

# A General Theorem on the Stability of a Class of Functional Equations including Quartic-Cubic-Quadratic-Additive Equations

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**Abstract.** We prove general stability theorems for  $n$ -dimensional quartic-cubic-quadratic-additive type functional equations of the form

$$\sum_{i=1}^{\ell} c_i f(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) = 0$$

by applying the direct method. These stability theorems can save us much trouble of proving the stability of relevant solutions repeatedly appearing in the stability problems for various functional equations.

**AMS Subject Classification:** 39B82, 39B52

**Key Words:** generalized Hyers-Ulam stability; functional equation;  $n$ -dimensional quartic-cubic-quadratic-additive type functional equation; direct method.

## 1 Introduction

The stability problem for functional equations started with the famous question of Ulam [15]: *Under what conditions does there exist an additive function near an approximately additive function?* In 1941, Hyers [4] gave a partial solution to the question of Ulam under the assumption that relevant functions are defined on Banach spaces. Indeed, (modified) Hyers' theorem states that the following statement is true for all  $\varepsilon \geq 0$ : If a function  $f$  satisfies the inequality  $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$  for all  $x$ , then there exists an exact additive function  $F$  and a function  $K : [0, \infty) \rightarrow [0, \infty)$  such that  $\|f(x) - F(x)\| \leq K(\varepsilon)$  for all  $x$  and  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$ . In this case, the Cauchy additive functional equation,  $f(x+y) = f(x) + f(y)$ , is said to have (or satisfy) the Hyers-Ulam stability.

When the Hyers' theorem is true even if we replace  $\varepsilon$  and  $K(\varepsilon)$  by  $\varphi(x)$  and  $\Phi(x)$ , where  $\varphi$  and  $\Phi$  are functions not depending on  $f$  and  $F$  explicitly, the corresponding equation is said to have (or satisfy) the generalized Hyers-Ulam stability. These terminologies will also be applied for other functional equations.

Since the Hyers' paper, a number of mathematicians have extensively investigated the stability problems for several functional equations (see for example [1, 2, 3, 5, 6, 7, 14] and the references therein).

Given abelian groups  $G_1$  and  $G_2$  and any mapping  $f : G_1 \rightarrow G_2$ , we will set

$$\begin{aligned} Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y), \\ Cf(x, y) &:= f(2x + y) - 3f(x + y) + 3f(y) - f(-x + y) - 6f(x), \\ Q'f(x, y) &:= f(2x + y) - 4f(x + y) + 6f(y) - 4f(-x + y) + f(-2x + y) - 24f(x) \end{aligned}$$

for all  $x, y \in G_1$ . In connection with these notations, a mapping  $f : G_1 \rightarrow G_2$  is said to be a quartic mapping, a cubic mapping, a quadratic mapping or an additive mapping provided  $f$  satisfies the functional equation  $Q'f(x, y) = 0$ ,  $Cf(x, y) = 0$ ,  $Qf(x, y) = 0$  or  $Af(x, y) = 0$  for all  $x, y \in G_1$ , respectively. Those functional equations seem to be familiar to us because we can easily find out examples for the existence of their solutions. For example, the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = ax^4$ ,  $ax^3$ ,  $ax^2$  or  $ax$  is a solution of  $Q'f(x, y) = 0$ ,  $Cf(x, y) = 0$ ,  $Qf(x, y) = 0$  or  $Af(x, y) = 0$ , respectively.

If a mapping  $f : G_1 \rightarrow G_2$  can be represented by the sum of a quartic mapping, a cubic mapping, a quadratic mapping, and an additive mapping, then we call  $f$  a quartic-cubic-quadratic-additive mapping and vice versa. For example, the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = ax^4 + bx^3 + cx^2 + dx$  is a quartic-cubic-quadratic-additive mapping. A functional equation is said to be a quartic-cubic-quadratic-additive functional equation provided the set of all of its solutions is the same as the set of all quartic-cubic-quadratic-additive mappings.

Throughout this paper, we assume that  $V$  and  $W$  are real vector spaces,  $X$  a real normed space, and that  $Y$  is a real Banach space if there is no specification. Let  $\mathbb{N}_0$  be the set of all nonnegative integers,  $\mathbb{R}$  the set of all real numbers, and let  $\mathbb{Q}$  denote the set of all rational numbers.

In this paper, we prove some general stability theorems that can be easily applied to the (generalized) Hyers-Ulam stability of a large class of functional equations of the form

$$Df(x_1, x_2, \dots, x_n) = 0$$

which includes quartic-cubic-quadratic-additive functional equations. From now on, for any given mapping  $f : V \rightarrow W$ , we define  $Df : V^n \rightarrow W$  by

$$Df(x_1, x_2, \dots, x_n) := \sum_{i=1}^{\ell} c_i f(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \quad (1.1)$$

for all  $x_1, x_2, \dots, x_n \in V$ , where  $\ell$  and  $n$  are fixed integers larger than 1 and  $c_i, a_{ij}$  are fixed real constants throughout this paper.

Till now, we have followed out a routine and monotonous procedure for studying the stability problems of the quartic-cubic-quadratic-additive functional equations. However, the

stability theorems of this paper can save us much trouble of proving the stability of relevant solutions repeatedly appearing in the stability problems for various functional equations (see [8, 9, 10, 12, 13]).

## 2 Preliminaries

Throughout this section, let  $V$  and  $W$  be real vector spaces and let  $Y$  be a real Banach space. Given a mapping  $f : V \rightarrow W$  and a real number  $a$ , we will use the following notations:

$$\begin{aligned} f_1(x) &:= \frac{a^3 f_o(x) - f_o(ax)}{a^3 - a}, & f_2(x) &:= \frac{a^4 f_e(x) - f_e(ax)}{a^4 - a^2}, \\ f_3(x) &:= \frac{f_o(ax) - a f_o(x)}{a^3 - a}, & f_4(x) &:= \frac{f_e(ax) - a^2 f_e(x)}{a^4 - a^2}, \\ f_o(x) &:= \frac{f(x) - f(-x)}{2}, & f_e(x) &:= \frac{f(x) + f(-x)}{2} \end{aligned} \quad (2.1)$$

for all  $x \in V$ . We will now introduce some lemmas from [11] in the following Corollaries 2.1, 2.2, 2.3 and 2.4.

**Lemma 2.1** ([11, Corollary 3.1]) *Given a real constant  $k > 1$ , assume that a function  $\phi : V \setminus \{0\} \rightarrow [0, \infty)$  satisfies either the condition*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x) < \infty \quad (\text{for all } x \in V \setminus \{0\}) \quad (2.2)$$

or

$$\Phi(x) := \sum_{i=0}^{\infty} k^{4i} \phi\left(\frac{x}{k^i}\right) < \infty \quad (\text{for all } x \in V \setminus \{0\}). \quad (2.3)$$

Furthermore, suppose  $f : V \rightarrow Y$  is an arbitrary mapping. If a mapping  $F : V \rightarrow Y$  satisfies the inequality

$$\|f(x) - F(x)\| \leq \Phi(x) \quad (2.4)$$

for all  $x \in V \setminus \{0\}$  as well as the equalities

$$\begin{aligned} F_1(kx) &= kF_1(x), & F_2(kx) &= k^2 F_2(x), \\ F_3(kx) &= k^3 F_3(x), & F_4(kx) &= k^4 F_4(x) \end{aligned} \quad (2.5)$$

for all  $x \in V$ , then  $F$  is the unique mapping with the properties (2.4) and (2.5).

**Lemma 2.2** ([11, Corollary 3.2]) *Given a real number  $k > 1$ , assume that the functions  $\phi, \psi : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy each of the following conditions*

$$\begin{aligned} \sum_{i=0}^{\infty} k^i \psi\left(\frac{x}{k^i}\right) &< \infty, & \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi(k^i x) &< \infty, \\ \tilde{\Phi}(x) &:= \sum_{i=0}^{\infty} k^i \phi\left(\frac{x}{k^i}\right) < \infty, & \tilde{\Psi}(x) &:= \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \psi(k^i x) < \infty \end{aligned}$$

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for all  $x \in V \setminus \{0\}$  and  $f : V \rightarrow Y$  is an arbitrary mapping. If a mapping  $F : V \rightarrow Y$  satisfies the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \quad (2.6)$$

for all  $x \in V \setminus \{0\}$  as well as the equalities in (2.5) for all  $x \in V$ , then  $F$  is the unique mapping satisfying equalities in (2.5) for all  $x \in V$  and inequality (2.6) for all  $x \in V \setminus \{0\}$ .

**Lemma 2.3** ([11, Corollary 3.3]) *Given a real number  $k > 1$ , assume that the functions  $\phi, \psi : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy each of the following conditions*

$$\begin{aligned} \sum_{i=0}^{\infty} k^{2i} \psi\left(\frac{x}{k^i}\right) &< \infty, & \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi(k^i x) &< \infty, \\ \tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^{2i} \phi\left(\frac{x}{k^i}\right) &< \infty, & \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \psi(k^i x) &< \infty \end{aligned}$$

for all  $x \in V \setminus \{0\}$  and  $f : V \rightarrow Y$  is an arbitrary mapping. If a mapping  $F : V \rightarrow Y$  satisfies the inequality (2.6) for all  $x \in V \setminus \{0\}$  as well as the equalities in (2.5) for all  $x \in V$ , then  $F$  is the unique mapping satisfying the equalities in (2.5) for all  $x \in V$  and inequality (2.6) for all  $x \in V \setminus \{0\}$ .

