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GRAPHS AND BINARY SYSTEMS

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Abstract. In this paper, we observe that if X is a set and $(Bin(X), \square)$ is the semigroup of binary systems on X, then its center $ZBin(X)$ consists of the locally-zero-semigroups and that these can be modeled as (simple) graphs and conversely. Using this device we show that we may obtain many results of interest concerning groupoids by reinterpreting graph theoretical properties and at the same time results on graphs G may be obtained by considering them as elements of centers of the semigroups of binary systems $(Bin(X), \square)$ where $X = V(G)$, the vertex set of G.

1. Introduction

In a sequence of papers Nebeský [5, 6, 7] has sought to associate with graphs (V, E) groupoids $(V, *)$ with various properties and conversely. Although not identical in outlook there is some similarity with Nebeský's work and that contained in this paper. In particular, Nebeský defines a *travel groupoid* $(X, *)$ as a groupoid satisfying the axioms: $(u*v)*u = u$ and $(u*v)*v = u$ implies $u = v$. If one adds these two laws to the orientation property, then $(X, *)$ is an OP-travel-groupoid. In this case $u * v = v$ implies $v * u = u$, i.e., $uv \in E$ implies $vu \in E$, i.e., the digraph (X, E) is a (simple) graph if $uu \notin E$, with $u*u = u$. Also, if $u \neq v$, then $u*v = u$ implies $(u*v)*v = u*v = u$ is impossible, whence $u * v = v$ and $uv \in E$, so that (X, E) is a complete (simple) graph. On the other hand if (X, E) is a complete (simple) graph, then $u \neq v$ implies $uv \in E$ and $u * v = v$, whence $(X, *)$ is the right zero semigroup in any case. Given that $(u * v) * v = v$ as a consequence, it follows that $(u * v) * v = u$ implies $u = v$ and $(X, *)$ is an OP-travel-groupoid. As we shall note below, our point of view permits us to consider arbitrary (simple) graphs as particular groupoids. By using this model, it is possible to assign to groupoids of a particular (locally zero) type certain simple graphs as well. Using the viewpoint developed, we are then able to assign graph theoretical parameter to

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groupoids in a meaningful way. How this is done is the topic of what follows. We have concentrated on two such parameters, i.e., "covering" and "shortest distance". It is clear that a great deal of more work remains to be done and can be done in a straightforward manner.

2. Graphs and binary systems.

Given a set X, a *binary system* or *groupoid* on the set X is a mapping $* : X^2 \to X$, with the image $*(x, y)$ usually denoted in the fashion of a product, i.e., $*(x, y) = z$ becomes $x*y = z$. The collection of all binary systems $(X, *)$ defined on X is denoted by $Bin(X)$. Given two such binary systems $(X, *)$ and $(X, \bullet)$ a binary operation "$\Box$" on $Bin(X)$ is defined by

$$(X, *)\Box(X, \bullet) = (X, \Box)$$

where for all $x, y \in X$, $x\Box y = (x * y) \bullet (y * x)$.

It follows, as shown in Kim and Neggers ([2]), that the operation $\Box$ is associative, i.e., $(Bin(X), \Box)$ is a semigroup with identity, the *left-zero-semigroup on* X, i.e., the groupoid $(X, *)$ for which $x * y = x$ for all $x, y \in X$. Another element with interesting properties in the semigroup $(Bin(X), \Box)$ is the *right-zero-semigroup*, i.e., the groupoid $(X, *)$ for which $x * y = y$ for all $x, y \in X$.

Fayoumi ([1]) has shown that a groupoid $(X, *)$ commutes relative to the operation $\Box$ of $(Bin(X), \Box)$ if and only if any two element subset of $(X, *)$ is a subgroupoid which is either a left-zero-semigroup or a right-zero-semigroup. Thus, $(X, *)$ is an element of the center $ZBin(X)$ of $(Bin(X), \Box)$ if and only if for any pair of elements $x, y \in X, x * y = x, y * x = y$ or $x * y = y, y * x = x$. Therefore, among these groupoids one finds both the left-zero-semigroup on X and the right-zero-semigroup on X as extreme cases. Furthermore, it is easily seen that $(ZBin(X), \Box)$ is itself a semigroup with identity, a subsemigroup of the semigroup $(Bin(X), \Box)$.

