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Article

Efficient Large Sparse Arrays Synthesis by Means of Smooth Re-Weighted L1 Minimization

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Abstract: In this paper we present an efficient technique for the synthesis of very large sparse arrays, with arbitrary circularly symmetrical upper bounds for the pattern specifications. The algorithm, that is based on iterative smooth re-weighted L1 minimizations is very flexible, and is capable of achieving very good performances with respect to competitive algorithms. Furthermore, thanks to its efficiency, planar arrays of hundreds of wavelengths can be synthesized with limited computational effort.

Keywords: Antenna Arrays; Sparse arrays; Radar; Pattern Synthesis; Signal Processing

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1. Introduction

Antenna arrays are one of the most important technology that allows to respond to the challenges of the close future communication systems, particularly in the 5G and IoT frameworks [1–4]. In this perspective, the use of sparse, non regular, arrays is particularly appealing, since it has been shown that a sparse array is often capable of achieving the same performance of regular lattice ones, with a much lower number of radiating elements, and a strong reduction of the issue of grating lobes.

Unfortunately, the synthesis of a sparse array is a much tougher task with respect to the synthesis of regular lattice ones. This difference is due to the fact that the relationship between the radiated field and the excitation of the radiating elements is linear. This allows the use of efficient linear and convex programming techniques for the determination of the elements excitation once the position of the radiators is fixed. Instead, the relationship between the radiated field and the element's positions is non linear and non convex, and no optimal algorithms capable to solve this problem are currently known.

In the last decades, researchers have followed different approaches to face the synthesis of sparse arrays. They can be roughly summarized in two main categories. In the first one we have all evolutionary computation based methods [5–20], that exploit the capability of genetic algorithms, particle swarm optimization and similar techniques to face the multiple minima that arise from the aforementioned non-linearity and non-convexity.

A second category is represented by a number of different approaches in which the general synthesis problem is substituted by a simpler one, that can be faced by means of deterministic techniques [21–33].

Both approaches show advantages and disadvantages. The former class is advantaged from the fact that a well-designed evolutionary algorithm is capable of asymptotically find the optimal solution of a given problem. The main issue is represented by the word "asymptotically". The number of iteration needed to get "close" to the global optimum is an exponential function of the number of unknowns, so it is very hard to deal with evolutionary algorithms when the dimension of the desired array is very large: we get a solution, but is often impossible to say how much close to the optimum we are[34].

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For the latter category, instead, the execution is - generally speaking - fast, and we are usually allowed to solve, with relative ease, large problems. But, again, since the starting problem solved is a sub-optimal one, so we do not usually obtain the global optimum.

The aforementioned classification is obviously a very simplified one, since in the years hybrid and mixed strategies have been presented [35–41]; furthermore the recent development of compressive sensing [42], ad related algorithms, has provided effective and computationally efficient algorithms [43–53]. In particular, the synthesis of sparse linear arrays of hundreds of wavelengths are shown in [53], a result that to the best knowledge of the authors is currently unmatched.

The synthesis of planar arrays of more than a hundred of wavelengths is instead an open issue. This kind of problems is computationally very difficult to face, because of the huge number of unknowns, with evolutionary algorithms; no results have been shown in the open literature by using other kind of algorithms.

In this contribution we will propose a sub optimal, yet effective, approach to the solution of this kind of problems, to be used for circularly symmetric beam pattern specifications. In particular, the circular symmetry of the specifications will be used to simplify the layout: we will look for antenna arrays with the elements displaced in circular concentric rings, with a reduction of the number of unknowns to optimize for. Furthermore, we will exploit a smooth re-weighted ℓ_1 norm minimization, a compressive sensing inspired approach, to look for the minimum number of rings; finally, each ring will be populated with a number of elements as small as possible, to achieve an overall minimum number of radiators.

The paper will be organized as follow. First of all we will discuss the synthesis problem formulation (section 2), then we will propose the use of a compressive sensing like approach for the determination of the number of rings (section 3) and the population of rings with radiating elements (section 4). In section 5 we will provide some numerical results to validate the proposed approach; conclusions follow (section 6).

