Evaluation of the Komlos Conjecture Using Multi-Objective Optimization

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ABSTRACT

Komlos conjecture is about the existing of a constant upper bound over the dimension n of the function $K(n)$ defined by

$$K(n) = \max_{\{\vec{V_1},\dots,\vec{V_n}\}\in\left\{\vec{V}\in\mathbb{R}^n:\ \|\vec{V}\|_2\le 1\right\}^n}\left(\min_{\{\varepsilon_1,\dots,\varepsilon_n\}\in\{-1,1\}^n}\left\|\sum_{i=1}^n \varepsilon_i\vec{V_i}\right\|_\infty\right).$$

In this paper, the function $K(n)$ is evaluated first for lower dimensions, $n \le 5$, where it found that $K(2) = \sqrt{2}$, $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$, $K(4) = \sqrt{3}$, and $K(5) = \frac{4+\sqrt{142}}{9}$. For higher dimension, the function $f(n) = \sqrt{n - \lceil log_2(2^{n-1}/n)\rceil}$ is found to be a lower bound for the function $K(n)$, from where it is concluded that the Komlos conjecture is false i.e., the universal constant $k = \max_{n\in N} K(n)$ does not exist because of

$$\lim_{n\to\infty} K(n) \ge \lim_{n\to\infty}\sqrt{\log(n) - 1} = +\infty.$$

Keywords: Komlos Conjecture; optimization; discrepancy theory.

Introduction

J. Komlos has made the following conjecture: For a given dimension n, let $K(n)$ denote the minimum value such that: for any set of n Vectors $\vec{V_1}, \dots, \vec{V_n} \in \mathbb{R}^n$ with $\|\vec{V_i}\|_2 \le 1$,there exists weights $\varepsilon_i = +1\ or - 1$ such that

$$\left\|\sum_{i=1}^n \varepsilon_i\vec{V_i}\right\|_\infty \le K(n).$$

Kolmos has conjected the existence of a universal constant K such that $K(n) \le k$ for any dimension n. The l_2 and l_∞ norms in $\mathbb{R}^n$ are denoted by $\|.\|_2$ and $\|.\|_\infty$ respectively.

This conjecture was referred by Joel Spencer [8] in 1994, where he linked kolmos Conjecture to Spencer's famous Six Standard Deviation in 1985, see [9].

The main nontrivial result known, which is due to Joel Spencer [9], is that if $k \leq n$ then $\left\|\sum_{i=1}^{k} \varepsilon_i \vec{V}_i\right\|_\infty = O(\log(n))$. The main result of D. Hajela [4] was very close to disprove the Komlos conjecture, where precisely he has proved the following theorem:

THEOREM 1. Let $f(n)$ be a function that's goes to infinity when n goes to infinity with $f(n) = O(n)$ and let $0 < \lambda < 1/2$. Then form $n \geq n_0$ (where n_0 depends only on n and λ) and any $A \subseteq \{1, -1\}^n$ $with$ $|A| \leq 2^{n/f(n)}$, there are orthogonal vectors $x_1, \dots, x_n$ in R^n, $\|x_i\|_2 \leq 1$ for all $1 \leq i \leq n$, and such that

$$\|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n\|_\infty \geq \exp\left(\frac{\lambda \log\log f(n)}{\log\log\log f(n)}\right),$$

for all $(\varepsilon_1, \dots, \varepsilon_n) \in A$.

The previous theorem disproves the conjecture of Komlos over the set $A \subseteq \{1, -1\}^n$ where $|A| \leq 2^{n/f(n)}$. The proof of Theorem 1 is based on certain inequalities which arise in the geometry of convex bodies [1], [10], and [2].

Komlos Conjecture is also related to discrepancy theory, paper of J.Becka and T.Fiala [6], where it states that for a global constant K and for any $m \times n$ matrix A, whose columns are inside a unit ball, there exists a vector $X \in \{-1, +1\}^n$ such that $\|AX\|_\infty \leq K$.

The best progress in proving Komlos conjecture is a result given by Banaszczyk [11] who proved the bound

$$\min_{x \in \{-1,+1\}^n} \|AX\| \leq K\sqrt{\log(n)}$$

for a global constant.

This is the best known bound for Becka-Fiala conjecture as well [5].

Discrepancy is a challenging problem that has application in geometry, data analysis, and complexity theory. The books, J. Matousek [7], B. Chazelle [3], and J. Beck and V.T [5], provide references for a wide array of applications.

For lower dimension, the idea is to find a hypercube of minimum side of $2K$, where all vertices formed by different combinations of the weights, $\sum_{i=1}^{n} \varepsilon_i \vec{V}_i$, should be all inside the hypercube. Also It is not hard to show that $\sqrt{n}$ is an upper bound for the function $K(n)$. The proof can be carried out by funding find a particular weights ε_i^* such that all vectors $\overrightarrow{V_1}, \dots, \overrightarrow{V_n} \in \mathbb{R}^n$ with $\left\|\overrightarrow{V_i}\right\|_2 \leq 1$, so

$$\left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_\infty \leq K(n) \leq \sqrt{n}.$$

The below boulets are the details of the proof:

- We will prove first that $\left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_2 \leq \sqrt{n}$, which it is a sufficient condition to prove that $\left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_\infty \leq \sqrt{n}$.
- For dimension 2: the upper bound can be evaluated by using cosine rule as follows:

$$\begin{aligned} \left\|\varepsilon_1^*\vec{V}_1 + \varepsilon_2^*\vec{V}_2\right\|_2 &= \left\|\vec{V}_1 + \frac{\varepsilon_2^*}{\varepsilon_1^*}\vec{V}_2\right\|_2 \\ &= \sqrt{\left\|\vec{V}_1\right\|_2^2 + \left\|\vec{V}_2\right\|_2^2 - 2\left\|\vec{V}_1\right\|_2\left\|\vec{V}_2\right\|_2 \cos\left(\vec{V}_1, \vec{V}_2\right)} \\ &\leq \sqrt{2}, \end{aligned}$$

where the weight $\frac{\varepsilon_2^*}{\varepsilon_1^*}$ is chosen in order to have $\cos\left(\vec{V}_1, \vec{V}_2\right) \geq 0$.

- If we suppose that $\left\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i\right\|_2 \leq \sqrt{n-1}$, we need to prove that $\left\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i + \varepsilon_n^* \vec{V}_n\right\|_2 \leq \sqrt{n}$. Again by cosine rule, we can write the following:

$$\begin{aligned} \left\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i + \varepsilon_n^* \vec{V}_n\right\|_2 &\leq \sqrt{\left\|\vec{V}_n\right\|_2^2 + \left\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i\right\|_2^2} \\ &\leq \sqrt{1+n-1} \\ &\leq \sqrt{n}. \end{aligned}$$

where the weight ε_n^* is chosen in order to have $\cos\left(\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i,\ \varepsilon_n^* \vec{V}_n\right) \geq 0$

- From the principle of induction proof, it is concluded that all vectors $\vec{V_1}, \dots, \vec{V_n} \in \mathbb{R}^n$ with $\left\|\vec{V_i}\right\|_2 \leq 1$, the weights can find ε_i^* such that

$$\left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_2 \leq \sqrt{n}.$$

Since $\left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_\infty \leq \left\|\sum_{i=1}^{n} \varepsilon_i^* \vec{V}_i\right\|_2$, we can conclude that the function $K(n)$ has an upper bound of order $\sqrt{n}$.

We can extend the Komlos conjecture statement to the below lemma, where it summarizes very interesting properties related to special vectors, $\vec{V_i}^*$, $i = 1, \dots, n$, that cannot cancel each other further than $K(n)$.

Lemma 1: Let C^n be a set of vectors in $\mathbb{R}^n$ have l_2 norm at most 1, and we denote by V^* as a set of vectors in $\mathbb{R}^n$ that satisfies $V^* = \left\{\vec{V_1}^*, \dots, \vec{V_n}^*\right\} = \underset{\vec{V}_i \in C^n}{\operatorname{argmax}} \min_{\varepsilon_i} \left\|\sum_{i=1}^{n} \epsilon_i \vec{V_i}^*\right\|_\infty$

The set V^* satisfies the below properties:

i. For any vector $\vec{V_i}^*$ in V^* has l_2 norm equal to 1, $\left\|\vec{V_i}^*\right\|_2 = 1$.

ii. All the vertices have the same distance l_∞, i.e.,
$\underset{\varepsilon_i \in \{-1,+1\}}{\text{Min}} \left\|\sum_{i=1}^{n} \varepsilon_i \vec{V_i}^*\right\|_\infty = \underset{\varepsilon_i \in \{-1,+1\}}{\text{Max}} \left\|\sum \varepsilon_i \vec{V_i}^*\right\|_\infty.$

iii. $K(n)$ is strictly increasing sequence, i.e., for all integers $m > n$ implies that $K(n) < K(m)$.