**Lemma 2.4** ([11, Corollary 3.4]) *Assume that  $k > 1$  is a real number and the functions  $\phi, \psi : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy each of the following conditions*

$$\begin{aligned} \sum_{i=0}^{\infty} k^{3i} \psi\left(\frac{x}{k^i}\right) &< \infty, & \sum_{i=0}^{\infty} \frac{1}{k^{4i}} \phi(k^i x) &< \infty, \\ \tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^{3i} \phi\left(\frac{x}{k^i}\right) &< \infty, & \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{4i}} \psi(k^i x) &< \infty \end{aligned}$$

for all  $x \in V \setminus \{0\}$ . In addition, suppose  $f : V \rightarrow Y$  is an arbitrary mapping. If a mapping  $F : V \rightarrow Y$  satisfies the inequality (2.6) for all  $x \in V \setminus \{0\}$  as well as the equalities in (2.5) for all  $x \in V$ , then  $F$  is the unique mapping satisfying the equalities in (2.5) for all  $x \in V$  and inequality (2.6) for all  $x \in V \setminus \{0\}$ .

### 3 Main results

In this section, we assume that  $V$  is a real vector space and  $Y$  is a real Banach space if there is no specification. In the following theorems, we prove that there exists only one exact solution near every approximate solution to  $Df(x_1, x_2, \dots, x_n) = 0$ .

**Theorem 3.1** *Given a real constant  $a$  with  $|a| > 1$ , assume that the functions  $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy the conditions*

$$\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^i} < \infty \quad (3.1)$$

for all  $x \in V \setminus \{0\}$  and a function  $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$  satisfies the condition

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^i} < \infty \quad (3.2)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . If a mapping  $f : V \rightarrow Y$  satisfies  $f(0) = 0$ ,

$$\begin{aligned} \|f_e(a^2 x) - (a^2 + a^4)f_e(ax) + a^6 f_e(x)\| &\leq \mu(x), \\ \|f_o(a^2 x) - (a + a^3)f_o(ax) + a^4 f_o(x)\| &\leq \nu(x) \end{aligned} \quad (3.3)$$

for all  $x \in V \setminus \{0\}$ , and if  $f$  moreover satisfies the inequality

$$\|Df(x_1, x_2, \dots, x_n)\| \leq \varphi(x_1, x_2, \dots, x_n) \quad (3.4)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , then there exists a unique mapping  $F : V \rightarrow Y$  such that

$$DF(x_1, x_2, \dots, x_n) = 0 \quad (3.5)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , and

$$\begin{aligned} F_1(ax) &= aF_1(x), & F_2(ax) &= a^2 F_2(x), \\ F_3(ax) &= a^3 F_3(x), & F_4(ax) &= a^4 F_4(x) \end{aligned} \quad (3.6)$$

for all  $x \in V$ , and such that

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left( \frac{|a^{2i+2} - 1|\mu(a^i x)}{|a^4 - a^2||a|^{4i+4}} + \frac{|a^{2i+2} - 1|\nu(a^i x)}{|a^3 - a||a|^{3i+3}} \right) \quad (3.7)$$

for all  $x \in V \setminus \{0\}$ .

**Proof.** First, we define the mappings  $J_m f : V \rightarrow Y$  by

$$J_m f(x) := \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + \frac{f_2(a^m x)}{a^{2m}} + \frac{f_1(a^m x)}{a^m}$$

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for all  $x \in V$  and  $m \in \mathbb{N}_0$ . It then follows from (2.1) and (3.3) that

$$\begin{aligned}
& \|J_m f(x) - J_{m+l} f(x)\| \\
& \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
& = \sum_{i=m}^{m+l-1} \left\| \frac{f_e(a^{i+1}x) - a^2 f_e(a^i x)}{a^{4i}(a^4 - a^2)} - \frac{f_e(a^{i+2}x) - a^2 f_e(a^{i+1}x)}{a^{4i+4}(a^4 - a^2)} \right. \\
& \quad + \frac{f_o(a^{i+1}x) - a f_o(a^i x)}{a^{3i}(a^3 - a)} - \frac{f_o(a^{i+2}x) - a f_o(a^{i+1}x)}{a^{3i+3}(a^3 - a)} \\
& \quad + \frac{a^4 f_e(a^i x) - f_e(a^{i+1}x)}{a^{2i}(a^4 - a^2)} - \frac{a^4 f_e(a^{i+1}x) - f_e(a^{i+2}x)}{a^{2i+2}(a^4 - a^2)} \\
& \quad \left. + \frac{a^3 f_o(a^i x) - f_o(a^{i+1}x)}{a^i(a^3 - a)} - \frac{a^3 f_o(a^{i+1}x) - f_o(a^{i+2}x)}{a^{i+1}(a^3 - a)} \right\| \\
& = \sum_{i=m}^{m+l-1} \left\| - \frac{f_e(a^{i+2}x) - (a^2 + a^4)f_e(a^{i+1}x) + a^6 f_e(a^i x)}{a^{4i+4}(a^4 - a^2)} \right. \\
& \quad - \frac{f_o(a^{i+2}x) - (a + a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{a^{3i+3}(a^3 - a)} \\
& \quad + \frac{f_e(a^{i+2}x) - (a^2 + a^4)f_e(a^{i+1}x) + a^6 f_e(a^i x)}{a^{2i+2}(a^4 - a^2)} \\
& \quad \left. + \frac{f_o(a^{i+2}x) - (a + a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{a^{i+1}(a^3 - a)} \right\| \\
& \leq \sum_{i=m}^{m+l-1} \left( \frac{|a^{2i+2} - 1|\mu(a^i x)}{|a|^{4i+4}|a^4 - a^2|} + \frac{|a^{2i+2} - 1|\nu(a^i x)}{|a|^{3i+3}|a^3 - a|} \right) \tag{3.8}
\end{aligned}$$

for all  $x \in V \setminus \{0\}$ .

In view of (3.1) and (3.8) and since  $|a| > 1$ , the sequence  $\{J_m f(x)\}$  is a Cauchy sequence for all  $x \in V \setminus \{0\}$ . Since  $Y$  is complete and  $f(0) = 0$ , the sequence  $\{J_m f(x)\}$  converges for all  $x \in V$ . Hence, we can define a mapping  $F : V \rightarrow Y$  by

$$F(x) := \lim_{m \rightarrow \infty} J_m f(x) = \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + \frac{f_2(a^m x)}{a^{2m}} + \frac{f_1(a^m x)}{a^m} \right) \tag{3.9}$$

for all  $x \in V$ .

If we replace  $x$  with  $a^m x$  in the second inequality of (3.3) and divide the resulting inequality by  $|a|^m$ , then we get

$$\left\| a^2 \frac{f_o(a^{m+2}x)}{a^{m+2}} - (a^2 + a^4) \frac{f_o(a^{m+1}x)}{a^{m+1}} + a^4 \frac{f_o(a^m x)}{a^m} \right\| \leq \frac{\nu(a^m x)}{|a|^m}$$

for all  $x \in V \setminus \{0\}$  and  $m \in \mathbb{N}$ . In view of the second inequalities of (3.1), this implies that

$$\lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^m} \text{ converges for all } x \in V \tag{3.10}$$

because of our hypothesis that  $f(0) = 0$ . Analogously, by the first inequalities of (3.1) and (3.3), we get

$$\lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}} \text{ converges for all } x \in V. \quad (3.11)$$

By (2.1), (3.9), (3.10) and (3.11), we have

$$\begin{aligned} F_o(x) &= \lim_{m \rightarrow \infty} \left( \frac{f_3(a^m x)}{a^{3m}} + \frac{f_1(a^m x)}{a^m} \right) = \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^m}, \\ F_e(x) &= \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + \frac{f_2(a^m x)}{a^{2m}} \right) = \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}} \end{aligned} \quad (3.12)$$

for all  $x \in V$ .

In view of (2.1) and (3.12), we easily obtain

$$F_1(x) = \frac{a^3 F_o(x) - F_o(ax)}{a^3 - a} = \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^m}$$

for all  $x \in V$ . Hence, we get

$$F_1(ax) = \lim_{m \rightarrow \infty} \frac{f_o(a^{m+1}x)}{a^m} = a F_1(x)$$

for all  $x \in V$ .

Moreover, we obtain

$$F_2(x) = \frac{a^4 F_e(x) - F_e(ax)}{a^4 - a^2} = \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}}$$

for each  $x \in V$ . Thus, we obtain

$$F_2(ax) = \lim_{m \rightarrow \infty} \frac{f_e(a^{m+1}x)}{a^{2m}} = a^2 F_2(x)$$

for all  $x \in V$ .

Similarly, using (2.1), (3.10) and (3.12), we have

$$F_3(x) = \frac{F_o(ax) - a F_o(x)}{a^3 - a} = 0$$

for every  $x \in V$ . Thus, we have  $F_3(ax) = a^3 F_3(x)$  for all  $x \in V$ .

Furthermore, by (2.1), (3.11) and (3.12), we obtain

$$F_4(x) = \frac{F_e(ax) - a^2 F_e(x)}{a^4 - a^2} = 0$$

for all  $x \in V$ . Therefore, it holds that  $F_4(ax) = a^4 F_4(x)$  for all  $x \in V$ .

