

Article

Fourier Transform on the Homogeneous Space of 3D Positions and Orientations for Exact Solutions to PDEs

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Abstract: Fokker-Planck PDEs (incl. diffusions) for stable Lévy processes (incl. Wiener processes) on the joint space of positions and orientations play a major role in mechanics, robotics, image analysis, directional statistics and probability theory. Exact analytic designs and solutions are known in the 2D case, where they have been obtained using Fourier transform on $SE(2)$. Here we extend these approaches to 3D using Fourier transform on the Lie group $SE(3)$ of rigid body motions. More precisely, we define the homogeneous space of 3D positions and orientations $\mathbb{R}^3 \rtimes S^2 := SE(3)/(\{0\} \times SO(2))$ as the quotient in $SE(3)$. In our construction, two group elements are equivalent if they are equal up to a rotation around the reference axis. On this quotient we design a specific Fourier transform. We apply this Fourier transform to derive new exact solutions to Fokker-Planck PDEs of α -stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$. This reduces classical analysis computations and provides an explicit algebraic spectral decomposition of the solutions. We compare the exact probability kernel for $\alpha = 1$ (the diffusion kernel) to the kernel for $\alpha = \frac{1}{2}$ (the Poisson kernel). We set up SDEs for the Lévy processes on the quotient and derive corresponding Monte-Carlo methods. We verify that the exact probability kernels arise as the limit of the Monte-Carlo approximations.

Keywords: Fourier Transform; Rigid Body Motions; Partial Differential Equations; Lévy Processes; Lie Groups; Homogeneous Spaces; Stochastic Differential Equations

1. Introduction

The Fourier transform has had a tremendous impact on various fields of mathematics including analysis, algebra and probability theory. It has a broad range of applied fields such as signal and image processing, quantum mechanics, classical mechanics, robotics and system theory. Thanks to Jean-Baptiste Joseph Fourier (1768-1830), who published his pioneering work “Théory analytique de la chaleur” in 1822, the effective technique of using a Fourier transform to solve linear PDE-systems (with appropriate boundary conditions) for heat transfer evolutions on compact subsets Ω of \mathbb{R}^d was born. The Fourier series representations of the solutions helped to understand the physics of heat transfer. Due to the linearity of the evolution operator that maps the possibly discontinuous square integrable initial condition to the square integrable solution at a fixed time $t > 0$, one can apply a spectral decomposition which shows how each eigenfunction is dampened over time. Thanks to contributions of Johann Peter Gustav Lejeune Dirichlet (1805-1859), completeness of the Fourier basis could then be formalized for several boundary conditions. Indeed separation of variables (also known as ‘the Fourier method’) directly provides a Sturm-Liouville problem [1] and an orthonormal basis of eigenfunctions for $L_2(\Omega)$, which is complete due to compactness of the associated self-adjoint kernel operator. When dilating the subset Ω to the full space \mathbb{R}^d , the discrete set of eigenvalues start to fill \mathbb{R} and the discrete spectrum approximates a continuous spectrum (see e.g.[2]). Then a diffusion system on \mathbb{R}^d can be solved via a unitary Fourier transform on $L_2(\mathbb{R}^d)$, cf.[3].

Nowadays, in fields such as mechanics/robotics [4–7], mathematical physics/harmonic analysis [8], machine learning [9–13] and image analysis [14–19] it is worthwhile to extend the spatial domain of functions on $M = \mathbb{R}^d$ (or $M = \mathbb{Z}^d$) to groups $G = M \rtimes T$ that are the semi-direct product of an Abelian group M and another matrix group T . This requires a generalization of the Fourier transforms on the Lie group $(\mathbb{R}^d, +)$ towards the groups $G = \mathbb{R}^d \rtimes T$. Then the Fourier transform gives rise to an invertible decomposition of a square integrable function into irreducible representations. This is a powerful mechanism in view of the Schur's lemma [20,21] and spectral decompositions [22,23]. However, it typically involves regularity constraints [24, ch:3.6], [22, ch:6.6] on the structure of the dual orbits in order that Mackey's imprimitivity theory [25] can be applied to characterize *all* unitary irreducible representations (UIRs) of G . This sets the Fourier transform on the Lie group G , [22,24,26]. Here, we omit technicalities on regularity constraints on the dual orbits and the fact that G may not be of type I (i.e. the quasi-dual group of G may not be equal to the dual group of G , cf. [22, thm.7.6, 7.7], [27], [24, ch:3]), as this does not play a role in our case of interest.

We are concerned with the case $M = \mathbb{R}^3$ and $T = SO(3)$ where $G = SE(3) = M \rtimes SO(3)$ is the Lie group of 3D rigid body motions. It is a (type I) Lie group with an explicit Fourier transform \mathcal{F}_G where the irreducible representations are determined by regular dual orbits (which are spheres in the Fourier domain indexed by their radius $p > 0$) and an integer index $s \in \mathbb{Z}$, cf. [4,26].

In this article we follow the idea of Joseph Fourier: we apply the Fourier transform \mathcal{F}_G on the rigid body motion group $G = SE(3)$ to solve hypo-elliptic and elliptic heat flow evolutions respectively on the Lie group G . More precisely, we design a Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space of positions and orientations G/H with $H \equiv \{\mathbf{0}\} \times SO(2)$ to solve hypo-elliptic and elliptic heat flow evolutions on the homogeneous space G/H . We also simultaneously solve related PDEs (beyond the diffusion case) as we will explain below.

The idea of applying Fourier transforms to solve linear (hypo-elliptic) PDEs on non-commutative groups of the type $\mathbb{R}^d \rtimes T$ is common and has been studied by many researchers. For example, tangible probability kernels for heat transfer (and fundamental solutions) on the Heisenberg group were derived by Gaveau [28]. They can be derived by application [23, ch:4.1.1] of the Fourier transform on the Heisenberg group [22, ch:1]. This also applies to probability kernels for hypo-elliptic diffusions on $SE(2) = \mathbb{R}^2 \rtimes SO(2)$, where 3 different types (a Fourier series, a rapidly decaying series, and a single expression) of explicit solutions to probability kernels for (convection)-diffusions are derived in previous works by Duits et al. [29–32]. For a concise review see [33, ch:5.1]. Here, the two fundamental models for contour perception by respectively Mumford [34], Petitot [35] and Citti & Sarti [15] formed great sources of inspiration to study the hypo-elliptic diffusion problem on $SE(2)$.

The hypo-elliptic diffusion kernel formula in terms of a Fourier series representation was generalized to the much more wide setting of unimodular Lie groups by Agrachev, Boscain, Gauthier and Rossi [36]. This approach was then pursued by Portegies & Duits (with additional classical analysis techniques) to achieve explicit exact solutions to (hypo)-elliptic (convection)-diffusions on the particular $SE(3)$ case, see [37].

Here, we structure the results in [37] and we present new, simpler formulas (relying on an algebraic approach rather than a classical analysis approach). To this end we first set up a specific Fourier transform on the homogeneous space of positions and orientations in Theorem 1. Then we use it to derive explicit spectral decompositions of the evolution operator in Theorem 2, from which we deduce explicit new kernel expressions in Theorem 3. Finally, we generalize the exact solutions to other PDE systems beyond the diffusion case: We will simultaneously solve the Forward-Kolmogorov PDEs for α -stable Lévy processes on the homogeneous space of positions and orientations. Next we address their relevance in the fields of image analysis, robotics and probability theory.

In image analysis, left-invariant diffusion PDEs on $SE(3)$ have been widely used for crossing-preserving diffusion and enhancement of fibers in diffusion-weighted MRI images of brain white matter [38–43], or for crossing-preserving enhancements of 3D vasculature in medical images [18]. They extend classical works on multi-scale image representations [44–47] to Lie groups [48].

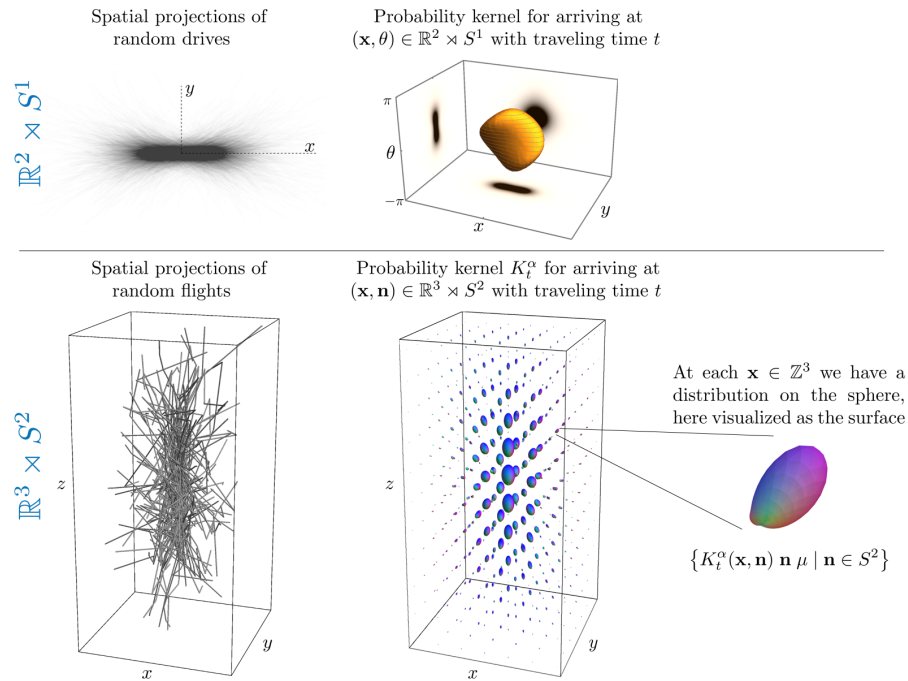


Figure 1. Various visualization of the diffusion process ($\alpha = 1$) on $\mathbb{R}^d \times S^{d-1}$, for $d = 2$ and $d = 3$. Top: random walks (or rather ‘drunk man’s drives’) and an iso-contour of the limiting diffusion kernel, for the case $d = 2$ studied in previous works (see e.g. [33,68,69]). Bottom: random walks (or rather ‘drunk man’s flights’) and a visualization of the limiting distribution for the case $d = 3$. This limiting distribution is a hypo-elliptic diffusion kernel $(\mathbf{x}, \mathbf{n}) \mapsto K_t^{\alpha=1}(\mathbf{x}, \mathbf{n})$ that we study in this article. We visualize the kernel $K_t^{\alpha=1}$ by a spatial grid of surfaces, where all surfaces are scaled by the same $\mu > 0$.

In robotics, they play a role via the central limit theorem [49] in work-space generation of robot arms [4, ch.12] and they appear indirectly in Kalman-filtering on $SE(3)$ for tracking [50] and camera motion estimation [51].

In probability theory, diffusion systems on Lie groups describe Brownian motions [52,53] and they appear as limits in central limit theorem on Lie groups [49].

Both in probability theory [54] and in image analysis [55–58], the spectral decomposition of the evolution operator also allows to simultaneously deal with important variants of the diffusion evolution. These variants of the heat-evolution are obtained by taking fractional powers $-(-\Delta)^\alpha$, cf. [59], of the minus Laplacian operator $\Delta = \text{div} \circ \text{grad}$ that generates the heat flow (due to Fick’s law and the Gauss divergence theorem), where $\alpha \in (0, 1]$.

This generalization allows for heavy tailed distributions of α -stable Lévy processes, that arise in a fundamental generalization [54] of the central limit theorem *where one drops the finite variance condition*. Here we note that recently an extension of the central limit on linear groups (such as $SE(3)$) has been achieved for finite 2nd order moments [49]. In engineering applications where (iterative group-) convolutions are applied [4, ch.12&13], [9,12,13,60–64] the ‘kernel width’ represents the spread of information or the scale of observing the signal. In case the applications allow for an underlying probabilistic model with finite variances, variance is indeed a good measure for ‘kernel width’. But often this is not the case. Probability kernels for stochastic Lévy processes (used in directional statistics [65], stock market modeling [66], natural image statistics [58]), modeling of point-spread functions in acquired images (e.g. in spectroscopy [67]), do require distributions with heavier tails than diffusion kernels. Therefore, ‘full width at half maximum’ is a more generally applicable measure for kernel width than variance, as it applies to all α -stable Lévy processes. The probability distributions for $\alpha < 1$ encode a longer range of interaction via their heavy tails and still allow for unlimitedly sharp kernels.

1.1. Structure of the Article

The structure of the article is as follows. In the first part of the introduction we briefly discussed the history of the Fourier transform, and its generalization to other groups that are the semi-direct product of the translation group and another matrix group, where we provided an overview of related works. Then we specified our domain of interest — the Fourier transform on the homogeneous space G/H of positions and orientations which is a Lie group quotient of the Lie group $G = SE(3)$ with a subgroup H isomorphic to $\{0\} \times SO(2)$. Then we addressed its application of solving PDE systems on G/H , motivated from applications in image analysis, robotics and probability theory.

There are 4 remaining subsections of the introduction. In Subsection 1.2 we provide basic facts on the homogeneous space G/H of positions and orientations and we provide preliminaries for introducing a Fourier transform on G/H . In Subsection 1.3 we formulate the PDEs of interest on G/H that we will solve. In Subsection 1.4 we formulate the corresponding PDEs on the group G . In Subsection 1.5 we provide a roadmap on the spectral decomposition of the PDE evolutions.

In Section 2, based on previous works, we collect the necessary prior information about the PDEs of interest and the corresponding kernels. We also describe how to extend the case $\alpha = 1$ (the diffusion case) to the general case $\alpha \in (0, 1]$.

In Section 3, we describe the Fourier transform on the Lie group $SE(3)$, where we rely on UIRs of $SE(3)$. In particular, by relating the UIRs to the dual orbits of $SO(3)$ and by using a decomposition w.r.t. an orthonormal basis of modified spherical harmonics, we recall an explicit formula for the inverse Fourier transform.

In Section 4, we present a Fourier transform $\mathcal{F}_{G/H}$ on the quotient $G/H = \mathbb{R}^3 \rtimes S^2$. Our construction requires an additional constraint — an input function must be bi-invariant w.r.t. subgroup H , as explained in Remark 2. This extra symmetry constraint is satisfied by the PDE kernels of interest. We prove a theorem, where we present: 1) a matrix representation for the Fourier transform on the quotient, 2) an explicit inversion formula and 3) a Plancherel formula.

In Section 5, we apply our Fourier transform on the quotient to solve the PDEs of interest. The solution is given by convolution of the initial condition with the specific kernels (which are the probability kernels of α -stable Lévy process). We find the exact formulas for the kernels in the frequency domain relying on a spectral decomposition of the evolution operator (involved in the PDEs). We show that this result can be obtained either via conjugation of the evolution operator with our Fourier transform on $\mathbb{R}^3 \rtimes S^2$ or (less efficiently) via conjugation of the evolution operator with the Fourier transform acting only on the spatial part \mathbb{R}^3 . Then we present a numerical scheme to approximate the kernels via Monte-Carlo simulation and we provide a comparison of the exact solutions and their approximations. Finally, in Section 6, we summarize our results and discuss their applications. In the appendices, we address the probability theory and stochastic differential equations (SDEs) regarding Lévy processes on $\mathbb{R}^3 \rtimes S^2$.

The main contributions of this article are:

- construction of $\mathcal{F}_{\mathbb{R}^3 \rtimes S^2}$ — the Fourier transform on the quotient $\mathbb{R}^3 \rtimes S^2$, in formula (39);
- matrix representation for $\mathcal{F}_{\mathbb{R}^3 \rtimes S^2}$, explicit inversion and Plancherel formulas, in Theorem 1;
- explicit spectral decompositions of PDE evolutions for α -stable Lévy process on $\mathbb{R}^3 \rtimes S^2$, both in the Fourier domains of $\mathbb{R}^3 \rtimes S^2$ and \mathbb{R}^3 , in Theorem 2; Here the new spectral decomposition in the Fourier domain of $\mathbb{R}^3 \rtimes S^2$ is simpler and involves ordinary spherical harmonics.
- the exact formulas for the probability kernels of α -stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$, in Theorem 3. This also includes new formulas for the heat kernels (the case $\alpha = 1$), that are more efficient than the heat kernels presented in previous work [37].
- simple formulation and verifications (Monte-Carlo simulations) of discrete random walks for α -stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$ in Proposition 2. The corresponding SDEs are in Appendix A.

1.2. Introduction to the Fourier Transform on the Homogeneous Space of Positions and Orientations

Let $G = SE(3)$ denote the Lie group of rigid body motions, equipped with group product:

$$g_1 g_2 = (\mathbf{x}_1, \mathbf{R}_1)(\mathbf{x}_2, \mathbf{R}_2) = (\mathbf{R}_1 \mathbf{x}_2 + \mathbf{x}_1, \mathbf{R}_1 \mathbf{R}_2), \quad \text{with } g_k = (\mathbf{x}_k, \mathbf{R}_k) \in G, \quad k = 1, 2. \quad (1)$$

Here $\mathbf{x}_k \in \mathbb{R}^3$ and $\mathbf{R}_k \in SO(3)$. Note that $SE(3) = \mathbb{R}^3 \rtimes SO(3)$ is a semi-direct product of \mathbb{R}^3 and $SO(3)$.

Definition 1. Let $B(\mathcal{H})$ denote the vector space of bounded linear operators on some Hilbert space \mathcal{H} . Within the space $B(\mathcal{H})$ we denote the subspace of bounded linear trace-class operators by

$$B_2(\mathcal{H}) = \left\{ A : \mathcal{H} \rightarrow \mathcal{H} \mid A \text{ linear and } \|A\|^2 := \text{trace}(A^* A) < \infty \right\}.$$

Definition 2. Consider a mapping $\sigma : G \rightarrow B(\mathcal{H}_\sigma)$, where \mathcal{H}_σ denotes the Hilbert space on which each σ_g acts. Then σ is a Unitary Irreducible Representation (UIR) of G if

1. $\sigma : G \rightarrow B(\mathcal{H}_\sigma)$ is a homomorphism,
2. $\sigma_g^{-1} = \sigma_g^*$ for all $g \in G$,
3. there does not exist a closed subspace V of \mathcal{H}_σ other than $\{0, \mathcal{H}_\sigma\}$ such that $\sigma_g V \subset V$.

We denote by \hat{G} the dual group of G . Its elements are equivalence classes of UIR's, where one identifies elements via $\sigma_1 \sim \sigma_2 \Leftrightarrow$ there exists a unitary linear operator v , s.t. $\sigma_1 = v \circ \sigma_2 \circ v^{-1}$.

Definition 3. The Fourier transform $\mathcal{F}_G(f) = ((\mathcal{F}_G f)(\sigma))_{\sigma \in \hat{G}}$ of a square-integrable, measurable and bounded function f on G is a measurable field of bounded operators indexed by unitary irreducible representations (UIR's) σ . Now \hat{G} can be equipped with a canonical Plancherel measure ν and the Fourier transform \mathcal{F}_G admits¹ an extension unitary operator from $\mathbb{L}_2(G)$ to the direct-integral space $\int_{\hat{G}}^{\oplus} B_2(\mathcal{H}_\sigma) d\nu(\sigma)$. This unitary extension [22, 4.25] (also known as 'Plancherel transform' [24, thm.3.3.1]) is given by

$$\begin{aligned} \mathcal{F}_G(f) &= \int_{\hat{G}}^{\oplus} \hat{f}(\sigma) d\nu(\sigma), \text{ with} \\ \hat{f}(\sigma) &= (\mathcal{F}_G f)(\sigma) = \int_G f(g) \sigma_{g^{-1}} dg \in B_2(\mathcal{H}_\sigma), \text{ for all } \sigma \in \hat{G}, \end{aligned} \quad (2)$$

for all $f \in \mathbb{L}_1(G) \cap \mathbb{L}_2(G)$.