What does all this have to do with graphs ?

Suppose that $(X, *)$ is an element of $ZBin(X)$. Suppose in addition that we construct a graph Γ_X as follows: $V(\Gamma_X) = X$ and $(x, y) \in E(\Gamma_X)$, the edge set of Γ_X provided that $x \neq y, x * y = x, y * x = y$. Thus, if $(x, y) \in E(\Gamma_X)$, then $(y, x) \in E(\Gamma_X)$ as well, and we identify $(x, y) = (y, x)$ as an *undirected edge* of Γ_X. Similarly, if $(x, y) \notin E(\Gamma_X)$, then by the fact that $(X, *)$ is an element of $ZBin(X)$, it follows that $x*y = y, y*x = x$ and $(x, y) = (y, x)$ determines the absence of an edge directed or otherwise. Since $x * x = x$ in any case, we do not consider this to be of particular graphical interest. The mapping $(X, *) \leftrightarrow \Gamma_X$ accomplishes the following:

Theorem 2.1. *If $G = (V(G), E(G))$ is a simple graph with vertex set $V(G) = X$ and edge set $E(G)$, then G determines a unique groupoid*

$(X,*)$ in the center of the semigroup $(Bin(X), \square)$ by defining the binary operation "$*$" as $x*y = x, y*x = y$ if $(x,y) = (y,x) \in E(G)$ and $x*y = y, y*x = x$ if $(x,y) = (y,x) \notin E(G)$.

Theorem 2.2. *If $G = (V(G), E(G))$ is a simple graph and $(X,*)$ is defined as in Theorem 1, then $V(\Gamma_X) = X = V(G)$ and $E(\Gamma_X) = E(G)$, so that $G = \Gamma_X$.*

Proof. The proofs of both theorems are straightforward. $\square$

Proposition 2.3. *If $(X,*)$ is the left-zero-semigroup, then Γ_X is the complete graph on X. If $|X| = n < \infty$, then $\Gamma_X = K_n$.*

Proof. From the definition it is clear that if $\{x,y\} \subseteq X, x \neq y$, then $x*y = x, y*x = y$, so that $(x,y) = (y,x) \in E(\Gamma_X)$, and the conclusion follows. $\square$

Proposition 2.4. *If $(X,*)$ is the right-zero-semigroup, then Γ_X is the null(empty) graph on X, i.e., $E(\Gamma_X) = \emptyset$, while $V(\Gamma_X) = X$.*

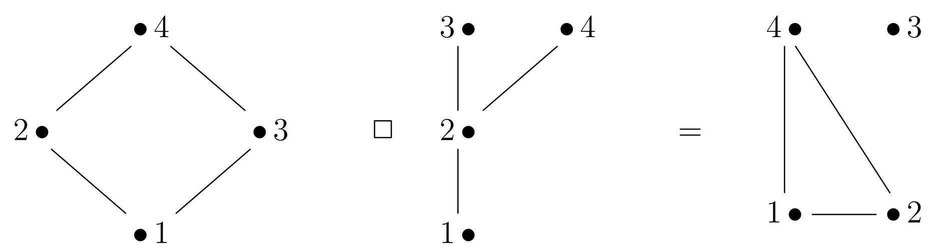
Proof. From the definition it is clear that if $\{x,y\} \subseteq X, x \neq y$, then $x*y = y, y*x = x$, so that $(x,y) = (y,x) \notin E(\Gamma_X)$, i.e., $E(\Gamma_X) = \emptyset$ and the conclusion follows. $\square$

To illustrate the close relationship between simple graphs and groupoids we note the following.

Theorem 2.5. *If $G = (V(G), E(G))$ is a simple graph and $(X,*), X = V(G)$, is the groupoid associated with G, then for the right-zero-semigroup $(X,\bullet)$ defined on X, $(X,\square) = (X,*)\square(X,\bullet) = (X,\bullet)\square(X,*)$ defines $(X,\square)$ as the complementary graph of G.*

