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$\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are substructural fuzzy logics

SanMin Wang

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Abstract Two representable substructural logics $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are logics for finite **UL** and **IUL**-algebras, respectively. In this paper, the standard completeness of $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ is proved by the method developed by Jenei, Montagna, Esteva, Gispert, Godo and Wang. This shows that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are substructural fuzzy logics.

Keywords Substructural fuzzy logics · Residuated lattices · Semilinear substructural logics · Standard completeness · Fuzzy logic

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1 Introduction

In [10], we constructed three representable substructural logics $\mathbf{UL}_\omega$, $\mathbf{IUL}_\omega$ and $\mathbf{HpsUL}^*_\omega$ by adding one simple axiom

$$\text{(FIN)}\ \ (A \to e) \leftrightarrow (A \odot A \to e)$$

to Metcalfe and Montagna's uninorm logic **UL**, involutive uninorm logic **IUL** [6], and a suitable extension $\mathbf{HpsUL}^*$ [7] of Metcalfe, Olivetti and Gabbay's pseudo-uninorm logic **HpsUL** [5], respectively. Especially, we showed that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are logics for finite **UL** and **IUL**-algebras, respectively.

In this paper, we prove that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are standard complete by Wang's constructions in [8] and [9], which are some generalizations of Jenei and Montagna-style approach for proving standard completeness for monoidal t-norm based logic **MTL** [4] and the proof of the standard completeness for **IMTL** given by Esteva, Gispert, Godo and Montagna in [2]. This shows that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are substructural fuzzy logics.

SanMin Wang
Faculty of Science, Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China
Tel.: +86-13655812543
Fax: +86-057186843224
E-mail: wangsanmin@hotmail.com

We have proved that **HpsUL*** is standard complete in [11]. However, we are unable to prove whether $\mathbf{HpsUL}^*_\omega$ is standard complete or, complete with respect to finite **HpsUL***-algebras and left them as open problems.

2 $\mathbf{HpsUL}^*_\omega$, $\mathbf{UL}_\omega$, $\mathbf{IUL}_\omega$ and algebras involved

The Hilbert system **HpsUL** is the logic of bounded representable residuated lattices, which is based on a countable propositional language with formulas built inductively as usual from a set of propositional variables, binary connectives $\odot, \to, \rightsquigarrow, \wedge, \vee$ and constants $e, f, \bot, \top$, with definable connectives:

$$\neg\varphi := \varphi \to f,$$

$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi),$$

$$\lambda_\chi(\varphi) := (\chi \to \varphi \odot \chi) \wedge e,$$

$$\rho_\chi(\varphi) := (\chi \rightsquigarrow \chi \odot \varphi) \wedge e.$$

Definition 1 **HpsUL** consists of the following axioms and rules [5]:

(A_1) $\vdash \varphi \to \varphi$
(A_2) $\vdash (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
(A_3) $\vdash \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$
(A_4) $\vdash (\varphi \rightsquigarrow (\psi \to \chi)) \to (\psi \to (\varphi \rightsquigarrow \chi))$
(A_5) $\vdash \psi \to (\varphi \to \varphi \odot \psi)$
(A_6) $\vdash (\psi \to (\varphi \to \chi)) \to (\varphi \odot \psi \to \chi)$
(A_7) $\vdash (\psi \rightsquigarrow \psi \odot (\psi \to \varphi)) \to (\psi \rightsquigarrow \varphi)$
(A_8) $\vdash (\varphi \wedge t) \odot (\psi \wedge t) \to \varphi \wedge \psi$
(A_9) $\vdash \varphi \wedge \psi \to \psi$
(A_{10}) $\vdash \varphi \wedge \psi \to \varphi$
(A_{11}) $\vdash (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$
(A_{12}) $\vdash \varphi \to \varphi \vee \psi$
(A_{13}) $\vdash \psi \to \varphi \vee \psi$
(A_{14}) $\vdash (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
(A_{15}) $\vdash e$
(A_{16}) $\vdash \varphi \to (e \to \varphi)$
(A_{17}) $\vdash \varphi \to \top$
(A_{18}) $\vdash \bot \to \varphi$
(PRL) $\vdash (\lambda_\chi(\varphi \vee \psi \to \varphi)) \vee (\rho_\chi(\varphi \vee \psi \to \psi))$
(MP) $\varphi, \varphi \to \psi \vdash \psi$
(ADJ_U) $\varphi \vdash \varphi \wedge e$
($PN_\to$) $\varphi \vdash \psi \to \varphi \odot \psi$
($PN_\rightsquigarrow$) $\varphi \vdash \psi \rightsquigarrow \psi \odot \varphi$