The Plancherel theorem states that $\|\mathcal{F}_G(f)\|_{\mathbb{L}_2(\hat{G})}^2 = \int_{\hat{G}} \|\mathcal{F}_G(f)(\sigma)\|_{B_2(\mathcal{H}_\sigma)}^2 d\nu(\sigma) = \int_G |f(g)|^2 dg = \|f\|_{\mathbb{L}_2(G)}^2$ for all $f \in \mathbb{L}_2(G)$, and we have the inversion formula $f = \mathcal{F}_G^{-1} \mathcal{F}_G f = \mathcal{F}_G^* \mathcal{F}_G f$. For details see [22, 24] and for detailed explicit computations see [4].

In this article, we will constrain and modify the Fourier transform \mathcal{F}_G on $G = SE(3)$ such that we obtain a suitable Fourier transform $\mathcal{F}_{G/H}$ defined on a homogeneous space²

$$\mathbb{R}^3 \rtimes S^2 := G/H \text{ with subgroup } H = \{\mathbf{0}\} \times \text{Stab}_{SO(3)}(\mathbf{a}) \quad (3)$$

of left cosets, where $\text{Stab}_{SO(3)}(\mathbf{a}) = \{\mathbf{R} \in SO(3) \mid \mathbf{R}\mathbf{a} = \mathbf{a}\}$ denotes the subgroup of $SO(3)$ that stabilizes an a priori reference axis $\mathbf{a} \in S^2$, say $\mathbf{a} = \mathbf{e}_z = (0, 0, 1)^T$. In the remainder of this article we set this choice $\mathbf{a} = \mathbf{e}_z$.

¹ Lie group $SE(3)$ is unimodular since the left-invariant Haar measure and the right-invariant Haar measures coincide. Furthermore, it is of type I (i.e. dual group and quasi dual group coincide) and thereby admits a Plancherel theorem [22, 24].

² Although the semi-direct product notation $\mathbb{R}^3 \rtimes S^2$ is formally not correct as S^2 is not a Lie group, it is convenient: It reminds that G/H denotes the homogeneous space of positions and orientations.

Remark 1. (notation and terminology)

Elements in (3) denote equivalence classes of rigid body motions $g = (\mathbf{x}, \mathbf{R}_\mathbf{n}) \in SE(3)$ that map $(\mathbf{0}, \mathbf{a})$ to (\mathbf{x}, \mathbf{n}) :

$$[g] =: (\mathbf{x}, \mathbf{n}) \in \mathbb{R}^3 \rtimes S^2 \quad \Leftrightarrow \quad g \odot (\mathbf{0}, \mathbf{a}) = (\mathbf{x}, \mathbf{n}),$$

under the (transitive) action

$$g \odot (\mathbf{x}', \mathbf{n}') = (\mathbf{R}\mathbf{x}' + \mathbf{x}, \mathbf{R}\mathbf{n}'), \quad \text{for all } g = (\mathbf{x}, \mathbf{R}) \in SE(3), (\mathbf{x}', \mathbf{n}') \in \mathbb{R}^3 \rtimes S^2. \quad (4)$$

Therefore, we simply denote the equivalence classes $[g]$ by (\mathbf{x}, \mathbf{n}) . This is similar to the conventional writing $\mathbf{n} \in S^2 = SO(3)/SO(2)$. Throughout this manuscript we refer to G/H as ‘the homogeneous space of positions and orientations’ and henceforth $\mathbf{R}_\mathbf{n}$ denotes any rotation that maps the reference axis \mathbf{a} into \mathbf{n} .

The precise definition of the Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space G/H will be presented in Section 4. It relies on the decomposition into unitary irreducible representations in (2), but we must take both a domain and a range restriction into account. This will be explained later in Section 4. Next we address an a priori domain constraint that is rather convenient than necessary.

Remark 2. We will constrain the Fourier transform $\mathcal{F}_{G/H}$ to

$$\mathbb{L}_2^{sym}(G/H) := \left\{ f \in \mathbb{L}_2(G/H) \mid \forall \mathbf{R} \in \text{Stab}_{SO(3)}(\mathbf{a}) : f(\mathbf{x}, \mathbf{n}) = f(\mathbf{R}\mathbf{x}, \mathbf{R}\mathbf{n}) \right\}. \quad (5)$$

This constraint is convenient in view of the PDEs of interest (and the symmetries of their kernels) that we will formulate in the next subsection, and that will solve via Fourier’s method in Section 5.

1.3. Introduction to the PDEs of Interest on the Quotient $\mathbb{R}^3 \rtimes S^2$

Our main objective is to use the Fourier transform $\mathcal{F}_{G/H}$ to solve the following PDEs on $\mathbb{R}^3 \rtimes S^2$:

$$\begin{cases} \frac{\partial}{\partial t} W_\alpha(\mathbf{x}, \mathbf{n}, t) = Q_\alpha W_\alpha(\mathbf{x}, \mathbf{n}, t), \\ W_\alpha(\mathbf{x}, \mathbf{n}, 0) = U(\mathbf{x}, \mathbf{n}), \end{cases} \quad (6)$$

where $(\mathbf{x}, \mathbf{n}) \in \mathbb{R}^3 \rtimes S^2$, $t \geq 0$, $\alpha \in (0, 1]$ and the generator

$$Q_\alpha := -(-Q)^\alpha \quad (7)$$

is expressed via

$$Q = D_{11} \|\mathbf{n} \times \nabla_{\mathbb{R}^3}\|^2 + D_{33} (\mathbf{n} \cdot \nabla_{\mathbb{R}^3})^2 + D_{44} \Delta_{\mathbf{n}}^{S^2},$$

with $D_{33} > D_{11} \geq 0$, $D_{44} > 0$, and with $\Delta_{\mathbf{n}}^{S^2}$ the Laplace-Beltrami operator on $S^2 = \{\mathbf{n} \in \mathbb{R}^3 \mid \|\mathbf{n}\| = 1\}$. Note, that the generator Q is a self-adjoint unbounded operator with domain

$$\mathcal{D}(Q) := \mathbb{H}_2(\mathbb{R}^3) \otimes \mathbb{H}_2(S^2),$$

where \mathbb{H}_2 denotes the Sobolev space \mathbb{W}_2^2 .

The semigroup for $\alpha = 1$ is a strongly continuous semigroup on $\mathbb{L}_2(\mathbb{R}^3 \rtimes S^2)$ with a closed generator, and by taking the fractional power of the generator one obtains another strongly continuous semigroup, as defined and explained in a more general setting in the work by Yosida [59, ch:11]. The fractional power is formally defined by

$$Q_\alpha W = -(-Q)^\alpha W := \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (Q - \lambda I)^{-1} (-QW) d\lambda \quad \text{for all } W \in \mathcal{D}(Q). \quad (8)$$

In Subsection 1.5 below, we will show that the common technical representation (8) is not really needed for our setting. In fact, it will be very easy to account for $\alpha \in (0, 1]$ in the solutions; by a spectral decomposition we only need to take fractional powers of certain eigenvalues in the Fourier domain. For the moment the reader may focus on the case $\alpha = 1$, where system (6) becomes an ordinary elliptic diffusion system which is hypo-elliptic (in the sense of Hörmander [70]) if $D_{11} = 0$.

The PDEs 6 have our interest as they are Forward-Kolmogorov equations for α -stable Lévy processes on G/H . See Appendix A for a precise formulation of discrete and continuous stochastic processes. This generalizes previous works on such basic processes [54,57] with applications in financial mathematics [71] and computer vision [58,72–74], from Lie group \mathbb{R}^3 to the Lie group quotient $\mathbb{R}^3 \rtimes S^2$.

See Figure 1 for a visualization of sample paths from the discrete stochastic processes explained in Appendix A. Roughly speaking, they represent ‘drunk man’s flights’ rather than ‘drunk man’s walks’.

1.4. Reformulation of the PDE on the Lie Group $SE(3)$

Now we reformulate and extend our PDEs (6) to the Lie group $G = SE(3)$ of rigid body motions, equipped with group product (1). This will help us to better recognize symmetries, as we will show in Subsection 2.1. To this end, the PDEs are best expressed in a basis of left-invariant vector fields $\{g \mapsto \mathcal{A}_i|_g\}_{i=1}^6$ on G . Such left-invariant vector fields are obtained by push forward from the left-multiplication $L_{g_1}g_2 := g_1g_2$ as

$$\mathcal{A}_i|_g = (L_g)_* \mathcal{A}_i \in T_g(G),$$

where $\mathcal{A}_i := \mathcal{A}_i|_e$ form an orthonormal basis for the Lie algebra $T_e(G)$. We choose such a basis typically such that the first 3 are spatial generators $A_1 = \partial_x, A_2 = \partial_y, A_3 = \partial_z = \mathbf{a} \cdot \nabla_{\mathbb{R}^3}$ and the remaining 3 are rotation generators, in such a way that A_6 is the generator of a counter-clockwise rotation around the reference axis \mathbf{a} . This means that for all $\tilde{U} \in C^1(G)$ and $g \in G$ one has

$$\mathcal{A}_i \tilde{U}(g) = \lim_{t \downarrow 0} \frac{\tilde{U}(g e^{tA_i}) - \tilde{U}(g)}{t}, \quad (9)$$

where $A \mapsto e^A$ denotes the exponent that maps Lie algebra element $A \in T_e(G)$ to the corresponding Lie group element. The explicit formulas for the left-invariant vector fields in Euler-angles (requiring 2 charts) can be found in Appendix B, or in [4,75].

Now we can re-express the PDEs (6) on the group $G = SE(3)$ as follows:

$$\begin{cases} \frac{\partial}{\partial t} \tilde{W}_\alpha(g, t) = \tilde{Q}_\alpha \tilde{W}_\alpha(g, t), & g \in G, t \geq 0 \\ \tilde{W}_\alpha(g, 0) = \tilde{U}(g), & g \in G, \end{cases} \quad (10)$$

where the generator

$$\tilde{Q}_\alpha := -(-\tilde{Q})^\alpha \quad (11)$$

is again a fractional power ($\alpha \in (0, 1]$) of the diffusion generator \tilde{Q} given by

$$\tilde{Q} = D_{11}(\mathcal{A}_1^2 + \mathcal{A}_2^2) + D_{33}\mathcal{A}_3^2 + D_{44}(\mathcal{A}_4^2 + \mathcal{A}_5^2), \quad (12)$$

where $\mathcal{A}_i^2 = \mathcal{A}_i \circ \mathcal{A}_i$ for all $i \in \{1, \dots, 5\}$. The initial condition in (10) is given by

$$\tilde{U}(g) = \tilde{U}(\mathbf{x}, \mathbf{R}) = U(\mathbf{x}, \mathbf{R}\mathbf{a}).$$

Similar to the previous works [37,76] one has

$$\tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) = W_\alpha(\mathbf{x}, \mathbf{R}\mathbf{a}, t), \quad (13)$$

198 that holds for all $t \geq 0$, $(\mathbf{x}, \mathbf{R}) \in SE(3)$.

Remark 3. The above formula (13) relates the earlier PDE formulation (6) on the quotient G/H to the PDE formulation (10) on the group G . It holds since we have the relations

$$\begin{aligned}\mathcal{A}_6 \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) &= 0, \\ (\mathcal{A}_5^2 + \mathcal{A}_4^2) \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) &= \Delta^{S^2} W_\alpha(\mathbf{x}, \mathbf{R}\mathbf{a}, t), \\ \mathcal{A}_3 \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}\mathbf{n}, t) &= \mathbf{n} \cdot \nabla_{\mathbb{R}^3} W_\alpha(\mathbf{x}, \mathbf{n}, t), \\ (\mathcal{A}_1^2 + \mathcal{A}_2^2) \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) &= \left(\Delta^{\mathbb{R}^3} - \mathcal{A}_3^2 \right) \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) = \|\mathbf{n} \times \nabla_{\mathbb{R}^3}\|^2 W_\alpha(\mathbf{x}, \mathbf{R}\mathbf{a}, t)\end{aligned}$$

so that the generator of the PDE (10) on G and the generator of the PDE (6) on G/H indeed stay related via

$$\tilde{Q}_\alpha \tilde{W}_\alpha(\mathbf{x}, \mathbf{R}, t) = Q_\alpha W_\alpha(\mathbf{x}, \mathbf{R}\mathbf{a}, t) \text{ for all } t \geq 0.$$

199 1.5. A Preview on the Spectral Decomposition of the PDE Evolution Operator and the Inclusion of α

Let U be in the domain of the generator Q_α given by (7), of our evolution (6). For a formal definition of this domain we refer to [77, Eq. 9]. Let its spatial Fourier transform be given by

$$\bar{U}(\omega, \mathbf{n}) = [\mathcal{F}_{\mathbb{R}^3} U(\cdot, \mathbf{n})](\omega) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} U(\mathbf{x}, \mathbf{n}) e^{-i\omega \cdot \mathbf{x}} d\mathbf{x}. \quad (14)$$

To the operator Q_α we associate the corresponding operator $-(-\mathcal{B})^\alpha$ in the spatial Fourier domain by

$$-(-\mathcal{B})^\alpha = \left(\mathcal{F}_{\mathbb{R}^3} \otimes 1_{\mathbb{L}_2(S^2)} \right) \circ Q_\alpha \circ \left(\mathcal{F}_{\mathbb{R}^3}^{-1} \otimes 1_{\mathbb{H}_{2\alpha}(S^2)} \right). \quad (15)$$

Then direct computations show us that

$$-(-\mathcal{B})^\alpha \bar{U}(\omega, \mathbf{n}) = [-(-\mathcal{B}_\omega)^\alpha \bar{U}(\omega, \cdot)](\mathbf{n}), \text{ for all } \mathbf{n} \in S^2, \quad (16)$$

where for each fixed $\omega \in \mathbb{R}^3$, the operator $-(-\mathcal{B}_\omega)^\alpha : \mathbb{H}_{2\alpha}(S^2) \rightarrow \mathbb{L}_2(S^2)$ is given by

$$-(-\mathcal{B}_\omega)^\alpha = - \left(-D_{44} \Delta_{\mathbf{n}}^{S^2} + D_{11} \|\omega \times \mathbf{n}\|^2 + D_{33} (\omega \cdot \mathbf{n})^2 \right)^\alpha. \quad (17)$$

In this article, we shall employ Fourier transform techniques to derive a complete orthonormal basis (ONB) of eigenfunctions

$$\left\{ \Phi_\omega^{l,m} \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \text{ with } |m| \leq l \right\}, \quad (18)$$

in $\mathbb{L}_2(S^2)$ for the operator $-(-\mathcal{B}_\omega) := -(-\mathcal{B}_\omega)^{\alpha=1}$. Then clearly, this basis is also an ONB of eigenfunctions for $-(-\mathcal{B}_\omega)^\alpha$, as we only need to take the fractional power of the eigenvalues. Indeed once eigenfunctions (18) and the eigenvalues

$$\mathcal{B}_\omega \Phi_\omega^{l,m} = \lambda_r^{l,m} \Phi_\omega^{l,m}, \text{ with } r = \|\omega\|, \quad (19)$$

are known, the exact solution of (6) is given by (shift-twist) convolution on $\mathbb{R}^3 \rtimes S^2$ as defined below

$$\begin{aligned} W_\alpha(\mathbf{x}, \mathbf{n}, t) &= (K_t^\alpha * U)(\mathbf{x}, \mathbf{n}) := \int_{S^2} \int_{\mathbb{R}^3} K_t^\alpha(\mathbf{R}_{\mathbf{n}'}^T(\mathbf{x} - \mathbf{x}'), \mathbf{R}_{\mathbf{n}'}^T \mathbf{n}) U(\mathbf{x}', \mathbf{n}') \, d\mathbf{x}' d\mu_{S^2}(\mathbf{n}') \\ &= \int_{\mathbb{R}^3} \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\langle \overline{U}(\boldsymbol{\omega}, \cdot), \Phi_{\boldsymbol{\omega}}^{l,m}(\cdot) \right\rangle_{\mathbb{L}_2(S^2)} \Phi_{\boldsymbol{\omega}}^{l,m}(\mathbf{n}) e^{-(\lambda_r^{l,m})^\alpha t} e^{i\mathbf{x} \cdot \boldsymbol{\omega}} d\boldsymbol{\omega}, \end{aligned} \quad (20)$$

with the probability kernel given by

$$\begin{aligned} K_t^\alpha(\mathbf{x}, \mathbf{n}) &= \left[\mathcal{F}_{\mathbb{R}^3}^{-1} \left(\overline{K_t^\alpha}(\cdot, \mathbf{n}) \right) \right](\mathbf{x}), \\ \text{with } \overline{K_t^\alpha}(\boldsymbol{\omega}, \mathbf{n}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \overline{\Phi_{\boldsymbol{\omega}}^{l,m}(\mathbf{a})} \Phi_{\boldsymbol{\omega}}^{l,m}(\mathbf{n}) e^{-(\lambda_r^{l,m})^\alpha t}. \end{aligned}$$

Here the inner product in $\mathbb{L}_2(S^2)$ is given by

$$\langle y_1(\cdot), y_2(\cdot) \rangle_{\mathbb{L}_2(S^2)} := \int_{S^2} y_1(\mathbf{n}) \overline{y_2(\mathbf{n})} \, d\mu_{S^2}(\mathbf{n}). \quad (21)$$

where μ_{S^2} is the usual Lebesgue measure on the sphere S^2 .

Remark 4. The eigenvalues $\lambda_r^{l,m}$ only depend on $r = \|\boldsymbol{\omega}\|$ due to the symmetry $\Phi_{\mathbf{Rn}}^{l,m}(\mathbf{Rn}) = \Phi_{\boldsymbol{\omega}}^{l,m}(\mathbf{n})$ that one directly recognizes from (17) and (19).

Remark 5. The kernels K_t^α are the probability density kernels of stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$, see Appendix A.1. Therefore, akin to the \mathbb{R}^n -case [54,58], we refer to them as the α -stable Lévy kernels on $\mathbb{R}^3 \rtimes S^2$.

2. Symmetries of the PDEs of Interest

Next we employ PDE-formulation (10) on the group $G = SE(3)$ to summarize the symmetries for the probability kernels $K_t^\alpha : \mathbb{R}^3 \rtimes S^2 \rightarrow \mathbb{R}^+$. For details see [78] and [37].

2.1. PDE symmetries

Consider PDE system (10) on the group $G = SE(3)$. Due to left-invariance (or rather left-covariance) of the PDE, linearity of the map $\tilde{U}(\cdot) \mapsto \tilde{W}_\alpha(\cdot, t)$, and the Dunford-Pettis theorem [79], the solutions are obtained by group convolution with a kernel $\tilde{K}_t^\alpha \in \mathbb{L}_1(G)$:

$$\tilde{W}_\alpha(g, t) = (\tilde{K}_t^\alpha * \tilde{U})(g) := \int_G \tilde{K}_t^\alpha(h^{-1}g) \tilde{U}(h) \, dh, \quad (22)$$

where we took the convention that the probability kernel acts from the left. In the special case $U = \delta_e$ with unity element $e = (\mathbf{0}, \mathbf{I})$ we get $\tilde{W}_\alpha(g, t) = \tilde{K}_t^\alpha(g)$.