2. Problem formulation

For the sake of simplicity we will consider the case of isotropic radiators; most of the considerations that we will discuss in the remaining of the paper can be extended to cases in which we take into account the element factor with a minimum effort.

Let us now consider the Array Factor of number N of sources placed on the $\{x, y\}$ plane:

$$F(\theta, \phi) = \sum_{n=1}^{N} a_n e^{j\beta(x_n \sin\theta \cos\phi + y_n \sin\theta \sin\phi)} \tag{1}$$

where $\beta = 2\pi/\lambda$ is the free space wavenumber, λ is the wavelength, $\{x_n, y_n\}$ are the coordinates of the $n-$th radiating element and a_n is its excitation.

The sparse array synthesis problem consists in finding the position and excitation of the N radiating element in order to satisfy a chosen array factor specification. As an example we could require that

$$|F(\theta, \phi)| \leq M_{UP}(\theta, \phi) \tag{2}$$

$$F(0, 0) = 1 \tag{3}$$

that represent the typical specifications of a pattern with arbitrary upper bounds.

Some considerations are now in order. First, the number N of radiators could not be specified, and it could also be one of the outputs of the optimization, since such a number is often required to be as small as possible. Second, in some specific applications we would like all the radiating element share the same excitations (i.e. "uniform excitation" arrays) or we suppose that the power delivered to each radiator, that could be also of two or more different kinds, is equal (i.e. “isophoric” arrays).

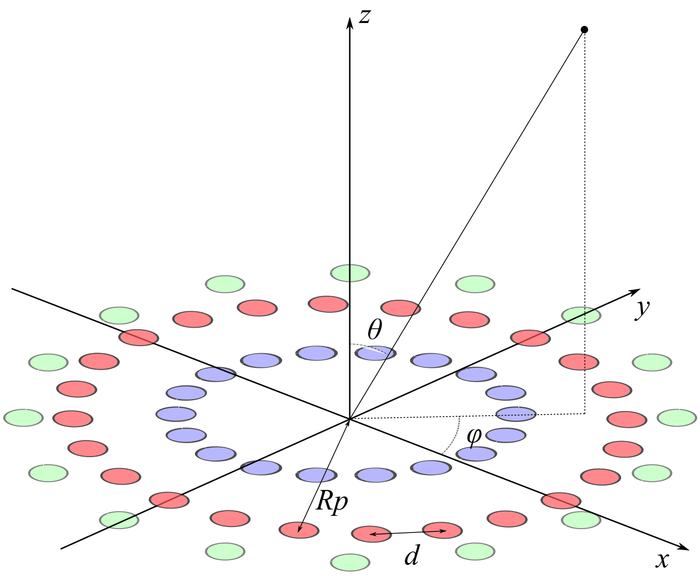

Figure 1. Scheme of a circular concentric ring array.

As stated in the introduction, the general solution of this kind of problem, even in the case of convex constraints for the pattern specifications, leads to a non-linear and non-convex problem, that is usually very hard to solve, and no deterministic techniques for its optimal solution are known.

It is now worth recalling that, when the array elements are arranged into circles, equally spaced along each ring1, the array factor (1) can be recast in a very convenient way:

$$F(\theta,\phi) = \sum_{p=1}^{P} A_p N_p \sum_{m=-\infty}^{+\infty} J_{mN_p}(\beta R_p \sin\theta) e^{jmN_p \Xi(\theta,\phi)} \tag{4}$$

where P is the overall number of rings, N_p is the number of antennas along the $p-$th ring, A_p is the common excitation of the elements belonging to the $p-$th ring of radius R_p, $J_h(*)$ is the $n-$th order Bessel function of the first kind and $\Xi(\theta,\phi)$ is a function of the angular variables, inessential for our purposes [54].