Proof. Suppose that $(x,y) = (y,x) \in E(G)$. Then $x*y = x, y*x = y$. Hence $x\square y = (x*y)\bullet(y*x) = y*x = y$ and $y\square x = (y*x)\bullet(x*y) = x*y = x$. Thus $(x,y) = (y,x) \notin E(\Gamma_{(X,\square)})$. Similarly, if $(x,y) = (y,x) \notin E(G)$, then $(x,y) = (y,x) \in E(\Gamma_{(X,\square)})$, so that $(X,\square)$ is the opposite groupoid of $(X,*)$, with $x\square y = y*x$, while $\Gamma_{(X,\square)}$ is the complementary graph of G, with $(x,y) = (y,x) \in E(\Gamma_{(X,\square)})$ if and only if $(x,y) = (y,x) \notin E(G)$. $\square$

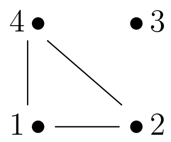
Example 2.6. The operation "$\square$" on $Bin(X)$ induces an operation "$\square$" on the graphs with vertex set X as well as already illustrated in the proof of Theorem 2.5. Let $X = \{1,2,3,4\}$ and consider two simple graphs on the vertex set X. Then we have



From the fact that $(Bin(X), \square)$ is a semigroup, it follows that this product is associative as well. As groupoids we have tables and a resultant(product) table:

$$
\begin{array}{c|cccc}
* & 1 & 2 & 3 & 4 \\ \hline
1 & 1 & 1 & 1 & 4 \\
2 & 2 & 2 & 3 & 2 \\
3 & 3 & 2 & 3 & 3 \\
4 & 1 & 4 & 4 & 4
\end{array}
\ \square\
\begin{array}{c|cccc}
\bullet & 1 & 2 & 3 & 4 \\ \hline
1 & 1 & 1 & 3 & 4 \\
2 & 2 & 2 & 2 & 2 \\
3 & 1 & 3 & 3 & 4 \\
4 & 1 & 4 & 3 & 4
\end{array}
\ =\
\begin{array}{c|cccc}
\square & 1 & 2 & 3 & 4 \\ \hline
1 & 1 & 1 & 3 & 1 \\
2 & 2 & 2 & 3 & 2 \\
3 & 1 & 2 & 3 & 4 \\
4 & 4 & 4 & 3 & 4
\end{array}
$$

Hence reconstructing the graph $\Gamma_{(X,\square)}$ we obtain



as pictured.

3. Properties of graphs as properties of groupoids.

If $G = (V(G), E(G))$ is a simple graph, then the *covering number* $\gamma(G)$ is the cardinal number of a minimal covering set which is smallest among these cardinal numbers. A covering set is a set of vertices such that any vertex not in the set is connected to an element in the set via an edge. A covering set is *minimal* if no vertex can be deleted from the set and still maintain the property of being a covering set. Thus, e.g., $\gamma(K_n) = \gamma(K_{1,n-1}) = 1$, since $\gamma(G) = 1$ for any n-graph containing a $K_{1,n-1}$, i.e., a star or a complete bipartite graph partitioned into two classes containing 1 and $n-1$ elements respectively. The number $\gamma(G)$ is an example of a graph parameter which can be directly taken over by groupoids. Indeed, we shall consider an element x of a groupoid to *cover* an element y of a groupoid if $x * y = x$, and mutually y covers x if $y * x = y$ as well. In the setting of graphs and the constructions made above, it follows that if $G = (X, E(G)), X = V(G)$, is a simple graph, then if $x * y = x$ it follows that $y * x = y$ as well whenever $(x, y) = (y, x) \in E(G)$. Thus, for simple graphs the two notions are equivalent. However, in the context of groupoids they are not the same.

Example 3.1. Suppose that $(X, \leq, 0)$ is a poset with minimal element 0. The standard BCK-algebra $(X, *, 0)$ for this poset is defined by setting $x * y = 0$ if $x \leq y$ and $x * y = x$ otherwise. Now $x * 0 = x$ and $0 * x = 0$, for all $x \in X$ and thus 0 mutually covers every element x of X. On the other hand, if $0 \notin \{x, y\}$ and $x \neq y$, then if $x \not\leq y, x * y = x$ and x covers y. If $y \not\leq x$, then $y * x = y$ and x and y mutually cover each other. Thus, we may consider the Hasse-diagram of the poset in terms of this relationship. For example, a poset $(X := \{0, 1, 2, 3, 4\}, \leq)$

