Article

(q, σ, τ)-differential graded algebras

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Abstract: We propose a notion of (q, σ, τ)-differential graded algebra, which generalizes the notions of (σ, τ)-differential graded algebra and q-differential graded algebra. We construct two examples of (q, σ, τ)-differential graded algebra, where the first one is constructed by means of generalized Clifford algebra with two generators (reduced quantum plane), where we use a (σ, τ)-twisted graded q-commutator. In order to construct the second example, we introduce a notion of (σ, τ)-pre-cosimplicial algebra.

Keywords: q-differential graded algebra, (σ, τ)-differential graded algebra, generalized Clifford algebra, pre-cosimplicial complex.

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1. Introduction

Skew-derivations or σ-derivations are generalized derivations obtained by twisting the Leibniz rule by means of an algebra map. They have been considered by physicists to study quantum groups and to obtain q-deformations of algebra of vector fields like Virasoro algebra and also Heisenberg algebras (oscillator algebras), see References [4,5,7,8,14]. The main example is given by Jackson derivative and lead for example to q-deformation of sl_2, Witt algebra and Virasoro algebra. A natural generalization consists of (σ, τ)-derivations involving two twist maps [18]. It turns out that when using σ-derivations the commutator bracket does no longer satisfies Jacobi condition. This was a starting point of studying Hom-type algebras, where the usual identities are twisted by homomorphisms [13,17]. Later development in the field of research of Hom-Lie algebras led to the introduction of a notion of (σ, τ)-differential graded algebra [19], which generalizes the notion of differential graded algebra.

It is well known that the concept of differential graded algebra is based on the equation $d^2 = 0$, where d is a differential of differential graded algebra. In order to generalize the concept of differential graded algebra, one can consider instead of $d^2 = 0$ a more general equation $d^N = 0$, where N is an integer greater or equal to 2. This generalization was proposed and studied in Reference [15]. Later this generalization was developed in Reference [9], where the authors proposed a notion of q-differential graded algebra, where q is a primitive Nth root of unity. It is worth to mention that particularly in the case of $q = -1$ (primitive square root of unity, $N = 2$) the definition of q-differential graded algebra gives the definition of differential graded algebra. Later it was shown that q-differential graded algebras can be applied in the field theories [12] and in noncommutative geometry to develop a generalization of a notion of connection [2].

In this paper we propose and study a notion of (q, σ, τ)-differential algebra, where q is a primitive Nth root of unity and σ, τ two degree zero endomorphisms of a graded algebra. This notion generalizes the notion of (σ, τ)-differential graded algebra, introduced in Reference [19], as well as the notion of q-differential graded algebra, which was studied in [1,3,11]. We construct several examples of (q, σ, τ)-differential graded algebras by means of generalized Clifford algebra with two generators (also called a reduced quantum plane) and by means of pre-cosimplicial algebra.

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2. First order (σ,τ)-differential calculus with right partial derivatives

Let $\mathfrak{A}$ be an associative unital algebra over $\mathbb{K}$, where $\mathbb{K}$ is either the field of real $\mathbb{R}$ or complex numbers $\mathbb{C}$. Let σ,τ be two algebra endomorphisms of $\mathfrak{A}$.

Definition 2.1. *A first order (σ,τ)-differential calculus over an algebra $\mathfrak{A}$ is the triple $(\mathfrak{A},d,\mathfrak{M})$, where $\mathfrak{M}$ is an $(\mathfrak{A},\mathfrak{A})$-bimodule and $d:\mathfrak{A}\to\mathfrak{M}$ is a linear mapping, which satisfies the (σ,τ)-Leibniz rule*

$$d(uv) = d(u)\,\tau(v) + \sigma(u)\,d(v). \tag{1}$$

Particularly if σ,τ are the identity transformations of algebra $\mathfrak{A}$, i.e. $\sigma=\tau=\mathrm{id}_{\mathfrak{A}}$, then the notion of first order (σ,τ)-differential calculus amounts to the notion of first order differential calculus over an associative unital algebra.