The main advantage of the formulation (4) is due to the fact that the Bessel functions, other than the zero order ones, tend to become very small under very simple conditions; in particular it is well known that $|J_a(b)| << 1$ when $b << a$, so looking to the worst case of $m = 1$ if

$$\beta R_p \sin\theta << N_p \approx 2\pi R_p / d \tag{5}$$

where d is the distance between two radiating elements on the same ring. Recasting (5) gives:

$$d << \lambda / \sin\theta \tag{6}$$

This means that as long as the elements belonging to the same ring are not too widely spaced, the array factor can be conveniently approximated by:

$$F(\theta,\phi) \approx F(\theta) = \sum_{p=1}^{P} A_p N_p J_0(\beta R_p \sin\theta) \tag{7}$$

This approximation is particularly useful, since the synthesis can be done analysing the θ angular direction only, with a significant computational improvement. Obviously, this simplification is applicable only to the synthesis of radiation pattern with a circular simmetry (i.e. $M_{UP}(\theta,\phi) = M_{UP}(\theta)$).

In the next section we will discuss how we can use a compressive sensing approach to synthesize very large sparse arrays using exploiting (7).

3. Smooth Re-Weighted L1 Minimization

For sake of simplicity, in the next section we will call $e_p = A_p N_p$ the ring excitation. The synthesis problem will be then recast as finding the number of rings, their radii and excitation in order to satisfy the given pattern specifications; the choice of the balancing between A_p and N_p will depend on the antenna architecture specifications (i.e. constant excitation A_p for isophoric arrays).

Let us now consider a dummy ring array, made of a huge number of closely spaced rings, introducing a dense equispaced position $\nu-$element vector $\tilde{\mathbf{r}}$ in the range $[0, R]$, with an inter-element spacing δ much smaller than the wavelength λ, and a vector of observation points $\mathbf{w} = \{w_1, ...w_\zeta\} \in [0, w_{max}]$, so that

$$\mathbf{F} = \mathbf{A}\tilde{\mathbf{e}} \tag{8}$$

provides the relationship between the array factor sampled in $w_h = \sin\theta_h$ and the vector of ring excitations $\tilde{\mathbf{e}}$, achieved by the radiation matrix $\mathbf{A}$, whose h,k element is $a_{h,k} = J_0(\beta r_k w_h)$.

The radial and angular discretization of the array factor allow us to solve the synthesis problem by solving the following constrained minimization:

$$\begin{aligned} &\textit{minimize} && \|\tilde{\mathbf{e}}\|_0 && (9)\\ &\textit{subject to} && |\mathbf{A}\tilde{\mathbf{e}}| \leq \mathbf{M}_{UP} && (10)\\ & && \mathbf{A_0}\tilde{\mathbf{e}} = 1 && (11) \end{aligned}$$

where $\|\mathbf{v}\|_0$ represents the so called ℓ_0 norm of $\mathbf{v}$ vector, $\mathbf{M}_{UP}$ is the sampled version of the upper mask, $\mathbf{A_0}$ is the array factor in the direction of the desired maximum of the main beam. Once this minimization is solved, the sparse vector of the solution will give us the radii and excitations of the rings of our sparse array (2)-(3).

Unfortunately, there are no known deterministic optimal techniques to solve the ℓ_0 norm minimization, but as shown in [52,55–57] the ℓ_0 norm minimization can be conveniently solved in a computationally efficient way by using a sequence of weighted ℓ_1 minimizations. We could then try to iteratively solve the following convex problem:

$$\begin{aligned} &\textit{minimize} && \|\mathbf{g}^p \circ \tilde{\mathbf{e}}^p\|_1 && (12)\\ &\textit{subject to} && |\mathbf{A}\tilde{\mathbf{e}}^p| < \mathbf{M}_{UP} && (13)\\ & && \mathbf{A_0}\tilde{\mathbf{e}}^p = 1 && (14) \end{aligned}$$

where $\mathbf{g}^p \circ \tilde{\mathbf{e}}^p$ is the Hadamard entrywise product of the two vectors $\mathbf{g}^p$ and $\tilde{\mathbf{e}}^p$, $\|\mathbf{v}\|_1$ is the ℓ_1 norm of $\mathbf{v}$ vector, given by $\sum_k |v_k|$, p is the iteration index and $\mathbf{g}^p = [g_1^p, \cdots, g_\nu^p]$ is a proper weighting vector.