A proof of the previous lemma will be publish soon in order to prove that $K(n) \sim \sqrt{log_2(n)}$.

The following sections are consecrated to evaluate the function K for a different dimension, the exact value of K will be calculated for a dimension less or equals to 5 and a lower bound will evaluated for any dimension n.

Evaluation of $K(2)$:

It is obvious to see that the constant $K(1)$ for dimension one is equal to one, and it is quite easy to calculate $K(2)$ by using some basic rules in geometry.

To find the value of $K(2)$, it will be useful to analyze the parallelogram formed by four vertices centered at the origin, resulting from the four combinations $\mp\vec{V}_1 \mp \vec{V}_2$ (see Figure1)

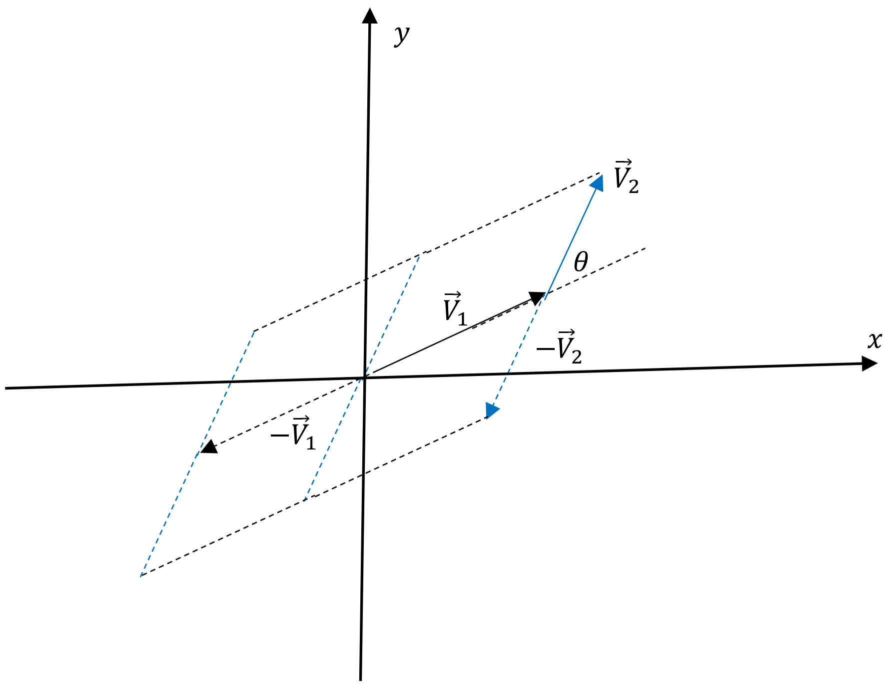

Figure 1. A parallelogram formed by four vertices, $\vec{V}_1 + \vec{V}_2, \vec{V}_1 - \vec{V}_2, -\vec{V}_1 + \vec{V}_2, -\vec{V}_1 - \vec{V}_2$.

By using the cosine rule, we can find the length of the big and the small diagonals respectively as follows:

$$\begin{cases} L^2 = \|\vec{V}_1\|_2 + \|\vec{V}_2\|_2 + 2\|\vec{V}_1\|_2\|\vec{V}_2\|_2 \cos(\theta) \\ l^2 = \|\vec{V}_1\|_2 + \|\vec{V}_2\|_2 - 2\|\vec{V}_1\|_2\|\vec{V}_2\|_2 \cos(\theta) \end{cases},$$

where θ is the acute angle between the two vectors $\vec{V}_1$ and $\vec{V}_2$.

We can notice that the small diagonal l has $\sqrt{2}$ as an upper bound, i.e.,

$$\sqrt{\|\vec{V}_1\|_2 + \|\vec{V}_2\|_2 - 2\|\vec{V}_1\|_2\|\vec{V}_2\|_2 \cos(\theta)} \leq \sqrt{\|\vec{V}_1\|_2 + \|\vec{V}_2\|_2} \leq \sqrt{2}\,.$$

As it was mentioned before, the two weights, ε_1 and ε_2 , can be chosen in such a way the length of $\varepsilon_1\vec{V}_1 + \varepsilon_2\vec{V}_2$ is smaller than the length of diagonal l, in which it implies that for all vectors $\vec{V}_i$ inside the circle of center (0,0) and Radius =1, we can find ε_1 and ε_2 such that $\|\varepsilon_1\vec{V}_1 + \varepsilon_2\vec{V}_2\|_\infty \leq K(n) \leq \sqrt{2}$.

In different method the prove of $K(2) \leq \sqrt{2}$ can be carried out by using technic of proof by contradiction. Let's assume the case where the vertices A, B, C, and D are located outside of the red square of side $2\sqrt{2}$ as it is shown in Figure 2.

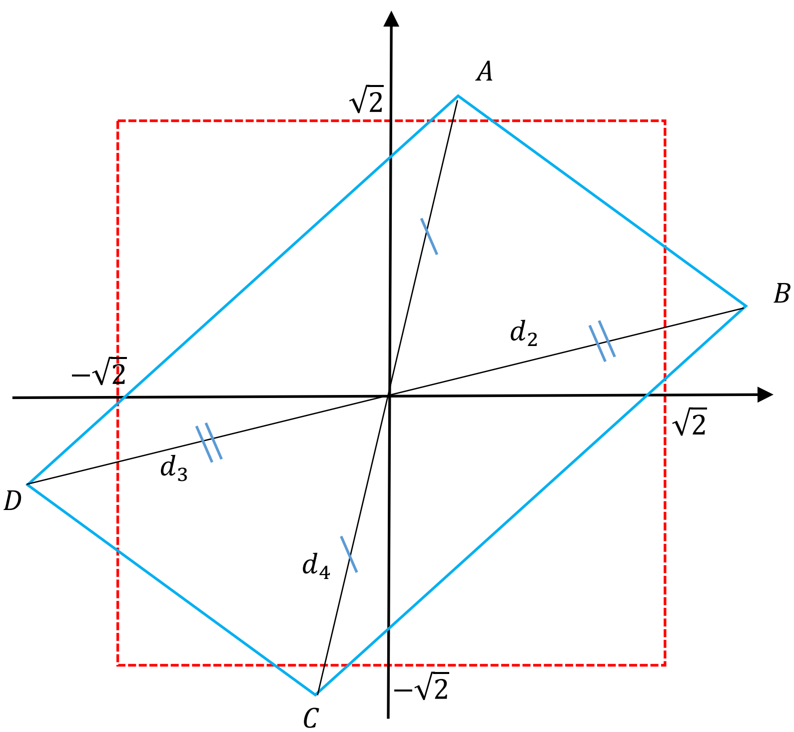

Figure 2. ABCD is a parallelogram with for vertices located outside of square whose side is $\sqrt{2}$ *has* $\min_i(d_i) \geq \sqrt{2}$.

The possibility of having all vertices,$\mp\vec{V}_1 \mp \vec{V}_2$, outside the red square in Figure 2 is impossible! Because it contradicts with the fact that small diagonal length is at less or eqal t0 $\sqrt{2}$.

From previous proof, we can conclude that $K(2) \leq \sqrt{2}$, and it is enough to find a particular case where $\min_{\varepsilon_i}\|\varepsilon_1\vec{V}_1 + \varepsilon_2\vec{V}_2\|_\infty = \sqrt{2}$ in order to prove that $K(2) = \sqrt{2}$.