Now we assume that $(\mathfrak{A},d,\mathfrak{M})$ is a first order (σ,τ)-differential calculus, where $\mathfrak{M}$ is a free finite right $\mathfrak{A}$-module of rank r with a basis $e^1,e^2,\ldots,e^r$, i.e. any element u of $\mathfrak{M}$ can be uniquely written as

$$u=\sum_{i=1}^{r} e^i\,u_i,\quad u_i\in\mathfrak{A}.$$

Then a structure of $(\mathfrak{A},\mathfrak{A})$-bimodule of $\mathfrak{M}$ is uniquely determined by the commutation relations

$$v\,e^i=\sum_{j=1}^{r} e^j\,R^i_j(v), \tag{2}$$

where the linear mappings $R^i_j:\mathfrak{A}\to\mathfrak{A}$ satisfy

$$R^i_j(uv)=R^k_j(u)\,R^i_k(v). \tag{3}$$

It is useful to compose the rth order square matrix $R=(R^i_j)$ such that a linear mapping R^i_j is its entry at the intersection of ith column and jth row. Then (3) can be written in the matrix form as follows $R(uv)=R(u)\,R(v)$. Hence the linear mapping $R:\mathfrak{A}\to M_r(\mathfrak{A})$, where $M_r(\mathfrak{A})$ is the algebra of rth order matrices over an algebra $\mathfrak{A}$, is the algebra homomorphism. Define the right partial derivatives $\partial_i:\mathfrak{A}\to\mathfrak{A}$ (in a basis e^i) by the formula

$$du=\sum_i e^i\,\partial_i(u). \tag{4}$$

Proposition 2.2. *If $(\mathfrak{A},d,\mathfrak{M})$ is a first order (σ,τ)-differential calculus over an algebra $\mathfrak{A}$ and $\mathfrak{M}$ is a free finite right $\mathfrak{A}$-module with a basis $e^1,e^2,\ldots,e^n$, whose $(\mathfrak{A},\mathfrak{A})$-bimodule structure is determined by the commutation relations (2), then the right partial derivatives, defined in (4), satisfy*

$$\partial_i(uv)=\partial_i(u)\,\tau(v)+\sum_j R^j_i(\sigma(u))\,\partial_j(v). \tag{5}$$

Proof. According to the definition of right partial derivatives, we can write

$$d(uv)=\sum_i e^i\partial_i(uv).$$

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On the other hand, making use of (σ,τ)-Leibniz rule, we get

$$\begin{aligned} d(uv) &= d(u)\,\tau(v)+\sigma(u)\,d(v) = \sum_i (e^i\,\partial_i(u))\,\tau(v)+\sigma(u)\,\sum_j e^j\,\partial_j(v) \\ &= \sum_i (e^i\,\partial_i(u))\,\tau(v)+\sum_{i,j} e^i\,R_i^j(\sigma(u))\,\partial_j(v) = \sum_i e^i\,(\partial_i(u))\,\tau(v)+R_i^j(\sigma(u))\,\partial_j(v)). \end{aligned}$$

□

A first order (σ,τ)-differential calculus $(\mathfrak{A},d,\mathfrak{M})$, where $\mathfrak{M}$ is a free right $\mathfrak{A}$-module of rank r, whose $(\mathfrak{A},\mathfrak{A})$-bimodule structure is determined by the commutation rule (2) and the right derivatives are defined by (4), will be referred to as a first order (σ,τ)-differential calculus with right partial derivatives. If $(\mathfrak{A},d,\mathfrak{M})$ is a first order (σ,τ)-differential calculus with right partial derivatives, an algebra $\mathfrak{A}$ is generated by variables $x^1,x^2,\ldots,x^r$ and $dx^1,dx^2,\ldots,dx^r$ is the basis for a free right $\mathfrak{A}$-module $\mathfrak{M}$, then this first order (σ,τ)-differential calculus will be referred to as a coordinate first order (σ,τ)-differential calculus with right partial derivatives.

3. (q,σ,τ)-differential graded algebra

Let $\mathcal{A}=\oplus_n\mathcal{A}^n$ be a graded associative unital algebra and σ,τ be degree zero endomorphisms of $\mathcal{A}$. The degree of a homogeneous element u will be denoted by $|u|$. In what follows q will be a primitive Nth root of unity, where $N\geq 2$. For instance, we can take $q=\exp(2\pi i/N)$. We give the following definition:

Definition 3.1. *$\mathcal{A}$ is said to be a (q,σ,τ)-differential graded algebra if $\mathcal{A}$ is endowed with a degree one linear mapping $d:\mathcal{A}^n\to\mathcal{A}^{n+1}$ such that it satisfies the following conditions:*

a) d commutes with endomorphisms σ,τ, i.e. $\sigma\circ d=d\circ\sigma, \tau\circ d=d\circ\tau$,
b) d satisfies the (q,σ,τ)-Leibniz rule

$$d(uv)=d(u)\,\tau(v)+q^{|u|}\,\sigma(u)\,d(v),$$

c) $d^N(u)=0$ for any element $u\in\mathcal{A}$.