As shown in [52], by means of a non local, "smooth" weighting vector it is possible to reduce the step of the radial discretization, preserving the small "clusters" that arise in the solution of the ℓ_1 norm minimization. In particular we have used:

$$g_k^p = \frac{1}{\max(z_k^{p-1}, \eta)} \tag{15}$$

where $\tilde{\mathbf{z}}^{p-1} = [z_1^{p-1}, \cdots, z_\nu^{p-1}]$ and

$$\tilde{\mathbf{z}}^{p-1} = |\tilde{\mathbf{e}}^{p-1}| * \mathbf{d} \tag{16}$$

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and $\mathbf{d}$ is a smoothing vector of positive numbers, and the convolution "$*$" returns only the central part of it, of the same size as $|\tilde{\mathbf{e}}^{p-1}|$.

Once the algorithm has reached a satisfactory solution, it is very easy to extract the ring radii R_p and excitations e_p from vector $\tilde{\mathbf{e}}$, by means of the refined clustering approach described in [52].

4. Ring population from the calculated excitation

As stated in section 2, the ring excitation represents the product of the number of elements and the relative excitation of the radiators belonging to a certain ring. The same ring excitation could be achieved by different combinations of the two factors; the specific choice we need to perform to synthesize the array layout will depend on a particular "ring population strategy".

In this section we will discuss two different ring population strategies: the former, to be used in equal amplitude and isophoric arrays; the latter to be used in arrays with variable excitation of the radiating elements.

The key point in both strategies is reducing as much as possible the number of radiators, trying always to limit the effect of the higher order Bessel terms of (4).

For isophoric arrays we need first to identify the index of the "minimally populated ring" (MPR), the ring that presents the lowest ratio of the excitation e_p and the ring radius R_p. Once the MPR has been identified, the number of radiators N_{MPR} on such ring can be chosen in order to verify:

$$J_{N_{MPR}}(\beta R_{MPR} \sin\theta) < \frac{T}{e_{MPR}} \tag{17}$$

where T is a threshold that depends on the required side lobe level (usually one hundredth of the SLL is sufficient). The value of N_{MPR} can be determined numerically, and then the number of radiators on all the other rings can be obtained as:

$$N_p = e_p \frac{N_{MPR}}{e_{MPR}} \tag{18}$$

The ring population strategy for arrays with variable elements excitation consists, instead, in finding for each ring the minimum number of radiators that allows limiting the effect of higher order Bessel terms:

$$J_{N_p}(\beta R_p \sin\theta) < \frac{T}{e_p} \tag{19}$$

once such a number is found the element excitation can be achieved by

$$A_p = \frac{e_p}{N_p} \tag{20}$$

Obviously, those two strategies aim to the reduction of the number of radiators, and slightly modified versions could be used if other specifications need to be met (for instance, a fixed overall number of radiators). Moreover, the aforementioned strategies can be easily modified to include non-overlapping constraints for the radiators.

The synthesis approach consists in the following steps:

1. Choose the values of δ, $\mathbf{d}$ and η to use.
2. Iteratively solve the problem (12)-(14); usually less than twenty iterations allow the convergence;
3. we extract the ring radii and excitations from the sparse vector $\tilde{\mathbf{e}}$;
4. we populate the ring according to one of the discussed strategies.

5. Numerical Examples

In this section we will provide some examples that will show the effectiveness of the proposed approach. For all the tests we have used an inter-ring spacing of $\delta = \lambda/20$, the smoothing vector is

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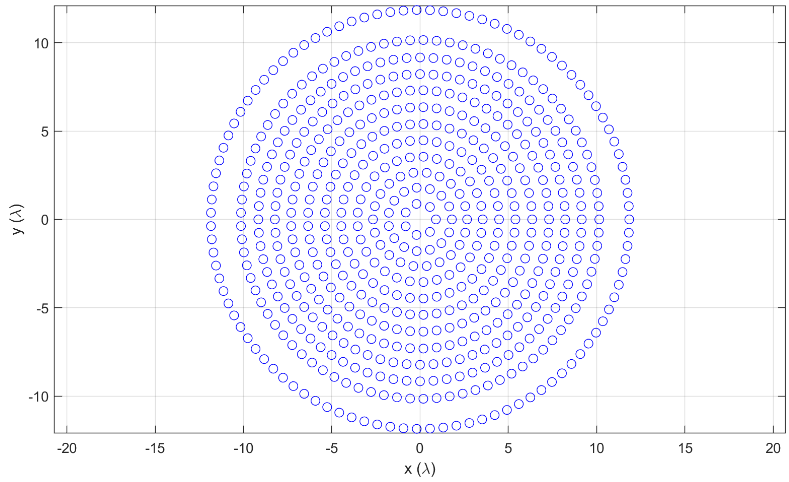

Figure 2. Layout of the 597 element sparse array. Each radiator is represented by a circle of half wavelength diameter.