Definition 2 [6, 7] A logic is a schematic extension (extension for short) of **HpsUL** if it results from **HpsUL** by adding axioms in the same language. In particular,

- **HpsUL*** is **HpsUL** plus (WCM) $\vdash (\varphi \rightsquigarrow e) \to (\varphi \to e)$;
- **UL** is **HpsUL** plus $\vdash \varphi \odot \psi \to \psi \odot \varphi$;
- **IUL** is **UL** plus $\vdash \neg\neg\varphi \to \varphi$.

Definition 3 New extensions of **HpsUL** are defined as follows.

- $\mathbf{HpsUL}^*_\omega$ is $\mathbf{HpsUL}^*$ plus (FIN) $\vdash (\varphi \to e) \leftrightarrow (\varphi \odot \varphi \to e)$;
- $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are **UL** and **IUL** plus (FIN), respectively.

Let $\mathbf{L} \in \{\mathbf{HpsUL}^*, \mathbf{UL}, \mathbf{IUL}, \mathbf{HpsUL}^*_\omega, \mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$ in the remainder of this section. A proof in $\mathbf{L}$ of a formula φ from a set Γ of formulas is defined as usual. We write $\Gamma \vdash_{\mathbf{L}} \varphi$ if such a proof exists.

Definition 4 [5] An **HpsUL**-algebra is a bounded residuated lattice $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top\rangle$ with universe A, binary operations $\wedge, \vee, \cdot, \to, \rightsquigarrow$, and constants $e, f, \bot, \top$ such that:

(i) $\langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice with top element $\top$ and bottom element $\bot$;

(ii) $\langle A, \cdot, e\rangle$ is a monoid;

(iii) $\forall x, y, z \in A, x \cdot y \leqslant z$ iff $x \leqslant y \rightsquigarrow z$ iff $y \leqslant x \to z$;

(iv) $\forall x, y, u, v \in A, (\lambda_u(x \vee y \to x)) \vee (\rho_v(x \vee y \to y)) = e$, where, for any $a, b \in A$, $\lambda_a(b) := (a \to b \cdot a) \wedge e, \rho_a(b) := (a \rightsquigarrow a \cdot b) \wedge e$.

We use the convention that $\cdot$ binds stronger than other binary operations and we shall often omit $\cdot$; we will thus write xy instead of $x \cdot y$, for example. Suitable classes of algebras of extensions of **HpsUL** are defined as follows.

Definition 5 [5, 7, 10] Let $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top\rangle$ be an **HpsUL**-algebra. For $\mathbf{L}$ an extension of **HpsUL**, $\mathcal{A}$ is an $\mathbf{L}$-algebra if all axioms of $\mathbf{L}$ are valid in $\mathcal{A}$. An $\mathbf{L}$-chain is an $\mathbf{L}$-algebra that is linearly ordered. In particular:

- $\mathcal{A}$ is an $\mathbf{HpsUL}^*$-algebra if the weak commutativity (Wcm) holds: $xy \le e$ iff $yx \le e$ for all $x, y \in A$;
- $\mathcal{A}$ is an **UL**-algebra if $xy = yx$ for all $x, y \in A$;
- $\mathcal{A}$ is an **IUL**-algebra if it is an **UL**-algebra such that $\neg\neg x = x$ for all $x \in A$;
- $\mathcal{A}$ is an $\mathbf{HpsUL}^*_\omega$-algebra ($\mathbf{UL}_\omega$ or $\mathbf{IUL}_\omega$-algebra) if it is an $\mathbf{HpsUL}^*$-algebra (**UL** or **IUL**-algebra) such that the following identity (Fin) holds:

$x \to e = x^2 \to e$ for all $x \in A$.