It is worth to mention that particularly if we take $\sigma=\tau=\mathrm{id}_{\mathcal{A}}$, then the notion of (q,σ,τ)-differential graded algebra amounts to the notion of q-differential graded algebra [1,2,11**? ?**]. If we choose $N=2$, then $q=-1$ and the definition of (q,σ,τ)-differential graded algebra gives a definition of (σ,τ)-differential algebra, which was introduced in Reference [19]. Finally if in the above definition we choose both $q=-1 (N=2)$ and $\sigma=\tau=\mathrm{id}_{\mathcal{A}}$, then Definition (3.1) gives the definition of differential graded algebra.

Let $\mathcal{A}=\oplus_n\mathcal{A}^n$ be a (q,σ,τ)-differential graded algebra. Obviously $\mathcal{A}^0\subset\mathcal{A}$ is the subalgebra of $\mathcal{A}$. Next it is easy to see that every subspace of homogeneous elements $\mathcal{A}^n$ can be considered as $(\mathcal{A}^0,\mathcal{A}^0)$-bimodule, where the left (right) $\mathcal{A}^0$-module structure of $\mathcal{A}^n$ is determined by the left (right) multiplication by degree zero elements, i.e.

$$(u,v)\in\mathcal{A}^0\times\mathcal{A}^n\to u\,v\in\mathcal{A}^n,\ \ (v,u)\in\mathcal{A}^n\times\mathcal{A}^0\to v\,u\in\mathcal{A}^n.$$

Hence the triple $(\mathcal{A}^0,d,\mathcal{A}^1)$ is the first order (σ,τ)-differential calculus, because $\mathcal{A}^1$ is the $(\mathcal{A}^0,\mathcal{A}^0)$-bimodule and $d:\mathcal{A}^0\to\mathcal{A}^1$ satisfies in this case the (σ,τ)-Leibniz rule

$$d(uv)=d(u)\,\tau(v)+\sigma(u)\,d(v),\quad u,v\in\mathcal{A}^0.$$

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4. Construction of (q,σ,τ)-differential graded algebra by means of graded q-commutator

In this section we show that given an associative unital graded algebra one can construct a (q,σ,τ)-differential graded algebra with the help of graded q-commutator, where q is a primitive Nth root of unity.

Let $\mathcal{A} = \oplus_n \mathcal{A}^n$ be a graded algebra over $\mathbb{C}$, $\mathbf{1}$ be its unit element and σ, τ be two degree zero endomorphisms of $\mathcal{A}$.

Theorem 4.1. *Let $\xi \in \mathcal{A}^1$ be an element of degree one. Define the degree one linear mapping $d : \mathcal{A}^n \to \mathcal{A}^{n+1}$ by the following formula*

$$d(u) = \xi\,\tau(u) - q^{|u|}\sigma(u)\,\xi. \tag{6}$$

If

a) q is a primitive Nth root of unity,
b) $\xi^N = \lambda\,\mathbf{1}$, where λ is non-zero complex number,
c) $\sigma(\xi) = \tau(\xi) = \xi, \quad \sigma\circ\tau = \tau\circ\sigma, \quad \sigma^N = \tau^N$,

then a graded algebra $\mathcal{A}$ endowed with the degree one linear mapping d is the (q,σ,τ)-differential graded algebra.

Proof. First we prove $\sigma\circ d = d\circ\sigma$. For any homogeneous $u \in \mathcal{A}$, we have

$$\begin{aligned}\sigma\circ d(u) &= \sigma(\xi\,\tau(u) - q^{|u|}\sigma(u)\,\xi) = \sigma(\xi)\,\sigma\circ\tau(u) - q^{|u|}\,\sigma^2(u)\,\sigma(\xi)\\ &= \xi\,\tau\circ\sigma(u) - q^{|u|}\sigma^2(u)\,\xi = d\circ\sigma(u).\end{aligned}$$