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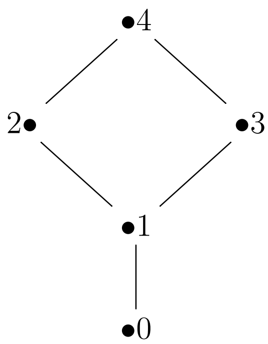

has a standard BCK-algebra table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

In terms of the covering relation we obtain:

$$\begin{aligned}\mathcal{R}_C = \{&(0,1),(0,2),(0,3),(0,4),(1,0),(2,0),(2,1),(2,3),\\ &(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\},\end{aligned}$$

while the mutual covering relation yields

$$\mathcal{R}_{MC} = \{(0,1),(1,0),(0,2),(0,3),(3,0),(0,4),(4,0),(2,3),(3,2)\}$$

For more information on BCK/BCI-algebras we refer to [3, 4].

Following the analogy pathway, it is clear that in this case, if $\gamma_C(X,*,0)$ is the cardinal of a minimum covering set (i.e., a minimal covering set of smallest cardinal number) then $\gamma_C(X,*,0) = 1$. If $\gamma_{MC}(X,*,0)$ is the cardinal number of a minimum mutual covering set, then $\gamma_{MC}(X,*,0) = 1$ as well.

Proposition 3.2. *If $(X,*)$ is a groupoid, then*

$$\gamma_C(X,*) \le \gamma_{MC}(X,*)$$

Proof. Suppose that $S \subseteq X$ is a mutual covering set for $(X,*)$. Then S is also a covering set for $(X,*)$. Hence, if S is a minimum mutual covering set for $(X,*)$, then it is a covering set for $(X,*)$ and hence it has a cardinal at least as large as $\gamma_C(X,*)$. □

Corollary 3.3. *If $\gamma_{MC}(X,*) = 1$, then $\gamma_C(X,*) = 1$.*

Proposition 3.4. *If $(X,*,0)$ is a BCK-algebra, then $\gamma_{MC}(X,*,0) = 1$.*

Proof. This follows from the fact that $0 * x = 0, x * 0 = x$ for all $x \in X$. Hence $\{0\}$ is a mutual covering set. □

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A *d-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (i) $x*x=0$, (ii) $0*x=0$, (iii) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y\in X$. For more detailed information we refer to [8, 9].

Proposition 3.5. *If $(X,*,0)$ is a d-algebra, then $\gamma_C(X,*,0)=1$.*

Proof. This follows from the fact that $0*x=0$ for all $x\in X$. □

Proposition 3.6. *If $G=(X,E(G)), X=V(G)$, is a simple graph and $(X,*)$ is the associated groupoid, then $\gamma_C(X,*)=\gamma_{MC}(X,*)=\gamma(G)$, the covering number of G.*

Proposition 3.7. *If $(X,*,e)$ is a group, then $\gamma_C(X,*,e)-1=\gamma_{MC}(X,*,e)=|X|$, the cardinal of X.*

Proof. If $x*y=x$, then $y=e$. Hence, it follows that the smallest covering set of $(X,*,e)$ is $X\backslash\{e\}$. Now, if $x*y=x$ and $y*x=y$, then $x=y=e$. Hence $X\backslash\{e\}$ is not a mutual covering set, i.e., the smallest mutual covering set is X itself, i.e., $\gamma_{MC}(X,*,e)=\gamma_C(X,*,e)+1$ as claimed. □

Proposition 3.8. *Let $(X,*)$ be a groupoid with $\gamma_{MC}(X,*)=1$. Define a set $X_1:=\{u|\{u\}$ is a minimum mutual covering set $\}$. Then $(X_1,*)$ is a subsemigroup of the groupoid $(X,*)$ and a left-zero-semigroup.*

Proof. Suppose $u,v\in X_1$. Then $u*v=u, v*u=v$, since $u\in X_1$. It follows that $(X_1,*,)$ is a left-zero-semigroup as claimed and as such a subsemigroup of $(X,*)$. □