Lt's considerthe case where $\vec{V}_1 = \frac{1}{\sqrt{2}}\binom{+1}{+1}$ and $\vec{V}_2 = \frac{1}{\sqrt{2}}\binom{-1}{+1}$, so for any possible values of ε_1 and ε_2, we can calculate the value $\min_{\varepsilon_i}\|\varepsilon_1\vec{V}_1 + \varepsilon_2\vec{V}_2\|_\infty$ as follows:

$$\begin{aligned}\max\left(\frac{|\varepsilon_1 + \varepsilon|}{\sqrt{2}}, \frac{|\varepsilon_1 - \varepsilon_2|}{\sqrt{2}}\right) &= \max\left(\frac{\left|1 + {}^{\varepsilon_2}/_{\varepsilon_1}\right|}{\sqrt{2}}, \frac{\left|1 - {}^{\varepsilon_2}/_{\varepsilon_1}\right|}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\max\left(1 + \left|{}^{\varepsilon_2}/_{\varepsilon_1}\right|, 1 - \left|{}^{\varepsilon_2}/_{\varepsilon_1}\right|\right) \\ &= \frac{2}{\sqrt{2}}.\end{aligned}$$

Therefore, we can conclude that $K(2) = \sqrt{2}$.

Evaluation of $K(3)$:

Given the vector space $\mathbb{R}^3$, the span of the set S of finite vectors is defined as the set of all linear combinations of the vectors in S, noted as follows:

$$Span(S) = \left\{\sum_{i=1}^{k} \alpha_i\vec{V}_i\, ; k \in \mathbb{N}, \vec{V}_i \in S, \alpha_i \in \mathbb{R}\right\}.$$

The calculation of $K(3)$ will be splited to several cases related to different configurations of the three vectors $\vec{V}_1, \vec{V}_2$, and $\vec{V}_3$ in $\mathbb{R}^3$.

<u>Case 1</u>: $\vec{V}_3 \perp Span(\vec{V}_2, \vec{V}_1)$ and $Span(\vec{V}_2, \vec{V}_1)$ = x-y plane.

As the vector $\vec{V}_3$ is parallel to y-axe, then without losing generality, we can write the following:

$$\min_{\varepsilon_i}\|\vec{\varepsilon}_3\vec{V}_3 + \vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1\|_\infty = \min_{\varepsilon_i}\|\vec{V}_3 + \vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1\|_\infty$$

by consequence,

$$\min_{\varepsilon_i}\|\vec{V}_3 + \vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1\|_\infty = max\left\{\|\vec{V}_3\|_2, \min_{\varepsilon_i}\|\vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1\|_\infty\right\}.$$

From previous section, we know that the constant $K(2) = \sqrt{2}$ and from the fact that $\vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1 \in x$-$y$ plane, we have

$$\begin{aligned}max\left\{\|\vec{V}_3\|_2, \min_{\varepsilon_i}\|\vec{\varepsilon}_2\vec{V}_2 + \vec{\varepsilon}_1\vec{V}_1\|_\infty\right\} &\le max\left\{\|\vec{V}_3\|_2, \sqrt{2}\right\} \\ &\le \sqrt{2}.\end{aligned}$$

By considering $\vec{V}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}+1\\+1\\0\end{pmatrix}$, $\vec{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\+1\\0\end{pmatrix}$, and $\vec{V}_3 = \begin{pmatrix}0\\0\\+1\end{pmatrix}$, $K(3)$ can be calculated as follows:

$$\left\|\sum_{i=1}^{3} \varepsilon_i \vec{V_i}\right\|_\infty = \max\left(\frac{|\varepsilon_1+\varepsilon_2|}{\sqrt{2}}, \frac{|\varepsilon_1-\varepsilon_2|}{\sqrt{2}}, |\varepsilon_3|\right)$$
$$= \max\left(\frac{|1+\varepsilon_2|}{\sqrt{2}}, \frac{|1-\varepsilon_2|}{\sqrt{2}}, 1\right)$$
$$= \sqrt{2}.$$

Therefore, under the case 1, the value $K(3)$ is equal to $\sqrt{2}$.

Case 2: $Span(\vec{V}_1, \vec{V}_2) = x\text{-}y$ plane.

We split the vector $\vec{V}_3$ as follows:

$\vec{V}_3 = \vec{V}_{31} \oplus \vec{V}_{32}$, where $\vec{V}_{31} \perp$ XY-plane. Without losing generality, the value of the weight ε_3 can be fixed to 1 in our calculation.

Therefore, for all vectors $\vec{V_1}, \vec{V_2}, \vec{V_3} \in \mathbb{R}^3$ with $\|\vec{V_i}\|_2 \leq 1$

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^{3} \varepsilon_i \vec{V_i}\right\|_\infty = \min_{\varepsilon_i}\left\|\vec{V}_3 + \sum_{i=1}^{2} \varepsilon_i \vec{V_i}\right\|_\infty$$
$$= \max\left\{\|\vec{V}_{31}\|_2, \min_{\varepsilon_i}\left\|\vec{V}_{32} + \sum_{i=1}^{2} \varepsilon_i \vec{V_i}\right\|_\infty\right\}$$

From previous equation, we can see that the calculation is moved from dimension 3 to dimension 2 by just calculating the following:

For all vectors $\vec{V_1}, \vec{V_2}, \overrightarrow{V_{32}} \in \mathbb{R}^2$ with $\|\vec{V_i}\|_2 \leq 1$, the below maximum is needed to be calculated

$$\max_{\vec{V_i}\in\mathbb{R}^3:\, \|\vec{V_i}\|_2 \leq 1} \min_{\varepsilon_i}\left\|\vec{V}_{32} + \sum_{i=1}^{2} \varepsilon_i \vec{V_i}\right\|_\infty$$

where $\vec{V}_{32} = \begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}$, and without losing generality, the two assumption $\alpha^2 + \beta^2 \leq 1$ and $0 \leq \alpha \leq \beta \leq 1$ can be added.

To evaluate the value $K(3)$, the question about the possibility to have all the vertices, $\vec{V}_{32} \pm \vec{V}_2 \pm \vec{V}_1$, outside the square of side $2\sqrt{2}$, as it shown in Figure 3, needs to be checked.

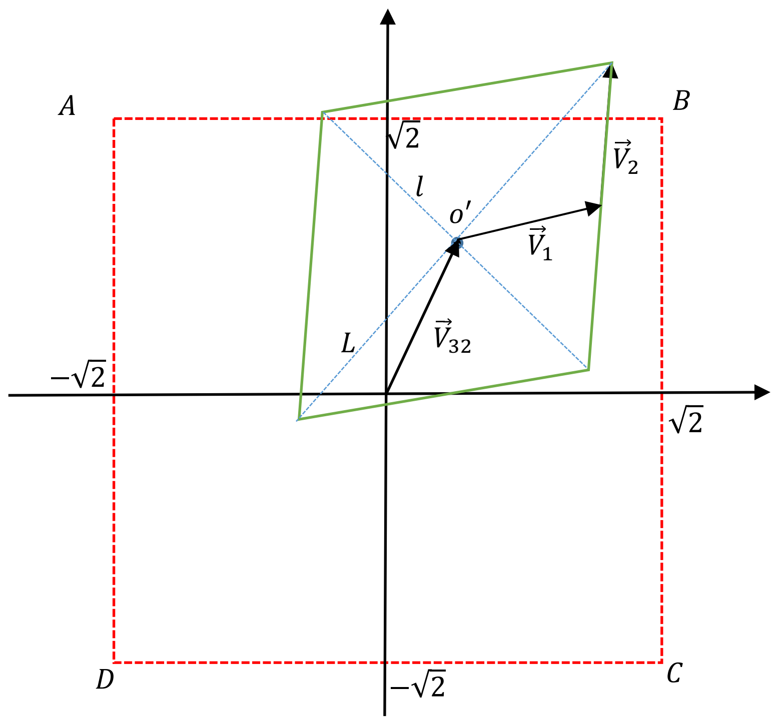

Figure 3 *Without losing generality, the vector* $\vec{V}_{32}$ *can be consider with slope bigger than one,* $m \geq 1$*. The two distances* l *and L are the length of the small and the big diagonal respectively.*

From the Figure 4, the small diagonal, l, of parallelogram centered at the point O' is at most equal to $\sqrt{2}$, consequently, we have to focus only on the green area, highlighted in Figure 5, the possible location of two opposite vertices that form the two small diagonal l.