Definition 6 [5] Let $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top\rangle$ be an $\mathbf{L}$-algebra. (i) An $\mathcal{A}$-valuation v is a homomorphism from the term algebra determined by formulas in $\mathbf{L}$ to $\mathcal{A}$; (ii) A formula φ is valid in $\mathcal{A}$ if $v(\varphi) \geqslant e$ holds for any $\mathcal{A}$-valuation v; (iii) The relation of semantic consequence $\Gamma \vDash_{\mathcal{A}} \varphi$ holds if each $\mathcal{A}$-evaluation that validates all formulae in a theory Γ validates φ as well.

Theorem 1 [5] *$\Gamma \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \vDash_{\mathcal{A}} \varphi$ for every $\mathbf{L}$-chain $\mathcal{A}$, i.e., $\mathbf{L}$ is a presentable substructural logic.*

Lemma 1 *Let $\mathcal{A}$ be an $\mathbf{HpsUL}^*_\omega$-chain and, $s, t, u \in A$. Then*

(i) $st \leqslant e$ iff $st^2 \leqslant e$;
(ii) $stu = s$ implies $st = s$ and $su = s$;
(iii) $stu = u$ implies $su = u$ and $tu = u$;
(iv) $st = e$ implies $s = t = e$.

Proof Only (ii) is proved as follows and, others see [10]. If $tu \leqslant e$ then $tut \leqslant e$ and $utu \leqslant e$ by (1) and (Wcm). Thus $stut \leqslant s$ and $stutu \leqslant st$. Hence $st \leqslant s$ and $s \leqslant st$. Therefore $st = s$. The case of $tu > e$ can be proved in the same way.

Clearly, Lemma 1 holds for all $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$-algebras.

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3 Wang's Construction and Standard completeness

In this section, let $\mathbf{L}_\omega \in \{\mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$, $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \rightsquigarrow, e, f, \bot, \top\rangle$ be a finite or countable linearly ordered $\mathbf{L}_\omega$-algebra and s, t, u be arbitrary elements of A.

Definition 7 [7,8] Let $\mathcal{A}$ be an $\mathbf{UL}_\omega$-algebra. For each $s \in A$, t is the immediate predecessor of s in A if (i) $t \in A$, $t < s$; (ii) $\forall u \in A, u < s$ implies $u \leqslant t$. For each $s \in A$, let s^- denote the immediate predecessor of s in A if it exists, otherwise take $s^- = s$.

Let $X = \{(s,1) : s \in A\} \cup \{(s,q) : s \in A, s > s^-, q \in Q \cap (0,1)\}$, we define: $(s,q) \leqslant (t,r)$ iff either $s <_S t$, or $s = t$ and $q \leqslant r$ and,

$$\begin{aligned} I_1 &:= \{(s,t) : s,t \in A, st = s \neq t, s > s^- t\} \\ I_2 &:= \{(s,t) : s,t \in A, st = t \neq s, t > st^-\} \\ I_3 &:= \{(s,t) : s,t \in A, st = t = s, s > st^-\} \\ I_4 &:= \{(s,t) : s,t \in A, (st \neq t \text{ and } st \neq s) \text{ or} \\ &\qquad (st = s^- t = s) \text{ or } (st = st^- = t)\}. \end{aligned}$$

Now define, for $(s,q),(t,r) \in X$:

$$(s,q) \circ (t,r) = \begin{cases} (s,q) & (s,t) \in I_1, \\ (t,r) & (s,t) \in I_2, \\ (s,q) \wedge_X (t,r) & (s,t) \in I_3, \\ (st,1) & (s,t) \in I_4, \end{cases}$$

where $\wedge_X$ and $\vee_X$ is meant $\min_X$ and $\max_X$ with respect to $\leqslant_X$, respectively. We will omit index if it does not cause confusion.