Analogously we can prove $\tau\circ d = d\circ\tau$. Starting with the right-hand side of (q,σ,τ)-Leibniz rule (Definition 3.1) we get

$$\begin{aligned}d(u)\,\tau(v) + q^{|u|}\,\sigma(u)\,d(v) &= (\xi\,\tau(u) - q^{|u|}\sigma(u)\,\xi)\tau(v) + q^{|u|}\,\sigma(u)(\xi\,\tau(v) - q^{|v|}\,\sigma(v)\,\xi)\\ &= \xi\,\tau(uv) - q^{|u|}\sigma(u)\,\xi\,\tau(v) + q^{|u|}\sigma(u)\,\xi\,\tau(v) - q^{|u|+|v|}\sigma(uv)\,\xi = d(uv),\end{aligned}$$

and the (q,σ,τ)-Leibniz rule is proved. For Nth power of d we have the following power expansion (see Reference [3])

$$d^N(u) = \sum_{i=0}^{N} (-1)^i\,p_i \begin{bmatrix} N \\ i \end{bmatrix}_q \xi^{N-i}\,\tau^{N-i}\circ\sigma^i(u)\,\xi^i, \tag{7}$$

where $p_i = q^{i|u|+\mu(i)}$ and $\mu(i) = \frac{i(i-1)}{2}$. According to the assumption q is a primitive Nth root of unity, which implies for the quantum Newton binomial coefficients

$$\begin{bmatrix} N \\ i \end{bmatrix}_q = 0, \;\; i = 1,2,\ldots,N-1,$$

and the terms in power expansion (7) labelled by $i = 1,2,\ldots,N-1$ vanish. Thus there are only two non-trivial terms in (7)

$$d^N(u) = \lambda\,(\tau^N(u) + (-1)^N\,q^{N(N-1)/2}\sigma^N(u)).$$

If N is an odd positive integer then

$$d^N(u) = \lambda\,(\tau^N(u) - (q^N)^{\frac{N-1}{2}}\sigma^N(u)) = \lambda(\tau^N(u) - \sigma^N(u)) = 0.$$

If N is an even positive integer then

$$d^N(u) = \lambda(\tau^N(u) + (q^{\frac{N}{2}})^{N-1}) = \lambda(\tau^N(u) + (-1)^{N-1}\sigma^N(u)) = \lambda(\tau^N(u) - \sigma^N(u)) = 0.$$

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□

In order to construct a matrix example of (q,σ,τ)-differential graded algebra, we apply Theorem 4.1 to a generalized Clifford algebra. We recall you that a generalized Clifford algebra is an associative unital algebra over $\mathbb{C}$ generated by variables $x^1, x^2, \ldots, x^n$, which are subjected to the relations

$$x^i\,x^j = q\,x^j\,x^i \;\;(i<j),\;\; (x^i)^N = \mathbf{1},\;\; i,j = 1,2,\ldots,n, \tag{8}$$

where q is a primitive Nth root of unity and $\mathbf{1}$ is the unit element of generalized Clifford algebra. A generalized Clifford algebra will be denoted by $\mathfrak{C}^N_n$, where n, N are independent integers, n is the number of generators and $N \geq 2$ is an exponent at which Nth power of every generator equals to identity element $\mathbf{1}$.

We consider the generalized Clifford algebra $\mathfrak{C}^N_2$ with two generators $x = x^1, \xi = x^2$. Then from relations (8) it follows

$$x\,\xi = q\,\xi\,x,\;\; x^N = \xi^N = \mathbf{1}. \tag{9}$$

The associative unital algebra generated by two variables x, ξ, which are subjected to the relations (9), is also called the algebra of functions on a reduced quantum plane. This algebra has the matrix representation by Nth order complex matrices. Indeed we can identify the generators x, ξ with the following Nth order matrices

$$x = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & q^{-1} & 0 & \ldots & 0 \\ 0 & 0 & q^{-2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & q^{-(N-1)} \end{pmatrix},\;\; \xi = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{10}$$

because these matrices satisfy the relations (9). Thus we have the matrix representation for the generalized Clifford algebra with two generators. It is worth to mention that the matrices (10) generate the whole algebra of Nth order complex matrices $M_N(\mathbb{C})$.

In order to define a structure of graded algebra on $\mathfrak{C}^N_2$ we attribute degree zero to the unit element $\mathbf{1}$ and the generator x, degree one to the generator ξ and extend this degree to any product of the generators x, ξ by defining the degree of a product as the sum of degrees of its cofactors. Then the whole algebra $\mathfrak{C}^N_2$ splits into the direct sum of subspaces of homogeneous elements and a subspace of elements of degree k will be denoted by $\mathfrak{C}^{N,k}_2$, where k runs over the residue classes modulo N, i.e. $k = 0, 1, \ldots, N-1$. Obviously the subalgebra of elements of degree zero $\mathfrak{C}^{N,0}_2$ will be generated by the generator x, i.e. $\mathfrak{C}^{N,0}_2$ is the algebra of polynomials of x. We consider the generator x as an analog of coordinate function of one-dimensional space. Thus the algebra of "functions" is the algebra of polynomials of x. In order to emphasize that we consider the elements of $\mathfrak{C}^{N,0}_2$ as analogs of functions, we will denote the elements of $\mathfrak{C}^{N,0}_2$ by $f(x), g(x), h(x)$.