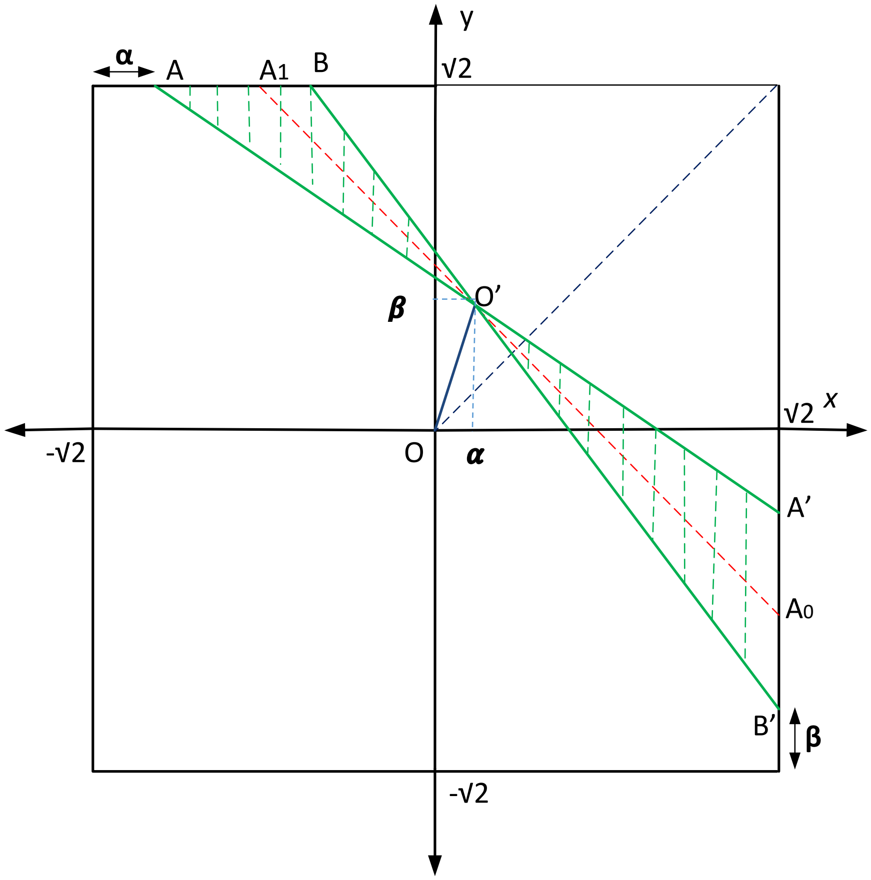

Figure 4 Green Area is the only possible location of the vertices, $\overrightarrow{V_{32}} \mp \overrightarrow{V_1} \mp \overrightarrow{V_2}$ *, in order to have maybe* $\left\|\overrightarrow{V_{32}} \mp \overrightarrow{V_1} \mp \overrightarrow{V_2}\right\|_{\infty} > \sqrt{2}$

The distance between the point O' and the midpoint of any two adjacent vertices is equals to either $\left\|\vec{V}_1\right\|$ or $\left\|\vec{V}_2\right\|$, which implies that the impossibility to have, on one side of square, two vertices outside of square, refer to Figure 4.

This impossibility can be proved by highlighting the fact that the distance between any point inside the area S_1 and any point inside the area S_2 is bigger than or equal to 1, see Figure 5.

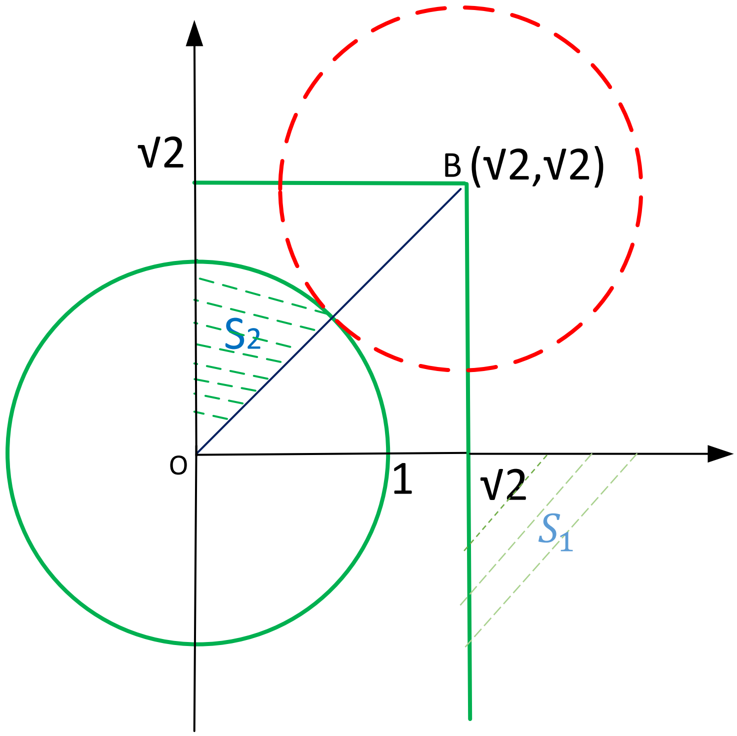

Figure 5. Distance between any point in the area S_1 and any point in the area S_2 has minimum distance of 1, $S_1 = \{x \geq \sqrt{2}\ and\ y \leq 0\}$ and $S_2 = \{x^2 + y^2 \leq 1\ and\ 0 \leq x \leq y \leq 1\}$, where the red and green circles have radius of 1 and centred at the point B and O respectively.

Therefore, under the case 2, the constant $K(3)$ is upper bounded by $\sqrt{2}$.

To conclude that $K(3) = \sqrt{2}$, it is enough to check the function $K(3)$ for $\vec{V}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}+1\\+1\\0\end{pmatrix}$, $\vec{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\+1\\0\end{pmatrix}$, and $\vec{V}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\+1\end{pmatrix}$, where

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^{3}\varepsilon_i\vec{V}_i\right\|_\infty = \sqrt{2}.$$

<u>Case 3:</u> General case

By symmetry, without losing generality, we can consider the weight $\varepsilon_3 = 1$ in our calculations, as it is proven below

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^{3}\varepsilon_i\vec{V}_i\right\|_\infty = \min_{\varepsilon_i}\left\|\varepsilon_3\sum_{i=1}^{3}\frac{\varepsilon_i}{\varepsilon_3}\vec{V}_i\right\|_\infty$$
$$= \min_{\varepsilon_i}\left\|\vec{V}_3+\sum_{i=1}^{2}\frac{\varepsilon_i}{\varepsilon_3}\vec{V}_i\right\|_\infty$$
$$= \min_{\varepsilon_j}\left\|\vec{V}_3+\sum_{j=1}^{2}\varepsilon_j\vec{V}_i\right\|_\infty .$$

The vector $\sum_{i=1}^{2}\varepsilon_i\vec{V}_i$ will be evaluated over two perpendicular spaces, x-y plane and z -axe, and a link between the two space will be found in order to maximize the $\left\|\sum_{i=1}^{3}\varepsilon_i\vec{V}_i\right\|_\infty$.

The projection of the vector $\vec{V}_i$ over the space x-y plane, $Proj_{\,XY-plan}(\vec{V}_i)$, is denoted by vector $\vec{U}_i$.

From case 2 , we have proven that it is not possible to have all vertices, $\vec{U}_3 \pm \vec{U}_2 \pm \vec{U}_1$, outside the square of side $2\sqrt{2}$ centered at the origin. An important question rises about the possibility to increase the l_∞ norm beyond $\sqrt{2}$ for two vertices and compensate the l_∞ norms of the two other vertices by l_∞ norm over z-axe?