Thanks to the fundamental relation (13) that holds in general, we have in particular that

$$\forall_{t \geq 0} \forall_{(\mathbf{x}, \mathbf{R}) \in G} : \tilde{K}_t^\alpha(\mathbf{x}, \mathbf{R}) = K_t^\alpha(\mathbf{x}, \mathbf{Ra}). \quad (23)$$

Furthermore, the PDE system given by (10) is invariant under $\mathcal{A}_i \mapsto -\mathcal{A}_i$, and, since inversion on the Lie algebra corresponds to inversion on the group, the kernels must satisfy

$$\forall_{t \geq 0} \forall_{g \in G} : \tilde{K}_t^\alpha(g) = \tilde{K}_t^\alpha(g^{-1}), \quad (24)$$

and for the corresponding kernel on the quotient this means

$$\forall_{t \geq 0} \forall_{(\mathbf{x}, \mathbf{n}) \in G/H} : K_t^\alpha(\mathbf{x}, \mathbf{n}) = K_t^\alpha(-\mathbf{R}_\mathbf{n}^T \mathbf{x}, \mathbf{R}_\mathbf{n}^T \mathbf{a}). \quad (25)$$

Finally, we see invariance of the PDE w.r.t. right actions of the subgroup H . This is due to the isotropy of the generator \tilde{Q}_α in the tangent subbundles $\text{span}\{\mathcal{A}_1, \mathcal{A}_2\}$ and $\text{span}\{\mathcal{A}_4, \mathcal{A}_5\}$. This due to (A11) in Appendix B. Note that invariance of the kernel w.r.t. right action of the subgroup H and invariance of the kernel w.r.t. inversion (24) also implies invariance of the kernel w.r.t. left-actions of the subgroup H , since $(g^{-1}(h')^{-1})^{-1} = h'g$ for all $h' \in H$ and $g \in G$. Therefore, we have

$$\begin{aligned} \forall_{t \geq 0} \forall_{g \in G} \forall_{h, h' \in H} : \quad & \tilde{K}_t^\alpha(g h) = \tilde{K}_t^\alpha(g) = \tilde{K}_t^\alpha(h' g), \\ \forall_{t \geq 0} \forall_{(\mathbf{x}, \mathbf{n}) \in G/H} \forall_{\bar{\alpha} \in [0, 2\pi)} : \quad & K_t^\alpha(\mathbf{x}, \mathbf{n}) = K_t^\alpha(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{n}). \end{aligned} \quad (26)$$

Remark 6. (notations, see also the list of abbreviations at the end of the article)

To avoid confusion between the Euler angle $\bar{\alpha}$ and the α indexing the α -stable Lévy distribution, we put an overline for this specific angle. Henceforth $\mathbf{R}_{\mathbf{v}, \psi}$ denotes a counter-clockwise rotation over axis \mathbf{v} with angle ψ . This applies in particular to the case where the axis is the reference axis $\mathbf{v} = \mathbf{a} = (0, 0, 1)^T$ and $\psi = \bar{\alpha}$. Recall that $\mathbf{R}_{\mathbf{n}}$ (without an angle in the subscript) denotes any 3D rotation that maps reference axis \mathbf{a} onto \mathbf{n} . We write the symbol $\hat{\cdot}$ above a function to indicate its Fourier transform on G and G/H ; we use the symbol $\bar{\cdot}$ for strictly spatial Fourier transform; the symbol $\tilde{\cdot}$ above a function/operator to indicate that it is defined on the group G and the function/operator without symbols when it is defined on the quotient G/H .

2.2. Obtaining the kernels with $D_{11} > 0$ from the kernels with $D_{11} = 0$

In [37, cor.2.5] it was deduced that for $\alpha = 1$ the elliptic diffusion kernel ($D_{11} > 0$) directly follows from the hypo-elliptic diffusion kernel ($D_{11} = 0$) in the spatial Fourier domain via

$$\bar{K}_t^{1, \text{elliptic}}(\omega, \mathbf{n}) = e^{-r^2 D_{11} t} \bar{K}_t^{1, \text{hypo-elliptic}}\left(\sqrt{\frac{D_{33} - D_{11}}{D_{33}}} \omega, \mathbf{n}\right), \quad \text{with } r = \|\omega\|, \quad 0 \leq D_{11} < D_{33}.$$

For the general case $\alpha \in (0, 1]$ the transformation from the case $D_{11} = 0$ to the case $D_{11} > 0$ is achieved by replacing $(-\lambda_r^{l,m})^\alpha \mapsto (-\lambda_r^{l,m} + r^2 D_{11})^\alpha$ and $r \mapsto r \sqrt{\frac{D_{33} - D_{11}}{D_{33}}}$ in formula (20) for the kernel. Henceforth, we set $D_{11} = 0$.

3. The Fourier Transform on $SE(3)$

The group $G = SE(3)$ is a unimodular Lie group (of type I) with (left- and right-invariant) Haar measure $dg = dx d\mu_{SO(3)}(\mathbf{R})$ being the product of the Lebesgue measure on \mathbb{R}^3 and the Haar-Measure $\mu_{SO(3)}$ on $SO(3)$. Then for all $f \in \mathbb{L}_1(G) \cap \mathbb{L}_2(G)$ the Fourier transform $\mathcal{F}_G f$ is given by (2). For more details see [22,24,26]. One has the inversion formula:

$$f(g) = (\mathcal{F}_G^{-1} \mathcal{F}_G f)(g) = \int_{\hat{G}} \text{trace} \{ (\mathcal{F}_G f)(\sigma) \sigma_g \} d\nu(\sigma) = \int_{\hat{G}} \text{trace} \{ \hat{f}(\sigma) \sigma_g \} d\nu(\sigma). \quad (27)$$

In our Lie group case of $SE(3)$ we identify all unitary irreducible representations $\sigma^{p,s}$ having non-zero dual measure with the pair $(p, s) \in \mathbb{R}^+ \times \mathbb{Z}$. This identification is commonly applied, see e.g. [4]. Using the method [26, Thm.2.1], [25] of induced representations, all unitary irreducible representations (UIR's) of G , up to equivalence, with non-zero Plancherel measure are given by:

$$\begin{aligned} \sigma = \sigma^{p,s} : SE(3) &\rightarrow B(\mathbb{L}_2(pS^2)), \quad p > 0, s \in \mathbb{Z}, \\ \left(\sigma_{(\mathbf{x}, \mathbf{R})}^{p,s} \phi \right)(\mathbf{u}) &= e^{-i \mathbf{u} \cdot \mathbf{x}} \phi(\mathbf{R}^{-1} \mathbf{u}) \Delta_s \left(\mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \mathbf{R} \mathbf{R}_{\frac{\mathbf{R}^{-1} \mathbf{u}}{p}} \right), \quad \mathbf{u} \in pS^2, \phi \in \mathbb{L}_2(pS^2), \end{aligned} \quad (28)$$

where pS^2 denotes a 2D sphere of radius $p = \|\mathbf{u}\|$; Δ_s is a unitary irreducible representation of $SO(2)$ (or rather of the stabilizing subgroup $\text{Stab}_{SO(3)}(\mathbf{a}) \subset SO(3)$ isomorphic to $SO(2)$) producing a scalar.

In (28), $\mathbf{R}_{\frac{\mathbf{u}}{p}}$ denotes a rotation that maps \mathbf{a} onto $\frac{\mathbf{u}}{p}$. So direct computation

$$\mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \mathbf{R} \mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \mathbf{a} = \mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \mathbf{R} \mathbf{R}^{-1} \left(\frac{\mathbf{u}}{p} \right) = \mathbf{a}$$

shows us that it is a rotation around the z -axis (recall $\mathbf{a} = \mathbf{e}_z$), say about angle $\bar{\alpha}$. This yields character $\Delta_s \left(\mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \mathbf{R} \mathbf{R}_{\frac{\mathbf{u}}{p}}^{-1} \right) = e^{-is\bar{\alpha}}$, for details see [4, ch.10.6]. Thus, we can rewrite (28) as

$$\left(\sigma_{(\mathbf{x}, \mathbf{R})}^{p,s} \phi \right) (\mathbf{u}) = e^{-i(\mathbf{u} \cdot \mathbf{x} + s\bar{\alpha})} \phi(\mathbf{R}^{-1} \mathbf{u}), \quad \text{where } (\mathbf{x}, \mathbf{R}) \in G, \mathbf{u} \in pS^2, \phi \in \mathbb{L}_2(pS^2).$$

Mackey's theory [25] relates the UIR $\sigma^{p,s}$ to the dual orbits pS^2 of $SO(3)$. Thereby, the dual measure ν can be identified with a measure on the family of dual orbits of $SO(3)$ given by $\{pS^2 \mid p > 0\}$, and

$$\left(\mathcal{F}_G^{-1} \hat{f} \right) (g) = \int_{\hat{G}} \text{trace} \left\{ \hat{f}(\sigma^{p,s}) \sigma_g^{p,s} \right\} d\nu(\sigma^{p,s}) = \int_{\mathbb{R}^+} \text{trace} \left\{ \hat{f}(\sigma^{p,s}) \sigma_g^{p,s} \right\} p^2 dp,$$

230 for all $p > 0, s \in \mathbb{Z}$. For details see [24, ch. 3.6].

The matrix elements of $\hat{f} = \mathcal{F}_G f$ w.r.t. an orthonormal basis of modified spherical harmonics $\{Y_s^{l,m}(p^{-1} \cdot)\}$, with $|m|, |s| \leq l$, see [4, ch.9.8], for $\mathbb{L}_2(pS^2)$ are given by

$$\hat{f}_{l,m,l',m'}^{p,s} := \int_G f(g) \left\langle \sigma_{g^{-1}}^{p,s} Y_s^{l',m'}(p^{-1} \cdot), Y_s^{l,m}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} dg, \quad (29)$$

231 where the \mathbb{L}_2 inner product is given by $\langle y_1(\cdot), y_2(\cdot) \rangle_{\mathbb{L}_2(pS^2)} := \langle y_1(p \cdot), y_2(p \cdot) \rangle_{\mathbb{L}_2(S^2)}$, recall (21).

232 For an explicit formula for the modified spherical harmonics $Y_s^{l,m}$ see [4], where they are denoted
233 by $h_{m,s}^l$. The precise technical analytic expansion of the modified spherical harmonics is not important
234 for this article. The only properties of $Y_s^{l,m}$ that we will need are gathered in the next proposition.

Proposition 1. *The modified spherical harmonics $Y_s^{l,m}$ have the following properties:*

1) for $s = 0$ or $m = 0$ they coincide with standard spherical harmonics $Y^{l,m}$, cf. [80, eq.4.32] :

$$Y_{s=0}^{l,m} = Y^{l,m} \text{ and } Y_s^{l,0} = (-1)^s Y^{l,s}, \text{ where } Y^{l,m}(\mathbf{n}(\beta, \gamma)) = \frac{\epsilon_m}{\sqrt{2\pi}} P_l^m(\cos \beta) e^{im\gamma},$$

with $\mathbf{n}(\beta, \gamma) = (\cos \gamma \sin \beta, \sin \gamma \sin \beta, \cos \beta)^T$, with spherical angles $\beta \in [0, \pi], \gamma \in [0, 2\pi)$,

with P_l^m the normalized associated Legendre polynomial and $\epsilon_m = (-1)^{\frac{1}{2}(m+|m|)}$;

2) they have a specific rotation transformation property in view of (28):

$$\sigma_{(\mathbf{0}, \mathbf{R})}^{p,s} Y_s^{l,m} = \sum_{m'=-l}^l \mathcal{D}_{m'm}^l(\mathbf{R}) Y_s^{l,m'}, \text{ where } \mathcal{D}_{m'm}^l(\cdot) \text{ denotes the Wigner D-function [81];}$$

3) for each $s \in \mathbb{Z}$ fixed they form a complete orthonormal basis for $\mathbb{L}_2(S^2)$:

$$\left\langle Y_s^{l,m}(\cdot), Y_s^{l',m'}(\cdot) \right\rangle_{\mathbb{L}_2(S^2)} = \delta^{l,l'} \delta^{m,m'} \text{ for all } m, m' \in \mathbb{Z}, l, l' \in \mathbb{N}_0, \text{ with } |m| \leq l, |m'| \leq l', l, l' \geq |s|.$$

For details and relation between different Euler angle conventions, see [4, ch:9.4.1]. In our convention of ZYZ-Euler angles, see Appendix B, one has

$$\mathcal{D}_{m'm}^l(\mathbf{R}_{\mathbf{e}_z, \bar{\alpha}} \mathbf{R}_{\mathbf{e}_y, \beta} \mathbf{R}_{\mathbf{e}_z, \gamma}) = e^{-im'\bar{\alpha}} P_{m'm}^l(\cos \beta) e^{-im\gamma}, \quad (30)$$

235 with $P_{m'm}^l$ a generalized associated Legendre polynomial given in [4, eq.9.21].

Moreover, we have inversion formula ([4, Eq.10.46]):

$$f(g) = \frac{1}{2\pi^2} \sum_{s \in \mathbb{Z}} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^l \int_0^{\infty} \hat{f}_{l,m,l',m'}^{p,s} \left(\sigma_g^{p,s} \right)_{l',m',l,m} p^2 dp, \quad (31)$$

with matrix coefficients (independent of f) given by

$$\left(\sigma_g^{p,s} \right)_{l',m',l,m} = \left\langle \sigma_g^{p,s} Y_s^{l,m}(p^{-1} \cdot), Y_s^{l',m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)}. \quad (32)$$

Note that $\sigma^{p,s}$ is a UIR so we have

$$\left(\sigma_{g^{-1}}^{p,s} \right)_{l',m',l,m} = \overline{\left(\sigma_g^{p,s} \right)_{l,m,l',m'}}. \quad (33)$$

4. A Specific Fourier Transform on the Homogeneous Space $\mathbb{R}^3 \rtimes S^2$ of Positions and Orientations

Now that we have introduced the notation of Fourier transform on the Lie group $G = SE(3)$, we will define the Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space $G/H = \mathbb{R}^3 \rtimes S^2$. Afterwards, in the subsequent section, we will solve the Forward-Kolmogorov/Fokker-Planck PDEs (6) via application of this transform. Or more precisely, via conjugation with Fourier transform $\mathcal{F}_{G/H}$.

4.1. The Homogeneous Space $\mathbb{R}^3 \rtimes S^2$

Throughout this manuscript we shall rely on a Fourier transform on the homogeneous space of positions and orientations that is defined by the partition of left-cosets: $\mathbb{R}^3 \rtimes S^2 := G/H$, given by (3).

Note that subgroup H can be parameterized as follows:

$$H = \{h_{\bar{\alpha}} := (\mathbf{0}, \mathbf{R}_{\mathbf{a},\bar{\alpha}}) \mid \bar{\alpha} \in [0, 2\pi)\}, \quad (34)$$

where we recall, that $\mathbf{R}_{\mathbf{a},\bar{\alpha}}$ denotes a (counter-clockwise) rotation around the reference axis $\mathbf{a} = \mathbf{e}_z$. The reason behind this construction is that the group $SE(3)$ acts transitively on $\mathbb{R}^3 \rtimes S^2$ by $(\mathbf{x}', \mathbf{n}') \mapsto g \odot (\mathbf{x}', \mathbf{n}')$ given by (4). Recall that by the definition of the left-cosets one has

$$H = \{\mathbf{0}\} \times SO(2), \text{ and } g_1 \sim g_2 \Leftrightarrow g_1^{-1}g_2 \in H.$$

The latter equivalence simply means that for $g_1 = (\mathbf{x}_1, \mathbf{R}_1)$ and $g_2 = (\mathbf{x}_2, \mathbf{R}_2)$ one has

$$g_1 \sim g_2 \Leftrightarrow \mathbf{x}_1 = \mathbf{x}_2 \text{ and } \exists_{\bar{\alpha} \in [0, 2\pi)} : \mathbf{R}_1 = \mathbf{R}_2 \mathbf{R}_{\mathbf{a},\bar{\alpha}}.$$

The equivalence classes $[g] = \{g' \in SE(3) \mid g' \sim g\}$ are often just denoted by (\mathbf{x}, \mathbf{n}) as they consist of all rigid body motions $g = (\mathbf{x}, \mathbf{R}_{\mathbf{n}})$ that map reference point $(\mathbf{0}, \mathbf{a})$ onto $(\mathbf{x}, \mathbf{n}) \in \mathbb{R}^3 \rtimes S^2$:

$$g \odot (\mathbf{0}, \mathbf{a}) = (\mathbf{x}, \mathbf{n}), \quad (35)$$

where we recall $\mathbf{R}_{\mathbf{n}}$ is any rotation that maps $\mathbf{a} \in S^2$ onto $\mathbf{n} \in S^2$.

4.2. Fourier Transform on $\mathbb{R}^3 \rtimes S^2$

Now we can define the Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space G/H . Prior this, we specify a class of functions where this transform acts.

Definition 4. Let $p > 0$ be fixed and $s \in \mathbb{Z}$. We denote

$$\mathbb{L}_2^{sym}(pS^2) = \left\{ f \in \mathbb{L}_2(pS^2) \mid \forall_{\bar{\alpha} \in [0, 2\pi)} \sigma_{h_{\bar{\alpha}}}^{p,s} f = f \right\}$$

the subspace of spherical functions that have the prescribed axial symmetry, w.r.t. the subgroup H , recall (34).

Definition 5. We denote the orthogonal projection from $\mathbb{L}_2(pS^2)$ onto the closed subspace $\mathbb{L}_2^{sym}(pS^2)$ by \mathbb{P}_p^{sym} .

Definition 6. To the group representation $\sigma^{p,s} : SE(3) \rightarrow B(\mathbb{L}_2(pS^2))$ given by (28), we relate a “representation” $\bar{\sigma}^{p,s} : \mathbb{R}^3 \rtimes S^2 \rightarrow B(\mathbb{L}_2(pS^2))$ on $\mathbb{R}^3 \rtimes S^2$, defined by

$$\bar{\sigma}_{[g]}^{p,s} := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \sigma_{h_{\bar{\alpha}} g h_{\bar{\alpha}}}^{p,s} d\bar{\alpha} d\tilde{\alpha} = \mathbb{P}_p^{sym} \circ \sigma_g^{p,s} \circ \mathbb{P}_p^{sym}. \quad (36)$$

Definition 7. A function $\tilde{U} : G \rightarrow \mathbb{C}$ is called axially symmetric if

$$\tilde{U}(\mathbf{x}, \mathbf{R}) = \tilde{U}(\mathbf{x}, \mathbf{R} \mathbf{R}_{\mathbf{a}, \bar{\alpha}}) \quad \text{for all } \bar{\alpha} \in [0, 2\pi) \text{ and all } (\mathbf{x}, \mathbf{R}) \in G. \quad (37)$$

To each function $U : G/H \rightarrow \mathbb{C}$ we relate an axially symmetric function $\tilde{U} : G \rightarrow \mathbb{C}$ by

$$\tilde{U}(\mathbf{x}, \mathbf{R}) := U(\mathbf{x}, \mathbf{R} \mathbf{a}). \quad (38)$$

Definition 8. We define the Fourier transform of function U on $G/H = \mathbb{R}^3 \rtimes S^2$ by

$$\hat{U}(\bar{\sigma}^{p,s}) = (\mathcal{F}_{G/H} U)(\bar{\sigma}^{p,s}) := \mathbb{P}_p^{sym} \circ \mathcal{F}_G \tilde{U}(\sigma^{p,s}) \circ \mathbb{P}_p^{sym}. \quad (39)$$

Standard properties of the Fourier transform \mathcal{F}_G on $SE(3)$ such as the Plancherel theorem and the inversion formula [4,26] naturally carry over to $\mathcal{F}_{G/H}$ with ‘simpler formulas’. This is done by a domain and range restriction via the projection operators \mathbb{P}_p^{sym} in (39). The reason for the specific construction (39) will become clear from the next lemmas, and the ‘simpler formulas’ for the Plancherel and inversion formulas are then summarized in a subsequent theorem, where we constrain ourselves to the case $m = m' = 0$ in the formulas. The operator \mathbb{P}_p^{sym} that is most right in (39) constrains the basis $Y_s^{l,m}$ to $m = 0$, whereas the operator \mathbb{P}_p^{sym} that is most left in (39) constrains the basis $Y_s^{l',m'}$ to $m' = 0$.

