60
very high	50	60	100

In Figs. 4, 5, 6 we show the graphs of the fuzzy sets of the fuzzy partitions for the domains U_{x1}, U_{x2}, U_y, respectively.

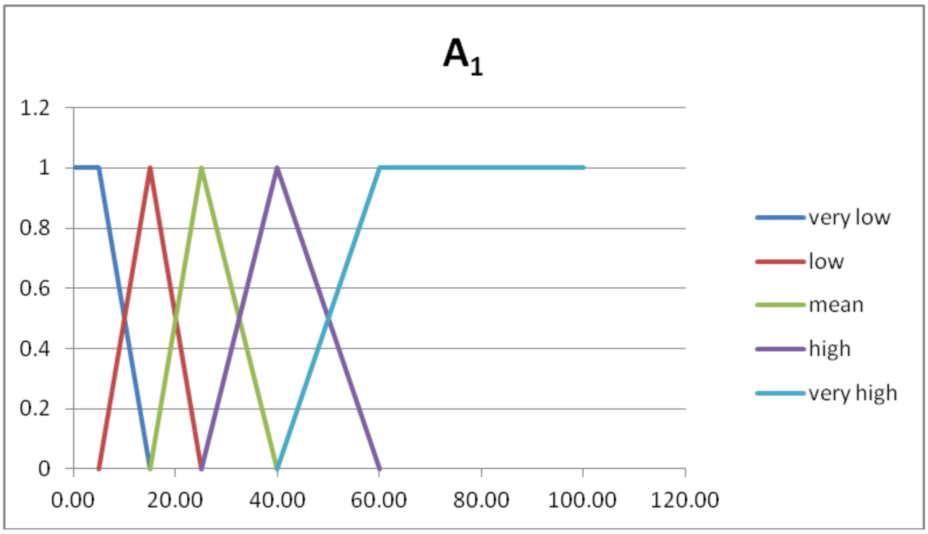

Fig. 4. Graph of the fuzzy sets of the fuzzy partition for U_{x1}

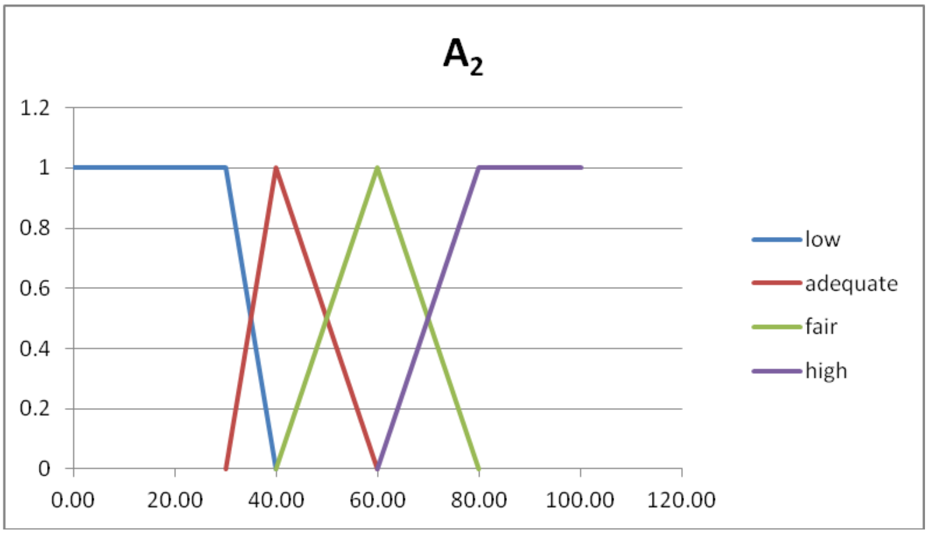

Fig. 5. Graph of the fuzzy sets of the fuzzy partition for Ux2

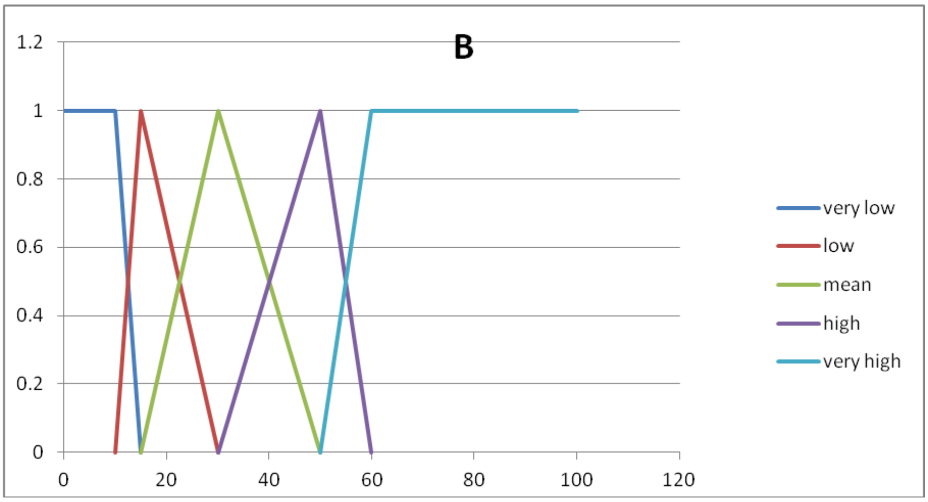

Fig. 6 Graph of the fuzzy sets of the fuzzy partition for Uy

The expert considers the following fuzzy rules:

Rule 1 → IF A_1= very low AND A_2= low THEN B = very high
Rule 2 → IF A_1= low AND A_2= low THEN B = high
Rule 3 → IF A_1= mean AND A_2= adequate THEN B = mean
Rule 4 → IF A_1= mean AND A_2= fair THEN B = mean
Rule 5 → IF A_1= mean AND A_2= high THEN B = low
Rule 6 → IF A_1= high AND A_2= fair THEN B = low
Rule 7 → IF A_1= high AND A_2= high THEN B = very low
Rule 8 → IF A_1= very high AND A_2= high THEN B = very low

In Table 11 we show the value of the index I calculated for any fuzzy rule (column I rule), by setting p = 2. For each pair $(A_w^{(1)}, B_z)$ and $(A_w^{(2)}, B_z)$ in the rule, we show the values of E, H, I.

The results in Table 11 show that the final indices of the fuzzy rules are greater than the threshold I_{th} = 0.7, except for the fuzzy rules 1 and 2.

Table 11. Values of the index I obtained for p = 2

rule	pair	p=2			
		E	**H**	**I**	**I rule**
Rule 1	(A1 = very low, B = very high)	32.00	0.00	0.32	0.32
	(A2 = low, B = very high)	84.50	0.00	0.84	
Rule 2	(A1 = low, B = high)	64.24	2.67	0.61	0.61
	(A2 = low, B = high)	88.88	0.00	0.89	
Rule 3	(A1 = mean, B = mean)	84.65	1.20	0.83	0.80
	(A2 = adequate, B = mean)	82.92	2.67	0.80	