Lemma 2 *Let $\mathcal{A}$ be an $\mathbf{UL}_\omega$-algebra. Then $(s,q)\circ(t,r) \leqslant (e,1)$ iff $(s,q)\circ(t,r)\circ(t,r) \leqslant (e,1)$ for all $(s,q),(t,r)$ in X.*

Proof Let $(s,q)\circ(t,r) \leqslant (e,1)$. Since $(s,q)\circ(t,r) = (st, \diamondsuit)$ for some $\diamondsuit \in \{q,r,1\}$ by Definition 7, then $st \leqslant e$, Thus $stt \leqslant e$ by (Fin). Hence $(s,q)\circ(t,r)\circ(t,r) \leqslant (e,1)$. The sufficiency part of the lemma can be proved in the same way.

Definition 8 [2,9] Let $\mathcal{A}$ be an $\mathbf{IUL}_\omega$-algebra. Let

$$I^* := \{(s,t) : s,t \in A, s^- < s, t^- < t, t = \neg s^-\},$$

$$I^{**} := \{(s,t) : s,t \in A, ss = s^- s = s = t\}.$$

$\forall (s,q),(t,r) \in X$, define

$$(s,q) \bigtriangleup (t,r) = \begin{cases} (s,q)\circ(t^-,1) \vee (s^-,1)\circ(t,r) & if\ (s,t) \in I^*, q + r \leqslant 1, \\ (s,q \vee r)\circ(s^-,1) & if\ (s,t) \in I^{**}, \\ (s,q)\circ(t,r) & otherwise. \end{cases}$$

Lemma 3 *Let $\mathcal{A}$ be an $\mathbf{IUL}_\omega$-chain and $s,t \in A$. (i) If $st^- \neq s$, $st^- \leqslant e$, $s^- t \leqslant e$ then $st^- t \leqslant e$; (ii) If $st^- = s^- t^-$ and $s^- t \leqslant e$ then $st^- t \leqslant e$; (iii) $(s,q) \bigtriangleup (t,r) \leqslant (s,q)\circ(t,r)$.*

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Proof (i) If $st \leqslant e$ then $stt \leqslant e$ by Lemma 1(i) and thus $st^-t \leqslant stt \leqslant e$. If $t \leqslant e$ then $st^-t \leqslant t \leqslant e$ by $st^- \leqslant e$. Thus, let $st > e$ and $t > e$ in the following.

$t^- \geq e$ by $t > e$. $t^- \neq e$ by $st^- \neq s$. Then $t^- > e$. Thus $st^- \geq s$. Hence $st^- > s$ by $st^- \neq s$. $st^- \neq e$ by Lemma 1(iv) and $t^- > e$. Therefore $st^- < e$ by $st^- \leqslant e$. Then $st^- < e < t^-$. Thus $st^- < t^-$. Hence $s < e$.

Suppose that $st \leqslant t^-$. Then $sst \leqslant st^- \leqslant e$. Thus $st \leqslant e$ by Lemma 1(i), a contradiction and hence $st > t^-$. Therefore $st \geqslant t$. $st \leqslant t$ by $s < e$. Then $st = t$.

Suppose that $s^-t \geqslant s$ then $s^-tt \geqslant st > e$. Thus $s^-t > e$ by Lemma 1(i), a contradiction and hence $s^-t < s$.

Therefore $s^-t \leqslant s^-$. $s^-t \geqslant s^-$ by $t > e$. Then $s^-t = s^-$. Then $s^-st = s^-$ by $st = t$. Thus $s^-s = s^-$ by Lemma 1(ii).

Suppose that $ss = s$ then $st^- = sst^- \leqslant s$, a contradiction with $st^- > s$ and hence $ss < s$ by $ss \leqslant s$. Then $ss \leqslant s^-$.

Thus $s^- = s^-s \leqslant ss \leqslant s^-$. Hence $ss = s^-$. Then $(ss)t = s^-t = s^-$ and $s(st) = st = t$. Thus $s^- = t$ by $(ss)t = s(st)$, a contradiction with $s^- < e < t$. Thus the case of $st > e$ and $t > e$ does not exist. This completes the proof of (i).