Corollary 3.9. *If $(X,*,0)$ is a BCK-algebra, then $X_1=\{0\}$.*

Proof. It follows immediately from Propositions 3.4 and 3.8. □

4. Distances of graphs with groupoids.

Given a groupoid $(X,*)$ we consider the *shortest distance* $d(x,y)$ to be $n+1$ if for $u_1,u_2,\cdots,u_n$ we have $x*u_1=x, u_i*u_{i+1}=u_i, i=1,2,\cdots,n-1, u_n*y=u_n$, where $x\neq y, x,y\in X$, and there is no set with fewer elements having this property. The *mutual shortest distance* $md(x,y)$ is $n+1$ for elements $x\neq y, x,y\in X$, if for $u_1,u_2,\cdots,u_n$ we have $x*u_1=x, u_1*x=u_1, u_i*u_{i+1}=u_i, u_{i+1}*u_i=u_{i+1}, i=1,2,\cdots,n-1, u_n*y=u_n, y*u_n=y$, and there is no set with fewer elements having this property. We set $d(x,x)=md(x,x)=0$ for any $x\in X$.

Proposition 4.1. *If $(X,*,0)$ is a BCK-algebra, then $d(x,y)\leq 2$ for all $x,y\in X$.*

Proof. If $x\neq y$, then $x*0=x, 0*y=0$ and thus $d(x,y)\leq 2$. □

Example 4.2. If $(X, \leq, 0)$ is a poset with standard BCK-algebra $(X, *, 0)$, then $x \not\leq y$ implies $x * y = x$ and thus $d(x, y) = 1$ since $x \neq y$. If $x \not\leq y, y \not\leq x$, then $x * y = x$ and $y * x = y$, so that $md(x, y) = 1$ in that situation as well.

Proposition 4.3. *If $(X, *)$ is any groupoid then, for any $x, y, z \in X$,*

(i) $d(x, y) \leq md(x, y)$,
(ii) $d(x, z) \leq d(x, y) + d(y, z)$,
(iii) $md(x, y) \leq md(x, y) + md(y, z)$,
(iv) $md(x, y) = md(y, x)$.

Proof. It follows immediately from the definition. □

Given a groupoid $(X, *)$ and $x, y \in X$, the *cycle number* $c(x, y)$ is the sum $c(x, y) = d(x, y) + d(y, x)$. Since $md(x, y) = md(y, x)$, the *mutual cycle number* $mc(x, y)$ is simply $2md(x, y)$.

Proposition 4.4. *If $(X, *)$ is any groupoid with the property that $(x * y) * z = (x * z) * y$, then $d(x * y, z) \leq d(x, z)$.*

Proof. Suppose $d(x, z) = n+1$, with intermediate elements $u_1, u_2, \cdots, u_n$. Then $(x * y) * u_1 = (x * u_1) * y = x * y$, $(u_1 * y) * u_2 = (u_1 * u_2) * y = u_1 * y, \cdots, (u_n * y) * z = (u_n * z) * y = u_n * y$, so that $u_1 * y, \cdots, u_n * y$ could serve as a set of intermediate elements for $x * y$ and z. The proposition follows. □

Given a poset $(X, \leq)$, it is said to be *strongly connected* if $d(x, y) < \infty$ for all $x, y \in X$. It is said to be *strongly mutually connected* if $md(x, y) < \infty$ for all $x, y \in X$. By Proposition 4.3-(i), $d(x, y) \leq md(x, y)$ so that strongly mutually connected groupoids are also strongly connected.

Proposition 4.5. *If $(X, *)$ is a commutative (abelian) groupoid, i.e., $x * y = y * x$ for all $x, y \in X$, then $x * y = x$ implies $y * x = x$. Hence, if $x \neq y$, then $md(x, y) = \infty$.*

Proof. If $x \neq y$ and $u_1, \cdots, u_n$ is a set of intermediate elements for $md(x, y) = n$, then $x * u_1 = x, u_1 * x = u_1$ implies $x = u_1$ and by continuation $x = u_1 = u_2 = \cdots = u_n = y$, a contradiction. It follows that $md(x, y) = \infty$. □

Example 4.6. Let $DG = (\{x, y\}, x \to y)$ be the digraph with diagram $x \bullet \longrightarrow \bullet\, y$. Then the associated groupoid $(X, *)$ has the Cayley table:

*	x	y
x	x	x
y	x	y

. It follows that $(X, *)$ is commutative and $d(x, y) = 1, d(y, x) = md(x, y) = \infty$. If $DG = (\{x, y, z\},\ x \bullet \longrightarrow \bullet\, y\ ,\ y \bullet \longrightarrow \bullet\, z\ ,$

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$z \bullet \longrightarrow \bullet x$) then $(X,*)$ has the Cayley-table

$*$	x	y	z
x	x	x	z
y	x	y	y
z	z	y	z

so that $(X,*)$ is a commutative groupoid. We have $d(x,y)=d(y,z)=1, d(x,z)=d(y,x)=d(z,y)=2$ so that $(X,*)$ is strongly connected, but not strongly mutually connected.