Lemma 1. (axial symmetry) Let $\tilde{U} : G \rightarrow \mathbb{C}$ be axially symmetric. Then

1. it relates to a unique function $U : G/H \rightarrow \mathbb{C}$ via $U(\mathbf{x}, \mathbf{n}) = \tilde{U}(\mathbf{x}, \mathbf{R}_{\mathbf{n}})$;
2. the matrix coefficients

$$\hat{U}_{l,m,l',m'}^{p,s} = [\mathcal{F}_G \tilde{U}(\sigma^{p,s})]_{l,m,l',m'} \text{ of linear operator } \mathcal{F}_G \tilde{U}(\sigma^{p,s})$$

- relative to the modified spherical harmonic basis $\{Y_s^{l,m}\}$ vanish if $m \neq 0$.
3. the matrix coefficients

$$\hat{U}_{l,m,l',m'}^{p,s} = [\mathcal{F}_{G/H} U(\bar{\sigma}^{p,s})]_{l,m,l',m'} \text{ of linear operator } \mathcal{F}_{G/H} U(\bar{\sigma}^{p,s})$$

- relative to the modified spherical harmonic basis $\{Y_s^{l,m}\}$ vanish if $m \neq 0$ or $m' \neq 0$;

Conversely, if $\tilde{U} = \mathcal{F}_G^{-1}(\hat{U})$ and

$$\forall p > 0 \forall l \in \mathbb{N}_0 \forall s \in \mathbb{Z}, \text{ with } |s| \leq l \quad \forall m' \in \mathbb{Z}, \text{ with } |m'| \leq l \quad \forall m \neq 0 : \hat{U}_{l,m,l',m'}^{p,s} = 0, \quad (40)$$

then \tilde{U} satisfies the axial symmetry (37).

Proof. Item 1: Uniqueness of U follows by the fact that the choice of $\mathbf{R}_{\mathbf{n}}$ of some rotation that maps \mathbf{a} onto \mathbf{n} does not matter. Indeed $U(\mathbf{x}, \mathbf{n}) = \tilde{U}(\mathbf{x}, \mathbf{R}_{\mathbf{n}} \mathbf{a}, \bar{\alpha}) = \tilde{U}(\mathbf{x}, \mathbf{R}_{\mathbf{n}})$.

Item 2: Assumption (37) can be rewritten as $\tilde{U}(g) = \tilde{U}(gh_{\bar{\alpha}})$ for all $h_{\bar{\alpha}} \in H, g \in G$. This gives:

$$\begin{aligned}
 \hat{U}_{l,m,l',m'}^{p,s} &= \left\langle (\mathcal{F}_G \tilde{U})(Y_s^{l',m'}(p^{-1}\cdot)), Y_s^{l,m}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} \\
 &= \int_G \tilde{U}(g) \left\langle \sigma_g^{p,s} Y_s^{l',m'}(p^{-1}\cdot), Y_s^{l,m}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} dg \\
 &= \int_G \tilde{U}(g) \left\langle Y_s^{l',m'}(p^{-1}\cdot), \sigma_g^{p,s} Y_s^{l,m}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} dg \\
 &= \int_G \tilde{U}(gh_{\bar{\alpha}}) \left\langle Y_s^{l',m'}(p^{-1}\cdot), \sigma_{gh_{\bar{\alpha}}}^{p,s} Y_s^{l,m}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} d(gh_{\bar{\alpha}}) \\
 &= \int_G \tilde{U}(g) \left\langle Y_s^{l',m'}(p^{-1}\cdot), \sigma_g^{p,s} \circ \sigma_{h_{\bar{\alpha}}}^{p,s} Y_s^{l,m}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} d(gh_{\bar{\alpha}}) \\
 &= e^{-im\bar{\alpha}} \hat{U}_{l,m,l',m'}^{p,s} \text{ for all } \bar{\alpha} \in [0, 2\pi),
 \end{aligned} \tag{41}$$

where we recall that σ is a UIR and that the Haar-measure on G is bi-invariant. In the first step we used the 3rd property whereas in the final step we used the 2nd property of Proposition 1 together with

$$\mathcal{D}_{m'm}^l(\mathbf{R}_{\mathbf{a},\bar{\alpha}}) = e^{-im\bar{\alpha}} \delta_{m'm} \text{ so that } \sigma_{h_{\bar{\alpha}}}^{p,s} Y_s^{l,m}(p^{-1}\cdot) = e^{-im\bar{\alpha}} Y_s^{l,m}(p^{-1}\cdot). \tag{42}$$

264 We conclude that $(1 - e^{-im\bar{\alpha}}) \hat{U}_{l,m,l',m'}^{p,s} = 0$ for all $\bar{\alpha} \in [0, 2\pi)$ so $m \neq 0 \Rightarrow \hat{U}_{l,m,l',m'}^{p,s} = 0$.

Item 3: Due to the 2nd property in Proposition 1 we have

$$\sigma_{(0,\mathbf{R})}^{p,s} Y_s^{l,m}(p^{-1}\cdot) = \sum_{m'=-l}^l \mathcal{D}_{m'm}^l(\mathbf{R}) Y_s^{l,m'}(p^{-1}\cdot).$$

Thereby the projection \mathbb{P}_p^{sym} is given by

$$\mathbb{P}_p^{sym} \left(\sum_{l=0}^{\infty} \sum_{m=-l}^l \alpha_{l,m} Y_s^{l,m} \right) = \sum_{l=0}^{\infty} \alpha_{l,0} Y_s^{l,0}. \tag{43}$$

265 Now the projection \mathbb{P}_p^{sym} that is applied first in (39) filters out $m = 0$ as the only possible nonzero
266 component. The second projection filters out $m' = 0$ as the only possible nonzero component.

Conversely, if (40) holds one has by inversion formula (31) that

$$\tilde{U}(g) = \frac{1}{2\pi^2} \sum_{s \in \mathbb{Z}} \sum_{l=|s|}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{m'=-l'}^{l'} \int_0^{\infty} \hat{U}_{l,0,l',m'}^{p,s} \left(\sigma_g^{p,s} \right)_{l',m',l,0} p^2 dp,$$

so then the final result follows by the identity

$$\left(\sigma_{gh_{\bar{\alpha}}}^{p,s} \right)_{l',m',l,0} = \left(\sigma_g^{p,s} \right)_{l',m',l,0}. \tag{44}$$

So it remains to show why Eq.(44) holds. It is due to $\sigma_{(\mathbf{x},\mathbf{R})}^{p,s} = \sigma_{(\mathbf{x},\mathbf{I})}^{p,s} \circ \sigma_{(0,\mathbf{R})}^{p,s}$ and (42), as one has

$$\sigma_{gh_{\bar{\alpha}}}^{p,s} = \sigma_{(\mathbf{x},\mathbf{R})(0,\mathbf{R}_{\mathbf{a},\bar{\alpha}})}^{p,s} = \sigma_{(\mathbf{x},\mathbf{R}\mathbf{R}_{\mathbf{a},\bar{\alpha}})}^{p,s} = \sigma_{(\mathbf{x},\mathbf{R})}^{p,s} \circ \sigma_{(0,\mathbf{R}_{\mathbf{a},\bar{\alpha}})}^{p,s}, \text{ and } Y_s^{l,0}(p^{-1}\mathbf{R}_{\mathbf{a},\bar{\alpha}}^{-1}\cdot) = Y_s^{l,0}(p^{-1}\cdot) \tag{45}$$

267 and thereby Eq.(44) follows by Eq.(32). \square

Lemma 2. If $\tilde{K} \in \mathbb{L}_2(G)$ is real-valued and satisfies the axial symmetry (37), and moreover the following holds

$$\tilde{K}(g^{-1}) = \tilde{K}(g) \tag{46}$$

268 then the Fourier coefficients satisfy $\hat{K}_{l,m,l',m'}^{p,s} = \overline{\hat{K}_{l',m',l,m}^{p,s}}$ and they vanish for $m \neq 0$ and for $m' \neq 0$.

269 Proof. The proof follows by Eq.(33) and inversion invariance of the Haar measure on G , see [77]. \square

270 The next lemma shows that (46) is a sufficient but not a necessary condition for the Fourier
271 coefficients to vanish for both the cases $m' \neq 0$ and $m \neq 0$.

Lemma 3. Let $\tilde{K} \in \mathbb{L}_2(G)$ and $K \in \mathbb{L}_2(G/H)$ be related by (38). Then we have the following equivalences:

$$\begin{aligned} K(\mathbf{x}, \mathbf{n}) &= K(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{n}), \quad \text{for all } \bar{\alpha} \in [0, 2\pi), (\mathbf{x}, \mathbf{n}) \in G/H \\ &\quad \Updownarrow \\ \tilde{K}(gh) &= \tilde{K}(g) = \tilde{K}(hg), \quad \text{for all } g \in G, h \in H \\ &\quad \Updownarrow \\ \text{The Fourier coefficients } \hat{K}_{l,m,l',m'}^{p,s} &\text{ vanish for } m \neq 0 \text{ and for } m' \neq 0. \end{aligned} \quad (47)$$

Proof. We show $a \Rightarrow b \Rightarrow c \Rightarrow a$ to get $a \Leftrightarrow b \Leftrightarrow c$.

$a \Rightarrow b$: Denoting $h = h_{\bar{\alpha}} = (\mathbf{0}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}})$, $g = (\mathbf{x}, \mathbf{R})$, we have

$$\begin{aligned} \forall_{\bar{\alpha}, \bar{\alpha}' \in [0, 2\pi)} \forall_{\mathbf{x} \in \mathbb{R}^3} \forall_{\mathbf{R} \in SO(3)} : \tilde{K}(gh_{\bar{\alpha}}) &= \tilde{K}(\mathbf{x}, \mathbf{R} \mathbf{R}_{\mathbf{a}, \bar{\alpha}}) = K(\mathbf{x}, \mathbf{R} \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{a}) = K(\mathbf{x}, \mathbf{R} \mathbf{a}) = \tilde{K}(\mathbf{x}, \mathbf{R}) = \tilde{K}(g) \\ &= K(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{R} \mathbf{a}) = \tilde{K}(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{R}) = \tilde{K}(h_{\bar{\alpha}} g). \end{aligned}$$

$b \Rightarrow c$: By Lemma 1 we know that the Fourier coefficients vanish for $m \neq 0$. Next we show they also vanish for $m' \neq 0$. Similar to (45) we have

$$\sigma_{h_{\bar{\alpha}} g}^{p,s} = \sigma_{(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{R})}^{p,s} = \sigma_{(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{I})}^{p,s} \circ \sigma_{(\mathbf{0}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{R})}^{p,s}, \quad (48)$$

which gives the following relation for the matrix-coefficients:

$$\begin{aligned} \left(\sigma_{g=(\mathbf{x}, \mathbf{R})}^{p,s} \right)_{l', m', l, m} &= \sum_{j=-l}^l \left\langle \sigma_{(\mathbf{x}, \mathbf{I})}^{p,s} Y_s^{l,j}(p^{-1} \cdot), Y_s^{l', m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} \mathcal{D}_{jm}^l(\mathbf{R}) \Rightarrow \\ \left(\sigma_{h_{\bar{\alpha}} g}^{p,s} \right)_{l', m', l, m} &= \sum_{j=-l}^l e^{-i(m'-j)\bar{\alpha}} \left\langle \sigma_{(\mathbf{x}, \mathbf{I})}^{p,s} Y_s^{l,j}(p^{-1} \cdot), Y_s^{l', m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} e^{-ij\bar{\alpha}} \mathcal{D}_{jm}^l(\mathbf{R}) \Rightarrow \\ \left(\sigma_{h_{\bar{\alpha}} g}^{p,s} \right)_{l', m', l, m} &= e^{-im'\bar{\alpha}} \left(\sigma_g^{p,s} \right)_{l', m', l, m}. \end{aligned} \quad (49)$$

The implication can be directly verified by Proposition 1, (30), (48), and

$$\begin{aligned} \left\langle Y_s^{l', m'}(p^{-1} \cdot), \sigma_{(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{x}, \mathbf{I})}^{p,s} Y_s^{l,j}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} &= \int_{pS^2} e^{-ip(\mathbf{x} \cdot \mathbf{R}_{\mathbf{a}, \bar{\alpha}}^T \mathbf{u})} Y_s^{l,j}(\mathbf{u}) \overline{Y_s^{l', m'}(\mathbf{u})} d\mu_{pS^2}(\mathbf{u}) \\ &= \int_{pS^2} e^{-ip(\mathbf{x} \cdot \mathbf{v})} Y_s^{l,j}(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{v}) \overline{Y_s^{l', m'}(\mathbf{R}_{\mathbf{a}, \bar{\alpha}} \mathbf{v})} d\mu_{pS^2}(\mathbf{v}). \end{aligned}$$

From (49) we deduce that:

$$\begin{aligned} \hat{K}_{l,m,l',m'}^{p,s} &= \int_G \tilde{K}(g) \left\langle \sigma_g^{p,s} Y_s^{l,m}(p^{-1} \cdot), Y_s^{l', m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} dg \\ &= \int_G \tilde{K}(h_{\bar{\alpha}} g) \left\langle \sigma_{h_{\bar{\alpha}} g}^{p,s} Y_s^{l,m}(p^{-1} \cdot), Y_s^{l', m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} d(h_{\bar{\alpha}} g) \\ &= \int_G \tilde{K}(g) \left\langle \sigma_g^{p,s} Y_s^{l,m}(p^{-1} \cdot), \sigma_{h_{\bar{\alpha}}^{-1}}^{p,s} Y_s^{l', m'}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)} dg = e^{+im'\bar{\alpha}} \hat{K}_{l,m,l',m'}^{p,s}, \end{aligned}$$

272 which holds for all $\bar{\alpha} \in [0, 2\pi)$. Thereby if $m' \neq 0$ then $\hat{K}_{l,m,l',m'}^{p,s} = 0$.

273 $c \Rightarrow a$: By inversion formula (31), where the only contributing terms have $m = 0$ and $m' = 0$, we
274 see that $\tilde{K}(gh) = \tilde{K}(hg) = \tilde{K}(g)$ for all $h = (\mathbf{0}, \mathbf{R}_{\mathbf{a}, \bar{\alpha}})$. Thereby \tilde{K} is axially symmetric and by Lemma 1
275 it relates to a unique kernel on G/H via $K(\mathbf{x}, \mathbf{n}) = \tilde{K}(\mathbf{x}, \mathbf{R}_{\mathbf{n}})$ and the result follows by (26). \square

Now that we characterized all functions $K \in \mathbb{L}_2(G/H)$ for which the Fourier coefficients $\hat{K}_{l,m,l',m'}^{p,s}$ vanish for $m \neq 0$ and $m' \neq 0$ in the above lemma, we will considerably simplify the inversion and Plancherel formula for Fourier transform \mathcal{F}_G on the group $G = SE(3)$ to the Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space $G/H = \mathbb{R}^3 \rtimes S^2$ in the next theorem. This will be important in our objective of deriving the kernels for the linear PDEs (6) that we address in the next section.

Theorem 1. (matrix-representation for $\mathcal{F}_{G/H}$, explicit inversion and Plancherel formula)

Let $K \in \mathbb{L}_2^{sym}(G/H)$ and $\tilde{K} \in \mathbb{L}_2(G)$ be related by (38). Then the matrix elements of $\mathcal{F}_{G/H}K$ are given by

$$\begin{aligned} \hat{K}_{l',0,l,0}^{p,s} &= \int_G \tilde{K}(g) \left(\sigma_g^{p,s} \right)_{l',0,l,0} dg, \\ \text{with } \left(\sigma_g^{p,s} \right)_{l',0,l,0} &= \sum_{j=-l}^l [l',0 | p,s | l,j] (\mathbf{x}) \mathcal{D}_{j0}^l(\mathbf{R}) \quad \text{for all } g = (\mathbf{x}, \mathbf{R}) \in G. \end{aligned}$$

The constants $[l',0 | p,s | l,j] (\mathbf{x}) := \left\langle \sigma_{(\mathbf{x},\mathbf{I})}^{p,s} Y_s^{l,j}(p^{-1}\cdot), Y_s^{l',0}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)}$ admit an analytic expression in terms of elementary functions [4, Eq.10.34] and the Wigner D-functions (30).

Furthermore, we have the following Plancherel and inversion formula:

$$\begin{aligned} \|K\|_{\mathbb{L}_2(G/H)}^2 &= \|\mathcal{F}_{G/H}K\|^2 = \sum_{s \in \mathbb{Z}} \int_{\mathbb{R}^+} \|(\mathcal{F}_{G/H}K)(\bar{\sigma}^{p,s})\|^2 p^2 dp = \int_{\mathbb{R}^+} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} |\hat{K}_{l,0,l',0}^{p,s}|^2 p^2 dp, \\ K(\mathbf{x}, \mathbf{n}) &= \left(\mathcal{F}_{G/H}^{-1} \mathcal{F}_{G/H} K \right) (\mathbf{x}, \mathbf{n}) = \sum_{s \in \mathbb{Z}} \int_{\mathbb{R}^+} \text{trace} \left\{ (\mathcal{F}_{G/H}K)(\bar{\sigma}^{p,s}) \bar{\sigma}_{(\mathbf{x},\mathbf{n})}^{p,s} \right\} p^2 dp \\ &= \frac{1}{2\pi^2} \sum_{s \in \mathbb{Z}} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \int_{\mathbb{R}^+} \hat{K}_{l,0,l',0}^{p,s} \left(\bar{\sigma}_{(\mathbf{x},\mathbf{n})}^{p,s} \right)_{l',0,l,0} p^2 dp, \end{aligned}$$

with matrix coefficients given by (for analytic formulas see [4, eq.10.35])

$$\begin{aligned} \left(\bar{\sigma}_{(\mathbf{x},\mathbf{n})}^{p,s} \right)_{l',0,l,0} &= \left(\sigma_g^{p,s} \right)_{l',0,l,0} = \left\langle \sigma_g^{p,s} Y_s^{l,0}(p^{-1}\cdot), Y_s^{l',0}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} \\ &= \left\langle \sigma_g^{p,s} Y^{l,s}(p^{-1}\cdot), Y^{l',s}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} \quad \text{for } g = (\mathbf{x}, \mathbf{R}_n). \end{aligned} \quad (50)$$

Proof. The above formulas are a direct consequence of Lemma 3 and the Plancherel and inversion formulas (see [26], [4, ch:10.8]) for Fourier transform on $SE(3)$. Recall that a coordinate-free definition of $\bar{\sigma}^{p,s}$ is given in (36). Its matrix coefficients are given by (50), where we recall the 1st item of Prop. 1 and where we note that they are independent on the choice of $\mathbf{R}_n \in SO(3)$ mapping \mathbf{a} onto \mathbf{n} . \square

Corollary 1. Let $K_1, K_2 \in \mathbb{L}_2^{sym}(G/H)$. Then for shift-twist convolution on $G/H = \mathbb{R}^3 \rtimes S^2$ given by

$$(K_1 * K_2)(\mathbf{x}, \mathbf{n}) = \int_{S^2} \int_{\mathbb{R}^3} K_1(\mathbf{R}_n^T(\mathbf{x} - \mathbf{x}'), \mathbf{R}_n^T \mathbf{n}) K_2(\mathbf{x}', \mathbf{n}') d\mathbf{x}' d\mu_{S^2}(\mathbf{n}')$$

we have $\mathcal{F}_{G/H}(K_1 * K_2) = (\mathcal{F}_{G/H}K_1) \circ (\mathcal{F}_{G/H}K_2)$.