Figure 3. Plot of the $\phi-$cuts of the array factor for the 597 element array.

$\mathbf{d} = [0.1, 0.5, 0.99, 1, 0.99, 0.5, 0.1]$ and the tolerance factor is $\eta = \max(|\tilde{\mathbf{x}}^p|/100)$. All the minimizations have been solved by means of CVX [58] in Matlab.

5.1. Large array with variable excitation

In the first example we will consider the synthesis of a sparse array with the same specification of the 718 elements array of [39]; this sparse array is able to radiate a pattern with a side lobe level lower than -37.05dB for $w \geq 0.074$. By means of the proposed approach we are able to synthesize a sparse array of 597 elements organized in 12 rings (Fig. 2), that satisfies the same specifications with a reduction of the radiating elements number of about 17% with respect to the reference layout.

In particular, the parameters of the found array are provided in table 1. The minimum, average and maximum spacing between the rings are, respectively, 0.860λ, 0.995λ and 1.700λ. The dynamical ratio of the excitations, the ratio of the maximum and minimum excitation of the elements, is 16.5dB. The array factor of the synthesized array is displayed in figures 3 and 4. It is interesting to observe from Fig.3 that the $\phi-$ cuts are perfectly superimposed up to about $56°$, confirming the fact that the Bessel function terms other than the zero order ones are correctly limited by the right choice of the element number per ring.

The calculation of the layout is very quick, ten iterations are needed and each iteration took about 10 seconds on an Intel i7 8700k processor.

5.2. Small Isophoric Sparse Array

In the second example we will consider the synthesis of a sparse array with the same specifications of the 185 elements array of [51]; this sparse arrays has been synthesized by means of an approach similar to the one proposed in this paper, but does not employ the smooth weighting function. The reference sparse array is able to radiate a pattern with a side lobe level lower than -23.51dB and shows a first null beamwidth (FNBW) of $14.2°$. By means of our approach we are able to synthesize a sparse array of 167 elements organized in 6 rings (Fig. 5), that satisfies the same SLL specifications with a reduction of beamwidth (in our case the FNBW is $13.5°$) and a reduction of the radiating elements number of about 10%.

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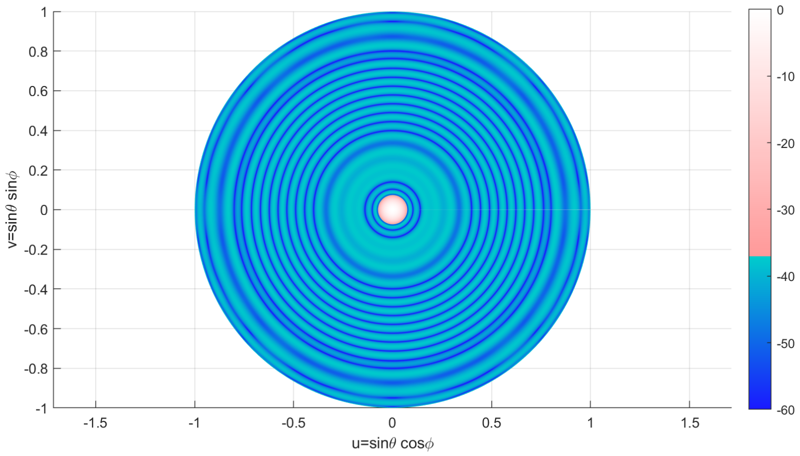

Figure 4. Imagemap of the array factor in the u, v plane for the 597 element array. The dual tone (pink/cyan) colormap does not show any violation of the SLL constraint out of the main beam.