To answer the previous question, we need to find z-coordinates of three vectors $\vec{V}_i$ that satisfy the following statement:

For each possible weight's vector $(1,\varepsilon_1,\varepsilon_2)$ where $\left\|\vec{U}_3+\sum_{j=1}^{2}\varepsilon_j\vec{U}_i\right\|_\infty < \sqrt{2}$ then

$$\left\|Z_3+\sum_{j=1}^{2}\varepsilon_j Z_i\right\|_\infty = \left|Z_3+\sum_{j=1}^{2}\varepsilon_j Z_i\right| > \sqrt{2}\ .$$

To summarize the above idea, we create an example of vectors $\vec{V}_i$, where $\left\|\sum_{i=1}^{3}\varepsilon_i\vec{V}_i\right\|_\infty > \sqrt{2}$, as follows:

$$\vec{V}_1=\begin{pmatrix}-x_1\\ y_1\\ z_1\end{pmatrix}, \vec{V}_2=\begin{pmatrix}-x_2\\ y_2\\ -z_2\end{pmatrix}\text{, and } \vec{V}_3=\begin{pmatrix}x_3\\ y_3\\ z_3\end{pmatrix},$$

where x_i , y_i, z_i are all non-negative value with $\left\|\vec{V}_i\right\|_2 = \sqrt{{x_i}^2+\ {y_i}^2+\ {z_i}^2} \le 1$,

We assume the following equations

$$if\ \begin{cases}\left\|\vec{V}_3-\vec{V}_2-\vec{V}_1\right\|_\infty = x_1+x_2+x_3 = K_1\\ \left\|\vec{V}_3+\vec{V}_2+\vec{V}_1\right\|_\infty = y_1+y_2+y_3 = K_2\end{cases}$$

$$then\ \begin{cases}\left\|\vec{V}_3-\vec{V}_2+\vec{V}_1\right\|_\infty = z_1+z_2+z_3 = K_3\\ \left\|\vec{V}_3+\vec{V}_2-\vec{V}_1\right\|_\infty = |z_3-z_1-z_2| = K_4.\end{cases}$$

By symmetry, we can consider the constants $K_1\ = K_2$ and $K_3\ = K_4$.

Then the system that is needed to be solved is summarized by the following equations:

$$\begin{cases} x_1 + x_2 + x_3 = K_1 \\ y_1 + y_2 + y_3 = K_1 \\ z_1 + z_2 + z_3 = K_3 \\ z_1 + z_2 - z_3 = K_3 \end{cases}$$

From the last two equations, we conclude that

$$z_3 = 0$$

By symmetry, we can conclude that

$$x_3 = y_3$$

Since $\|\vec{V}_3\| \leq 1$, it is convenient to increase x_3& y_3 as much as we can in order to maximum the value of K_1, where it can be found when the coordinates of $\vec{V}_3$ are :

$$x_3 = y_3 = \frac{1}{\sqrt{2}}$$

Therefore, the system will be simplified again as follows

$$\begin{cases} x_1 + x_2 = K_1 - \frac{\sqrt{2}}{2} \\ y_1 + y_2 = K_1 - \frac{\sqrt{2}}{2} \\ z_1 + z_2 = K_3 \end{cases}$$

Again by symmetry, we can consider the following equations:

$$x_1 = y_1 = \alpha$$

$$x_2 = y_2 = \beta$$

$$z_1 = z_2 = \gamma$$

In order to maximize K and by symmetry, we need to impose that $K_1 = K_3$, then the final system that need to be solved is as follows:

$$\begin{cases} \alpha + \beta = K - \frac{\sqrt{2}}{2} \\ \gamma = \frac{K}{2} \\ \alpha^2 + \beta^2 + \gamma^2 \leq 1 , \end{cases}$$

the last inequality comes from the constraint that $\|\vec{V}_i\|_2 \leq 1$, for *i*=1,2.

Again, without losing generality, we can assume that $\alpha = \beta$, The maximum value of K can be calculated by

$$\begin{cases} \alpha = \frac{K}{2} - \frac{\sqrt{2}}{4} \\ \gamma = \frac{K}{2} \\ 2\alpha^2 + \gamma^2 = 1 \end{cases}$$

So, we end up to solve the below quadratic equation

$$2\left(\frac{K}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K}{2}\right)^2 = 1$$

After simplification, we find:

$$3K^2 - 2\sqrt{2}K - 3 = 0$$

The solution of previous quadratic equation is when the value K is equal to $\frac{\sqrt{2}+\sqrt{11}}{3}$.

Hence

$$K(3) \geq \frac{\sqrt{2}+\sqrt{11}}{3}.$$

A simulation is used to answer the question if the value $K(3)$ is eqal to $\frac{\sqrt{2}+\sqrt{11}}{3}$ or not. A cylindrical Coordinate has been used in our simulation to check most of the cases, the possible coordinate's values of the vector $\vec{V_i}$ are summarized as follows:

$$x = r\, cos(\theta)\sin(\alpha)$$

$$y = r\, sin(\theta)\sin(\alpha)$$

$$z = r\, cos\,(\alpha)$$

where $\theta = [start\ value: step: end\ value] = [0: 0.001: 2\pi]$, $\alpha = [start\ value: step: end\ value] = [0: 0.001: 2\pi]$, and $r = [start\ value: step: end\ value] = [0: 0.01: 1]$.

The simulation has showed that the value $K(3)$ is equl to $\frac{\sqrt{2}+\sqrt{11}}{3}$ i.e. $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$.

The constant $K(4)$

Before giving the approach for dimension 4, we will review the evaluation $K(3)$ for dimension 2 and 3 in a different ways in order to be generalized later on.

For dimension 2, we denote by $\vec{V_1} = \binom{\alpha_1}{\beta_1}$ and $\vec{V_2} = \binom{\alpha_2}{\beta_2}$ the two particular vectors that verified

$$\min_{\varepsilon_1}\left\|\vec{V}_1+\varepsilon_2\vec{V}_2\right\|_\infty=K(2).$$

By symmetry, we can assume that

$$\left\|\vec{V}_1+\vec{V}_2\right\|_\infty=\alpha_1+\alpha_2=K(2)$$

$$\left\|\vec{V}_1-\vec{V}_2\right\|_\infty=\beta_1-\beta_2=K(2)$$

From the definition of $K(2)$, to get the maximum value of it, the coordinates of the two vectors should be non-negatives values except the coordinate β_2 should be a negative value.

By symmetry, we denote $\alpha_1 = \alpha_2=\alpha$ & $\beta_1=-\beta_2=\beta$.

to find $K(2)$, it is a enough to solve the following system:

$$\begin{cases}2\alpha=K(2)\\2\beta=K(2)\end{cases}$$

under the constraint $\alpha^2+\beta^2\le 1$.

The maximum $K(2)$ can be found by considering $\alpha^2+\beta^2=1$, so the previous system is equivalent to the following quadratic equation

$$\left(\frac{K(2)}{2}\right)^2+\left(\frac{K(2)}{2}\right)^2=1$$

Therefore

$$K(2)=\sqrt{2}.$$

For dimension 3 , we would like to find $\vec{V}_1=\begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\end{pmatrix}$, $\vec{V}_2=\begin{pmatrix}\alpha_2\\\beta_2\\\gamma_2\end{pmatrix}$ and $\vec{V}_3=\begin{pmatrix}\alpha_3\\\beta_3\\\gamma_3\end{pmatrix}$ that verify

$$K(3)=\min_{\varepsilon_1,\varepsilon_2}\left\|\vec{V}_3+\varepsilon_2\vec{V}_2+\varepsilon_1\vec{V}_1\right\|_\infty.$$

All possible cases of the vector $\vec{\varepsilon}=\begin{pmatrix}\varepsilon_1\\\varepsilon_2\\1\end{pmatrix}$ will be gathered under a matrix named A_3, where its rows r_i form all cases of the vector $\vec{\varepsilon}$.

The matric A_3 is defined as follows:

$$A_3=\begin{bmatrix}+1&+1&+1\\+1&-1&+1\\+1&+1&-1\\+1&-1&-1\end{bmatrix}.$$

The four rows are not independent vectors because it is noted that $r_4 = r_2+r_3-r_1$.

By symmetry, we can assume the following equations

$$\left\|\vec{V}_3+\vec{V}_2+\vec{V}_1\right\|_\infty=|\alpha_3+\alpha_2+\alpha_1|=K(3)$$

$$\left\|\vec{V}_3+\vec{V}_2-\vec{V}_1\right\|_\infty=|\beta_3+\beta_2-\beta_1|=K(3)$$

$$\left\|\vec{V}_3-\vec{V}_2+\vec{V}_1\right\|_\infty=|\gamma_3-\gamma_2+\gamma_1|=K(3)$$

$$\left\|\vec{V}_3-\vec{V}_2-\vec{V}_1\right\|_\infty=|\gamma_3-\gamma_2-\gamma_1|=K(3)$$

In order to maximize the value of $K(3)$, it is suitable to consider the coordinates β_1, γ_1, and γ_2 as a negative values, so the coordinate of the three vectors $\vec{V}_i$ will be summarized as following

$$\vec{V}_1=\begin{pmatrix}\alpha_1\\-\beta_1\\-\gamma_1\end{pmatrix},\ \vec{V}_2=\begin{pmatrix}\alpha_2\\\beta_2\\-\gamma_2\end{pmatrix}\text{ and }\vec{V}_3=\begin{pmatrix}\alpha_3\\\beta_3\\\gamma_3\end{pmatrix},$$

where all parameters, $(\alpha_i\,,\beta_i\,,\gamma_i)$ are non-negative values.