Proof. Set $\tilde{K}_1(g) = K_1(g \odot (\mathbf{0}, \mathbf{a}))$. Standard Fourier theory [5] gives $\mathcal{F}_G(\widetilde{K_1 * K_2}) = \mathcal{F}_G(\tilde{K}_1 * \tilde{K}_2)$, so

$$\begin{aligned} \mathcal{F}_{G/H}(K_1 * K_2) &\stackrel{\text{def}}{=} \mathbb{P}_p^{sym} \circ \mathcal{F}_G(\widetilde{K_1 * K_2}) \circ \mathbb{P}_p^{sym} \\ &= \mathbb{P}_p^{sym} \circ \mathcal{F}_G(\tilde{K}_1) \circ \mathcal{F}_G(\tilde{K}_2) \circ \mathbb{P}_p^{sym} \\ &= \mathbb{P}_p^{sym} \circ \mathcal{F}_G(\tilde{K}_1) \circ \mathbb{P}_p^{sym} \circ \mathbb{P}_p^{sym} \circ \mathcal{F}_G(\tilde{K}_2) \circ \mathbb{P}_p^{sym} \\ &= (\mathcal{F}_{G/H}K_1) \circ (\mathcal{F}_{G/H}K_2), \end{aligned}$$

where the 1st equality is given by (39) and the 3rd equality follows by Lemma 3 and (43). \square

5. Application of the Fourier Transform on $\mathbb{R}^3 \rtimes S^2$ for Explicit Solutions of the Fokker-Plank PDEs of α -stable Lévy Processes on $\mathbb{R}^3 \rtimes S^2$

Our objective is to solve PDE system (6) on the homogeneous space of positions and orientations G/H . Recall that we extended this PDE system to G in (10). As the cases $D_{11} > 0$ follow from the case $D_{11} = 0$ (recall Subsection 2.2), we consider the case $D_{11} = 0$ in this section. From the symmetry consideration in Section 2 it follows that the solution of (10) is given by $\tilde{W}_\alpha(g, t) = (\tilde{K}_t^\alpha * \tilde{U})(g)$ with a probability kernel $\tilde{K}_t^\alpha : G \rightarrow \mathbb{R}^+$, whereas the solution of (6) is given by

$$W_\alpha(\mathbf{x}, \mathbf{n}, t) = (K_t^\alpha * U)(\mathbf{x}, \mathbf{n}) := \int_{S^2} \int_{\mathbb{R}^3} K_t^\alpha(\mathbf{R}_{\mathbf{n}'}^T(\mathbf{x} - \mathbf{x}'), \mathbf{R}_{\mathbf{n}}^T \mathbf{n}) U(\mathbf{x}', \mathbf{n}') d\mathbf{x}' d\mu_{S^2}(\mathbf{n}'),$$

where the kernels K_t^α are invariant with respect to left-actions of the subgroup H , recall Eq. (26). This invariance means that the condition for application of the Fourier transform $\mathcal{F}_{G/H}$ on $\mathbb{R}^3 \rtimes S^2$ is satisfied (recall Lemma 3) and we can indeed employ Theorem 1 to keep all our computations, spectral decompositions and Fourier transforms in the 5D homogeneous space $\mathbb{R}^3 \rtimes S^2 = G/H$ rather than a technical and less direct approach [37] in the 6D group $G = SE(3)$.

Remark 7. For the underlying probability theory, and sample paths of discrete random walks of the α -Stable Lévy stochastic processes we refer to Appendix A. To get a general impression of how Monte Carlo simulations of such stochastic processes can be used to approximate the exact probability kernels K_t^α , see Fig. 1. In essence, such a stochastic approximation is computed by binning the endpoints of the random walks. A brief mathematical explanation will follow in Subsection 5.2.

For now let us ignore the probability theory details and let us first focus on deriving exact analytic solutions to (6) and its kernel K_t^α via Fourier transform $\mathcal{F}_{G/H}$ on $G/H = \mathbb{R}^3 \rtimes S^2$.

5.1. Exact Kernel Representations by Spectral Decomposition in the Fourier Domain

Let us consider evolution (6) for α -stable Lévy process on the quotient $G/H = \mathbb{R}^3 \rtimes S^2$. Then the mapping from the initial condition $W(\cdot, 0) = U(\cdot) \in \mathbb{L}_2(G/H)$ to the solution $W(\cdot, t)$ at a fixed time $t \geq 0$ is a bounded linear mapping. It gives rise to a strongly continuous (holomorphic) semigroup [59]. We will conveniently denote the bounded linear operator on $\mathbb{L}_2(G/H)$ as follows:

$$W_\alpha(\cdot, t) = (e^{tQ_\alpha} U)(\cdot), \quad \text{for all } t \geq 0. \quad (51)$$

In the next main theorem we will provide a spectral decomposition of the operator using both a direct sum and a direct integral decomposition. Note that definitions of direct integral decompositions (and the underlying measure theory) can be found in [24, ch:3.3&3.4].

5.1.1. Eigenfunctions and Preliminaries

In order to formulate the main theorem we need some preliminaries and formalities. First of all let us define $\overline{\mathcal{F}}_{\mathbb{R}^3} : \mathbb{L}_2(\mathbb{R}^3 \rtimes S^2) \rightarrow \mathbb{L}_2(\mathbb{R}^3 \rtimes S^2)$ by

$$(\overline{\mathcal{F}}_{\mathbb{R}^3} U)(\omega, \mathbf{n}) := [\mathcal{F}_{\mathbb{R}^3} U(\cdot, \mathbf{n})](\omega). \quad (52)$$

Recall (15). Then we re-express the generator in the spatial Fourier domain:

$$\begin{aligned} -(-\mathcal{B})^\alpha &= \overline{\mathcal{F}}_{\mathbb{R}^3} \circ Q_\alpha \circ \overline{\mathcal{F}}_{\mathbb{R}^3}^{-1} \Rightarrow \\ -(-\mathcal{B}_\omega)^\alpha &= - \left(-D_{33} (i\omega \cdot \mathbf{n})^2 - D_{44} \Delta_{\mathbf{n}}^{S^2} \right)^\alpha \\ &= - \left(D_{33} r^2 \left(\mathbf{a} \cdot (\mathbf{R}_{r^{-1}\omega}^T \mathbf{n}) \right)^2 - D_{44} \Delta_{\mathbf{n}}^{S^2} \right)^\alpha \\ &= - \left(D_{33} r^2 \cos^2(\beta^\omega) - D_{44} \Delta_{\mathbf{n}}^{S^2} \right)^\alpha, \quad \text{with } r = \|\omega\|, \alpha \in (0, 1], \end{aligned} \quad (53)$$

where β^ω denotes the angle between \mathbf{n} and $r^{-1}\omega$, see Fig. 2. This re-expression is the main reason for the following definitions.

Instead of the modified spherical Harmonics $Y_s^{l,m}$ in Prop. 1, which are commonly used as a standard basis to represent each operator in the Fourier transform on $SE(3)$, we will use our generalized spherical harmonics, depending on a spatial frequency vector as this is in accordance with (53).

Definition 9. Let $l \in \mathbb{N}_0$. Let $m \in \mathbb{Z}$ such that $|m| \leq l$. Let $\omega \in \mathbb{R}^3$ be a frequency vector. We define

$$Y_\omega^{l,m}(\mathbf{n}) = Y^{l,m}(\mathbf{R}_{r^{-1}\omega}^T \mathbf{n}), \quad \text{with } r = \|\omega\|, \mathbf{n} \in S^2, \quad (54)$$

where we take the rotation which maps \mathbf{a} onto $r^{-1}\omega$ whose matrix representation in the standard basis is:

$$\mathbf{R}_{r^{-1}\omega} = \left(\frac{(\omega \times \mathbf{a}) \times \omega}{\|(\omega \times \mathbf{a}) \times \omega\|} \mid \frac{\omega \times \mathbf{a}}{\|\omega \times \mathbf{a}\|} \mid r^{-1}\omega \right) \text{ for } r^{-1}\omega \neq \mathbf{a}, \text{ and } \mathbf{R}_\mathbf{a} = \mathbf{I}, \text{ and } \mathbf{R}_0 = \mathbf{I}.$$

Recall the standard spherical angle formula $\mathbf{n}(\beta, \gamma) = (\sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta)^T$ from Proposition 1. These are Euler-angles relative to the reference axis $\mathbf{a} = \mathbf{e}_z$. For the Euler-angles relative to the (normalized) frequency $r^{-1}\omega$ one has (see also Fig. 2):

$$\mathbf{n}^\omega(\beta^\omega, \gamma^\omega) = \mathbf{R}_{r^{-1}\omega} \mathbf{n}(\beta^\omega, \gamma^\omega). \quad (55)$$

Definition 10. Let $l \in \mathbb{N}_0$. Let $m \in \mathbb{Z}$ such that $|m| \leq l$. We define the functions $\Phi_\omega^{l,m} \in \mathbb{L}_2(S^2)$ by

$$\Phi_\omega^{l,m}(\mathbf{n}) = \sum_{j=0}^{\infty} \frac{d_j^{l,m}(r)}{\|\mathbf{d}^{l,m}(r)\|} Y_\omega^{|m|+j,m}(\mathbf{n}), \quad (56)$$

where $r = \|\omega\|$ and $\mathbf{d}^{l,m}(r) := \left(d_j^{l,m}(r) \right)_{j=0}^{\infty}$ are coefficients such that

$$\Phi_\omega^{l,m}(\mathbf{n}^\omega(\beta^\omega, \gamma^\omega)) = S_\rho^{l,m}(\cos \beta^\omega) \frac{e^{im\gamma^\omega}}{\sqrt{2\pi}}, \quad \text{with } \rho = r \sqrt{\frac{D_{33}}{D_{44}}},$$

where $S_\rho^{l,m}(\cdot)$ denotes the \mathbb{L}_2 -normalized spheroidal wave function.

Remark 8. The spheroidal wave function arises from application of the method of separation on operator \mathcal{B}_ω in (53) where basic computations (for details see [37]) lead to the following singular Sturm-Liouville problem:

$$(Ly)(x) = \frac{d}{dx} \left[p(x) \frac{dy(x)}{dx} \right] + q(x)y(x) = -\lambda(r) y(x), \quad x = \cos \beta^\omega \in [-1, 1]. \quad (57)$$

with $p(x) = (1 - x^2)$, $q(x) = -\rho^2 x^2 - \frac{m^2}{1-x^2}$, and again $\rho = r \sqrt{D_{33}/D_{44}}$. In this formulation, $p(x)$ vanishes at the boundary of the interval, which makes our problem a singular Sturm-Liouville problem. It is sufficient to require boundedness of the solution and its derivative at the boundary points to have nonnegative, distinct,

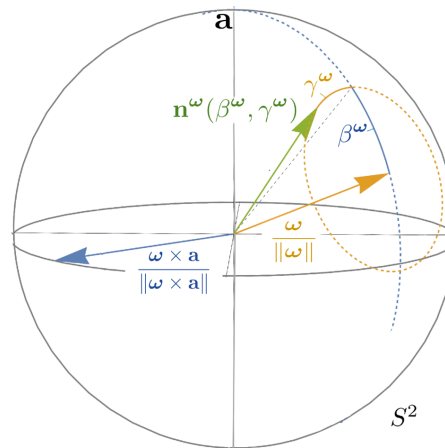


Figure 2. For $\omega \neq \mathbf{a}$, we parameterize every orientation \mathbf{n} (green) by rotations around $r^{-1}\omega$ (orange) and $\frac{\omega \times \mathbf{a}}{\|\omega \times \mathbf{a}\|}$ (blue). In other words, $\mathbf{n}^\omega(\beta^\omega, \gamma^\omega) = \mathbf{R}_{r^{-1}\omega, \gamma^\omega} \mathbf{R}_{\frac{\omega \times \mathbf{a}}{\|\omega \times \mathbf{a}\|}, \beta^\omega}(r^{-1}\omega)$.

315 simple eigenvalues $\lambda_r^{l,m}$ and existence of a countable, complete orthonormal basis of eigenfunctions $\{y_j\}_{j=0}^\infty$ [82]
 316 for the spheroidal wave equation.

As a result standard Sturm-Liouville theory (that applies the spectral decomposition theorem for compact self-adjoint operators to a kernel operator that is the right-inverse of L), provides us (for each ω fixed) a complete orthonormal basis of eigenfunctions $\{\Phi_\omega^{l,m}\}$ in $\mathbb{L}_2(S^2)$ with eigenvalues of our (unbounded) generators:

$$-(-\mathcal{B}_\omega)^\alpha \Phi_\omega^{l,m} = -(\lambda_r^{l,m})^\alpha \Phi_\omega^{l,m}, \quad \text{for all } |m| \leq l. \quad (58)$$

317 **Remark 9.** Define $\mathcal{Y}_{l,m}(\beta, \gamma) := Y^{l,m}(\mathbf{n}(\beta, \gamma))$. Then (54), (55) imply $Y_\omega^{l,m}(\mathbf{n}^\omega(\beta^\omega, \gamma^\omega)) = \mathcal{Y}_{l,m}(\beta^\omega, \gamma^\omega)$.

Remark 10. The matrix-representation of $-(-\mathcal{B}_\omega)^\alpha$ w.r.t. orthonormal basis $\{Y_\omega^{|m|+j,m}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}}$ equals

$$\bigoplus_{m \in \mathbb{Z}} -(D_{33}r^2 \mathbf{M}^m + D_{44} \mathbf{\Lambda}^m)^\alpha,$$

where $\mathbf{\Lambda}^m := \text{diag}\{l(l+1)\}_{l=|m|}^\infty = \text{diag}\{(|m|+j)(|m|+j+1)\}_{j=0}^\infty$, $r = \|\omega\|$ and where \mathbf{M}^m is the tri-diagonal matrix (that can be computed analytically [37, eq. 106]) given by

$$(\cos \beta)^2 Y^{|m|+j,m}(\mathbf{n}(\beta, \gamma)) = \sum_{j'=0}^\infty \left((\mathbf{M}^m)^T \right)_{j,j'} Y^{|m|+j',m}(\mathbf{n}(\beta, \gamma)). \quad (59)$$

As a result, we see from (56), (58) that the coefficients $\mathbf{d}^{l,m}(r)$ for our eigenfunctions are eigenvectors of a matrix

$$-(D_{33}r^2 \mathbf{M}^m + D_{44} \mathbf{\Lambda}^m) \mathbf{d}^{l,m}(r) = \lambda_r^{l,m} \mathbf{d}^{l,m}(r), \quad \text{for } l \geq |m|. \quad (60)$$

318 This matrix (and its diagonalization) will play a central role for our main spectral decomposition theorem both in
 319 the spatial Fourier domain and in the Fourier domain of the homogeneous space of positions and orientations.

320 5.1.2. The Explicit Spectral Decomposition of the Evolution Operators

321 In Theorem 2 we will present the explicit spectral decompositions both in the spatial Fourier
 322 domain and in the Fourier domain of the homogeneous space of positions and orientations.

Prior to this theorem we explain the challenges that appear when we apply $\mathcal{F}_{G/H}$ to the PDE of interest (6) on the quotient G/H . In order to get a grip on the evolution operator and the corresponding kernel, we set the initial condition equal to a delta distribution at the origin, i.e. we consider

$$U = \delta_{(0,\mathbf{a})} \Rightarrow W_\alpha(\cdot, t) = e^{tQ_\alpha} U = e^{-t(-Q)^\alpha} \delta_{(0,\mathbf{a})} = K_t^\alpha.$$

In this case, the necessary condition (47) in Lemma 3 for application of $\mathcal{F}_{G/H}$ is indeed satisfied, due to the symmetry property of the kernel, given by (26). Now due to linearity

$$\mathcal{F}_{G/H} \circ e^{tQ_\alpha} \circ \mathcal{F}_{G/H}^{-1} = e^{t(\mathcal{F}_{G/H} \circ Q_\alpha \circ \mathcal{F}_{G/H}^{-1})},$$

323 we just need to study the generator in the Fourier domain.

For the moment we set $\alpha = 1$ (the hypo-elliptic diffusion case) and return to the general case later on (recall Subsection 1.5 and Subsection 2.2). Then it follows that (for details see [37, App.D])

$$\begin{aligned} \left(\mathcal{F}_{G/H} \circ Q \circ \mathcal{F}_{G/H}^{-1} \hat{K}_t^1 \right) (\bar{\sigma}^{p,s}) &= \left(-D_{33} (\mathbf{a} \cdot \mathbf{u})^2 + D_{44} \Delta_{\mathbf{u}}^{pS^2} \right) \hat{K}_t^1 (\bar{\sigma}^{p,s}), \\ \text{with the kernel } \hat{K}_t^1 &:= \mathcal{F}_{G/H} K_t^1(\cdot). \end{aligned} \quad (61)$$

Here $\Delta_{\mathbf{u}}^{pS^2}$ denotes the Laplace-Beltrami operator on a sphere $pS^2 = \{\mathbf{u} \in \mathbb{R}^3 \mid \|\mathbf{u}\| = p\}$ of radius $p > 0$. We recall that $\mathbf{u} \in pS^2$ is the variable of the functions on which $\bar{\sigma}^{p,s}$ acts. Recall Eq.(28). So the first part in the righthand side of (61) denotes a multiplier operator \mathcal{M} given by

$$(\mathcal{M}\phi)(\mathbf{u}) := -(\mathbf{a} \cdot \mathbf{u})^2 \phi(\mathbf{u}), \text{ for all } \phi \in \mathbb{L}_2(pS^2), \text{ and almost every } \mathbf{u} \in pS^2.$$

As a result we obtain the following PDE system for \hat{K}_t^α (now for general $\alpha \in (0, 1]$):

$$\begin{cases} \frac{\partial}{\partial t} \hat{K}_t^\alpha (\bar{\sigma}^{p,s}) &= - \left(-D_{33} \mathcal{M} - D_{44} \Delta_{\mathbf{u}}^{pS^2} \right)^\alpha \hat{K}_t^\alpha (\bar{\sigma}^{p,s}) \\ \hat{K}_0^\alpha (\bar{\sigma}^{p,s}) &= 1_{\mathbb{L}_2(pS^2)}. \end{cases}$$

Remark 11. There is a striking analogy between the operators $\mathcal{F}_{G/H} \circ Q_\alpha \circ \mathcal{F}_{G/H}^{-1}$ and $\overline{\mathcal{F}}_{\mathbb{R}^3} \circ Q_\alpha \circ \overline{\mathcal{F}}_{\mathbb{R}^3}^{-1}$ given by (53), where the role of $r\mathbf{R}_{\omega/r}^T \mathbf{n}$ corresponds to \mathbf{u} . This correspondence ensures that the multipliers of the multiplier operators in the generator coincide and that the roles of p and r coincide:

$$\mathbf{u} = r\mathbf{R}_{r^{-1}\omega}^T \mathbf{n} \Rightarrow (\mathbf{a} \cdot \mathbf{u})^2 = r^2 (\mathbf{R}_{r^{-1}\omega}^T \mathbf{a} \cdot \mathbf{n})^2 = (\omega \cdot \mathbf{n})^2 \text{ and } \|\mathbf{u}\| = p = r = \|\omega\|.$$

Lemma 4. Let $t \geq 0$ and $p > 0$ be fixed. The matrix-representation of operator $e^{t(D_{33} \mathcal{M} + D_{44} \Delta_{\mathbf{u}}^{pS^2})} : \mathbb{L}_2(pS^2) \rightarrow \mathbb{L}_2(pS^2)$ w.r.t. the orthonormal basis of spherical harmonics $\left\{ Y^{l=|s|+j, s}(p^{-1} \cdot) \right\}_{j \in \mathbb{N}_0, s \in \mathbb{Z}}$ equals

$$\bigoplus_{s \in \mathbb{Z}} e^{-t(D_{33} p^2 \mathbf{M}^s + D_{44} \Lambda^s)}. \quad (62)$$

Proof. Recall (59) that defines matrix \mathbf{M}^m (for analytic formulas of this tri-diagonal matrix see [37]). This may be re-written as follows:

$$(\mathbf{a} \cdot \mathbf{n})^2 Y^{|m|+j, m}(\mathbf{n}) = \sum_{j'=0}^{\infty} \left((\mathbf{M}^m)^T \right)_{j, j'} Y^{|m|+j', m}(\mathbf{n}).$$

Now fix $s \in \mathbb{Z}$ and set $m = s$ and $\mathbf{n} = p^{-1}\mathbf{u}$ and we have:

$$\left\langle \left(D_{33}\mathcal{M} + D_{44}\Delta_{\mathbf{u}}^{pS^2} \right) Y^{l,s}(p^{-1}\cdot), Y^{l',s}(p^{-1}\cdot) \right\rangle_{\mathbb{L}_2(pS^2)} = -p^2 D_{33}(\mathbf{M}^s)_{j',j} - D_{44}l(l+1)\delta_{jj'},$$

where again $l = |s| + j$, $l' = |s| + j'$ and $j, j' \in \mathbb{N}_0$.