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Now, it follows from (1.1), (2.1) and (3.9) that

$$\begin{aligned}
 DF(x_1, x_2, \dots, x_n) &= \sum_{i=1}^{\ell} c_i F(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\
 &= \lim_{m \rightarrow \infty} \sum_{i=1}^{\ell} c_i \left( \frac{f_4(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{4m}} + \frac{f_3(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{3m}} \right. \\
 &\quad \left. + \frac{f_2(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{2m}} + \frac{f_1(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^m} \right) \\
 &= \lim_{m \rightarrow \infty} \sum_{i=1}^{\ell} c_i \left( \frac{f_e(a^{m+1}(a_{i1}x_1 + \dots + a_{in}x_n)) - a^2 f_e(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{4m}(a^4 - a^2)} \right. \\
 &\quad + \frac{f_o(a^{m+1}(a_{i1}x_1 + \dots + a_{in}x_n)) - a f_o(a^m(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{3m}(a^3 - a)} \\
 &\quad + \frac{a^4 f_e(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) - f_e(a^{m+1}(a_{i1}x_1 + \dots + a_{in}x_n))}{a^{2m}(a^4 - a^2)} \\
 &\quad \left. + \frac{a^3 f_o(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) - f_o(a^{m+1}(a_{i1}x_1 + \dots + a_{in}x_n))}{a^m(a^3 - a)} \right) \\
 &= \lim_{m \rightarrow \infty} \left( \frac{Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n) - a^2 Df_e(a^m x_1, \dots, a^m x_n)}{a^{4m}(a^4 - a^2)} \right. \\
 &\quad + \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - a Df_o(a^m x_1, \dots, a^m x_n)}{a^{3m}(a^3 - a)} \\
 &\quad + \frac{a^4 Df_e(a^m x_1, \dots, a^m x_n) - Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n)}{a^{2m}(a^4 - a^2)} \\
 &\quad \left. + \frac{a^3 Df_o(a^m x_1, \dots, a^m x_n) - Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n)}{a^m(a^3 - a)} \right)
 \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ .

Hence, in view of (3.2) and (3.4), we have

$$\begin{aligned}
 &\|DF(x_1, x_2, \dots, x_n)\| \\
 &\leq \lim_{m \rightarrow \infty} \left( \frac{(1 + |a|^{m+1} + |a|^{2m} + |a|^{3m+1})\varphi_e(a^{m+1}x_1, a^{m+1}x_2, \dots, a^{m+1}x_n)}{|a|^{4m}|a^4 - a^2|} \right. \\
 &\quad \left. + \frac{(|a|^2 + |a|^{m+2} + |a|^{2m+4} + |a|^{3m+4})\varphi_e(a^m x_1, \dots, a^m x_n)}{|a|^{4m}|a^4 - a^2|} \right) \\
 &= 0
 \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , where  $\varphi_e(x_1, \dots, x_n) := \frac{1}{2}(\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n))$ , i.e.,  $DF(x_1, x_2, \dots, x_n) = 0$  for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . Moreover, if we put  $m = 0$  and let  $l \rightarrow \infty$  in (3.8), then we obtain the inequality (3.7).

We note that the equalities

$$\begin{aligned} F_1(|a|x) &= |a|F_1(x), & F_2(|a|x) &= |a|^2F_2(x), \\ F_3(|a|x) &= |a|^3F_3(x), & F_4(|a|x) &= |a|^4F_4(x) \end{aligned}$$

are true in view of (3.6). (When  $a < 0$ , we have

$$F_1(|a|x) = F_1(-ax) = \frac{a^3F_o(-ax) - F_o(-a^2x)}{a^3 - a} = -\frac{a^3F_o(ax) - F_o(a^2x)}{a^3 - a},$$

since  $F_o$  is odd. By the definition of  $F_1$  (see (2.1)) and since we already showed that  $F_1(ax) = aF_1(x)$  for all  $x \in V$ , we further get

$$F_1(|a|x) = -F_1(ax) = -aF_1(x) = |a|F_1(x)$$

and so on.)

On account of Lemma 2.1 with  $k = |a|$ , the mapping  $F : V \rightarrow Y$  is the unique mapping satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \left( \frac{|a^{2i+2} - 1|\mu(a^i x)|}{|a^4 - a^2||a|^{4i+4}} + \frac{|a^{2i+2} - 1|\nu(a^i x)|}{|a^3 - a||a|^{3i+3}} \right) \\ &\leq \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a^2 - 1||a|^i} + \frac{\nu(a^i x)}{|a^2 - 1||a|^i} \right) \\ &\leq \sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x) \end{aligned}$$

holds for all  $x \in V \setminus \{0\}$ , where  $k := |a|$  and  $\phi(x) := \frac{1}{|a^2 - 1|}(\mu(x) + \mu(-x) + \nu(x) + \nu(-x))$ .  $\square$

In the following theorem, let  $V$  and  $Y$  be a real vector space and a real Banach space, respectively.

**Theorem 3.2** *Given a real constant  $a$  with  $|a| > 1$ , assume that the functions  $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy the conditions*

$$\sum_{i=0}^{\infty} |a|^{4i} \mu\left(\frac{x}{a^i}\right) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^{4i} \nu\left(\frac{x}{a^i}\right) < \infty \quad (3.13)$$

for all  $x \in V \setminus \{0\}$  and a function  $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$  satisfies the condition

$$\sum_{i=0}^{\infty} |a|^{4i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \quad (3.14)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . If a mapping  $f : V \rightarrow Y$  satisfies  $f(0) = 0$ , inequalities in (3.3) for all  $x \in V \setminus \{0\}$ , and if  $f$  satisfies inequality (3.4) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , then

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there exists a unique mapping  $F : V \rightarrow Y$  satisfying (3.5) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$  and (3.6) for all  $x \in V$ , and such that

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left( \frac{|a^{4i} - a^{2i}|}{|a^4 - a^2|} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \right) \quad (3.15)$$

for all  $x \in V \setminus \{0\}$ .

**Proof.** First, we define the mappings  $J_m f : V \rightarrow Y$  by

$$J_m f(x) := a^{4m} f_4\left(\frac{x}{a^m}\right) + a^{3m} f_3\left(\frac{x}{a^m}\right) + a^{2m} f_2\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right)$$

for all  $x \in V$  and  $m \in \mathbb{N}_0$ . It then follows from (2.1) and (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+l} f(x)\| \\ & \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ & = \sum_{i=m}^{m+l-1} \left\| a^{4i} f_4\left(\frac{x}{a^i}\right) + a^{3i} f_3\left(\frac{x}{a^i}\right) + a^{2i} f_2\left(\frac{x}{a^i}\right) + a^i f_1\left(\frac{x}{a^i}\right) \right. \\ & \quad \left. - a^{4i+4} f_4\left(\frac{x}{a^{i+1}}\right) - a^{3i+3} f_3\left(\frac{x}{a^{i+1}}\right) - a^{2i+2} f_2\left(\frac{x}{a^{i+1}}\right) - a^{i+1} f_1\left(\frac{x}{a^{i+1}}\right) \right\| \\ & \leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left\| a^{4i} \left( f_e\left(\frac{a^2 x}{a^{i+1}}\right) - (a^2 + a^4) f_e\left(\frac{ax}{a^{i+1}}\right) + a^6 f_e\left(\frac{x}{a^{i+1}}\right) \right) \right. \\ & \quad \left. - a^{2i} \left( f_e\left(\frac{a^2 x}{a^{i+1}}\right) - (a^2 + a^4) f_e\left(\frac{ax}{a^{i+1}}\right) + a^6 f_e\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ & \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| a^{3i} \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right. \\ & \quad \left. - a^i \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ & \leq \sum_{i=m}^{m+l-1} \left( \frac{|a^{4i} - a^{2i}|}{|a^4 - a^2|} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \right) \end{aligned} \quad (3.16)$$

for all  $x \in V \setminus \{0\}$  and  $l, m \in \mathbb{N}_0$ .

On account of (3.13) and (3.16), the sequence  $\{J_m f(x)\}$  is a Cauchy sequence for all  $x \in V \setminus \{0\}$ . Since  $Y$  is complete and  $f(0) = 0$ , the sequence  $\{J_m f(x)\}$  converges for all  $x \in V$ . Hence, we can define a mapping  $F : V \rightarrow Y$  by

$$\begin{aligned} F(x) & := \lim_{m \rightarrow \infty} J_m f(x) \\ & = \lim_{m \rightarrow \infty} \left( a^{4m} f_4\left(\frac{x}{a^m}\right) + a^{3m} f_3\left(\frac{x}{a^m}\right) + a^{2m} f_2\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right) \right) \end{aligned} \quad (3.17)$$

for all  $x \in V$ . Moreover, if we put  $m = 0$  and let  $l \rightarrow \infty$  in (3.16), we obtain the inequality (3.15).

As we saw in the proof of the previous theorem, it follows from (3.3) and (3.13) that

$$\lim_{m \rightarrow \infty} a^{3m} f_o\left(\frac{x}{a^m}\right) \quad \text{and} \quad \lim_{m \rightarrow \infty} a^{4m} f_e\left(\frac{x}{a^m}\right) \quad \text{converge for all } x \in V. \quad (3.18)$$

By (2.1) and (3.18), we get

$$\begin{aligned} F_o(x) &= \lim_{m \rightarrow \infty} \left( a^{3m} f_3\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right) \right) = \lim_{m \rightarrow \infty} a^{3m} f_o\left(\frac{x}{a^m}\right), \\ F_e(x) &= \lim_{m \rightarrow \infty} \left( a^{4m} f_4\left(\frac{x}{a^m}\right) + a^{2m} f_2\left(\frac{x}{a^m}\right) \right) = \lim_{m \rightarrow \infty} a^{4m} f_e\left(\frac{x}{a^m}\right) \end{aligned} \quad (3.19)$$

for all  $x \in V$ .