Let σ, τ be two degree zero endomorphisms of the generalized Clifford algebra $\mathfrak{C}^N_2$ such that they commute, $\sigma(\xi) = \tau(\xi) = \xi$, $\sigma^N = \tau^N$ and $\tau(x) - q\,\sigma(x)$ is invertible element of $\mathfrak{C}^{N,0}_2$. According to Theorem (4.1), we define the differential $d_\xi : \mathfrak{C}^{N,k}_2 \to \mathfrak{C}^{N,k+1}_2$ by the formula

$$d_\xi(u) = \xi\,\tau(u) - q^{|u|}\,\sigma(u)\,\xi, \tag{11}$$

where u is a homogeneous element of generalized Clifford algebra $\mathfrak{C}^N_2$ and $|u|$ is its degree. Since all the assumptions of Theorem (4.1) are fulfilled, the algebra of Nth order complex matrices $M_N(\mathbb{C})$ endowed with the structure of $\mathbb{Z}_N$-graded algebra, which is based on $|x| = 0, |\xi| = 1$, and with the differential d_ξ is the (q,σ,τ)-differential graded algebra.

We conclude this section by considering the structure of first order differential calculus of matrix (q,σ,τ)-differential graded algebra $\mathfrak{C}_2^N$. It is easy to show that $(\mathfrak{C}_2^{N,0}, d_\xi, \mathfrak{C}_2^{N,1})$ is the coordinate first order (σ,τ)-differential calculus with right derivative. Indeed, we have

$$d_\xi(x) = \xi\,\tau(x) - \sigma(x)\xi = \xi\,\big(\tau(x) - q\,\sigma(x)\big).$$

Because we assume that $\tau(x) - q\,\sigma(x)$ is invertible element of $\mathfrak{C}_2^{N,0}$, the differential $d_\xi(x)$ of coordinate function x can serve as the basis for the right $\mathfrak{C}_2^{N,1}$-module. The commutation relation, which determines the $\mathfrak{C}_2^{N,0}$-bimodule structure of $\mathfrak{C}_2^{N,1}$, has the form

$$f(x)\,dx = dx\,R(f(x)),$$

where $R : f(x) \in \mathfrak{C}_2^{N,0} \to f(qx) \in \mathfrak{C}_2^{N,0}$ is the automorphism of algebra. Thus, according to (4), the differential d_ξ induces the right derivative

$$d_\xi f(x) = d_\xi x\,\frac{df(x)}{dx}, \tag{12}$$

where f is a polynomial of x. According to Proposition (2.2), this derivative satisfies

$$\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}\tau(g(x)) + \sigma(f(qx))\,\frac{dg(x)}{dx}. \tag{13}$$

Since σ,τ are linear mappings, the left-hand side of (12) can be written as

$$\begin{aligned} d_\xi f(x) &= \xi\,\tau(f(x)) - \sigma(f(x))\,\xi = \xi\big(f(\tau(x)) - f(\sigma(qx))\big) \\ &= \xi\,\big(\tau(x) - q\,\sigma(x)\big)\frac{f(\tau(x)) - f(\sigma(qx))}{\tau(x) - \sigma(qx)} = d_\xi x\,\frac{f(\tau(x)) - f(\sigma(qx))}{\tau(x) - \sigma(qx)}, \end{aligned}$$

where

$$\big(\tau(x) - \sigma(qx)\big)^{-1} = \frac{1}{\tau(x) - \sigma(qx)}$$

is inverse of $\tau(x) - \sigma(qx)$. From this it follows

$$\frac{df(x)}{dx} = \frac{f(\tau(x)) - f(\sigma(qx))}{\tau(x) - \sigma(qx)}. \tag{14}$$

The derivative (14) is called (σ,τ)-twisted Jackson q-derivative [13]. Thus the differential d_ξ, defined in (11), determines the coordinate first order (σ,τ)-differential calculus with (σ,τ)-twisted Jackson type q-derivative.