Finally, we note that operator $D_{33}\mathcal{M} + D_{44}\Delta_{\mathbf{u}}^{pS^2}$ is negative definite and maps each subspace $\text{span} \left\{ \{Y^{l,s}(p^{-1}\cdot)\}_{l=|s|}^\infty \right\}$ for fixed $s \in \mathbb{Z}$ onto itself, which explains direct sum decomposition (62). \square

Next we formulate the main result, where we apply a standard identification of tensors $\mathbf{a} \otimes \mathbf{b}$ with linear maps:

$$\mathbf{x} \mapsto (\mathbf{a} \otimes \mathbf{b})(\mathbf{x}) = \langle \mathbf{x}, \mathbf{b} \rangle \mathbf{a}. \quad (63)$$

Theorem 2. We have the following spectral decompositions for the Forward-Kolomogorov evolution operator of α -stable Levy-processes on the homogeneous space $G/H = \mathbb{R}^3 \rtimes S^2$:

- In the Fourier domain of the homogeneous space of positions and orientations we have:

$$\begin{aligned} & \mathcal{F}_{G/H} \circ e^{-t(-Q)^\alpha} \circ \mathcal{F}_{G/H}^{-1} \\ &= \int_{\mathbb{R}^+} \bigoplus_{s \in \mathbb{Z}} \sum_{l, l'=|s|}^\infty \left[e^{-(D_{33}p^2\mathbf{M}^s + D_{44}\Lambda^s)^\alpha t} \right]_{l, l'} \left(Y^{l,s}(p^{-1}\cdot) \otimes Y^{l',s}(p^{-1}\cdot) \right) p^2 dp \\ &= \int_{\mathbb{R}^+} \bigoplus_{s \in \mathbb{Z}} \sum_{l=|s|}^\infty e^{-(\lambda_p^{l,s})^\alpha t} \left(\Phi_{p\mathbf{a}}^{l,s}(p^{-1}\cdot) \otimes \Phi_{p\mathbf{a}}^{l,s}(p^{-1}\cdot) \right) p^2 dp \end{aligned} \quad (64)$$

- In the spatial Fourier domain we have

$$\begin{aligned} & \left(\overline{\mathcal{F}}_{\mathbb{R}^3} \circ e^{-t(-Q)^\alpha} \circ \overline{\mathcal{F}}_{\mathbb{R}^3}^{-1} \overline{U} \right) (\omega, \cdot) = \overline{W}(\omega, \cdot, t) \\ &= \sum_{m \in \mathbb{Z}} \sum_{l, l'=|m|}^\infty \left[e^{-(D_{33}r^2\mathbf{M}^m + D_{44}\Lambda^m)^\alpha t} \right]_{l, l'} \left(Y_\omega^{l,m} \otimes Y_\omega^{l',m} \right) (\overline{U}(\omega, \cdot)) \\ &= \sum_{m \in \mathbb{Z}} \sum_{l=|m|}^\infty e^{-(\lambda_r^{l,m})^\alpha t} \left(\Phi_\omega^{l,m} \otimes \Phi_\omega^{l,m} \right) (\overline{U}(\omega, \cdot)) \end{aligned} \quad (65)$$

where $\overline{W}(\omega, \cdot, t) = \overline{\mathcal{F}}_{\mathbb{R}^3} W(\omega, \cdot, t)$ and $\overline{U}(\omega, \cdot) = \overline{\mathcal{F}}_{\mathbb{R}^3} U(\omega, \cdot)$, recall (52).

In both cases the normalized eigenfunctions $\Phi_\omega^{l,m}$ are given by (56) in Definition 10. The eigenvalues $\lambda_r^{l,m}$ are the eigenvalues of the spheroidal wave equation as explained in Remark 8.

Proof. The first identity (64) follows by:

$$\begin{aligned} \mathcal{F}_{G/H} \circ e^{-t(-Q)^\alpha} \circ \mathcal{F}_{G/H}^{-1} &= e^{t(\mathcal{F}_{G/H} \circ (-Q)^\alpha \circ \mathcal{F}_{G/H}^{-1})} \\ &\stackrel{[37, \text{App.D}] \& \text{Thm. 1}}{=} \int_{\mathbb{R}^+} e^{-t \left(-D_{33}\mathcal{M} + D_{44}\Delta_{\mathbf{u}}^{pS^2} \right)^\alpha} p^2 dp \\ &\stackrel{\text{Lemma 4 \& Thm. 1}}{=} \int_{\mathbb{R}^+} \bigoplus_{s \in \mathbb{Z}} \sum_{l, l'=|s|}^\infty \left[e^{-t(D_{33}p^2\mathbf{M}^s + D_{44}\Lambda^s)^\alpha} \right]_{l, l'} \left(Y^{l,s}(p^{-1}\cdot) \otimes Y^{l',s}(p^{-1}\cdot) \right) p^2 dp \\ &\stackrel{(56)}{=} \int_{\mathbb{R}^+} \bigoplus_{s \in \mathbb{Z}} \sum_{l=|s|}^\infty e^{-(\lambda_p^{l,s})^\alpha t} \left(\Phi_{p\mathbf{a}}^{l,s}(p^{-1}\cdot) \otimes \Phi_{p\mathbf{a}}^{l,s}(p^{-1}\cdot) \right) p^2 dp. \end{aligned}$$

In the last equality we use the fact that $\Phi_{\mathbf{a}}^{l,m} = Y^{l,m}$. By applying identification (63), one observes that formula (65) is a reformulation of (20), that has already been derived for $\alpha = 1$ in previous work by the first author with J.M.Portegies [37, Thm.2.3 & Eq.31]. The key idea behind the derivation, the expansion and the completeness of the eigenfunctions $\{\Phi_\omega^{l,m}\}$ is summarized in Remark 8. The general case $\alpha \in (0, 1]$ then directly follows by Subsection 1.5. \square

Recently, exact formulas for the (hypo-elliptic) heat-kernels on $G = SE(3)$ and on $G/H = \mathbb{R}^3 \rtimes S^2$ (i.e. the case $\alpha = 1$) have been published in [37]. In the next theorem we

- extend these results to the kernels of PDE (6), which are Forward Kolmogorov equations of α -stable Lévy process with $\alpha \in (0, 1]$,
- provide a structured alternative formula via the transform $\mathcal{F}_{G/H}$ characterized in Theorem 1.

Theorem 3. We have the following formulas for the probability kernels of α -stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$:

- Via conjugation with $\mathcal{F}_{\mathbb{R}^3 \rtimes S^2}$:

$$K_t^\alpha(\mathbf{x}, \mathbf{n}) = \frac{1}{(2\pi)^2} \int_0^\infty \sum_{s \in \mathbb{Z}} \sum_{l=|s|}^\infty e^{-(\lambda_p^{l,s})^\alpha t} \left[\bar{\sigma}_{(\mathbf{x}, \mathbf{n})}^{p,s} \right]_{l,0,l,0} p^2 dp. \quad (66)$$

where $\left[\bar{\sigma}_{(\mathbf{x}, \mathbf{n})}^{p,s} \right]_{l,0,l,0} = \left\langle \sigma_{(\mathbf{x}, \mathbf{R}_n)}^{p,s} \Phi_{p\mathbf{a}}^{l,s}(p^{-1} \cdot), \Phi_{p\mathbf{a}}^{l,s}(p^{-1} \cdot) \right\rangle_{\mathbb{L}_2(pS^2)}$ can be derived analytically (see [77, Rem.16]).

- Via conjugation with $\bar{\mathcal{F}}_{\mathbb{R}^3}$:

$$K_t^\alpha(\mathbf{x}, \mathbf{n}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left(\sum_{l=0}^\infty \sum_{m=-l}^l e^{-(\lambda_{\|\omega\|}^{l,m})^\alpha t} \overline{\Phi_\omega^{l,m}(\mathbf{a})} \Phi_\omega^{l,m}(\mathbf{n}) \right) e^{i\mathbf{x} \cdot \omega} d\omega. \quad (67)$$

Proof. Formula (66) follows by

$$K_t^\alpha(\mathbf{x}, \mathbf{n}) = (e^{tQ_\alpha} \delta_{(0,\mathbf{a})})(\mathbf{x}, \mathbf{n}) = \left(\mathcal{F}_{G/H}^{-1} \circ e^{t\mathcal{F}_{G/H} \circ Q_\alpha \circ \mathcal{F}_{G/H}^{-1}} \circ \mathcal{F}_{G/H} \delta_{(0,\mathbf{a})} \right)(\mathbf{x}, \mathbf{n}).$$

Now $(\mathcal{F}_{G/H} \delta_{(0,\mathbf{a})})(\sigma^{p,s}) = 1_{\mathbb{L}_2(pS^2)}$ implies $((\mathcal{F}_{G/H} \delta_{(0,\mathbf{a})})(\sigma^{p,s}))_{l,0,l',0} = \delta_{ll'}$ so that the result follows by setting $U = \delta_{(0,\mathbf{a})}$ (or more precisely, by taking U a sequence that is a bounded approximation of the unity centered around $(0, \mathbf{a})$ in Theorem 2, where we recall the inversion formula from the first part of Theorem 1.

Formula (67) follows similarly by

$$K_t^\alpha(\mathbf{x}, \mathbf{n}) = (e^{tQ_\alpha} \delta_{(0,\mathbf{a})})(\mathbf{x}, \mathbf{n}) = \left(\bar{\mathcal{F}}_{\mathbb{R}^3}^{-1} \circ e^{t\bar{\mathcal{F}}_{\mathbb{R}^3} \circ Q_\alpha \circ \bar{\mathcal{F}}_{\mathbb{R}^3}^{-1}} \circ \bar{\mathcal{F}}_{\mathbb{R}^3} \delta_{(0,\mathbf{a})} \right)(\mathbf{x}, \mathbf{n}).$$

Now $(\bar{\mathcal{F}}_{\mathbb{R}^3} \delta_{(0,\mathbf{a})})(\sigma^{p,s}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \delta_{\mathbf{a}}$ and the result follows from the second part of Theorem 1 (again by taking U a sequence that is a bounded approximation of the unity centered around $(0, \mathbf{a})$). \square

Remark 12. There also exist Gaussian estimates for the heat kernel $K_t^{\alpha=1}$ that use a weighted modulus on the logarithm on G , [83]. Such Gaussian estimates can account for the quotient structure G/H [78], and can be reasonably close [84, Fig.4.4] to the exact solutions for practical parameter settings in applications [41,85].

5.2. Monte-Carlo Approximations of the Kernels

A stochastic approximation for the kernel K_t^α is computed by binning the endpoints of discrete random walks simulating α -stable processes on the quotient $\mathbb{R}^3 \rtimes S^2$ that we will explain next. Let us first consider the case $\alpha = 1$. For $M \in \mathbb{N}$ fixed, we have the discretization

$$\begin{cases} \mathbf{X}_M = \mathbf{X}_0 + \sum_{k=1}^M \sqrt{\frac{tD_{33}}{M}} \epsilon_k \mathbf{N}_{k-1}, \\ \mathbf{N}_M = \left(\prod_{k=1}^M \mathbf{R}_{\mathbf{a}, \gamma_k} \mathbf{R}_{\mathbf{e}_{y, \beta_k} \sqrt{\frac{tD_{44}}{M}}} \right) \mathbf{N}_0 = \left(\mathbf{R}_{\mathbf{a}, \gamma_M} \mathbf{R}_{\mathbf{e}_{y, \beta_M} \sqrt{\frac{tD_{44}}{M}}} \circ \dots \circ \mathbf{R}_{\mathbf{a}, \gamma_1} \mathbf{R}_{\mathbf{e}_{y, \beta_1} \sqrt{\frac{tD_{44}}{M}}} \right) \mathbf{N}_0, \end{cases} \quad (68)$$

with $\epsilon_k \sim G_{t=1}^{\mathbb{R}} \sim \mathcal{N}(0, \sigma = \sqrt{2})$ stochastically independent Gaussian distributed on \mathbb{R} with $t = 1$; with uniformly distributed $\gamma_k \sim \text{Unif}(\mathbb{R}/(2\pi\mathbb{Z}) \equiv [-\pi, \pi))$; and $\beta_k \sim g$, where $g: \mathbb{R} \rightarrow \mathbb{R}^+$ equals $g(r) = \frac{|r|}{2} e^{-\frac{r^2}{4}}$ in view of the theory of isotropic stochastic processes on Riemannian manifolds by Pinsky [86]. By the central limit theorem for independently distributed variables *with finite variance* it is only the variances of the distributions for the random variables g and $G_{t=1}^{\mathbb{R}}$ that matter. One may also take

$$\epsilon_k \sim \sqrt{3} \text{Unif}\left[-\frac{1}{2}, \frac{1}{2}\right] \text{ and } \beta_k \sim \sqrt{6} \text{Unif}\left[-\frac{1}{2}, \frac{1}{2}\right] \text{ or } \epsilon_k \sim G_{t=1}^{\mathbb{R}} \text{ and } \beta_k \sim G_{t=2}^{\mathbb{R}}.$$

354 These processes are implemented recursively, for technical details and background see Appendix A.

Proposition 2. The discretization (68) can be re-expressed, up to order $\frac{1}{M}$ for $M \gg 0$, as follows:

$$(\mathbf{X}_M, \mathbf{N}_M) \sim \mathbf{G}_M \odot (\mathbf{0}, \mathbf{a}), \text{ with } \mathbf{G}_M = \left(\prod_{k=1}^M e^{\sum_{i=3}^5 \sqrt{\frac{tD_{ii}}{M}} \epsilon_k^i A_i} \right) \mathbf{G}_0, \quad (69)$$

355 with $\epsilon_k^i \sim G_{t=1}^{\mathbb{R}}$ stochastically independent normally distributed variables with $t = \frac{1}{2}\sigma^2 = 1$, and $D_{44} = D_{55}$.

Proof. In our construction, β_k and γ_k can be seen as the polar radius and the polar angle (on a periodic square $[-\pi, \pi] \times [-\pi, \pi]$) of a Gaussian process with $t = 1$ on a plane spanned by rotational generators A_4 and A_5 . The key ingredient to obtain (69) from (68) is given by the following relation:

$$e^{u \cos v A_5 - u \sin v A_4} = e^{v A_6} e^{u A_5} e^{-v A_6}, \text{ for all } u, v \in \mathbb{R}, \quad (70)$$

356 which we use for $u = \beta_k \sqrt{\frac{tD_{44}}{M}}$ and $v = \gamma_k \sqrt{\frac{tD_{44}}{M}}$.

The second ingredient is given by the Campbell-Baker-Hausdorff-Dynkin formula:

$$\text{for all } a_i = O\left(\frac{1}{\sqrt{M}}\right) \text{ and for } M \text{ large, we have } e^{a_3 A_3} e^{a_4 A_4} e^{a_5 A_5} = e^{(a_3 A_3 + a_4 A_4 + a_5 A_5)(1 + O(\frac{1}{M}))},$$

357 that allows to decompose the stochastic process in $SE(3)$ into its spatial and angular parts. \square

For the binning we divide \mathbb{R}^3 into cubes c_{ijk} , $i, j, k \in \mathbb{Z}$, of size $\Delta s \times \Delta s \times \Delta s$:

$$c_{ijk} := \left[\left(i - \frac{1}{2}\right)\Delta s, \left(i + \frac{1}{2}\right)\Delta s \right] \times \left[\left(j - \frac{1}{2}\right)\Delta s, \left(j + \frac{1}{2}\right)\Delta s \right] \times \left[\left(k - \frac{1}{2}\right)\Delta s, \left(k + \frac{1}{2}\right)\Delta s \right]. \quad (71)$$

We divide S^2 into bins B_l , $l = \{1, \dots, b\}$ for $b \in \mathbb{N}$, with surface area σ_{B_l} and maximal surface area σ_B . The number of random walks in a simulation with traveling time t that have their end point $\mathbf{x}_M \in c_{ijk}$ with their orientation $\mathbf{n}_M \in B_l$ is denoted with $\#_t^{ijkl}$. Furthermore, we define the indicator function

$$1_{c_{ijk}, B_l}(\mathbf{x}, \mathbf{n}) := \begin{cases} 1 & \mathbf{x} \in c_{ijk}, \mathbf{n} \in B_l, \\ 0 & \text{otherwise.} \end{cases}$$

When the number of paths $N \rightarrow \infty$, the number of steps in each path $M \rightarrow \infty$ and the bin sizes tend to zero, the obtained distribution converges to the exact kernel:

$$\lim_{N \rightarrow \infty} \lim_{\Delta s, \sigma_B \rightarrow 0} \lim_{M \rightarrow \infty} p_t^{\Delta s, \sigma_B, N, M}(\mathbf{x}, \mathbf{n}) = K_t^{\alpha=1}(\mathbf{x}, \mathbf{n}), \quad (72)$$

$$\text{with } p_t^{\Delta s, \sigma_B, N, M}(\mathbf{x}, \mathbf{n}) = \sum_{l=1}^b \sum_{i,j,k \in \mathbb{Z}} 1_{c_{ijk}, B_l}(\mathbf{x}, \mathbf{n}) \frac{\#_t^{ijkl}}{M(\Delta s)^3 \sigma_{B_l}}.$$

Spatial projections of random flights in $\mathbb{R}^3 \times S^2$:

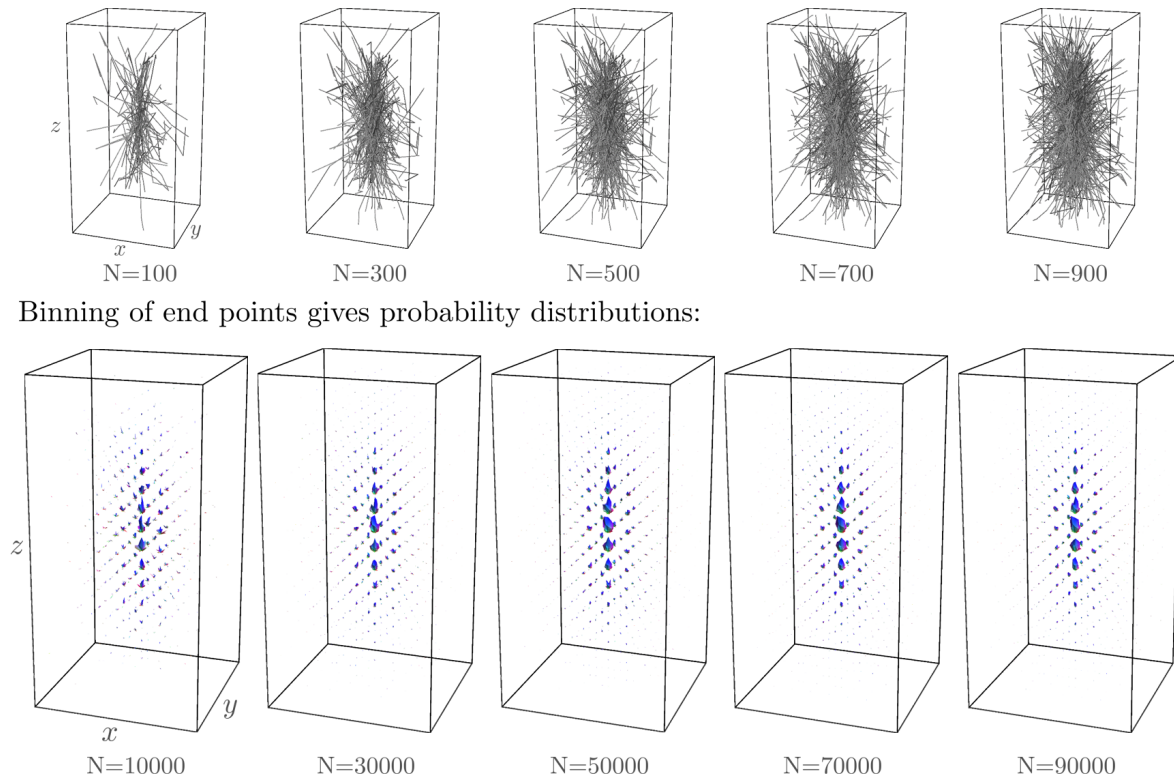


Figure 3. Top: Spatial projections in \mathbb{R}^3 of N sample paths of the discrete random walks (or rather ‘drunk man’s flights’) in $\mathbb{R}^3 \times S^2$ for $\alpha = 1$, given by (68), for increasing N (with $\sigma = \frac{4\pi}{252}$, $\Delta s = 1$, $M = 40$). Bottom: Convergence of the Monte-Carlo simulation kernel (72) for $\alpha = 1$ and $N \rightarrow \infty$. As N increases the Monte-Carlo simulation converges towards the exact solution. For a comparison of the exact diffusion kernel (66) and its Monte-Carlo approximation (72), see Fig.5.