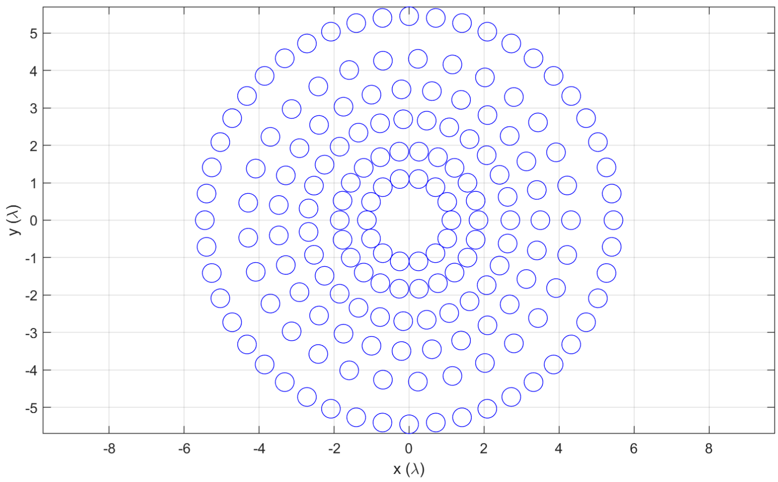

Figure 5. Layout of the 167 elements sparse array. Each radiator is represented by a circle of half wavelength diameter.

Table 1. Parameters of the synthesized 597 element array.

#	R_p (λ)	N_p	A_p
1	0.9	7	0.693
2	1.805	15	1
3	2.665	22	0.943
4	3.55	29	0.794
5	4.476	37	0.72
6	5.403	45	0.631
7	6.352	53	0.52
8	7.309	61	0.464
9	8.222	68	0.368
10	9.161	76	0.269
11	10.15	85	0.277
12	11.85	99	0.15

Table 2. Parameters of the synthesized 167 element array.

#	R_p (λ)	N_p
1	1.127	14
2	1.85	22
3	2.7	27
4	3.5	27
5	4.317	29
6	5.45	48

In particular, the parameters of the found array are provided in table 2. The minimum, average and maximum spacing between the rings are, respectively, 0.72273λ, 0.86454 λ and 1.1333λ. The array factor of the synthesized array is displayed in figures 6 and 7.

It is worth noting that the calculation of the layout is almost instantaneous, only three iterations are needed and each iteration took less than 2 seconds on an Intel i7 8700k processor.

5.3. Very Large Isophoric Sparse Array

In this section we instead will discuss the synthesis of a large sparse array with constant element excitation, with a diameter of about 290λ. To the best knowledge of the authors, there are no results available in the open literature with sparse arrays of these dimensions, confirming the capability of the proposed approach to handle synthesis problems of very high size.

The particular array sought should be able to radiate a pencil beam with a side lobe level lower than -30dB for $w \geq 0.005$, and to scan it on the whole earth surface from GEO orbit, always satisfying the SLL constraint within the Earth's surface (that is seen a solid angle of about 8.5° (deg) radius). For this reason the synthesis will be performed in the angular region with $w_{max} = 0.287$, following the guidelines in [31].

By means of the proposed approach we are able to synthesize a sparse array of 3516 elements organized in 17 rings, see figure 8, that perfectly satisfies the same chosen specifications.

In particular, the parameters of the found array are provided in table 3. The minimum, average and maximum spacing between the rings are, respectively, 3.350λ, 7.804λ and 17.559λ. The array factor of the synthesized array is displayed in figures 9 and 10; in figure 11 we also show a scanned beam at the edge of the Earth's surface, as seen from GEO orbit.