To calculate $K(3)$, it is enough to solve the below system:

$$\begin{cases}\alpha_3+\alpha_2+\alpha_1=K(3)\\\beta_3+\beta_2+\beta_1=K(3)\\\gamma_3+\gamma_2+\gamma_1=K(3)\\-\gamma_3+\gamma_2+\gamma_1=K(3)\end{cases}$$

Under the constraints

$${\alpha_1}^2+{\beta_1}^2+{\gamma_1}^2\leq 1$$

$${\alpha_2}^2+{\beta_2}^2+{\gamma_2}^2\leq 1$$

$${\alpha_3}^2+{\beta_3}^2+{\gamma_3}^2\leq 1$$

From the last two equations of the system we can conclude that $\gamma_3=0$.

By symmetry also, we can assume that

$$\alpha_1=\alpha_2=\beta_2=\beta_1=\beta$$

$$\gamma_2=\gamma_1=\gamma$$

$$\alpha_3=\beta_3=\alpha$$

Therefore,

$$\begin{cases}2\beta+\alpha=K(3)\\\quad 2\gamma=K(3)\end{cases}$$

Under the constraints

$$2\beta^2+\alpha^2\leq 1$$

$$2\alpha^2\leq 1$$

In order to maximize the value of $K(3)$, the two constraints can be considered as

$$2\beta^2 + \alpha^2 = 1$$

$$2\alpha = 1$$

Then the system will be simplified as follows

$$\begin{cases} 2\beta = K(3) - \dfrac{\sqrt{2}}{2} \\ 2\gamma = K(3) \end{cases}$$

The below quadratic equation is needed to be solved to calculate the value $K(3)$,

$$2\left(\frac{K(3)}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K(3)}{2}\right)^2 = 1\ .$$

After simplification, the quadratic equation can be as follows:

$$3K(3)^2 - 2\sqrt{2}K(3) - 3 = 0$$

As a consequence, it concludes that

$$K(3) = \frac{\sqrt{2} + \sqrt{11}}{3}.$$

The particular vectors that cannot canceled each other further than $K(3)$ are define as follows:

$$\vec{V}_1 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{-K(3)}{4} + \frac{\sqrt{2}}{4} \\ -\frac{K(3)}{2} \end{pmatrix},\ \vec{V}_2 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ -\frac{K(3)}{2} \end{pmatrix} \text{ and } \vec{V}_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}.$$

Note that these particular three vectors are not unique solution of $\arg_{\vec{V}_i} \min_{\varepsilon_i} \left\| \sum \varepsilon_i \vec{V}_i \right\|_\infty$.

The idea is to generalize the previous approach in evaluating the function $K(n)$, for that let's denote by $V = \{\vec{V}_4, \vec{V}_3, \vec{V}_2, \vec{V}_1\}$ the set of particular vectors that satisfy the below equation

$$K(4) = \min_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \left\| \vec{V}_4 + \sum_{i=1}^{3} \varepsilon_i \vec{V}_i \right\|_\infty .$$

The previous matrix A_3 can be extended to matrix A_4 in order to fit all possible 4-dimension vectors $(1,\ \varepsilon_1, \varepsilon_2, \varepsilon_3)$ as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 \end{bmatrix},$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$, and $\dim\big(span(r_1, r_2, r_3)\big) = 3$.

The idea is to well assign each row r_i to one of fourth dimension in order to avoid zero coordinate in $\vec{V_i}$, which it is a consequence of maximizing the value of $K(4)$, i.e., the axes where l_∞ norm of $\vec{V_4} + \sum_{i=1}^{3} \varepsilon_i \vec{V_i}$ is located will be distributed over all possible combinations of $(1,\ \varepsilon_1, \varepsilon_2, \varepsilon_3)$ in a way to maximize the value $K(4)$.

The below diagram, in Figure 6, identifies which coordinate will be eliminated, being zero, when we associate two rows to same axes.

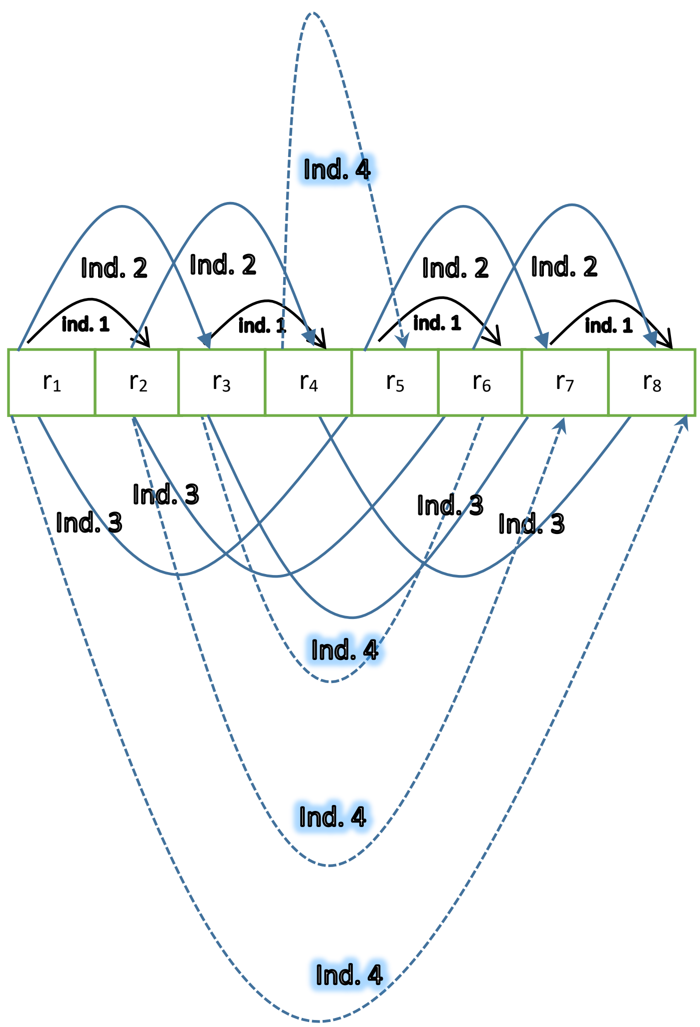

Figure 6 A link between gathering two rows as system of equation and the index variable that will be eliminated.

From previous diagram, the rows are gathered as follows

- (r_1, r_2): $\left\|\vec{V}_4 \pm \vec{V}_3 + \vec{V}_2 + \vec{V}_1\right\|_\infty = \left|V_4^1 \pm V_3^1 + V_2^1 + V_1^1\right| = K(4)$
- (r_5, r_7): $\left\|\vec{V}_4 + \vec{V}_3 \pm \vec{V}_2 - \vec{V}_1\right\|_\infty = \left|V_4^2 + V_3^2 \pm V_2^2 - V_1^2\right| = K(4)$
- (r_4, r_8): $\left\|\vec{V}_4 - \vec{V}_3 - \vec{V}_2 \pm \vec{V}_1\right\|_\infty = \left|V_4^3 - V_3^3 - V_2^3 \pm V_1^3\right| = K(4)$
- (r_3, r_6): $\left\|\pm\vec{V}_4 + \vec{V}_3 - \vec{V}_2 + \vec{V}_1\right\|_\infty = |\pm V_4^4 + V_3^4 - V_2^4 + V_1^4| = K(4)$,

where V_i^j is the jth coordinate of vector $\vec{V}_i$.

In order tomaximize the value $K(4)$, the coordinate's sign of $\vec{V}_i$ can be found as follows:

$$\vec{V}_1 = \begin{pmatrix} \alpha_1 \\ -\beta_2 \\ \gamma_2 \\ w_2 \end{pmatrix},\ \vec{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ -\gamma_2 \\ -w_2 \end{pmatrix},\ \vec{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ -\gamma_3 \\ w_4 \end{pmatrix} \text{ and } \vec{V}_4 = \begin{pmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \\ w_4 \end{pmatrix},$$

where α_i , β_i , γ_i and w_i are non-negative values.