5. Construction of (q,σ,τ)-differential graded algebra by means of (σ,τ)-pre-cosimplicial algebra

In this section we introduce a notion of (σ,τ)-pre-cosimplicial algebra and show that a (q,σ,τ)-differential algebra can be constructed with the help of a (σ,τ)-pre-cosimplicial algebra.

First we recall you the notion of a pre-cosimplicial vector space [16]. A pre-cosimplicial vector space $(\mathfrak{A}, \mathfrak{f}_i)$ is a positive graded vector space $\mathfrak{A} = \oplus_{n\geq 0}\mathfrak{A}^n$ together with coface homomorphisms (linear mappings of vector spaces) $\mathfrak{f}_i : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$, where i runs from 0 to $n+1$, such that

$$\mathfrak{f}_j \circ \mathfrak{f}_i = \mathfrak{f}_i \circ \mathfrak{f}_{j-1}, \quad i < j. \tag{15}$$

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Thus every pair of vector spaces $\mathfrak{A}^n, \mathfrak{A}^{n+1}$ is equipped with the $n+2$ coface homomorphisms $\mathfrak{f}_0, \mathfrak{f}_1, \ldots, \mathfrak{f}_{n+1}$, where $\mathfrak{f}_i : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$. For example, in the case of vector spaces $\mathfrak{A}^1, \mathfrak{A}^2$ there are three coface homomorphisms $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2 : \mathfrak{A}^1 \to \mathfrak{A}^2$, which satisfy

$$\mathfrak{f}_1 \circ \mathfrak{f}_0 = \mathfrak{f}_0^2, \quad \mathfrak{f}_2 \circ \mathfrak{f}_0 = \mathfrak{f}_0 \circ \mathfrak{f}_1, \quad \mathfrak{f}_2 \circ \mathfrak{f}_1 = \mathfrak{f}_1^2.$$

The following definition generalizes the notion of a pre-cosimplicial algebra, which can be found in Reference [10].

Definition 5.1. *Let σ, τ be two degree zero endomorphisms of a pre-cosimplicial vector space $(\mathfrak{A}, \mathfrak{f}_i)$ such that they commute with coface homomorphisms, i.e.*

$$\sigma \circ \mathfrak{f}_i = \mathfrak{f}_i \circ \sigma, \quad \tau \circ \mathfrak{f}_i = \mathfrak{f}_i \circ \tau.$$

A pre-cosimplicial vector space $(\mathfrak{A}, \mathfrak{f}_i)$ is said to be a (σ, τ)-pre-cosimplicial algebra if

1) *$\mathfrak{A} = \oplus_{n \geq 0} \mathfrak{A}^n$ is a graded algebra,*
2) *σ, τ are degree zero endomorphisms of a graded algebra $\mathfrak{A}$,*
3) *for any homogeneous elements $u, v \in \mathfrak{A}$ and any integer $i \in \{0, 1, \ldots, |u| + |v| + 1\}$ we have*

$$\mathfrak{f}_i(uv) = \begin{cases} \mathfrak{f}_i(u)\,\tau(v), & \text{if } |u| \geq i, \\ \sigma(u)\,\mathfrak{f}_{i-|u|}(v), & \text{if } 0 \leq |u| < i, \end{cases} \qquad \mathfrak{f}_{|u|+1}(u)\tau(v) = \sigma(u)\,\mathfrak{f}_0(v). \tag{16}$$

Particularly if we take $\sigma = \tau = \mathrm{id}_{\mathfrak{A}}$ then the above definition reduces to the definition of a pre-cosimplicial algebra.

Theorem 5.2. *Let $(\mathfrak{A}, \mathfrak{f}_i)$ be a (σ, τ)-pre-cosimplicial algebra. Define the degree one linear mapping $d : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$ by*

$$d = \sum_{k=0}^{n} q^k\, \mathfrak{f}_k - q^n\, \mathfrak{f}_{n+1}, \tag{17}$$

where q is a primitive Nth root of unity. Then a (σ, τ)-pre-cosimplicial algebra $\mathfrak{A}$ endowed with the degree one linear mapping d is the (q, σ, τ)-differential graded algebra.

Proof. According to Definition (5.1), degree zero endomorphisms σ, τ of a positive graded vector space $\mathfrak{A}$ commute with coface homomorphisms $\mathfrak{f}_i$ and this immediately implies that σ, τ commute with d. It is proved in Reference [11] that for any pre-cosimplicial vector space $\mathfrak{A}$ the degree one linear mapping d, defined in (17), satisfies $d^N = 0$, i.e. according to the terminology adopted in Reference [11], d is N-differential. Since the right-hand side of the formula for d does not depend on degree zero endomorphisms σ, τ, the same result holds in the case of (σ, τ)-pre-cosimplicial algebra. Hence we only need to prove that d satisfies the (q, σ, τ)-Leibniz rule.