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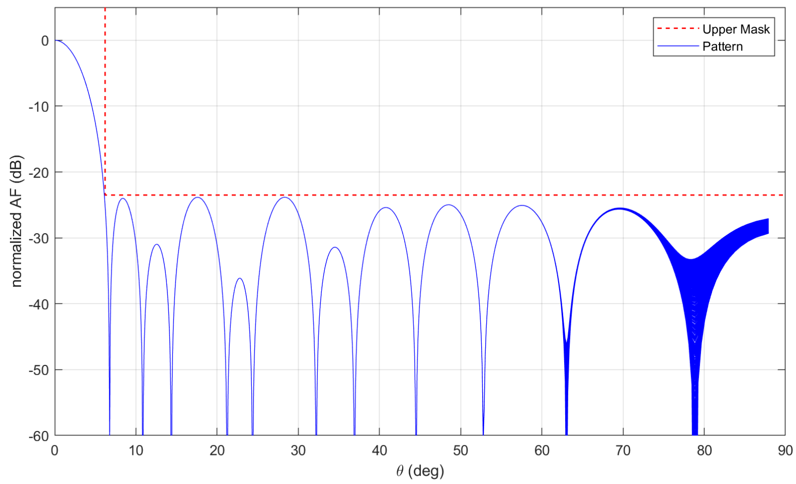

Figure 6. Plot of the $\phi-$cuts of the array factor for the 167 element array.

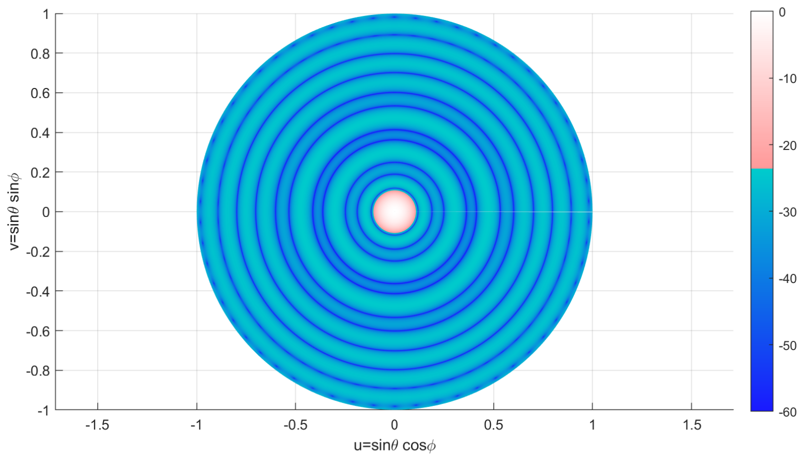

Figure 7. Imagemap of the array factor in the u, v plane for the 167 element array. The dual tone (pink/cyan) colormap does not show any violation of the SLL constraint out of the main beam.

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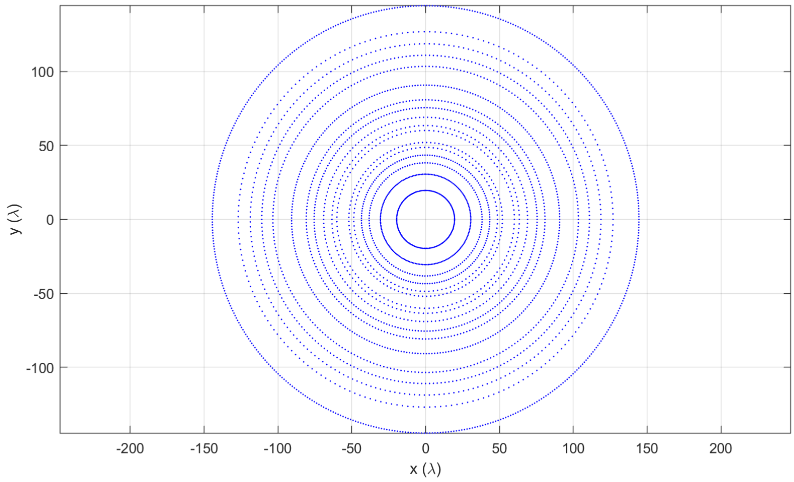

Figure 8. Layout of the 3516 element sparse array. Each radiator is represented by a circle of half wavelength diameter.

Table 3. Parameters of the synthesized 3516 element array.

#	R_p (λ)	N_p
1	19.602	144
2	30.594	192
3	38.25	139
4	43.45	167
5	48.647	103
6	52	117
7	60.1	119
8	63.45	130
9	69.048	162
10	75.5	238
11	80.84	232
12	90.75	301
13	103.45	272
14	110.95	264
15	118.721	223
16	126.9	215
17	144.459	498

In this case the calculation of the layout is slower than the previous cases, 41 iterations are needed and each iteration took about 70 seconds on an Intel i7 8700k processor, but considering the dimension of the achieved layout this is indeed an excellent result.