(ii) It follows from $s^-t \leqslant e$ that $s^-tt \leqslant e$ by Lemma 1(i). Then $st^-t = s^-t^-t \leqslant s^-tt \leqslant e$ by $st^- = s^-t^-$ and thus $st^-t \leqslant e$.

(iii) See Proposition 3.7 (2) of [9].

Lemma 4 *Let $\mathcal{A}$ be a finite $\mathbf{IUL}_\omega$-algebra. Then $(s,q) \triangle (t,r) \leqslant (e,1)$ if and only if $(s,q) \triangle (t,r) \triangle (t,r) \leqslant (e,1)$ for all $(s,q),(t,r)$ in X.*

Proof Let $(s,q) \triangle (t,r) \leqslant (e,1)$. There are three cases to be considered.

Case 1. $(s,t) \in I^*$ and $q + r \leqslant 1$. Then $(s,q) \triangle (t,r) = (s,q) \circ (t^-,1) \vee (s^-,1) \circ (t,r) \leqslant (e,1)$. Thus $st^- \leqslant e$, $s^-t \leqslant e$. Then $s^-tt \leqslant e$ by Lemma 1(i). If $(s,q) \triangle (t,r) = (s^-,1) \circ (t,r)$ then $(s,q) \triangle (t,r) \triangle (t,r) = ((s^-,1) \circ (t,r)) \triangle (t,r) \leqslant ((s^-,1) \circ (t,r)) \circ (t,r) \leqslant (s^-tt,1) \leqslant (e,1)$ by Lemma 3(iii). Let $(s,q) \triangle (t,r) = (s,q) \circ (t^-,1)$ in the following. If $(s,q) \circ (t^-,1) = (s,q)$ then $(s,q) \triangle (t,r) \triangle (t,r) = (s,q) \triangle (t,r) \leqslant (e,1)$. Otherwise $st^- \neq s$ or $st^- = s^-t^-$. Then $st^-t \leqslant e$ by Lemma 3(i) and 3(ii). Thus $(s,q) \triangle (t,r) \triangle (t,r) = ((s,q) \circ (t^-,1)) \triangle (t,r) \leqslant ((s,q) \circ (t^-,1)) \circ (t,r) \leqslant (st^-t,1) \leqslant (e,1)$.

Case 2. $(s,t) \in I^{**}$ then $ss = s^-s = s = t$ and $(s,q) \triangle (t,r) = (s, q \vee r) \circ (s^-,1) \leqslant (e,1)$. Thus $ss^- \leqslant e$. Hence $ss^-s \leqslant e$ by Lemma 1(i) and (Wcm). Therefore $(s,q) \triangle (t,r) \triangle (t,r) = ((s,q \vee r) \circ (s^-,1)) \triangle (s,r) \leqslant ((s,q \vee r) \circ (s^-,1)) \circ (s,r) \leqslant (ss^-s,1) \leqslant (e,1)$.

Case 3. $(s,q) \triangle (t,r) = (s,q) \circ (t,r) \leqslant (e,1)$ then $st \leqslant e$. Thus $stt \leqslant e$ by Lemma 1(i). Hence, by Lemma 3(iii), $(s,q) \triangle (t,r) \triangle (t,r) \leqslant (s,q) \circ (t,r) \circ (t,r) \leqslant (stt,1) \leqslant (e,1)$.

By a similar procedure, we prove that $(s,q) \triangle (t,r) \leqslant (e,1)$ if $(s,q) \triangle (t,r) \triangle (t,r) \leqslant (e,1)$.

Lemma 5 *Let Let $\mathcal{A}$ be an $\mathbf{HpsUL}^*_\omega$-algebra, X and the binary operation $\circ$ on X be as in Definition 7. The following conditions hold:*

(a) X is densely ordered, and has a maximum $\top_X = (\top,1)$ and a minimum $\bot_X = (\bot,1)$.