Let $u \in \mathfrak{A}^k$ be a homogeneous element of degree k and $v \in \mathfrak{A}$. Our aim is to prove

$$d(uv) = d(u)\,\tau(v) + q^k\,\sigma(u)\,d(v). \tag{18}$$

Making use of the formula (17) and the formula (16), we can write the left-hand side of (q, σ, τ)-Leibniz rule as follows

$$\begin{aligned} d(uv) &= \mathfrak{f}_0(uv) + q\,\mathfrak{f}_1(uv) + \ldots + q^k\,\mathfrak{f}_k(uv) + q^{k+1}\,\mathfrak{f}_{k+1}(uv) + \ldots + q^n\,\mathfrak{f}_n(uv) - q^n\,\mathfrak{f}_{n+1}(uv) \\ &= f_0(u)\,\tau(v) + q\,f_1(u)\,\tau(v) + \ldots + q^k\,f_k(u)\,\tau(v) + q^{k+1}\sigma(u)\mathfrak{f}_1(v) \\ &\qquad + \ldots + q^n\,\sigma(u) f_{n-k}(v) - q^n\,\sigma(u) f_{n+1-k}(v). \end{aligned} \tag{19}$$

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The right-hand side of the same formula can be written in the form

$$
\begin{aligned}
d(u)\,\tau(v) \;+\; q^k\,\sigma(u)\,d(v) = \mathfrak{f}_0(u)\,\tau(v) + q\,\mathfrak{f}_1(u)\,\tau(v) + \ldots + q^k\,\mathfrak{f}_k(u)\,\tau(v) - \cancel{q^k\,\mathfrak{f}_{k+1}(u)\tau(v)} \\
+\cancel{q^k\,\sigma(u)\mathfrak{f}_0(v)} + q^{k+1}\sigma(u)\mathfrak{f}_1(v) + \ldots + q^{k+l}\sigma(u)\mathfrak{f}_{n-k}(v) - q^{k+l}\sigma(u)\mathfrak{f}_{n+1-k}(v), \quad (20)
\end{aligned}
$$

where the crossed out terms cancel each other because of the second relation in (16). Comparing (19) with (20), we see that their left-hand sides are equal and this ends the proof. □

Let $\mathcal{A}$ be an associative unital algebra, whose unit element will be denoted by $\mathbf{1}$, and σ,τ be two endomorphisms of $\mathcal{A}$. The tensor algebra $\mathfrak{T}(\mathcal{A}) = \oplus_{n\geq 0}\mathfrak{T}^n(\mathcal{A})$ is the graded algebra, where a subspace of elements of degree n is the tensor product $\otimes^{n+1}\mathcal{A}$, i.e. $\mathfrak{T}^n(\mathcal{A}) = \otimes^{n+1}\mathcal{A}$, and the algebra multiplication $(u,v)\to uv$, where $u = u_0\otimes u_1\otimes\ldots\otimes u_n$ and $v = v_0\otimes v_1\otimes\ldots\otimes v_m$ are homogeneous elements of degree n and m respectively, is defined by

$$uv = u_0\otimes u_1\otimes\ldots\otimes u_{n-1}\otimes u_n v_0\otimes v_1\otimes\ldots\otimes v_m. \quad (21)$$

We extend endomorphisms σ,τ to the tensor algebra $\mathfrak{T}(\mathcal{A})$ by

$$\sigma(u) = \sigma(u_0)\otimes\sigma(u_1)\otimes\ldots\otimes\sigma(u_n),\quad \tau(u) = \tau(u_0)\otimes\tau(u_1)\otimes\ldots\otimes\tau(u_n).$$

Obviously σ,τ are degree zero endomorphisms of graded algebra $\mathfrak{T}(\mathcal{A})$.