Proposition 4.7. *Let $(X,*)$ be a groupoid such that $x*y=x$ implies $y*x=y$. Then $d(x,y)=md(x,y)$ for all $x,y\in X$.*

Proof. Suppose that $d(x,y)=n$ and $u_1,\cdots,u_n$ is a set of intermediate elements. Then $x*u_1=x$ implies $u_1*x=u_1, u_i*u_{i+1}=u_i$ implies $u_{i+1}*u_i=u_{i+1}, u_n*z=u_n$ implies $z*u_n=z$. Hence $u_1,\cdots,u_n$ is also a set of intermediate elements for $md(x,y)$, i.e., $md(x,y)\le d(x,y)$. By Proposition 4.3-(i), $d(x,y)\le md(x,y)$ and thus $d(x,y)=md(x,y)$. $\square$

We shall consider the groupoids of Proposition 4.7 to be *undirected groupoids*.

Proposition 4.8. *If $G=(X,E(G)), X=V(G)$, is an (undirected) simple graph, then the associated groupoid $(X,*)$ is an undirected groupoid.*

Proof. It follows from the definition. $\square$

Example 4.9. Consider the groupoid $(X,*)$ with $X=\{a,b,c\}$ and Cayley-table:

$*$	a	b	c
a	a	a	b
b	b	b	b
c	b	c	c

. Then $a*c=c*a=b, a*b=a, b*a=b, c*b=c, b*c=b$, and $(X,*)$ is an undirected groupoid since $a*b=a$ implies $b*a=b$. However, it does not correspond to a simple graph. If we compute $md(a,b)=1, md(b,c)=1, md(a,c)=2$, we obtain an undirected graph as follows: $a\bullet$ —— $\bullet b$ —— $\bullet c$. Thus, the undirected groupoid $(X,*)$ is also strongly mutually connected.

5. Frame graphs with groupoids.

Given a groupoid $(X,*)$, we consider the *frame* of $(X,*)$ to be the collection of all pairs $\{x,y\}$ such that $md(x,y)=md(y,x)=1$. From the frame of $(X,*)$ we may construct a simple graph $F(X,*)$ with $VF(X,*)=\{x|x\in\{x,y\}$ for some pair in the frame $\}$, and $EF(X,*)=\{(x,y)=(y,x)|$ for $\{x,y\}$ a pair in the frame $\}$. $F(X,*)$ is the *frame graph* of $(X,*)$ and every groupoid has such a frame graph.

Given a groupoid $(X,*)$, we consider the *diframe* of $(X,*)$ to be the collection of all ordered pairs (x,y) such that $d(x,y)=1$. From the diframe of $(X,*)$ we may construct a simple digraph $DF(X,*)$ with

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$VDF(X,*) = \{x|(x,y)$ or (y,x) belongs to the diframe of $(X,*)$ for some $y \in X\}$, and $EDF(X,*) = \{(x,y) \mid (x,y)$: an ordered pair in the diframe $\}$. $DF(X,*)$ is the *diframe graph* of $(X,*)$ and every groupoid has such a diframe graph. The frame graph $F(X,*)$ is seen to be a subdigraph of the diframe graph $DF(X,*)$ in a natural way, i.e., the undirected edge $\{x,y\}$ of $F(X,*)$ generates a pair of directed edges (x,y) and (y,x) in $DF(X,*)$.

Given a groupoid $(X,*)$ with a frame $F(X,*)$, it generates a groupoid $(VF(X,*),\bullet)$, where $x,y \in VF(X,*)$ implies $x \bullet y = x, y \bullet x = y$ if $md(x,y) = md(y,x) = 1$ and $x \bullet y = y, y \bullet x = x$ otherwise, i.e., the groupoid $(VF(X,*),\bullet)$ is an element of $ZBin(VF(X,*))$, the *frame groupoid* of $(X,*)$.