In view of (2.1), (3.18) and (3.19), we have

$$F_1(x) = \frac{a^3 F_o(x) - F_o(ax)}{a^3 - a} = 0$$

for each  $x \in V$ . Therefore, it holds that  $F_1(ax) = aF_1(x)$  for any  $x \in V$ . Analogously, it follows from (2.1), (3.18) and (3.19) that  $F_2(x) = 0$  for every  $x \in V$ . Thus, we have  $F_2(ax) = a^2 F_2(x)$  for all  $x \in V$ .

Similarly, using (2.1), (3.18) and (3.19), we obtain

$$F_3(x) = \frac{F_o(ax) - aF_o(x)}{a^3 - a} = \lim_{m \rightarrow \infty} a^{3m} f_o\left(\frac{x}{a^m}\right)$$

for all  $x \in V$ . Hence, we easily see that  $F_3(ax) = a^3 F_3(x)$  for any  $x \in V$ . Moreover, similarly as the case of  $F_3$ , we obtain

$$F_4(x) = \frac{F_e(ax) - a^2 F_e(x)}{a^4 - a^2} = \lim_{m \rightarrow \infty} a^{4m} f_e\left(\frac{x}{a^m}\right)$$

and  $F_4(ax) = a^4 F_4(x)$  for any  $x \in V$ .

Now, using (1.1), (2.1) and (3.17), we have

$$\begin{aligned} DF(x_1, x_2, \dots, x_n) &= \sum_{i=1}^{\ell} c_i F(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\ &= \lim_{m \rightarrow \infty} \left( a^{4m} \sum_{i=1}^{\ell} c_i f_4\left(a_{i1}\frac{x_1}{a^m} + \dots + a_{in}\frac{x_n}{a^m}\right) \right. \\ &\quad \left. + a^{3m} \sum_{i=1}^{\ell} c_i f_3\left(a_{i1}\frac{x_1}{a^m} + \dots + a_{in}\frac{x_n}{a^m}\right) \right. \\ &\quad \left. + a^{2m} \sum_{i=1}^{\ell} c_i f_2\left(a_{i1}\frac{x_1}{a^m} + \dots + a_{in}\frac{x_n}{a^m}\right) \right) \end{aligned}$$

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$$\begin{aligned}
& + a^m \sum_{i=1}^{\ell} c_i f_1 \left( a_{i1} \frac{x_1}{a^m} + \cdots + a_{in} \frac{x_n}{a^m} \right) \\
= & \lim_{m \rightarrow \infty} \left[ a^{4m} \sum_{i=1}^{\ell} c_i \left( \frac{1}{a^4 - a^2} f_e \left( a_{i1} \frac{x_1}{a^{m-1}} + \cdots + a_{in} \frac{x_n}{a^{m-1}} \right) \right. \right. \\
& \quad \left. \left. - \frac{a^2}{a^4 - a^2} f_e \left( a_{i1} \frac{x_1}{a^m} + \cdots + a_{in} \frac{x_n}{a^m} \right) \right) \right. \\
& + a^{3m} \sum_{i=1}^{\ell} c_i \left( \frac{1}{a^3 - a} f_o \left( a_{i1} \frac{x_1}{a^{m-1}} + \cdots + a_{in} \frac{x_n}{a^{m-1}} \right) \right. \\
& \quad \left. \left. - \frac{a}{a^3 - a} f_o \left( a_{i1} \frac{x_1}{a^m} + \cdots + a_{in} \frac{x_n}{a^m} \right) \right) \right. \\
& + a^{2m} \sum_{i=1}^{\ell} c_i \left( \frac{a^4}{a^4 - a^2} f_e \left( a_{i1} \frac{x_1}{a^m} + \cdots + a_{in} \frac{x_n}{a^m} \right) \right. \\
& \quad \left. \left. - \frac{1}{a^4 - a^2} f_e \left( a_{i1} \frac{x_1}{a^{m-1}} + \cdots + a_{in} \frac{x_n}{a^{m-1}} \right) \right) \right. \\
& + a^m \sum_{i=1}^{\ell} c_i \left( \frac{a^3}{a^3 - a} f_o \left( a_{i1} \frac{x_1}{a^m} + \cdots + a_{in} \frac{x_n}{a^m} \right) \right. \\
& \quad \left. \left. - \frac{1}{a^3 - a} f_o \left( a_{i1} \frac{x_1}{a^{m-1}} + \cdots + a_{in} \frac{x_n}{a^{m-1}} \right) \right) \right] \\
= & \lim_{m \rightarrow \infty} \left[ \frac{a^{4m}}{a^4 - a^2} \left( Df_e \left( \frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) - a^2 Df_e \left( \frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right) \right. \\
& + \frac{a^{3m}}{a^3 - a} \left( Df_o \left( \frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) - a Df_o \left( \frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right) \\
& + \frac{a^{2m}}{a^4 - a^2} \left( a^4 Df_e \left( \frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) - Df_e \left( \frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) \right) \\
& \left. + \frac{a^m}{a^3 - a} \left( a^3 Df_o \left( \frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) - Df_o \left( \frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) \right) \right]
\end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . Hence, by (3.4) and (3.14), we obtain

$$\begin{aligned}
& \|DF(x_1, x_2, \dots, x_n)\| \\
\leq & \lim_{m \rightarrow \infty} \left[ \left( \frac{|a|^{4m} + |a|^{2m}}{|a^4 - a^2|} + \frac{|a|^{3m} + |a|^m}{|a^3 - a|} \right) \varphi_e \left( \frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}} \right) \right. \\
& \quad \left. + \left( \frac{|a|^{4m+2} + |a|^{2m+4}}{|a^4 - a^2|} + \frac{|a|^{3m+1} + |a|^{m+3}}{|a^3 - a|} \right) \varphi_e \left( \frac{x_1}{a^m}, \dots, \frac{x_n}{a^m} \right) \right] \\
= & 0
\end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , where  $\varphi_e(x_1, \dots, x_n) := \frac{1}{2}(\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n))$ .

Thus, it holds that  $DF(x_1, x_2, \dots, x_n) = 0$  for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ .

As we see in the proof of Theorem 3.1, we notice that

$$\begin{aligned} F_1(|a|x) &= |a|F_1(x), & F_2(|a|x) &= |a|^2F_2(x), \\ F_3(|a|x) &= |a|^3F_3(x), & F_4(|a|x) &= |a|^4F_4(x) \end{aligned}$$

for all  $x \in V$ .

According to Lemma 2.1, the mapping  $F : V \rightarrow Y$  satisfying equalities in (3.6) and inequality (3.15) is determined uniquely, since

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \left( \frac{|a^{4i} - a^{2i}|}{|a^4 - a^2|} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \frac{|a^{4i}|}{|a^3 - a|} \left( \mu\left(\frac{x}{a^{i+1}}\right) + \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} k^{4i} \phi\left(\frac{x}{k^i}\right) \end{aligned}$$

for all  $x \in V \setminus \{0\}$ , where  $k := |a|$  and  $\phi(x) := \frac{1}{|a^3 - a|} (\mu(\frac{x}{a}) + \mu(\frac{-x}{a}) + \nu(\frac{x}{a}) + \nu(\frac{-x}{a}))$ .  $\square$

We assume again that  $V$  is a real vector space and  $Y$  is a real Banach space if there is no specification.

**Theorem 3.3** *Given a real constant  $a$  with  $|a| > 1$ , assume that the functions  $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy all the conditions*

$$\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{2i}} < \infty, \quad \sum_{i=0}^{\infty} |a|^i \mu\left(\frac{x}{a^i}\right) < \infty, \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{2i}} < \infty, \quad \sum_{i=0}^{\infty} |a|^i \nu\left(\frac{x}{a^i}\right) < \infty \quad (3.20)$$

for all  $x \in V \setminus \{0\}$  and a function  $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$  satisfies the conditions

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{2i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \quad (3.21)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . If a mapping  $f : V \rightarrow Y$  satisfies  $f(0) = 0$  and inequalities of (3.3) for all  $x \in V \setminus \{0\}$  and if  $f$  satisfies inequality (3.4) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , then there exists a unique mapping  $F : V \rightarrow Y$  satisfying equality (3.5) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , equalities in (3.6) for all  $x \in V$ , and moreover

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a^4 - a^2|} \frac{|a^{2i+2} - 1|}{|a|^{4i+4}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \quad (3.22)$$

for all  $x \in V \setminus \{0\}$ .