Theorem 5.3. *For any $u = u_0\otimes u_1\otimes\ldots\otimes u_n\in\mathfrak{T}^n(\mathcal{A})$ define the linear mappings $\mathfrak{f}_k:\mathfrak{T}^n(\mathcal{A})\to\mathfrak{T}^{n+1}(\mathcal{A})$, where $k\in\{0,1,\ldots,n+1\}$, by*

$$
\begin{aligned}
\mathfrak{f}_0(u) &= \mathbf{1}\otimes\tau(u_0)\otimes\tau(u_1)\otimes\ldots\otimes\tau(u_n),\\
\mathfrak{f}_k(u) &= \sigma(u_0)\otimes\sigma(u_1)\otimes\ldots\otimes\sigma(u_{k-1})\otimes\mathbf{1}\otimes\tau(u_k)\otimes\ldots\otimes\tau(u_n),\quad k=1,2,\ldots,n-1,\\
\mathfrak{f}_{n+1}(u) &= \sigma(u_0)\otimes\sigma(u_1)\otimes\ldots\otimes\sigma(u_n)\otimes\mathbf{1}. \quad (22)
\end{aligned}
$$

Then $(\mathfrak{T}(\mathcal{A}),\mathfrak{f}_k)$ is the (σ,τ)-pre-cosimplicial algebra and $\mathfrak{f}_k$ are its coface homomorphisms. If we endow the (σ,τ)-pre-cosimplicial algebra $(\mathfrak{T}(\mathcal{A}),\mathfrak{f}_k)$ with the N-differential d, defined in (17), then $(\mathfrak{T}(\mathcal{A}),\mathfrak{f}_k)$ becomes the (q,σ,τ)-differential graded algebra.

Proof. Let $u = u_0\otimes u_1\otimes\ldots\otimes u_n$, $v = v_0\otimes v_1\otimes\ldots\otimes v_n$ be two homogeneous elements of tensor algebra $\mathfrak{T}(\mathcal{A})$. Then their product

$$uv = u_0\otimes u_1\otimes\ldots\otimes u_{n-1}\otimes u_n v_0\otimes v_1\otimes\ldots\otimes v_m$$

is the element of the subspace $\mathfrak{T}^{n+m+1}(\mathcal{A})$ and, consequently, we have $\mathfrak{f}_0,\mathfrak{f}_1,\ldots,\mathfrak{f}_{n+m+1}$ coface homomorphisms from the subspace $\mathfrak{T}^{n+m}(\mathcal{A})$ to the subspace $\mathfrak{T}^{n+m+1}(\mathcal{A})$. For the coface homomorphisms $\mathfrak{f}_0,\mathfrak{f}_1,\ldots,\mathfrak{f}_n$ we have to prove the formula

$$\mathfrak{f}_k(uv) = \mathfrak{f}_k(u)\,\tau(v),\ k=0,1,2,\ldots,n. \quad (23)$$

According to the definition of coface homomorphisms, we can write the left-hand side as follows

$$\mathfrak{f}_k(uv) = \sigma(u_0)\otimes\ldots\otimes\sigma(u_{k-1})\otimes\mathbf{1}\otimes\tau(u_k)\otimes\ldots\otimes\tau(u_n v_0)\otimes\ldots\otimes\tau(v_m).$$

The right-hand side can be written as follows

$$\mathfrak{f}_k(u)\tau(v) = \sigma(u_0)\otimes\ldots\sigma(u_{k-1})\otimes\mathbf{1}\otimes\tau(u_k)\otimes\ldots\otimes\tau(u_n)\tau(v_0)\otimes\ldots\otimes\tau(v_m).$$

Since τ is an endomorphism of algebra, we have $\tau(u_n v_0) = \tau(u_n)\tau(v_0)$ and the formula (23) is proved. For coface homomorphisms $\mathfrak{f}_{n+1}, \ldots, \mathfrak{f}_{n+m+1}$ we have to prove

$$\mathfrak{f}_k(uv) = \sigma(u)\mathfrak{f}_{k-n}(v),\ k = n+1, \ldots, n+m+1. \tag{24}$$

The proof of this formula is similar to the proof of the (23). Finally, we have to prove the second relation in (16), i.e.

$$\mathfrak{f}_{n+1}(u)\tau(v) = \sigma(u)\mathfrak{f}_0(v).$$

The left-hand side of this relation can be written as

$$\mathfrak{f}_{n+1}(u)\tau(v) = \sigma(u_0) \otimes \sigma(u_1) \otimes \ldots \otimes \sigma(u_n) \otimes \mathbf{1}\,\tau(v_0) \otimes \ldots \otimes \tau(v_m)),$$

and the right-hand side can be written as

$$\sigma(u)\mathfrak{f}_0(v) = \sigma(u_0) \otimes \sigma(u_1) \otimes \ldots \otimes \sigma(u_n)\,\mathbf{1} \otimes \tau(v_0) \otimes \ldots \otimes \tau(v_m)),$$

and we see that they are equal. □

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