The convergence is illustrated in Figure 3.

5.2.1. Monte-Carlo simulation for $\alpha \in (0, 1]$.

Let $q_{t,\alpha} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the temporal probability density given by the inverse Laplace transform

$$q_{t,\alpha}(\tau) = \mathcal{L}^{-1} \left(\lambda \rightarrow e^{-t\lambda^\alpha} \right) (\tau), \text{ with in particular:} \quad (73)$$

$$\text{for } \alpha = \frac{1}{2} \text{ it is } q_{t,\frac{1}{2}}(\tau) = \frac{t}{2\tau\sqrt{\pi\tau}} e^{-\frac{t^2}{4\tau}},$$

$$\text{for } \alpha \uparrow 1 \text{ we find } q_{t,\alpha}(\cdot) \rightarrow \delta_t \text{ in distributional sense.}$$

For explicit formulas in the general case $\alpha \in (0, 1]$ see [59]. Then one can deduce from Theorem 3 that

$$K_t^\alpha(\mathbf{x}, \mathbf{n}) = \int_0^\infty q_{t,\alpha}(\tau) K_\tau^{\alpha=1}(\mathbf{x}, \mathbf{n}) d\tau. \quad (74)$$

This allows us to directly use the Monte-Carlo simulations for the diffusion kernel $\alpha = 1$ for several time instances to compute a Monte-Carlo simulation of the α -stable Lévy kernels for $\alpha \in (0, 1]$. To this end we replace the Monte Carlo approximation (72) for $\alpha = 1$ in the above formula (74). See Figure 4, where we compare the diffusion kernel $K_t^{\alpha=1}$ to the Poisson kernel $K_t^{\alpha=\frac{1}{2}}$. See also Appendix A.2.1.

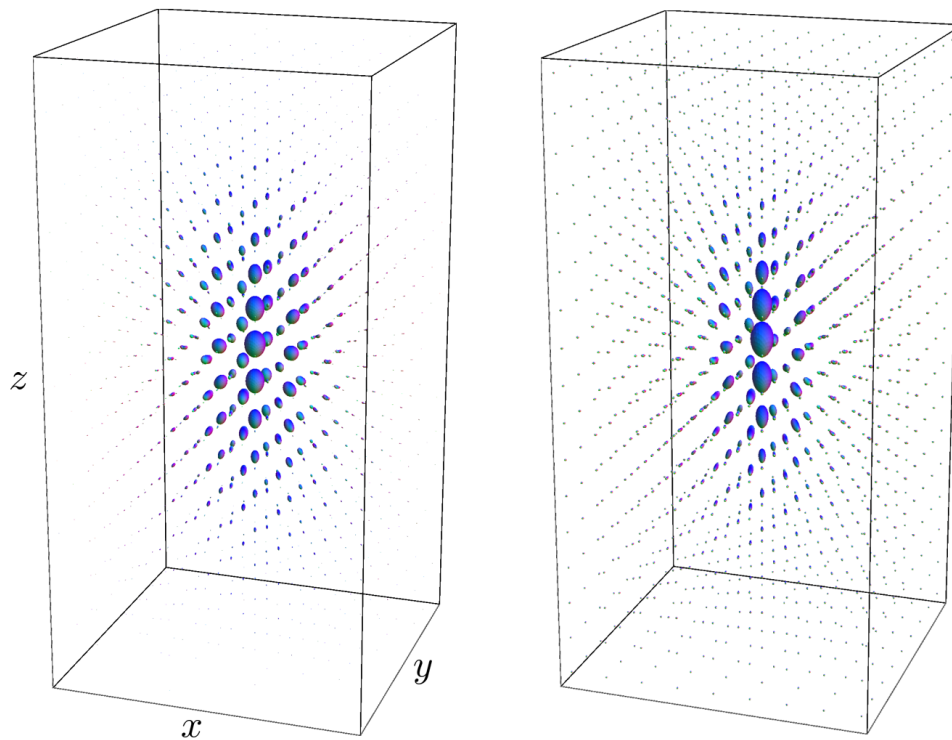


Figure 4. Left: The hypo-elliptic diffusion kernel (Eq.(66) for $\alpha = 1$ and $t = 2$). Right: the hypo-elliptic Poisson kernel (Eq.(66) for $\alpha = \frac{1}{2}$ and $t = 3.5$). Parameters settings: $D_{44} = 0.2, D_{33} = 1, D_{11} = 0$.

5.3. Comparison of Monte-Carlo Approximations of the Kernels to the Exact Solutions

In this section we compute the probability density kernels K_t^α via the analytic approach of Subsection 5.1.2 (Eq. 67, Thm. 3) and via the Monte-Carlo approximation of Subsection 5.2. The kernels are computed on a regular grid with each (x_i, y_j, z_k) at the center of the cubes c_{ijk} of (71) with $i, j = -3, \dots, 3, k = -5, \dots, 5$, and $\Delta s = 0.5$. The Monte-Carlo simulations also require spherical sampling which we did by a geodesic polyhedron that sub-divides each mesh triangle of an icosahedron into n^2 new triangles and projects the vertex points to the sphere. We set $n = 4$ to obtain 252 (almost) uniformly sampled points on S^2 .

The exact solution is computed using (truncated) spherical harmonics with $l \leq 12$. To obtain the kernel we first solve the solution in the spatial Fourier domain and then do an inverse spatial Fast Fourier Transform. The resulting kernel K_t^α (where we literally follow (67)) is only spatially sampled and provides for each (x_i, y_j, z_k) an analytic spherical distribution expressed in spherical harmonics.

For the Monte-Carlo approximation we follow the procedure as described in Subsection 5.2. The kernel K_t^α is obtained by binning the end points of random paths on the quotient $\mathbb{R}^3 \rtimes S^2$ (cf. Eq. (68)) and thereby approximate the limit in Eq. (72). Each path is discretized with $M = 40$ steps and in total $N = 10^{10}$ random paths were generated. The sphere S^2 is divided into 252 bins with an average surface area of $\sigma_{B_l} \approx \frac{4\pi}{252}$.

In Figures 1, 3, 4 and 5 we set $D_{33} = 1, D_{44} = 0.2$. In the comparison between the kernels $K_t^{\alpha=1}$ with $K_t^{\alpha=0.5}$ we set $t = 2$ and $t = 3.5$ respectively in order to match the full width at half maximum value of respectively the distributions. In Figures 1, 3 and 5 we set $\alpha = 1$ and $t = 2$. In Figures 1, 3, 4 we sample the grid (71) with $|i|, |j| \leq 4, |k| \leq 8$.

Figure 5 shows that the Monte-Carlo kernel closely approximates the exact solution and since the exact solutions can be computed at arbitrary spherical resolution, it provides a reliable way to validate numerical methods for α -stable Lévy processes on $\mathbb{R}^3 \rtimes S^2$.

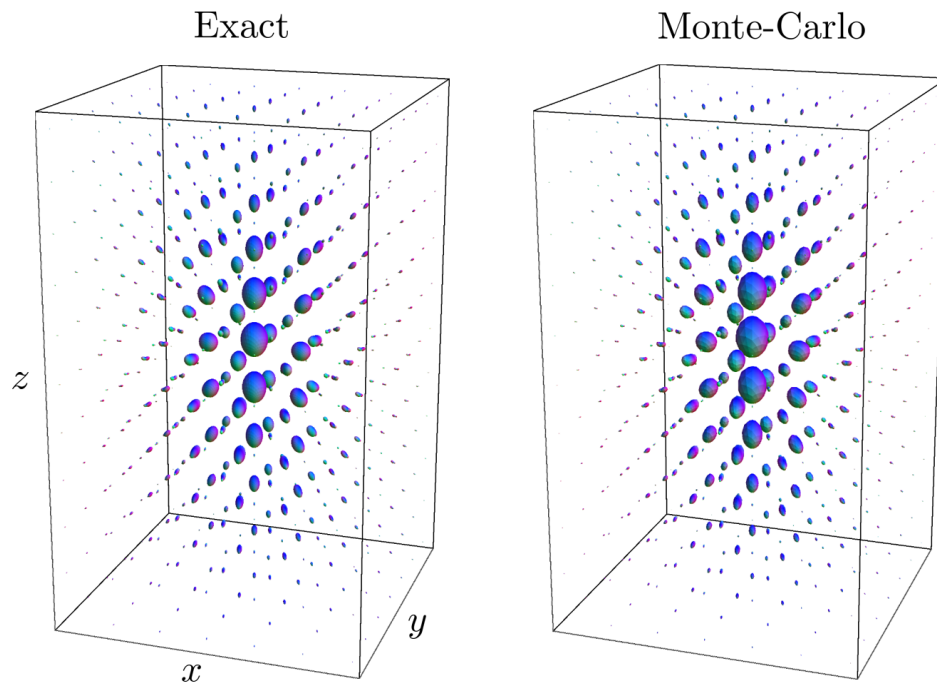


Figure 5. The exact kernel K_t^α and its Monte-Carlo approximation for $t = 2$, $\alpha = 1$, $D_{33} = 1$, $D_{44} = 0.2$.

6. Conclusions

We have set up a Fourier transform $\mathcal{F}_{G/H}$ on the homogeneous space of positions and orientations. The considered Fourier transform acts on functions that are bi-invariant with respect to the action of subgroup H . We provide explicit formulas (relative to a basis of modified spherical harmonics) for the transform, its inverse, and its Plancherel formula, in Theorem 1.

Then we used this Fourier transform to derive new exact solutions to the probability kernels of α -stable Lévy processes on G/H , including the diffusion PDE for Wiener processes which is the special case $\alpha = 1$. They are obtained by spectral decomposition of the evolution operator in Theorem 2.

New formulas for the probability kernels are presented in Theorem 3. There the general case $0 < \alpha < 1$ followed from the case $\alpha = 1$ by taking the fractional power of the eigenvalues. In comparison to previous formulas in [37] for the special case $\alpha = 1$ obtained via a spatial Fourier transform, we have more concise formulas with a more structured evolution operator in the Fourier domain of G/H , where we rely on ordinary spherical harmonics, and where we reduced the dimension of the manifold over which is integrated from 3 to 1 (as can be seen in Theorem 3).

We introduced stochastic differential equations (or rather stochastic integral equations) for the α -stable Lévy processes in Appendix A.1, and we provide simple discrete approximations where we rely on matrix exponentials in the Lie group $SE(3)$ in Proposition 2.

We verified the exact solutions and the stochastic process formulations, by Monte-Carlo simulations that confirm to give the same kernels, as is shown in Figure 5. We also observed the expected behavior that the probability kernels for $0 < \alpha < 1$ have heavier tails, as shown in Figure 4.

The PDEs and the probability kernels have a wide variety of applications in image analysis (crossing-preserving, contextual enhancement of diffusion-weighted MRI, cf. [38,39,42,85,87–89] or in crossing-preserving diffusions in 3D scalar images [18]), and in robotics [4,5,50] and in probability theory [49,54]. The generalizations to $\alpha \in (0, 1]$ allow for longer range interactions between local orientations (due to the heavy tails). This is also of interest in machine learning, where convolutional neural networks on the homogeneous space of positions and orientations [9,12] can be extended to 3D[60,61], which may benefit from the PDE descriptors and the Fourier transform presented here.

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Conflicts of Interest: "The authors declare no conflict of interest."

Abbreviations

The following abbreviations and symbols are used in this manuscript:

UIR	Unitary Irreducible Representation	
G	The rigid body motions group $SE(3)$	Eq. (1)
\mathbf{a}	The reference axis $\mathbf{a} = \mathbf{e}_z = (0, 0, 1)^T$	Eq. (3)
H	The subgroup that stabilizes $(\mathbf{0}, \mathbf{a})$	Eq. (3)
G/H	The homogeneous space of positions and orientations $\mathbb{R}^3 \rtimes S^2$	Eq. (3)
\bar{U}	The spatial Fourier transform of U	Eq. (14)
\hat{U}	The Fourier transform $\hat{U} = \mathcal{F}_{G/H} U$	Eq. (39)
α	Parameter of the α -Stable processes (indexing fractional power of the generator)	Eq. (10)
$\bar{\alpha}$	Rotation angle around reference axis $\mathbf{a} = \mathbf{e}_z = (0, 0, 1)$	Remark 6
$\sigma^{p,s}$	UIR of $G = SE(3)$	Eq. (28)
$\bar{\sigma}^{p,s}$	the action on the quotient corresponding to $\sigma^{p,s}$	Definition 6
\tilde{K}_t^α	The probability kernel on G	Eq. (22)
K_t^α	The probability kernel on G/H	Eq. (23)
\tilde{W}_α	Solution of the PDE on G	Eq. (10)
W_α	Solution of the PDE on G/H	Eq. (6)
\tilde{Q}_α	Evolution generator of the PDE on G	Eq. (11)
Q_α	Evolution generator of the PDE on G/H	Eq. (7)
$\mathbf{R}_\mathbf{n}$	Any rotation that maps \mathbf{a} onto \mathbf{n}	Remark 1
$\mathbf{R}_{\mathbf{v},\phi}$	A counter-clockwise rotation about axis \mathbf{v} with angle ϕ	Remark 1
\mathbf{P}_t	Lévy Processes on G/H	Definition A1
$\bar{\mathbf{P}}_t$	Lévy Processes on $\mathbb{R}^3 \times \mathbb{R}^3$	Eq. (A4)
$q_{t,\alpha}$	The kernel relating K_t^α and K_t^1	Eq. (73)
$Y^{l,m}$	The ordinary spherical harmonics	Proposition 1
$Y_s^{l,m}$	The modified spherical harmonics according to [4]	Proposition 1
$Y_\omega^{l,m}$	The generalized spherical harmonics according to [37]	Definition 9
$\Phi_\omega^{l,m}$	The spheroidal wave basis function for $\mathbb{L}_2(S^2)$	Definition 10
$(\bar{\alpha}, \beta, \gamma)$	ZYZ Euler angles.	(A12)

434 Appendix A Probability theory

435 Appendix A.1 Lévy Processes on $\mathbb{R}^3 \rtimes S^2$

In the next definition, we define Lévy processes on our manifold of interest $G/H = \mathbb{R}^3 \rtimes S^2$. Recall, that the action of $G = SE(3)$ on G/H is given by (4). As a prerequisite, we define the ‘difference’ of 2 random variables $\mathbf{P}_1 = (\mathbf{X}_1, \mathbf{N}_1)$ and $\mathbf{P}_2 = (\mathbf{X}_2, \mathbf{N}_2)$ in $\mathbb{R}^3 \rtimes S^2$:

$$\mathbf{G}_2^{-1} \odot \mathbf{P}_1 = (\mathbf{X}_2, \mathbf{R}_{\mathbf{N}_2})^{-1} \odot (\mathbf{X}_1, \mathbf{N}_1) = (\mathbf{R}_{\mathbf{N}_2}^T (\mathbf{X}_1 - \mathbf{X}_2), \mathbf{R}_{\mathbf{N}_2}^T \mathbf{N}_1), \quad (\text{A1})$$

436 where we relate random variables on G/H and in G via $\mathbf{P} = \mathbf{G} \odot (\mathbf{0}, \mathbf{a})$, according to (35).

437 We will assume that \mathbf{P}_1 and \mathbf{P}_2 are chosen such that the distribution of $\mathbf{G}_2^{-1} \odot \mathbf{P}_1$ is invariant
438 under the choice of rotation variable $\mathbf{R}_{\mathbf{N}_2} \in SO(3)$ which maps reference axis \mathbf{a} onto \mathbf{N}_2 . This is done
439 in view of the homogeneous space structure (3) and the fact that Lévy processes on Lie groups such as
440 $G = SE(3)$ require Lie group inversion in their definition (see, e.g. [90]).

441 **Definition A1.** A stochastic process $\{\mathbf{P}_t : t \geq 0\}$ on G/H is a Lévy process if the following conditions hold:

- 442 1. For any $n \geq 1$ and $0 \leq t_0 < t_1 < \dots < t_n$, the variables $\mathbf{P}_{t_0}, \mathbf{G}_{t_0}^{-1} \odot \mathbf{P}_{t_1}, \dots, \mathbf{G}_{t_{n-1}}^{-1} \odot \mathbf{P}_{t_n}$ are independent.
- 443 2. The distribution of $\mathbf{G}_s^{-1} \odot \mathbf{P}_{s+t}$ does not depend on $s \geq 0$.
- 444 3. $\mathbf{P}_0 = (\mathbf{0}, \mathbf{a})$ almost surely.
- 445 4. It is stochastically continuous, i.e. $\lim_{s \rightarrow t} P[d(\mathbf{P}_s, \mathbf{P}_t) > \varepsilon] = 0, \forall \varepsilon > 0$.
- 446 Here $d((\mathbf{x}_1, \mathbf{n}_1), (\mathbf{x}_2, \mathbf{n}_2)) = |\mathbf{x}_1 - \mathbf{x}_2|^2 + \arccos^2(\mathbf{n}_1 \cdot \mathbf{n}_2)$.

Let us consider the solutions

$$W_\alpha(\mathbf{x}, \mathbf{n}, t) = (K_t^\alpha * U)(\mathbf{x}, \mathbf{n})$$

of our linear PDEs of interest (6) for $\alpha \in (0, 1]$ fixed. Let us consider the case where $U \sim \delta_{(\mathbf{0}, \mathbf{a})}$, so that the solutions are the probability kernels K_t^α themselves. We consider the random variables \mathbf{P}_t^α such that their probability densities are given by

$$P(\mathbf{P}_t^\alpha = (\mathbf{x}, \mathbf{n})) = K_t^\alpha(\mathbf{x}, \mathbf{n}) \text{ for all } t \geq 0, (\mathbf{x}, \mathbf{n}) \in \mathbb{R}^3 \rtimes S^2. \quad (\text{A2})$$

447 **Proposition A1.** The stochastic process $\{\mathbf{P}_t^\alpha : t \geq 0\}$ is a Lévy processes on $\mathbb{R}^3 \rtimes S^2$.