**Proof.** We define the mappings  $J_m f : V \rightarrow Y$  by

$$J_m f(x) := \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + \frac{f_2(a^m x)}{a^{2m}} + a^m f_1\left(\frac{x}{a^m}\right)$$

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for all  $x \in V$  and  $m \in \mathbb{N}_0$ . By (2.1) and (3.3), we have

$$\begin{aligned}
& \|J_m f(x) - J_{m+l} f(x)\| \\
& \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
& = \sum_{i=m}^{m+l-1} \left\| \frac{f_e(a^{i+1}x) - a^2 f_e(a^i x)}{a^{4i}(a^4 - a^2)} - \frac{f_e(a^{i+2}x) - a^2 f_e(a^{i+1}x)}{a^{4i+4}(a^4 - a^2)} \right. \\
& \quad - \frac{f_e(a^{i+1}x) - a^4 f_e(a^i x)}{a^{2i}(a^4 - a^2)} + \frac{f_e(a^{i+2}x) - a^4 f_e(a^{i+1}x)}{a^{2i+2}(a^4 - a^2)} \\
& \quad + \frac{f_o(a^{i+1}x) - a f_o(a^i x)}{a^{3i}(a^3 - a)} - \frac{f_o(a^{i+2}x) - a f_o(a^{i+1}x)}{a^{3i+3}(a^3 - a)} \\
& \quad - \frac{a^i}{a^3 - a} \left( f_o\left(\frac{ax}{a^i}\right) - a^3 f_o\left(\frac{x}{a^i}\right) \right) \\
& \quad \left. + \frac{a^{i+1}}{a^3 - a} \left( f_o\left(\frac{ax}{a^{i+1}}\right) - a^3 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\
& \leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_e(a^2 \cdot a^i x) - (a^2 + a^4)f_e(a^{i+1}x) + a^6 f_e(a^i x)}{a^{4i+4}} \right. \\
& \quad \left. + \frac{f_e(a^2 \cdot a^i x) - (a^2 + a^4)f_e(a^{i+1}x) + a^6 f_e(a^i x)}{a^{2i+2}} \right\| \\
& \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_o(a^2 \cdot a^i x) - (a + a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{a^{3i+3}} \right. \\
& \quad \left. - a^i \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3)f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\
& \leq \sum_{i=m}^{m+l-1} \frac{\mu(a^i x)}{|a^4 - a^2|} \frac{|a^{2i+2} - 1|}{|a|^{4i+4}} + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \quad (3.23)
\end{aligned}$$

for all  $x \in V \setminus \{0\}$  and for all  $l, m \in \mathbb{N}_0$ .

In view of (3.20) and (3.23), the sequence  $\{J_m f(x)\}$  is a Cauchy sequence for all  $x \in V \setminus \{0\}$ . Since  $Y$  is complete and  $f(0) = 0$ , the sequence  $\{J_m f(x)\}$  converges for all  $x \in V$ . Hence, we can define a mapping  $F : V \rightarrow Y$  by

$$F(x) := \lim_{m \rightarrow \infty} J_m f(x) = \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + \frac{f_2(a^m x)}{a^{2m}} + a^m f_1\left(\frac{x}{a^m}\right) \right) \quad (3.24)$$

for all  $x \in V$ .

By (1.1), (2.1) and (3.24), we get

$$\begin{aligned}
& DF(x_1, x_2, \dots, x_n) \\
&= \sum_{i=1}^{\ell} c_i F(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\
&= \sum_{i=1}^{\ell} c_i \lim_{m \rightarrow \infty} \left( \frac{f_4(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n)}{a^{4m}} + \frac{f_3(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n)}{a^{3m}} \right. \\
&\quad \left. + \frac{f_2(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n)}{a^{2m}} + a^m f_1\left(\frac{a_{i1}x_1}{a^m} + \dots + \frac{a_{in}x_n}{a^m}\right) \right) \\
&= \lim_{m \rightarrow \infty} \frac{1}{a^{4m}} \sum_{i=1}^{\ell} c_i \frac{f_e(a_{i1}a^{m+1}x_1 + \dots + a_{in}a^{m+1}x_n) - a^2 f_e(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n)}{a^4 - a^2} \\
&\quad + \lim_{m \rightarrow \infty} \frac{1}{a^{3m}} \sum_{i=1}^{\ell} c_i \frac{f_o(a_{i1}a^{m+1}x_1 + \dots + a_{in}a^{m+1}x_n) - a f_o(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n)}{a^3 - a} \\
&\quad + \lim_{m \rightarrow \infty} \frac{1}{a^{2m}} \sum_{i=1}^{\ell} c_i \frac{a^4 f_e(a_{i1}a^m x_1 + \dots + a_{in}a^m x_n) - f_e(a_{i1}a^{m+1}x_1 + \dots + a_{in}a^{m+1}x_n)}{a^4 - a^2} \\
&\quad + \lim_{m \rightarrow \infty} \frac{a^m}{a^3 - a} \sum_{i=1}^{\ell} c_i \left( a^3 f_o\left(\frac{a_{i1}x_1}{a^m} + \dots + \frac{a_{in}x_n}{a^m}\right) - f_o\left(\frac{a_{i1}x_1}{a^{m-1}} + \dots + \frac{a_{in}x_n}{a^{m-1}}\right) \right) \\
&= \lim_{m \rightarrow \infty} \frac{Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n) - a^2 Df_e(a^m x_1, \dots, a^m x_n)}{a^{4m}(a^4 - a^2)} \\
&\quad + \lim_{m \rightarrow \infty} \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - a Df_o(a^m x_1, \dots, a^m x_n)}{a^{3m}(a^3 - a)} \\
&\quad + \lim_{m \rightarrow \infty} \frac{a^4 Df_e(a^m x_1, \dots, a^m x_n) - Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n)}{a^{2m}(a^4 - a^2)} \\
&\quad + \lim_{m \rightarrow \infty} \frac{a^m}{a^3 - a} \left( a^3 Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) - Df_o\left(\frac{ax_1}{a^m}, \dots, \frac{ax_n}{a^m}\right) \right)
\end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . Hence, by (3.4) and (3.21), we have

$$\begin{aligned}
& \|DF(x_1, x_2, \dots, x_n)\| \\
&\leq \lim_{m \rightarrow \infty} \left( \frac{(1 + |a|^{m+1} + |a|^{2m})\varphi_e(a^{m+1}x_1, a^{m+1}x_2, \dots, a^{m+1}x_n)}{|a|^{4m}|a^4 - a^2|} \right. \\
&\quad + \frac{(a^2 + |a|^{m+2} + |a|^{2m+4})\varphi_e(a^m x_1, \dots, a^m x_n)}{|a|^{4m}|a^4 - a^2|} \\
&\quad \left. + \frac{|a|^{m+3}}{|a^3 - a|}\varphi_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) + \frac{|a|^m}{|a^3 - a|}\varphi_e\left(\frac{ax_1}{a^m}, \dots, \frac{ax_n}{a^m}\right) \right) \\
&= 0
\end{aligned}$$

for any  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , where  $\varphi_e(x_1, \dots, x_n) := \frac{1}{2}(\varphi(x_1, \dots, x_n) + \varphi(-x_1, \dots, -x_n))$ ,

i.e.,  $DF(x_1, x_2, \dots, x_n) = 0$  for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . Moreover, if we put  $m = 0$  and let  $l \rightarrow \infty$  in (3.23), then we obtain the inequality (3.22).

As we did in the proofs of the previous theorems, on account of (3.3) and (3.20), we can show that

$$\lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}}, \quad \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^{3m}} \quad \text{and} \quad \lim_{m \rightarrow \infty} a^m f_o\left(\frac{x}{a^m}\right) \quad \text{converge for all } x \in V.$$

In view of (2.1), (3.24) and the last relations, we get

$$\begin{aligned} F_e(x) &= \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}}, \\ F_o(x) &= \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^{3m}} + \lim_{m \rightarrow \infty} a^m f_o\left(\frac{x}{a^m}\right) \end{aligned} \quad (3.25)$$

for all  $x \in V$ .

By using (2.1) and (3.25), we get

$$\begin{aligned} F_1(x) &= \frac{a^3}{a^3 - a} F_o(x) - \frac{1}{a^3 - a} F_o(ax) = \lim_{m \rightarrow \infty} a^m f_o\left(\frac{x}{a^m}\right), \\ F_2(x) &= \frac{a^4}{a^4 - a^2} F_e(x) - \frac{1}{a^4 - a^2} F_e(ax) = \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{2m}}, \\ F_3(x) &= \frac{1}{a^3 - a} F_o(ax) - \frac{a}{a^3 - a} F_o(x) = \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^{3m}}, \\ F_4(x) &= \frac{1}{a^4 - a^2} F_e(ax) - \frac{a^2}{a^4 - a^2} F_e(x) = 0 \end{aligned}$$

for all  $x \in V$ . Hence, we can verify that  $F_1(ax) = aF_1(x)$ ,  $F_2(ax) = a^2F_2(x)$ ,  $F_3(ax) = a^3F_3(x)$  and  $F_4(ax) = a^4F_4(x)$  for each  $x \in V$ .

Similarly as in the proof of Theorem 3.1, we can use (3.6) to show that

$$\begin{aligned} F_1(|a|x) &= |a|F_1(x), & F_2(|a|x) &= |a|^2F_2(x), \\ F_3(|a|x) &= |a|^3F_3(x), & F_4(|a|x) &= |a|^4F_4(x). \end{aligned}$$

Due to Lemma 2.2, the mapping  $F : V \rightarrow Y$  is the unique mapping satisfying equalities in (3.6) and inequality (3.7), since it follows from (3.22) that

$$\begin{aligned} \|f(x) - F(x)\| &\leq \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a^4 - a^2|} \frac{|a^{2i+2} - 1|}{|a|^{4i+4}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x) + \nu(a^i x)}{|a^2 - 1||a|^{2i}} + \frac{|a|^i}{|a^2 - 1|} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \left( \frac{1}{k^{2i}} \psi(k^i x) + k^i \phi\left(\frac{x}{k^i}\right) \right) \end{aligned}$$

for all  $x \in V \setminus \{0\}$ , where we set  $k := |a|$ ,  $\psi(x) := \frac{1}{|a^2 - 1|}(\mu(x) + \mu(-x) + \nu(x) + \nu(-x))$  and  $\phi(x) := \frac{1}{|a^2 - 1|}(\nu(\frac{x}{a}) + \nu(\frac{-x}{a}))$ .  $\square$

In the following theorem, let  $V$  be a real vector space and  $Y$  a real Banach space.