Example 5.1. Given the groupoid $(X,*)$ of Example 4.9 has a frame $a\bullet\!\!-\!\!-\!\!-\!\!\bullet b \!-\!\!-\!\!-\!\!\bullet c$ and thus $(VF(X,*),\bullet)$ has the Cayley table:

$*$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

. If, for the groupoid $(X,*)$, we consider the diframe graph $DF(X,*)$, then the associated algebraic structure is the groupoid $(VDF(X,*),\bullet)$ where $x*y = x$ if $(x,y) \in EDF(X,*)$, i.e., if $d(x,y) = 1$. Since $md(x,y) = md(y,x) = 1$ implies $d(x,y) = d(y,x) = 1$ as well, it follows that in such a case $x \to y$ and $y \to x$ represent the edge $x\bullet\!\!-\!\!-\!\!-\!\!\bullet y$. Hence, in this sense the frame graph, $F(X,*)$ is a subdigraph of the diframe graph $DF(X,*)$.

Example 5.2. Given the groupoid as in Example 4.9, the condition $a*c = c*ab$ yields $(a,c),(c,a)$ as not being elements of $EDF(X,*)$ and thus $a*c = c, c*a = a$ in $DF(X,*)$, i.e., $(VF(X,*),\bullet)$ and $(VDF(X,*),\bullet)$ have exactly the same Cayley table.

Example 5.3. Suppose that $(X,*)$ is the groupoid with $X = \{a,b,c\}$, and $DF(X,*)$ the digraph

$a\bullet \longleftrightarrow \bullet b \longrightarrow \bullet c$. Then $F(X,*)$ is the frame graph $a\bullet\!\!-\!\!-\!\!-\!\!\bullet b$.

The table for $(X,*)$ can be filled in as follows:

$*$	a	b	c
a	$\surd$	a	c
b	b	$\surd$	b
c	a	b	$\surd$

where $\surd$ can be filled with any element of X. For $(VDF(X,*),\bullet)$ the table is

$*$	a	b	c
a	a	a	c
b	b	b	b
c	a	b	c

. Also, for $(VF(X,*),\bullet)$ the Cayley table is

$\bullet$	a	b
a	a	a
b	b	b

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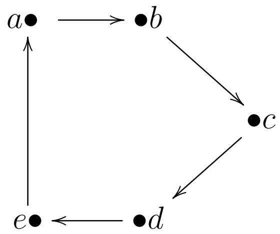
Example 5.4. Let $(X, *)$ be a groupoid with the following general table:

$*$	a	b	c	d	e
a	$\surd$	a	$\neg$ a	$\neg$ a	$\neg$ a
b	$\neg$ b	$\surd$	b	$\neg$ b	$\neg$ b
c	$\neg$ c	$\neg$ c	$\surd$	c	$\neg$ c
d	$\neg$ d	$\neg$ d	$\neg$ d	$\surd$	d
e	e	$\neg$ e	$\neg$ e	$\neg$ e	$\surd$

where $\neg x$ means that "not x", i.e., anything but x in X, and $\surd$ can be filled with any element of X. Then $DF(X, *)$ is the following graph:



i.e., for any $x, y \in \{a, b, c, d, e\}$, $c(x, y) = d(x, y) + d(y, x) = 5$. Also, $F(X, *)$ has $VF(X, *) = \emptyset$ since there is no pair $\{x, y\}$ with $(x, y), (y, x)$ elements of $EDF(X, *)$.

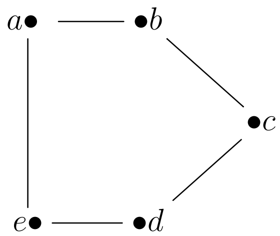
The groupoid $(VDF(X, *), \bullet)$ has the associated table:

$\bullet$	a	b	c	d	e
a	a	a	c	d	e
b	a	b	b	d	e
c	a	b	c	c	e
d	a	b	c	d	d
e	e	b	c	d	e

Example 5.5. Let $(X, *)$ be a groupoid with the following general table:

$*$	a	b	c	d	e
a	$\surd$	a	$\neg$ a	$\neg$ a	a
b	b	$\surd$	b	$\neg$ b	$\neg$ b
c	$\neg$ c	c	$\surd$	c	$\neg$ c
d	$\neg$ d	$\neg$ d	d	$\surd$	d
e	e	$\neg$ e	$\neg$ e	e	$\surd$

where $\neg x$ means that "not x", i.e., anything but x in X, and $\surd$ can be filled with any element of X. Then $F(X, *) = DF(X, *)$ is the simple graph:



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The corresponding groupoid $(VF(X,*),\bullet)$ (or $(VDF(X,*),\bullet)$) has the following multiplication table:

$\bullet$	a	b	c	d	e
a	a	a	c	d	a
b	b	b	b	d	e
c	a	c	c	c	e
d	a	b	d	d	d
e	e	b	c	e	e

The question is then how to represent the undirected cyclical nature of the groupoid $(X,*)$ in the most elegant way, such as we were able to do in Example 5.4.

Theorem 5.6. *Let $f:(X,*)\to(Y,\bullet)$ be an onto homomorphism for groupoids. Define a binary operation "$\rightsquigarrow$" on X by $x\rightsquigarrow y$ by $f(x*y)=f(x)$ for any $x,y\in X$. If we define $DEF_f(X,*) := \{(x,y)\,|\,x\rightsquigarrow y\}$, then the graph $DF(X,*)$ is a subdigraph of $DF_f(X,*)$.*

Proof. If $f:(X,*)\to(Y,\bullet)$ is an onto homomorphism for groupoids, then $x*y=x$ implies $f(x)\bullet f(y)=f(x)$. Thus, either $f(x)=f(y)$ or $x\bullet \longrightarrow \bullet y$ maps to $f(x)\bullet \longrightarrow \bullet f(y)$ from $DF(X,*)$ to $DF(Y,\bullet)$. Furthermore, if $x\bullet \longrightarrow \bullet y$ is an arrow (directed edge) in $DF(Y,\bullet)$, then $f(x)\bullet f(y)=f(x)$ if $u=f(x), v=f(y)$. Hence $f(x*y)=f(x)$, i.e., $x\rightsquigarrow y$. Thus, if we consider the digraph $DF_f(X,*)$ to have vertex-set $DVF_f(X,*)=X$ and edge-set $DEF_f(X,*):=\{(x,y)|x\rightsquigarrow y\}$, then f induces a mapping $\widehat{f}:DF_f(X,*)\to DF(Y,\bullet)$ by setting $\widehat{f}(x\rightsquigarrow y)=f(x)\to f(y)$. The mapping $\widehat{f}$ is a surjection of graphs (or a graph epimorphism). Hence the graph $DF(X,*)$ is naturally a subdigraph of $DF_f(X,*)$. $\square$

6. Comments.

From the constructions made above we see clearly that we may now infuse the theory of $(Bin(X),\square)$ with a multitude of graph theoretical notions. As a further illustration we may consider the eccentricity $e(x)$ of a vertex x of a groupoid $(X,*)$ as the maximum, $\max\{d(x,y)|x,y\in X\}$, and if the maximum is finite assign that number to the vertex x. From the eccentricity function $e:V(X)\to\mathbf{R}$, we may now derive the concepts of radius and diameter. Incidentally we may do the same with respect to mutual-distance to obtain the mutual-eccentricity function $me:V(X)\to\mathbf{R}$. After having done so, types of groupoids can be discussed in terms of these parameters. We have a selection of possibilities:

(1) Groupoids in $Bin(X)$ such that $\min e$(i.e., radius) $=\max e$(i.e., diameter) or equivalently e is a constant;

(2) Groupoids in $Bin(X)$ such that $e(x) = me(x)$ for $x \in A \subseteq X$, where A is an interesting subset of X, e.g., $A = X$;
(3) Groupoids in $Bin(X)$ with $\min e$ small but $\max e$ as large as possible;
(4) Groupoids in $Bin(X)$ with $e(x) + me(x) =$ constant K.

Obviously, there are many other interesting parameters of a graph theoretical nature which may be introduced. Hence it is clearly possible of interest in certain applications to pursue the subject further. In the interest of producing an article which is clear but manageable in size we have decided not to go into more detail in this paper.

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