Proof. First, address items 1, 2. On $G = SE(3)$, one has for two stochastically independent variables:

$$P(\mathbf{G}_1 \mathbf{G}_2 = g) = \int_G P(\mathbf{G}_2 = h^{-1}g) P(\mathbf{G}_1 = h) dh.$$

In particular, for $\mathbf{G}_1 = \mathbf{G}_t \sim \tilde{K}_t^\alpha$ and $\mathbf{G}_2 = \mathbf{G}_s \sim \tilde{K}_s^\alpha$ we have

$$\mathbf{G}_s \mathbf{G}_t \sim \tilde{K}_t^\alpha * \tilde{K}_s^\alpha = \tilde{K}_{t+s}^\alpha \quad \text{and} \quad \mathbf{G}_s^{-1} \mathbf{G}_{t+s} = \mathbf{G}_t \sim \tilde{K}_t^\alpha,$$

which is due to $e^{t\tilde{Q}_\alpha} \circ e^{s\tilde{Q}_\alpha} = e^{(t+s)\tilde{Q}_\alpha}$, recall (51). Similarly, on the quotient G/H we have

$$\mathbf{G}_s^{-1} \odot \mathbf{P}_{s+t} = \mathbf{P}_t \sim K_t^\alpha.$$

Furthermore, the choice of \mathbf{G}_s such that $\mathbf{G}_s \odot (\mathbf{0}, \mathbf{a}) = (\mathbf{0}, \mathbf{a})$ does not matter, since

$$P((\mathbf{0}, \mathbf{R}_{\mathbf{a}, \tilde{\mathbf{a}}})^{-1} \mathbf{G}_s^{-1} \odot \mathbf{P}_{s+t} = (\mathbf{x}, \mathbf{n})) = K_t^\alpha((\mathbf{0}, \mathbf{R}_{\mathbf{a}, \tilde{\mathbf{a}}}) \odot (\mathbf{x}, \mathbf{n})) = K_t^\alpha(\mathbf{x}, \mathbf{n}),$$

recall Eq. (26). Item 3 is obvious since we have $\mathbf{P}_0 = \delta_{(0,\mathbf{a})}$. Item 4 follows by strong continuity of the semigroup operators [59], [57, Thm. 2]. \square

Lemma A1. *The kernels K_t^α are infinitely divisible, i.e.*

$$K_t^\alpha * K_s^\alpha = K_{t+s}^\alpha \quad \text{for all } s, t \geq 0.$$

Proof. The infinite divisibility directly follows from Corollary 1 and $\mathcal{F}_{G/H}(K_t^\alpha * K_s^\alpha) = \mathcal{F}_{G/H}(K_t^\alpha) \circ \mathcal{F}_{G/H}(K_s^\alpha) = \mathcal{F}_{G/H}(K_{t+s}^\alpha)$ which is clear due to (66). \square

Remark A1. Recall that on \mathbb{R}^n a Lévy process \mathbf{X}_t is called α -stable if

$$a^{-\frac{1}{2\alpha}} \mathbf{X}_{at} \sim \mathbf{X}_t \quad \text{for all } a > 0. \quad (\text{A3})$$

This convention and property applies to all $n \in \mathbb{N}$, cf. [54]. Next we will come to a generalization of α -stability but then for the processes \mathbf{P}_t . Here an embedding of $\mathbb{R}^3 \rtimes S^2$ into $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ will be required in order to give a meaning to α -stability and a scaling relation on $\mathbf{P}_t = (\mathbf{X}_t, \mathbf{N}_t)$ that is similar to (A3).

Appendix A.2 SDE formulation of α -Stable Lévy Processes on $\mathbb{R}^3 \rtimes S^2$

Consider the Lévy processes $\{\mathbf{P}_t : t \geq 0\}$ on $\mathbb{R}^3 \rtimes S^2$ given by (A2). They give rise to the Forward Kolmogorov PDEs (6) in terms of their stochastic differential equation (SDE) according to the book of Hsu on Stochastic Analysis on Manifolds [53].

We will apply [53, Prop.1.2.4] on the embedding map $\Phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rtimes S^2$ given by

$$\Phi : (\mathbf{x}, \bar{\mathbf{n}}) \mapsto \Phi(\mathbf{x}, \bar{\mathbf{n}}) = \left(\mathbf{x}, \frac{\bar{\mathbf{n}}}{\|\bar{\mathbf{n}}\|} \right) = (\mathbf{x}, \mathbf{n}).$$

Note that $\Phi_* = \mathcal{D}\Phi = \left(I, \frac{1}{\|\bar{\mathbf{n}}\|} \left(I - \frac{\bar{\mathbf{n}}}{\|\bar{\mathbf{n}}\|} \otimes \frac{\bar{\mathbf{n}}}{\|\bar{\mathbf{n}}\|} \right) \right)$. Here I denotes the identity map on \mathbb{R}^3 .

Let us first concentrate on $\alpha = 1$. In this case, our PDE (6) becomes a diffusion PDE that is the forward Kolmogorov equation of a Wiener process $\mathbf{P}_t = (\mathbf{X}_t, \mathbf{N}_t)$ on $\mathbb{R}^3 \rtimes S^2$. Next we relate this Wiener process to a Wiener process $(\mathbf{W}_t^{(1)}, \bar{\mathbf{W}}_t^{(2)})$ in the embedding space $\mathbb{R}^3 \times \mathbb{R}^3$. We will write down the stochastic differential equation (SDE) and show that (68) boils down to discretization of the stochastic integral (in Itô sense) solving the SDE.

Next, we define $\bar{\mathbf{P}}_t = (\mathbf{X}_t, \bar{\mathbf{N}}_t)$ by the SDE in the embedding space:

$$d\bar{\mathbf{P}}_t = \bar{\mathbf{s}}|_{\bar{\mathbf{P}}_t} \circ d\mathbf{W}_t, \quad (\text{A4})$$

where $\mathbf{W}_t = (\mathbf{W}_t^{(1)}, \bar{\mathbf{W}}_t^{(2)})$, with $\mathbf{W}_t^{(1)}$ and $\bar{\mathbf{W}}_t^{(2)}$ being Wiener processes in \mathbb{R}^3 ; and where

$$\bar{\mathbf{s}}|_{\bar{\mathbf{P}}} (d\mathbf{x}, d\bar{\mathbf{n}}) = \begin{pmatrix} \bar{\mathbf{s}}^{(1)}|_{\bar{\mathbf{P}}} (d\mathbf{x}, d\bar{\mathbf{n}}) \\ \bar{\mathbf{s}}^{(2)}|_{\bar{\mathbf{P}}} (d\mathbf{x}, d\bar{\mathbf{n}}) \end{pmatrix} = \begin{pmatrix} \sqrt{D_{33}} \frac{\bar{\mathbf{N}}}{\|\bar{\mathbf{N}}\|} \left(\frac{\bar{\mathbf{N}}}{\|\bar{\mathbf{N}}\|} \cdot d\mathbf{x} \right) \\ \sqrt{D_{44}} d\bar{\mathbf{n}} \end{pmatrix}.$$

Here index (1) stands for the spatial part and (2) stands for the angular part.

Now we define a corresponding process on $\mathbb{R}^3 \rtimes S^2$:

$$\mathbf{P}_t = \Phi(\bar{\mathbf{P}}_t).$$

Then the SDE for $\mathbf{P}_t = (\mathbf{X}_t, \mathbf{N}_t)$ becomes (see [53, Prop.1.2.4])

$$d\mathbf{P}_t = d(\Phi \circ \bar{\mathbf{P}}_t) \Leftrightarrow \begin{cases} d\mathbf{X}_t = \bar{\mathbf{s}}^{(1)}|_{\bar{\mathbf{P}}_t} \circ d\mathbf{W}_t^{(1)}, \\ d\mathbf{N}_t = \mathbb{P}_{\langle \mathbf{N}_t \rangle^\perp} \bar{\mathbf{s}}^{(2)}|_{\bar{\mathbf{P}}_t} \circ d\bar{\mathbf{W}}_t^{(2)}, \end{cases}$$

where $\mathbf{N}_t = \frac{\bar{\mathbf{N}}_t}{\|\bar{\mathbf{N}}_t\|}$; and where $\mathbb{P}_{\langle \mathbf{N}_t \rangle^\perp} = (I - \mathbf{N}_t \otimes \mathbf{N}_t)$ denotes the orthogonal projection to the tangent plane perpendicular to \mathbf{N}_t .

Therefore, we have the following SDE on $\mathbb{R}^3 \rtimes S^2$:

$$\begin{cases} d\mathbf{X}_t = \sqrt{D_{33}} \mathbf{N}_t (\mathbf{N}_t \cdot d\mathbf{W}_t^{(1)}), \\ d\mathbf{N}_t = \sqrt{D_{44}} \mathbb{P}_{\langle \mathbf{N}_t \rangle^\perp} d\bar{\mathbf{W}}_t^{(2)} \end{cases} \quad (\text{A5})$$

So, integrating the SDE, we obtain the following stochastic integral (in Itô form):

$$\begin{cases} \mathbf{X}_t = \mathbf{X}_0 + \sqrt{D_{33}} \int_0^t \mathbf{N}_s (\mathbf{N}_s \cdot d\mathbf{W}_s^{(1)}) = \mathbf{X}_0 + \sqrt{D_{33}} \text{ms-lim}_{M \rightarrow \infty} \sum_{k=1}^M \mathbf{N}_{t_{k-1}} \left(\mathbf{N}_{t_{k-1}} \cdot \left(\mathbf{W}_{t_k}^{(1)} - \mathbf{W}_{t_{k-1}}^{(1)} \right) \right), \\ \mathbf{N}_t = \text{ms-lim}_{M \rightarrow \infty} \prod_{k=1}^M \exp_{S^2} \left(\sqrt{D_{44}} (I - \mathbf{N}_{t_{k-1}} \otimes \mathbf{N}_{t_{k-1}}) \left(\bar{\mathbf{W}}_{t_k}^{(2)} - \bar{\mathbf{W}}_{t_{k-1}}^{(2)} \right) \right) \mathbf{N}_0. \end{cases} \quad (\text{A6})$$

Here $\exp_{S^2}(V)\mathbf{n}_0$ denotes the exponential map on a sphere, i.e. its value is the end point (for $t = 1$) of a geodesic starting from $\mathbf{n}_0 \in S^2$ with the tangent vector $V \in T_{\mathbf{n}_0}S^2$. Note that in the formula above, the symbol \prod denotes the composition

$$\prod_{k=1}^M \exp_{S^2}(V_k)\mathbf{n}_0 = (\exp_{S^2}(V_M) \circ \dots \circ \exp_{S^2}(V_1)) \mathbf{n}_0.$$

Note that $\sqrt{D_{33}} \left(\mathbf{W}_{t_k}^{(1)} - \mathbf{W}_{t_{k-1}}^{(1)} \right) = \sqrt{D_{33}} \mathbf{W}_{t_k - t_{k-1}}^{(1)} = \sqrt{\frac{tD_{33}}{M}} \epsilon_k$, where $\epsilon_k \sim \mathbf{W}_1^{(1)}$, i.e. $\epsilon_k \sim G_{t=1}$.

For $M \in \mathbb{N}$ fixed, we propose a discrete approximation for the stochastic integrals in (A6):

$$\begin{cases} \mathbf{X}_M = \mathbf{X}_0 + \sum_{k=1}^M \sqrt{\frac{tD_{33}}{M}} \epsilon_k \mathbf{N}_{k-1}, \\ \mathbf{N}_M = \left(\prod_{k=1}^M \mathbf{R}_{\mathbf{a}, \gamma_k} \mathbf{R}_{\mathbf{e}_y, \beta_k \sqrt{\frac{tD_{44}}{M}}} \right) \mathbf{N}_0, \end{cases} \quad (\text{A7})$$

with $\epsilon_k \sim G_{t=1}^{\mathbb{R}} \sim \mathcal{N}(0, \sigma = \sqrt{2})$ stochastically independent Gaussian distributed on \mathbb{R} with $t = 1$; with uniformly distributed $\gamma_k \sim \text{Unif}(\mathbb{R}/(2\pi\mathbb{Z}) \equiv [-\pi, \pi))$; and with $\beta_k \sim g$, where $g : \mathbb{R} \rightarrow \mathbb{R}^+$ equals $g(\beta) = \frac{|\beta|}{2} e^{-\frac{\beta^2}{4}}$. The choice of g is done by application of the theory of isotropic stochastic processes on Riemannian manifolds by Pinsky [86], where we note that

$$G_t^{\mathbb{R}^2}(\beta \cos \gamma, \beta \sin \gamma) = g(\beta) \text{Unif}([-\pi, \pi))(\gamma), \quad \beta \in \mathbb{R}, \gamma \in [-\pi, \pi).$$

Now, in the numerical simulation we can replace g by $G_{t=2}^{\mathbb{R}}$ due to the central limit theorem on \mathbb{R} and

$$\text{Var}(\beta) = \int_{-\infty}^{\infty} \beta^2 g(\beta) d\beta = 2 \int_0^{\infty} \beta^2 g(\beta) d\beta = 2.$$

Appendix A.2.1 From the diffusion case $\alpha = 1$ to the general case $\alpha \in (0, 1]$

For the case $\alpha \in (0, 1]$ we define the (fractional) random processes by their probability densities

$$\begin{aligned} P(\mathbf{P}_t^\alpha = (\mathbf{x}, \mathbf{n})) &= \int_0^\infty q_{t, \alpha}(\tau) P(\mathbf{P}_\tau = (\mathbf{x}, \mathbf{n})) d\tau, \\ P(\bar{\mathbf{P}}_t^\alpha = (\mathbf{x}, \bar{\mathbf{n}})) &= \int_0^\infty q_{t, \alpha}(\tau) P(\bar{\mathbf{P}}_\tau = (\mathbf{x}, \bar{\mathbf{n}})) d\tau. \end{aligned} \quad (\text{A8})$$

recall that the kernel $q_{t,\alpha}(\tau)$ was given by (73). For Monte-Carlo simulations one can use (74), or alternatively use $\mathbf{P}_{t_M}^\alpha \approx \prod_{i=1}^M \mathbf{G}_{T_i} \odot \mathbf{P}_0$, for $M \gg 0$, where \mathbf{P}_0 is almost surely $(\mathbf{0}, \mathbf{a})$, with T_i a temporal random variable with $P(T_i = \tau) = q_{t_i,\alpha}(\tau)$, with $t_i = \frac{i}{M}t$ and \mathbf{G}_{t_i} given by (69).

Appendix A.2.2 α -Stability of the Lévy Process

Due to the absence of suitable dilations on G/H , we resort to the embedding space where α -stability is defined. The Lévy process $\{\bar{\mathbf{P}}_t^\alpha = (\mathbf{X}_t^\alpha, \bar{\mathbf{N}}_t^\alpha) \mid t \geq 0\}$ associated to the Lévy process $\{\mathbf{P}_t^\alpha = (\mathbf{X}_t^\alpha, \mathbf{N}_t^\alpha) \mid t \geq 0\}$ in $\mathbb{R}^3 \rtimes S^2$ is α -stable, i.e. for all $a, t > 0$ we have (by (A5) and (74))

$$a^{-\frac{1}{2\alpha}} \mathbf{X}_{at}^\alpha \sim \mathbf{X}_t^\alpha \text{ and } a^{-\frac{1}{2\alpha}} \bar{\mathbf{N}}_{at}^\alpha \sim \bar{\mathbf{N}}_t^\alpha.$$

Appendix B Left-Invariant Vector fields on SE(3) via 2 Charts

We need two charts to cover $SO(3)$. When using the following coordinates (ZYZ-Euler angles) for $SE(3) = \mathbb{R}^3 \rtimes SO(3)$ for the first chart:

$$g = (x, y, z, \mathbf{R}_{\mathbf{e}_z, \gamma} \mathbf{R}_{\mathbf{e}_y, \beta} \mathbf{R}_{\mathbf{e}_z, \bar{\alpha}}), \text{ with } \beta \in (0, \pi), \bar{\alpha}, \gamma \in [0, 2\pi), \quad (\text{A9})$$

formula (9) yields the following formulas for the left-invariant vector fields:

$$\begin{aligned} \mathcal{A}_1|_g &= (\cos \bar{\alpha} \cos \beta \cos \gamma - \sin \bar{\alpha} \sin \gamma) \partial_x + (\sin \bar{\alpha} \cos \gamma + \cos \bar{\alpha} \cos \beta \sin \gamma) \partial_y - \cos \bar{\alpha} \sin \beta \partial_z \\ \mathcal{A}_2|_g &= (-\sin \bar{\alpha} \cos \beta \cos \gamma - \cos \bar{\alpha} \sin \gamma) \partial_x + (\cos \bar{\alpha} \cos \gamma - \sin \bar{\alpha} \cos \beta \sin \gamma) \partial_y + \sin \bar{\alpha} \sin \beta \partial_z \\ \mathcal{A}_3|_g &= \sin \beta \cos \gamma \partial_x + \sin \beta \sin \gamma \partial_y + \cos \beta \partial_z, \\ \mathcal{A}_4|_g &= \cos \bar{\alpha} \cot \beta \partial_{\bar{\alpha}} + \sin \bar{\alpha} \partial_{\beta} - \frac{\cos \bar{\alpha}}{\sin \beta} \partial_{\gamma}, \\ \mathcal{A}_5|_g &= -\sin \bar{\alpha} \cot \beta \partial_{\bar{\alpha}} + \cos \bar{\alpha} \partial_{\beta} + \frac{\sin \bar{\alpha}}{\sin \beta} \partial_{\gamma}, \\ \mathcal{A}_6|_g &= \partial_{\bar{\alpha}}. \end{aligned} \quad (\text{A10})$$

We observe that

$$\mathcal{A}_{gh_{\bar{\alpha}}} \equiv (\mathbf{R}_{\mathbf{e}_z, \bar{\alpha}} \oplus \mathbf{R}_{\mathbf{e}_z, \bar{\alpha}})^T \mathcal{A}_g, \quad \text{where } \mathcal{A}_g = (\mathcal{A}_1|_g, \dots, \mathcal{A}_6|_g). \quad (\text{A11})$$

The above formula's do not hold for $\beta = \pi$ or $\beta = 0$. So we even lack expressions for our left-invariant vector fields at the unity element $(\mathbf{0}, \mathbf{I}) \in SE(3)$ when using the standard ZYZ-Euler angles. Therefore, one formally needs a second chart, for example the XYZ-coordinates in [75,78,91]:

$$g = (x, y, z, \mathbf{R}_{\mathbf{e}_x, \tilde{\gamma}} \mathbf{R}_{\mathbf{e}_y, \tilde{\beta}} \mathbf{R}_{\mathbf{e}_z, \bar{\alpha}}), \text{ with } \tilde{\beta} \in [-\pi, \pi), \bar{\alpha} \in [0, 2\pi), \tilde{\gamma} \in (-\pi/2, \pi/2), \quad (\text{A12})$$

formula (9) yields the following formulas for the left-invariant vector fields (only for $|\tilde{\beta}| \neq \frac{\pi}{2}$):

$$\begin{aligned} \mathcal{A}_1|_g &= \cos \bar{\alpha} \cos \tilde{\beta} \partial_x + (\cos \tilde{\gamma} \sin \bar{\alpha} + \cos \bar{\alpha} \sin \tilde{\beta} \sin \tilde{\gamma}) \partial_y + (\sin \bar{\alpha} \sin \tilde{\gamma} - \cos \bar{\alpha} \sin \tilde{\beta} \cos \tilde{\gamma}) \partial_z \\ \mathcal{A}_2|_g &= -\sin \bar{\alpha} \cos \tilde{\beta} \partial_x + (\cos \bar{\alpha} \cos \tilde{\gamma} - \sin \bar{\alpha} \sin \tilde{\beta} \sin \tilde{\gamma}) \partial_y + (\sin \bar{\alpha} \sin \tilde{\beta} \cos \tilde{\gamma} + \cos \bar{\alpha} \sin \tilde{\gamma}) \partial_z \\ \mathcal{A}_3|_g &= \sin \tilde{\beta} \partial_x - \cos \tilde{\beta} \sin \tilde{\gamma} \partial_y + \cos \tilde{\beta} \cos \tilde{\gamma} \partial_z, \\ \mathcal{A}_4|_g &= -\cos \bar{\alpha} \tan \tilde{\beta} \partial_{\bar{\alpha}} + \sin \bar{\alpha} \partial_{\tilde{\beta}} + \frac{\cos \bar{\alpha}}{\cos \tilde{\beta}} \partial_{\tilde{\gamma}}, \\ \mathcal{A}_5|_g &= \sin \bar{\alpha} \tan \tilde{\beta} \partial_{\bar{\alpha}} + \cos \bar{\alpha} \partial_{\tilde{\beta}} - \frac{\sin \bar{\alpha}}{\cos \tilde{\beta}} \partial_{\tilde{\gamma}}, \\ \mathcal{A}_6|_g &= \partial_{\bar{\alpha}}. \end{aligned} \quad (\text{A13})$$

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