**Theorem 3.4** Assume that the functions  $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy the conditions

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{3i}} < \infty, \quad \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^i}\right) < \infty, \\ \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{3i}} < \infty, \quad \sum_{i=0}^{\infty} |a|^{2i} \nu\left(\frac{x}{a^i}\right) < \infty \end{aligned} \quad (3.26)$$

for all  $x \in V \setminus \{0\}$  and a function  $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$  satisfies the conditions

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{3i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \quad (3.27)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . If a mapping  $f : V \rightarrow Y$  satisfies  $f(0) = 0$  and the inequality (3.3) for all  $x \in V \setminus \{0\}$  and if  $f$  satisfies (3.4) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , then there exists a unique mapping  $F : V \rightarrow Y$  satisfying the equality (3.5) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , the equalities in (3.6) for all  $x \in V$ , and

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{|a^4 - a^2|} \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a|^{4i+4}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\quad + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \end{aligned} \quad (3.28)$$

for all  $x \in V \setminus \{0\}$ .

**Proof.** We define the mappings  $J_m f : V \rightarrow Y$  by

$$J_m f(x) := \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + a^{2m} f_2\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right)$$

for all  $x \in V$  and  $m \in \mathbb{N}_0$ . Then it follows from (2.1) and (3.3) that

$$\begin{aligned} &\|J_m f(x) - J_{m+l} f(x)\| \\ &\leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ &= \sum_{i=m}^{m+l-1} \left\| \frac{f_4(a^i x)}{a^{4i}} - \frac{f_4(a^{i+1} x)}{a^{4i+4}} + \frac{f_3(a^i x)}{a^{3i}} - \frac{f_3(a^{i+1} x)}{a^{3i+3}} \right. \\ &\quad \left. + a^{2i} f_2\left(\frac{x}{a^i}\right) - a^{2i+2} f_2\left(\frac{x}{a^{i+1}}\right) + a^i f_1\left(\frac{x}{a^i}\right) - a^{i+1} f_1\left(\frac{x}{a^{i+1}}\right) \right\| \\ &\leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_e(a^2 \cdot a^i x) - (a^2 + a^4) f_e(a^{i+1} x) + a^6 f_e(a^i x)}{a^{4i+4}} \right. \\ &\quad \left. - a^{2i} \left( f_e\left(\frac{a^2 x}{a^{i+1}}\right) - (a^2 + a^4) f_e\left(\frac{ax}{a^{i+1}}\right) + a^6 f_e\left(\frac{x}{a^{i+1}}\right) \right) \right\| \end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_o(a^2 \cdot a^i x) - (a + a^3)f_o(a^{i+1}x) + a^4 f_o(a^i x)}{a^{3i+3}} \right. \\
& \quad \left. - a^i \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3)f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\
& \leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left( \frac{\mu(a^i x)}{|a|^{4i+4}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) \\
& \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \tag{3.29}
\end{aligned}$$

for all  $x \in V \setminus \{0\}$ .

In view of (3.26) and (3.29), the sequence  $\{J_m f(x)\}$  is a Cauchy sequence for all  $x \in V \setminus \{0\}$ . Since  $Y$  is complete and  $f(0) = 0$ , the sequence  $\{J_m f(x)\}$  converges for all  $x \in V$ . Hence, we can define a mapping  $F : V \rightarrow Y$  by

$$F(x) := \lim_{m \rightarrow \infty} J_m f(x) = \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + \frac{f_3(a^m x)}{a^{3m}} + a^{2m} f_2\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right) \right) \tag{3.30}$$

for all  $x \in V$ . Then, by oddness and evenness of  $F_o$  and  $F_e$ , it follows from (3.30) that

$$\begin{aligned}
F_o(x) &= \lim_{m \rightarrow \infty} \left( \frac{f_3(a^m x)}{a^{3m}} + a^m f_1\left(\frac{x}{a^m}\right) \right), \\
F_e(x) &= \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + a^{2m} f_2\left(\frac{x}{a^m}\right) \right)
\end{aligned}$$

for all  $x \in V$ . Moreover, if we put  $m = 0$  and let  $l \rightarrow \infty$  in (3.29), we obtain inequality (3.28).

In view of (3.3) and (3.26), it holds that

$$\lim_{m \rightarrow \infty} a^m f_o\left(\frac{x}{a^m}\right), \quad \lim_{m \rightarrow \infty} \frac{f_o(a^m x)}{a^{3m}}, \quad \lim_{m \rightarrow \infty} a^{2m} f_e\left(\frac{x}{a^m}\right), \quad \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{4m}}$$

converge for all  $x \in V$ . Using these facts, we get

$$\begin{aligned}
F_1(x) &= \frac{a^3}{a^3 - a} F_o(x) - \frac{1}{a^3 - a} F_o(ax) \\
&= \frac{a^3}{a^3 - a} \lim_{m \rightarrow \infty} \left( \frac{f_3(a^m x)}{a^{3m}} + a^m f_1\left(\frac{x}{a^m}\right) \right) \\
&\quad - \frac{1}{a^3 - a} \lim_{m \rightarrow \infty} \left( \frac{f_3(a^{m+1} x)}{a^{3m}} + a^m f_1\left(\frac{x}{a^{m-1}}\right) \right) \\
&= \frac{a^3}{a^3 - a} \lim_{m \rightarrow \infty} a^m f_1\left(\frac{x}{a^m}\right) - \frac{a}{a^3 - a} \lim_{m \rightarrow \infty} a^{m-1} f_1\left(\frac{x}{a^{m-1}}\right) \\
&= \lim_{m \rightarrow \infty} a^m f_1\left(\frac{x}{a^m}\right)
\end{aligned}$$

for all  $x \in V$ . Therefore,  $F_1(ax) = aF_1(x)$  for all  $x \in V$ . By the similar way, we can show that the equalities in (3.6) hold true for any  $x \in V$ .

On account of (1.1), (2.1) and (3.30), it holds that

$$\begin{aligned}
 & DF(x_1, x_2, \dots, x_n) \\
 &= \sum_{i=1}^{\ell} c_i F(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\
 &= \sum_{i=1}^{\ell} c_i \lim_{m \rightarrow \infty} \left( \frac{1}{a^{4m}} f_4(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) \right. \\
 &\quad \left. + \frac{1}{a^{3m}} f_3(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) \right. \\
 &\quad \left. + a^{2m} f_2\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^m}\right) \right. \\
 &\quad \left. + a^m f_1\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^m}\right) \right) \\
 &= \lim_{m \rightarrow \infty} \frac{1}{a^{4m}} \left( \frac{1}{a^4 - a^2} Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n) \right. \\
 &\quad \left. - \frac{a^2}{a^4 - a^2} Df_e(a^m x_1, \dots, a^m x_n) \right) \\
 &\quad + \lim_{m \rightarrow \infty} \frac{1}{a^{3m}} \left( \frac{1}{a^3 - a} Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) \right. \\
 &\quad \left. - \frac{a}{a^3 - a} Df_o(a^m x_1, \dots, a^m x_n) \right) \\
 &\quad + \lim_{m \rightarrow \infty} a^{2m} \left( \frac{a^4}{a^4 - a^2} Df_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \right. \\
 &\quad \left. - \frac{1}{a^4 - a^2} Df_e\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) \right) \\
 &\quad + \lim_{m \rightarrow \infty} a^m \left( \frac{a^3}{a^3 - a} Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \right. \\
 &\quad \left. - \frac{1}{a^3 - a} Df_o\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) \right)
 \end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ .

Using (3.4) and (3.27), we get

$$\begin{aligned}
 & \|DF(x_1, x_2, \dots, x_n)\| \\
 &\leq \lim_{m \rightarrow \infty} \left( \frac{1}{|a^4 - a^2||a|^{4m}} \varphi_e(a^{m+1}x_1, \dots, a^{m+1}x_n) \right. \\
 &\quad \left. + \frac{|a|^2}{|a^4 - a^2||a|^{4m}} \varphi_e(a^m x_1, \dots, a^m x_n) \right)
 \end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{|a^3 - a||a|^{3m}} \varphi_e(a^{m+1}x_1, \dots, a^{m+1}x_n) \\
& + \frac{|a|}{|a^3 - a||a|^{3m}} \varphi_e(a^m x_1, \dots, a^m x_n) \\
& + \frac{|a|^4 |a|^{2m}}{|a^4 - a^2|} \varphi_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) + \frac{|a|^{2m}}{|a^4 - a^2|} \varphi_e\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) \\
& + \frac{|a|^3 |a|^m}{|a^3 - a|} \varphi_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) + \frac{|a|^m}{|a^3 - a|} \varphi_e\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) \\
& = 0
\end{aligned}$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , i.e.,  $DF(x_1, x_2, \dots, x_n) = 0$  for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . By a similar way as the previous proofs, we notice that the equalities

$$\begin{aligned}
F_1(|a|x) &= |a|F_1(x), & F_2(|a|x) &= |a|^2F_2(x), \\
F_3(|a|x) &= |a|^3F_3(x), & F_4(|a|x) &= |a|^4F_4(x)
\end{aligned}$$

are true in view of (3.6).