The negative sign highlighted at the coordinate of $\vec{V}_i$ comes from rows $r_1, r_5, r_4, and\ r_3$. For instance, if we assume that $\left\|\vec{V}_4 + \sum_{i=1}^{3} \varepsilon_i \vec{V}_i\right\|_\infty = \left|\sum_{i=1}^{3} \varepsilon_i \alpha_i + \alpha_4\right|$, where $r_4 = (1, -1, -1, 1)$, in order to get maximum value of $K(4)$, it is preferable to consider the two coordinates α_2 and α_3 as negative values such that the equation $1(\alpha_1) - 1(\alpha_2) - 1(\alpha_3) + 1(\alpha_4) = K(4)$ will be equivalent to the equation

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = K(4).$$

The rows distribution can formulated by the following systems of equations

$$(r_1, r_2): \begin{cases} \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K \\ \alpha_4 - \alpha_3 + \alpha_2 + \alpha_1 = K \end{cases} \Rightarrow \alpha_3 = 0$$

$$(r_5, r_7): \begin{cases} \beta_4 + \beta_3 + \beta_2 + \beta_1 = K \\ \beta_4 + \beta_3 - \beta_2 + \beta_1 = K \end{cases} \Rightarrow \beta_2 = 0$$

$$(r_4, r_8): \begin{cases} \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K \\ \gamma_4 + \gamma_3 + \gamma_2 - \gamma_1 = K \end{cases} \Rightarrow \gamma_1 = 0$$

$$(r_3, r_6): \begin{cases} w_4 + w_3 + w_2 + w_1 = K \\ -w_4 + w_3 + w_2 + w_1 = K \end{cases} \Rightarrow w_4 = 0$$

The system can be simplified further by

$$\begin{cases} \alpha_4 + \alpha_2 + \alpha_1 = K \\ \beta_4 + \beta_3 + \beta_1 = K \\ \gamma_4 + \gamma_3 + \gamma_2 = K \\ w_3 + w_2 + w_1 = K \end{cases}$$

Under the below constraints

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ w_4^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \alpha_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases}$$

As before, the previous system of equations needs to be matched with coordinates of the four vectors in order to maximize the value of $K(4)$, then

$$[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_3] = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & \alpha_4 \\ -\beta_1 & 0 & \beta_3 & \beta_4 \\ 0 & -\gamma_2 & -\gamma_3 & \gamma_4 \\ w_1 & -w_2 & w_3 & 0 \end{bmatrix}.$$

By symmetry, we can assume that

$$\alpha_4 = \alpha_2 = \alpha_1 = \alpha$$

$$\beta_4 = \beta_3 = \beta_1 = \beta$$

$$\gamma_4 = \gamma_3 = \gamma_2 = \gamma$$

$$w_3 = w_2 = w_1 = w$$

Therefore

$$\begin{cases} \alpha = K/3 \\ \beta = K/3 \\ \gamma = K/3 \\ w = K/3 \end{cases}$$

To maximize K, the constraints can be assumed to be as follows:

$$\begin{aligned} 1 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= \alpha^2 + \beta^2 + w^2 \\ &= \alpha^2 + w^2 + \gamma^2 \\ &= w^2 + \beta^2 + \gamma^2 \end{aligned}$$

To find the value of $K(4)$, it is enough to solve the below quadrature equation

$$\alpha^2 + \beta^2 + \gamma^2 = 3\left(\frac{K}{3}\right)^2 = 1.$$

It implies that

$$K(4) \geq \sqrt{3},$$

and the coordinates of the particular set of vectors $\vec{V_i}$ are summarized under the below matrix

$$\left[\vec{V_1} \quad \vec{V_2} \quad \vec{V_3} \quad \vec{V_4}\right] = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

Note: Other distribution can formulated by the following the below configuration:

The matrix can be formulated differently as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & -1 & -1 & +1 \end{bmatrix},$$

From the below Table 1, the row distribution can configurated by the following systems of equations

$$(r_1, r_2): \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \\ -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \end{cases} \Longrightarrow \alpha_1 = 0$$

$$(r_6, r_8): \begin{cases} \beta_1 + \beta_2 + \beta_3 + \beta_4 = K \\ \beta_1 - \beta_2 + \beta_3 + \beta_4 = K \end{cases} \Longrightarrow \beta_2 = 0$$

$$(r_3, r_7): \begin{cases} \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = K \\ \gamma_1 + \gamma_2 - \gamma_3 + \gamma_4 = K \end{cases} \Longrightarrow \gamma_3 = 0$$

$$(r_4, r_5): \begin{cases} w_1 + w_2 + w_3 + w_4 = K \\ w_1 + w_2 + w_3 - w_4 = K \end{cases} \Longrightarrow w_4 = 0$$

Rows	alpha	beta	Gamma	Lambda	Index vector and coodinate= 0											Dimension		
1	1	1	1	1	a													
2	-1	1	1	1	a											dim=2		
3	1	-1	1	1						g							dim=3	
4	-1	-1	1	1											w	dim=2		
5	1	1	-1	1											w			dim=4
6	-1	1	-1	1			b									dim=2		
7	1	-1	-1	1						g							dim=3	
8	-1	-1	-1	1			b									dim=2		

Table 1 How to gather two rows of the matrix in order to eliminate a given index coordinate.

By finishing the calculation, we find that

$$K(4) \geq \sqrt{3}.$$

The Cauchy-Schwarz inequality, also known as the Cauchy–Bunyakovsky–Schwarz inequality, can be used to optimized the following system:

Maximizing the variable K, under the fourth objective functions:

$$\begin{cases} \alpha_2 + \alpha_3 + \alpha_4 = K \\ \beta_1 + \beta_3 + \beta_4 = K \\ \gamma_1 + \gamma_2 + \gamma_4 = K \\ w_1 + w_2 + w_3 = K \end{cases}$$

Under the below constraints:

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ w_3^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \gamma_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases}$$

The system can be modified by Cauchy as follows:

$$\begin{cases} K^2 = (\alpha_2 + \alpha_3 + \alpha_4)^{\,2} \leq (\alpha_2^2 + \alpha_3^2 + \alpha_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (\beta_1 + \beta_3 + \beta_4)^{\,2} \leq (\beta_1^2 + \beta_3^2 + \beta_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (\gamma_1 + \gamma_2 + \gamma_4)^{\,2} \leq (\gamma_1^2 + \gamma_2^2 + \gamma_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (w_1 + w_2 + w_3)^{\,2} \leq (w_1^2 + w_2^2 + w_3^2)(1^2 + 1^2 + 1^2) \end{cases}$$

By adding all the four equation will get

$$4K^2 \leq 3(\alpha_4^2 + \alpha_2^2 + \alpha_3^2 + \beta_4^2 + \beta_3^2 + \beta_1^2 + \gamma_4^2 + \gamma_2^2 + \gamma_1^2 + w_3^2 + w_2^2 + w_1^2),$$

from constraints, the maximum of the value K can be calculated as follows:

$$4K^2 = 12,$$

then the constant of the optimization is found to be as follows

$$K = \sqrt{3},$$

Then the komlos constant has a lower bound as follows

$$K(4) \geq \sqrt{3}.$$

The coordinate of the four vectors can be calculated from the equality of Cauchy-Schwarz inequality property that states that

$$\begin{cases} (\alpha_4+\alpha_2+\alpha_3)^{\,2} = (\alpha_4^2+\alpha_2^2+\alpha_3^2)(1^2+1^2+1^2) \Rightarrow \dfrac{\alpha_3}{1}=\dfrac{\alpha_2}{1}=\dfrac{\alpha_4}{1} \\ (\beta_4+\beta_3+\beta_1)^{\,2} = (\beta_4^2+\beta_3^2+\beta_1^2)(1^2+1^2+1^2) \Rightarrow \dfrac{\beta_1}{1}=\dfrac{\beta_3}{1}=\dfrac{\beta_4}{1} \\ (\gamma_4+\gamma_2+\gamma_1)^{\,2} = (\gamma_4^2+\gamma_1^2+\gamma_2^2)(1^2+1^2+1^2) \Rightarrow \dfrac{\gamma_1}{1}=\dfrac{\gamma_2}{1}=\dfrac{\gamma_4}{1} \\ (w_3+w_2+w_1)^{\,2} = (w_3^2+w_2^2+w_1^2)(1^2+1^2+1^2) \Rightarrow \dfrac{w_1}{1}=\dfrac{w_2}{1}=\dfrac{w_3}{1} \end{cases}$$

Therefore

$$\alpha_i = \beta_i = \gamma_i = w_i = \frac{\sqrt{3}}{3}.$$

The coordinate of the particular vectors $\overrightarrow{V_i}$ are summarized under the below matrix

$$[\overrightarrow{V_1} \quad \overrightarrow{V_2} \quad \overrightarrow{V_3} \quad \overrightarrow{V_4}] = \frac{\sqrt{3}}{3}\begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & -1 & +1 \\ +1 & -1 & 0 & +1 \\ -1 & -1 & +1 & 0 \end{bmatrix}.$$

In the case where the dimension is under the form of 2^m, for certain integer m, the optimization is perfect but for other cases of dimension an upper bound can be found for the constant *K* if Cauchy-Schwarz inequality is applied as above.