Rule 4	(A1 = mean, B = mean)	95.30	0.00	0.95	0.72
	(A2 = fair, B = mean)	76.58	5.68	0.72	
Rule 5	(A1 = mean, B = low)	88.59	2.00	0.87	0.87

	(A2 = high, B = low)	90.81	0.00	0.91	
Rule 6	(A1 = high, B = low)	90.60	2.00	0.89	0.89
	(A2 = high, B = low)	90.81	0.00	0.91	
Rule 7	(A1 = high, B = very low)	86.68	1.85	0.85	0.85
	(A2 = high, B = very low)	86.20	0.00	0.86	
Rule 8	(A1 = very high, B = very low)	100.00	0.00	1.00	0.91
	(A2 = high, B = very low)	90.81	0.00	0.91	

5. Conclusions

We present a new method that uses the fuzzy energy and fuzzy entropy measures to evaluate the strength of fuzzy rules set by an expert with respect to a set of data. We correlate the input and the output data via equation (5) where t is the Yager t-norm and calculate the corresponding relations which are solutions of (5) having the maximum energy and the minimum entropy.

After the processes of the creation of the fuzzy partitions of the input and output variable domains and of the significant fuzzy rule set by the expert, a normalized index of the strength of the fuzzy rule with respect to the data is measured for each fuzzy rule.

If this index is greater than a calibrated threshold, then the fuzzy rule is considered significant with respect to the data. We extend this approach to fuzzy rules in which there are two or more input variables. In this case we calculate the index of strength separately for each pair of input and output fuzzy sets and we assign as best index of strength for that rule having the minimum value of these indices. The results of some experiments are presented as well.

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