Using Lemma 2.3, we conclude that the mapping  $F : V \rightarrow Y$  is the unique mapping satisfying equalities in (3.6) and inequality (3.28), since

$$\begin{aligned}
& \|f(x) - F(x)\| \\
& \leq \frac{1}{|a^4 - a^2|} \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a|^{4i+4}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left( \frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\
& \leq \frac{1}{|a^2 - 1|} \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x) + \nu(a^i x)}{|a|^{3i}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + |a|^{2i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\
& \leq \sum_{i=0}^{\infty} \left( \frac{1}{k^{3i}} \psi(k^i x) + k^{2i} \phi\left(\frac{x}{k^i}\right) \right)
\end{aligned}$$

for all  $x \in V \setminus \{0\}$ , where we set  $k := |a|$ ,  $\psi(x) := \frac{1}{|a^2 - 1|}(\mu(x) + \mu(-x) + \nu(x) + \nu(-x))$  and  $\phi(x) := \frac{1}{|a^2 - 1|}(\mu(\frac{x}{a}) + \mu(\frac{-x}{a}) + \nu(\frac{x}{a}) + \nu(\frac{-x}{a}))$ .  $\square$

**Theorem 3.5** Assume that the functions  $\mu, \nu : V \setminus \{0\} \rightarrow [0, \infty)$  satisfy all the conditions

$$\sum_{i=0}^{\infty} |a|^{3i} \mu\left(\frac{x}{a^i}\right) < \infty, \quad \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{4i}} < \infty, \quad \sum_{i=0}^{\infty} |a|^{3i} \nu\left(\frac{x}{a^i}\right) < \infty, \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{4i}} < \infty \quad (3.31)$$

for all  $x \in V \setminus \{0\}$  and a function  $\varphi : (V \setminus \{0\})^n \rightarrow [0, \infty)$  satisfies the conditions

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \dots, a^i x_n)}{|a|^{4i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty \quad (3.32)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ . If a mapping  $f : V \rightarrow Y$  satisfies  $f(0) = 0$  and the inequality (3.3) for all  $x \in V \setminus \{0\}$  and if  $f$  satisfies (3.4) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , then there exists

a unique mapping  $F : V \rightarrow Y$  satisfying the equality (3.5) for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , the equalities in (3.6) for all  $x \in V$ , and

$$\|f(x) - F(x)\| \leq \frac{1}{|a^4 - a^2|} \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a|^{4(i+1)}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) + \sum_{i=0}^{\infty} \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \quad (3.33)$$

for all  $x \in V \setminus \{0\}$ .

**Proof.** We define the mappings  $J_m f : V \rightarrow Y$  by

$$J_m f(x) := \frac{f_4(a^m x)}{a^{4m}} + a^{3m} f_3\left(\frac{x}{a^m}\right) + a^{2m} f_2\left(\frac{x}{a^m}\right) + a^m f_1\left(\frac{x}{a^m}\right)$$

for all  $x \in V$  and  $m \in \mathbb{N}_0$ . It follows from (2.1) and (3.3) that

$$\begin{aligned} & \|J_m f(x) - J_{m+l} f(x)\| \\ & \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ & = \sum_{i=m}^{m+l-1} \left\| \frac{f_4(a^i x)}{a^{4i}} - \frac{f_4(a^{i+1} x)}{a^{4(i+1)}} + a^{2i} f_2\left(\frac{x}{a^i}\right) - a^{2(i+1)} f_2\left(\frac{x}{a^{i+1}}\right) \right. \\ & \quad \left. + a^{3i} f_3\left(\frac{x}{a^i}\right) - a^{3(i+1)} f_3\left(\frac{x}{a^{i+1}}\right) + a^i f_1\left(\frac{x}{a^i}\right) - a^{i+1} f_1\left(\frac{x}{a^{i+1}}\right) \right\| \\ & \leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_e(a^2 \cdot a^i x) - (a^2 + a^4) f_e(a^{i+1} x) + a^6 f_e(a^i x)}{a^{4i+4}} \right. \\ & \quad \left. - a^{2i} \left( f_e\left(\frac{a^2 x}{a^{i+1}}\right) - (a^2 + a^4) f_e\left(\frac{ax}{a^{i+1}}\right) + a^6 f_e\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ & \quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| a^{3i} \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right. \\ & \quad \left. - a^i \left( f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ & \leq \frac{1}{|a^4 - a^2|} \sum_{i=m}^{m+l-1} \left( \frac{\mu(a^i x)}{|a|^{4(i+1)}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) + \sum_{i=m}^{m+l-1} \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \quad (3.34) \end{aligned}$$

for all  $x \in V \setminus \{0\}$  and  $l, m \in \mathbb{N}_0$ .

On the other hand, by (3.3) and (3.31), we further see that

$$\lim_{m \rightarrow \infty} a^{3m} f_o\left(\frac{x}{a^m}\right), \quad \lim_{m \rightarrow \infty} a^{3m} f_e\left(\frac{x}{a^m}\right), \quad \lim_{m \rightarrow \infty} \frac{f_e(a^m x)}{a^{4m}} \quad \text{converge for all } x \in V. \quad (3.35)$$

In view of (3.31) and (3.34), the sequence  $\{J_m f(x)\}$  is a Cauchy sequence for all  $x \in V \setminus \{0\}$ . Since  $Y$  is complete and  $f(0) = 0$ , the sequence  $\{J_m f(x)\}$  converges for all  $x \in V$ . Hence,

considering (2.1) and (3.35), we can define a mapping  $F : V \rightarrow Y$  by

$$F(x) := \lim_{m \rightarrow \infty} J_m f(x) = \lim_{m \rightarrow \infty} \left( \frac{f_4(a^m x)}{a^{4m}} + a^{3m} f_3\left(\frac{x}{a^m}\right) \right) \quad (3.36)$$

for all  $x \in V$ . Moreover, if we put  $m = 0$  and let  $l \rightarrow \infty$  in (3.34), we obtain inequality (3.33).

On account of (2.1) and (3.36), the oddness and evenness of  $F_o$  and  $F_e$  imply that

$$F_o(x) = \lim_{m \rightarrow \infty} a^{3m} f_3\left(\frac{x}{a^m}\right) \quad \text{and} \quad F_e(x) = \lim_{m \rightarrow \infty} \frac{f_4(a^m x)}{a^{4m}}$$

for all  $x \in V$ . Using these facts, for example, we get

$$\begin{aligned} F_4(x) &= \frac{1}{a^4 - a^2} F_e(ax) - \frac{a^2}{a^4 - a^2} F_e(x) \\ &= \frac{1}{a^4 - a^2} \lim_{m \rightarrow \infty} \frac{f_4(a^{m+1}x)}{a^{4m}} - \frac{a^2}{a^4 - a^2} \lim_{m \rightarrow \infty} \frac{f_4(a^m x)}{a^{4m}} \\ &= \lim_{m \rightarrow \infty} \frac{f_4(a^m x)}{a^{4m}} \end{aligned}$$

for all  $x \in V$ . Therefore,  $F_4(ax) = a^4 F_4(x)$  for all  $x \in V$ . By the similar way, we can show that all the equalities in (3.6) hold true for any  $x \in V$ .

By using (1.1), (2.1) and (3.36), we have

$$\begin{aligned} DF(x_1, x_2, \dots, x_n) &= \sum_{i=1}^{\ell} c_i F(a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n) \\ &= \sum_{i=1}^{\ell} c_i \lim_{m \rightarrow \infty} \left( \frac{1}{a^{4m}} f_4(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) \right. \\ &\quad \left. + a^{3m} f_3\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^m}\right) \right) \\ &= \sum_{i=1}^{\ell} c_i \lim_{m \rightarrow \infty} \left( \frac{1}{(a^4 - a^2)a^{4m}} f_e(a^{m+1}(a_{i1}x_1 + \dots + a_{in}x_n)) \right. \\ &\quad \left. - \frac{a^2}{(a^4 - a^2)a^{4m}} f_e(a^m(a_{i1}x_1 + \dots + a_{in}x_n)) \right) \\ &\quad + \sum_{i=1}^{\ell} c_i \lim_{m \rightarrow \infty} \left( \frac{a^{3m}}{a^3 - a} f_o\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^{m-1}}\right) \right. \\ &\quad \left. - \frac{a^{3m+1}}{a^3 - a} f_o\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^m}\right) \right) \\ &= \lim_{m \rightarrow \infty} \left( \frac{1}{(a^4 - a^2)a^{4m}} Df_e(a^{m+1}x_1, \dots, a^{m+1}x_n) \right. \\ &\quad \left. - \frac{a^2}{(a^4 - a^2)a^{4m}} Df_e(a^m x_1, \dots, a^m x_n) \right) \\ &\quad + \lim_{m \rightarrow \infty} \left( \frac{a^{3m}}{a^3 - a} Df_o\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^{m-1}}\right) \right. \\ &\quad \left. - \frac{a^{3m+1}}{a^3 - a} Df_o\left(\frac{a_{i1}x_1 + \dots + a_{in}x_n}{a^m}\right) \right) \end{aligned}$$

$$- \frac{a^2}{(a^4 - a^2)a^{4m}} Df_e(a^m x_1, \dots, a^m x_n) \Bigg) \\ + \lim_{m \rightarrow \infty} \left( \frac{a^{3m}}{a^3 - a} Df_o\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) - \frac{a^{3m+1}}{a^3 - a} Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \right)$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ .

By (3.4) and (3.32), together with the last equality, we get

$$\|DF(x_1, x_2, \dots, x_n)\| = 0$$

for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ , i.e.,  $DF(x_1, x_2, \dots, x_n) = 0$  for all  $x_1, x_2, \dots, x_n \in V \setminus \{0\}$ .

We remark that the equalities

$$F_1(|a|x) = |a|F_1(x), \quad F_2(|a|x) = |a|^2F_2(x), \\ F_3(|a|x) = |a|^3F_3(x), \quad F_4(|a|x) = |a|^4F_4(x)$$

are true in view of (3.6).

Using Lemma 2.4, we conclude that the mapping  $F : V \rightarrow Y$  is the unique mapping satisfying equalities in (3.6) and inequality (3.33), since

$$\begin{aligned} \|f(x) - F(x)\| &\leq \frac{1}{|a^4 - a^2|} \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a|^{4(i+1)}} + |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) \right) + \sum_{i=0}^{\infty} \frac{|a^{3i} - a^i|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+1}}\right) \\ &\leq \sum_{i=0}^{\infty} \left( \frac{\mu(a^i x)}{|a|^{4i}|a^2 - 1|} + \frac{|a|^{3i}}{|a^2 - 1|} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a|^{3i}}{|a^2 - 1|} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &\leq \sum_{i=0}^{\infty} \left( \frac{1}{k^{4i}} \psi(k^i x) + k^{3i} \phi\left(\frac{x}{k^i}\right) \right) \end{aligned}$$

for all  $x \in V \setminus \{0\}$ , where we set  $k := |a|$ ,  $\phi(x) := \frac{1}{|a^2 - 1|} (\mu(\frac{x}{a}) + \mu(\frac{-x}{a}) + \nu(\frac{x}{a}) + \nu(\frac{-x}{a}))$  and  $\psi(x) := \frac{1}{|a^2 - 1|} (\mu(x) + \mu(-x))$ .  $\square$

## 4 Corollaries

By using Theorems 3.1, 3.2, 3.3, 3.4, and 3.5, we can prove the Hyers-Ulam-Rassias stability of the functional equation (1.1).

**Corollary 4.1** Assume that  $X$  is a real normed space,  $Y$  is a real Banach space and  $p, \theta, \eta, \xi$  are real constants such that  $p \notin \{1, 2, 3, 4\}$ ,  $|a| > 1$ ,  $\xi > 0$ ,  $\eta > 0$ , and  $\theta > 0$ . If a mapping  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and

$$\begin{aligned} \|f_e(a^2x) - (a^2 + a^4)f_e(ax) + a^6f_e(x)\| &\leq \eta\|x\|^p, \\ \|f_o(a^2x) - (a + a^3)f_o(ax) + a^4f_o(x)\| &\leq \xi\|x\|^p \end{aligned} \quad (4.1)$$

for all  $x \in X \setminus \{0\}$  and if  $f$  moreover satisfies the inequality

$$\|Df(x_1, x_2, \dots, x_n)\| \leq \theta(\|x_1\|^p + \dots + \|x_n\|^p) \quad (4.2)$$

for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , then there exists a unique mapping  $F : X \rightarrow Y$  satisfying (3.5) for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , and the equalities in (3.6) for all  $x \in X$ , as well as

$$\|f(x) - F(x)\| \leq \frac{\eta \|x\|^p}{|a^4 - |a|^p||a|^p - a^2|} + \frac{\xi \|x\|^p}{||a|^3 - |a|^p||a|^p - |a||} \quad (4.3)$$

for all  $x \in X \setminus \{0\}$ .

**Proof.** If we put  $\varphi(x_1, x_2, \dots, x_n) := \theta(\|x_1\|^p + \dots + \|x_n\|^p)$  for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , then  $\varphi$  satisfies (3.2), (3.14), (3.21), (3.27), and (4.1) when  $p < 1$ ,  $p > 4$ ,  $1 < p < 2$ ,  $2 < p < 3$ , and  $3 < p < 4$ , respectively.

Therefore, there is a unique mapping  $F : X \rightarrow Y$  satisfying (3.5) for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , the equalities in (3.6) for all  $x \in X$ , as well as

$$\|f(x) - F(x)\| \leq \begin{cases} \left( \frac{\eta}{|a|^2 - |a|^p} - \frac{\eta}{|a|^4 - |a|^p} + \frac{|a|\xi}{|a| - |a|^p} - \frac{|a|\xi}{|a|^3 - |a|^p} \right) \frac{\|x\|^p}{|a|^2|a^2 - 1|} & (\text{for } p < 1), \\ \left( \frac{\eta}{|a|^2 - |a|^p} - \frac{\eta}{|a|^4 - |a|^p} + \frac{|a|\xi}{|a|^p - |a|} + \frac{|a|\xi}{|a|^3 - |a|^p} \right) \frac{\|x\|^p}{|a|^2|a^2 - 1|} & (\text{for } 1 < p < 2), \\ \left( \frac{\eta}{|a|^p - |a|^2} + \frac{\eta}{|a|^4 - |a|^p} + \frac{|a|\xi}{|a|^p - |a|} + \frac{|a|\xi}{|a|^3 - |a|^p} \right) \frac{\|x\|^p}{|a|^2|a^2 - 1|} & (\text{for } 2 < p < 3), \\ \left( \frac{\eta}{|a|^p - |a|^2} + \frac{\eta}{|a|^4 - |a|^p} + \frac{|a|\xi}{|a|^p - |a|^3} - \frac{|a|\xi}{|a|^p - |a|} \right) \frac{\|x\|^p}{|a|^2|a^2 - 1|} & (\text{for } 3 < p < 4), \\ \left( \frac{\eta}{|a|^p - |a|^4} - \frac{\eta}{|a|^p - |a|^2} + \frac{|a|\xi}{|a|^p - |a|^3} - \frac{|a|\xi}{|a|^p - |a|} \right) \frac{\|x\|^p}{|a|^2|a^2 - 1|} & (\text{for } p > 4). \end{cases}$$

From the above inequalities, we get the desired inequality (4.3).  $\square$

**Corollary 4.2** Assume that  $X$  is a real normed space,  $Y$  is a real Banach space, and that  $p, \theta, \eta, \xi$  are real constants such that  $p \notin \{1, 2, 3, 4\}$ ,  $0 < |a| < 1$ ,  $\xi > 0$ , and  $\theta > 0$ . If a mapping  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and the inequalities in (4.1) for all  $x \in X \setminus \{0\}$ , as well as if  $f$  satisfies the inequality (4.2) for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , then there exists a unique mapping  $F : X \rightarrow Y$  satisfying (3.5) for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$  and the equalities in (3.6) for all  $x \in X$ , as well as (4.3) for all  $x \in X \setminus \{0\}$ .

**Proof.** Replacing  $x$  by  $\frac{x}{a^2}$  and multiplying  $\frac{1}{|a|^6}$  in the first inequality of (4.1), we get

$$\left\| \frac{1}{a^6} f_e(x) - \frac{a^2 + a^4}{a^6} f_e\left(\frac{x}{a}\right) + f_e\left(\frac{x}{a^2}\right) \right\| \leq \frac{\eta}{|a|^6} \left\| \frac{x}{a^2} \right\|^p$$

for all  $x \in X \setminus \{0\}$ . Analogously, replacing  $x$  by  $\frac{x}{a^2}$  and multiplying  $\frac{1}{|a|^4}$  in the second inequality of (4.1), we have

$$\left\| \frac{1}{a^4} f_o(x) - \frac{a + a^3}{a^4} f_o\left(\frac{x}{a}\right) + f_o\left(\frac{x}{a^2}\right) \right\| \leq \frac{\xi}{|a|^4} \left\| \frac{x}{a^2} \right\|^p$$

for all  $x \in X \setminus \{0\}$ .

Let  $b = \frac{1}{a}$ . Then we get the inequalities

$$\begin{aligned}\|f_e(b^2x) - (b^2 + b^4)f_e(bx) + b^6f_e(x)\| &\leq b^{2p}b^6\eta\|x\|^p, \\ \|f_o(b^2x) - (b + b^3)f_o(bx) + b^4f_o(x)\| &\leq b^{2p}b^4\xi\|x\|^p\end{aligned}$$

for all  $x \in X \setminus \{0\}$ . By Corollary 4.1, there exists a unique mapping  $F : X \rightarrow Y$  satisfying (3.5) for all  $x_1, x_2, \dots, x_n \in X \setminus \{0\}$ , the equalities in (3.6) for all  $x \in X$ , as well as

$$\begin{aligned}\|f(x) - F(x)\| &\leq \frac{|b|^{2p}|b|^6\eta\|x\|^p}{|b^4 - |b|^p||b|^p - b^2|} + \frac{|b|^{2p}|b|^4\xi\|x\|^p}{||b|^3 - |b|^p||b|^p - |b|}| \\ &= \frac{\eta\|x\|^p}{|a^4 - |a|^p||a|^p - a^2|} + \frac{\xi\|x\|^p}{||a|^3 - |a|^p||a|^p - |a|}| \end{aligned}$$

for all  $x \in X \setminus \{0\}$ . □

**Acknowledgments.** Soon-Mo Jung was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B03931061).

**Author Contributions:** Writing – original draft, Yang-Hi Lee and Soon-Mo Jung; Writing – review & editing, Yang-Hi Lee and Soon-Mo Jung.

**Conflicts of Interest:** The authors declare no conflict of interest.

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