(b) $\langle X, \circ, \leqslant_X, e_X \rangle$ is a linearly ordered monoid, where $e_X = (e,1)$.

(c) $\circ$ is left-continuous with respect to the order topology on $\langle X, \leqslant_X \rangle$.

(d) There is a map Φ from A into X such that Φ is an embedding of the structure $\langle A, \wedge, \vee, \cdot, e, \bot, \top\rangle$ into $\langle X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X\rangle$, and for all $s, t \in A, \Phi(s \to t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$ in $\langle X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X\rangle$, respectively.

(e) $\forall (s, q), (t, r) \in X, (s, q) \circ (t, r) \leq (e, 1)$ iff $(s, q) \circ (t, r) \circ (t, r) \leq (e, 1)$.

Proof Claim (e) has been proved by Lemma 2. As pointed out in [7], the associativity of $\circ$ is mainly dependent on Lemma 1(ii) and 1(iii). Other claims can be proved in the same way as that of [7, Theorem 4.5].

Lemma 6 *Every countable linearly ordered* $\mathbf{UL}_\omega^*$*-algebra can be embedded into a standard* $\mathbf{UL}_\omega^*$*-algebra.*

Proof Let $X, \mathcal{A}$, etc. be as in Definition 7. We can assume, without loss of generality, that $X = \mathrm{Q} \cap [0, 1]$. Now define for $\alpha, \beta \in [0, 1]$, $\alpha * \beta = \sup\{x \circ y : x, y \in X, x \leq \alpha, y \leq \beta\}$. The proof of the weak commutativity, the monotonicity, associativity, left-continuity, etc. of $*$ is the same as that of [7, Theorem 4.6]. The neutral element of $*$ is e_X in $\mathrm{Q} \cap [0, 1]$. By the left-continuity of $*$, the following property holds.

(P) $\alpha, \beta, \gamma \in [0, 1]$, $\alpha * \beta * \gamma = \sup\{x \circ y \circ z : x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \gamma\}$.

We prove that $\alpha * \beta \leq e_X$ iff $\alpha * \beta * \beta \leq e_X$ for any α, β in $[0, 1]$. Given $\alpha * \beta \leq e_X$ then $x \circ y \leq e_X$ for all $x, y \in X, x \leq \alpha, y \leq \beta$. Let $x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \beta$. Then $x \circ y \leq e_X, x \circ z \leq e_X$. Thus $x \circ y \circ y \leq e_X, x \circ z \circ z \leq e_X$ by Lemma 5(e). Hence $x \circ y \circ z \leq \max\{x \circ y \circ y, x \circ z \circ z\} \leq e_X$. Therefore $\alpha * \beta * \beta \leq e_X$ by (P). The sufficient part of the claim can be proved in the similar way.

By Lemma 1, Definition 8, Lemma 4, we can prove the claims similar to Lemma 5 and 6 for $\mathbf{IUL}_\omega$-algebras. As a consequence of these lemmas, and extending [4, Theorem 3.3] in the obvious way, we obtain the following standard completeness.

Theorem 2 $\mathbf{UL}_\omega$ *and* $\mathbf{IUL}_\omega$ *are complete with respect to the class of standard algebras involved.*

4 Concluding remarks

Roughly speaking, the methodological significance of Jenei and Montagna's proof is that it does not require a complete understanding of the structure of the **MTL**-algebras by embedding a countable **MTL**-algebras into a dense one. It is indeed different from the proof of the **BL**'s standard completeness given by Hajek, Cignoli, Esteva, Godo, Torrens et al in [1,3]. The validation of the structure X in Definitions 7, 8 and Theorems 3.6, 3.7 is dependent on Lemma 1(ii) which claims that $stu = s$ implies $st = s$. However, we are unable to prove the condition that $stu = t$ implies $st = t$ in $\mathbf{HpsUL}_\omega^*$. It seems that we need to introduce some more strong axioms into $\mathbf{HpsUL}_\omega^*$ to guarantee its completeness with respect to finite (or standard) $\mathbf{HpsUL}^*$-algebras.

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.
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