6. Discussion and Conclusions

A very efficient algorithm for the synthesis of sparse array has been presented.

The approach is capable of synthesizing sparse arrays with variable or constant excitation of the element, allowing the satisfaction of circularly symmetrical constraints for the upper bounds of the pattern.

The shown results are very good, the obtained layouts present a lower number of elements with respect to other algorithms in literature, when the same pattern specifications are met.

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Figure 9. Plot of the $\phi-$cuts of the array factor for the 3516 element array.

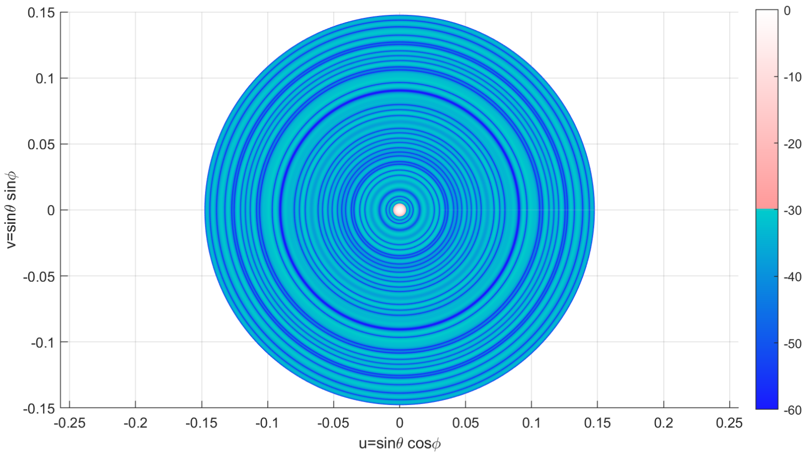

Figure 10. Imagemap of the array factor for the broadside beam in the u, v plane for the 3516 element array. Only the inner part of the u, v plane, relative to Earth's surface as seen from GEO orbit is displayed.

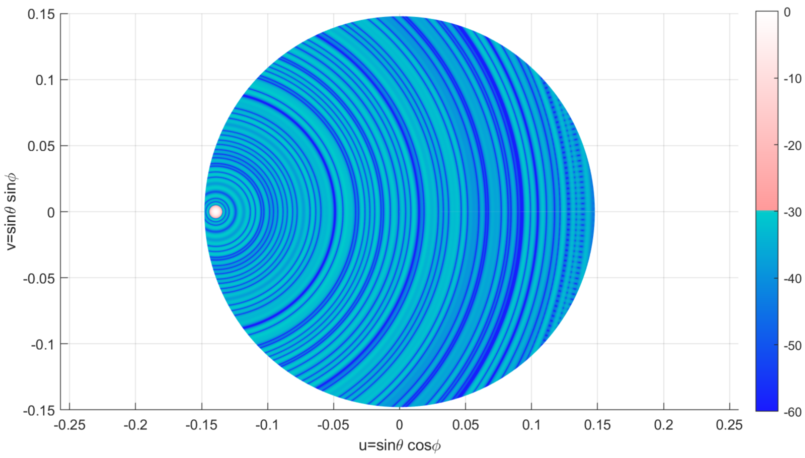

Figure 11. Imagemap of the array factor for the beam scanned to 8° deg in the u, v plane for the 3516 element array. Only the inner part of the u, v plane, relative to Earth's surface as seen from GEO orbit is displayed. The dual tone (pink/cyan) colormap does not show any violation of the SLL constraint out of the main beam.

Furthermore the very high computational efficiency of the approach allows to face also problems of very large size: in one of the examples a sparse planar array of about three hundred wavelengths is demonstrated, and in other numerical tests, not shown here for sake of brevity, we were able to synthesize even larger arrays.

In a future development we plan to extend the proposed synthesis method to the case of shaped beams.

Author Contributions: D.P., M.D.M. and G.P. conceived the method; D.P. wrote the numerical code and performed the array syntheses; M.D.M, M.L. and F.S. helped with the comparison with other approaches-; D.P., G.P., F.S. and M.D.M. wrote the paper.

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