Evaluation of $K(5)$

By using the same idea of the previous section, in dimension 4, we denote by $\overrightarrow{V_1}, \dots, \overrightarrow{V_5}$ as a special vectors satisfying

$$K(5) = \min_{\varepsilon_i} \left\| \overrightarrow{V_5} + \sum_{i=1}^{4} \varepsilon_i \overrightarrow{V_i} \right\|_\infty$$

All the different combination of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1)$ are summarized at the rows of the matrix A_5 defined as follows

$$A_5 = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & -1 & +1 \end{bmatrix}$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$ and $dim\{r_i, i = 1, \dots, 16\} = 5$.

The target is to distribute the 16 rows among to five dimensions, named $\{\propto, \beta, \gamma, \lambda, w\}$ in such way to minimize number of zeros in the 5 vectors, $\vec{V_i} = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \lambda_i \\ w_i \end{pmatrix}$, $i = 1, \dots, 5$.

The rows distribution is summarized as following:

- Four rows will be assigned to each axe except axe α, where one row is a linear combination of the others $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, it looks like each three independent rows will be assigned to one axe,
- Two rows will be assigned to axe α

Formulating the previous distribution of the 16 rows to the below 16 equations as follows:

<u>*For α-Axe:*</u> $\left\|\vec{\boldsymbol{V}}_5 + \sum_{i=1}^{4} \boldsymbol{\varepsilon}_i \vec{\boldsymbol{V}}_i\right\|_\infty = \sum_{i=1}^{4} |\boldsymbol{\varepsilon}_i \boldsymbol{\alpha}_i| + |\boldsymbol{\alpha}_5|$

$$\begin{cases} r_{15}: \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K(5) \\ r_{16}: \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 - \alpha_1 = K(5) \end{cases} \Longrightarrow \alpha_1 = 0$$

<u>*For β-Axe:*</u> $\left\|\vec{\boldsymbol{V}}_5 + \sum_{i=1}^{4} \boldsymbol{\varepsilon}_i \vec{\boldsymbol{V}}_i\right\|_\infty = \sum_{i=1}^{4} |\boldsymbol{\varepsilon}_i \boldsymbol{\beta}_i| + |\boldsymbol{\beta}_5|$

$$\begin{cases} r_1: \beta_5 + \beta_4 + \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_3: \beta_5 + \beta_4 + \beta_3 - \beta_2 + \beta_1 = K(5) \\ r_5: \beta_5 + \beta_4 - \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_7: \beta_5 + \beta_4 - \beta_3 - \beta_2 + \beta_1 = K(5) \end{cases} \Longrightarrow \beta_2 = \beta_3 = 0$$

Note that the last equations depend on the 3 first equations.

For γ-Axe: $\left\|\vec{\boldsymbol{V}}_5+\sum_{i=1}^{4}\boldsymbol{\varepsilon}_i\vec{\boldsymbol{V}}_i\right\|_\infty=\sum_{i=1}^{4}|\boldsymbol{\varepsilon}_i\,\boldsymbol{\gamma}_i|+|\,\boldsymbol{\gamma}_5|$

$$\begin{cases} r_2: \gamma_5+\gamma_4+\gamma_3+\gamma_2+\gamma_1=K(5)\\ r_6: \gamma_5+\gamma_4-\gamma_3+\gamma_2+\gamma_1=K(5)\\ r_{10}: \gamma_5-\gamma_4+\gamma_3+\gamma_2+\gamma_1=K(5)\\ r_{14}: \gamma_5-\gamma_4-\gamma_3+\gamma_2+\gamma_1=K(5)\end{cases}\Longrightarrow \gamma_4=\gamma_3=0$$

Note that the last equations depend on the 3 first equations.

For λ-Axe: $\left\|\vec{\boldsymbol{V}}_5+\sum_{i=1}^{4}\boldsymbol{\varepsilon}_i\vec{\boldsymbol{V}}_i\right\|_\infty=\sum_{i=1}^{4}|\boldsymbol{\varepsilon}_i\boldsymbol{\lambda}_i|+|\,\boldsymbol{\lambda}_5|$

$$\begin{cases} r_4: \lambda_5+\lambda_4+\lambda_3+\lambda_2+\lambda_1=K(5)\\ r_5: \lambda_5+\lambda_4-\lambda_3-\lambda_2-\lambda_1=-K(5)\\ r_{12}: \lambda_5-\lambda_4+\lambda_3+\lambda_2+\lambda_1=K(5)\\ r_{13}: \lambda_5-\lambda_4-\lambda_3-\lambda_2-\lambda_1=K(5)\end{cases}\Longrightarrow \lambda_4=\lambda_5=0$$

Note that the last equations depend on the 3 first equations.

For w-Axe: $\left\|\vec{\boldsymbol{V}}_5+\sum_{i=1}^{4}\boldsymbol{\varepsilon}_i\vec{\boldsymbol{V}}_i\right\|_\infty=\sum_{i=1}^{4}|\boldsymbol{\varepsilon}_i\boldsymbol{w}_i|+|\,\boldsymbol{w}_5|$

$$\begin{cases} r_6: w_5+w_4+w_3+w_2+w_1=K(5)\\ r_8: w_5+w_4+w_3-w_2+w_1=-K(5)\\ r_9: w_5-w_4-w_3+w_2+-w_1=K(5)\\ r_{11}: w_5-w_4-w_3-w_2-w_1=K(5)\end{cases}\Longrightarrow w_5=w_2=0.$$

Note that the last equations depend on the 3 first equations.

From the previous systems of equations, we can shape our five vectors $\vec{V_i}$ in order to maximize $K(5)$ as follows

$$\begin{bmatrix}\vec{V_1} & \vec{V_2} & \vec{V_3} & \vec{V}_4 & \vec{V}_5\end{bmatrix}=\begin{bmatrix}0 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \alpha_5\\ \beta_1 & 0 & 0 & \beta_4 & \beta_5\\ -\gamma_1 & \gamma_2 & 0 & 0 & \gamma_5\\ -\lambda_1 & -\lambda_2 & \lambda_3 & 0 & 0\\ -w_1 & 0 & -w_3 & w_4 & 0\end{bmatrix},$$

where α_i, β_i , γ_i , λ_i , and w_i are non-negative values.

The negative sign highlighted at the coordinate of $\vec{V_i}$ comes from rows r_{15}, r_1, r_2, r_4 *and* r_6 , for instance if we assume that $\left\|\vec{V}_5+\sum_{i=1}^{4}\varepsilon_i\vec{V}_i\right\|_\infty=\left|\sum_{i=1}^{4}\varepsilon_i\alpha_i+\alpha_5\right|$ where $r_5=(1,-1,-1,-1,1)$ and our target is to maximize the value of $K(5)$, then it is preferable to considerα_2, α_3, and α_4 are negative values such that the equation $1(\alpha_1)-1(\alpha_2)-1(\alpha_3)-1(\alpha_4)+1(\alpha_5)=K(5)$ will be equivalent to the below equation,

$$|\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_3|+|\alpha_5|=K(5)\,,$$

for notation simplification notation, we write any negative parameter as -α_i.

To calculate the constant $K(5)$, we need to solve the below system

$$\begin{cases} \alpha_2+\alpha_3+\alpha_4+\alpha_5=K(5) \\ \beta_1+\beta_4+\beta_5=K(5) \\ \gamma_1+\gamma_2+\gamma_5=K(5) \\ \lambda_1+\lambda_2+\lambda_3=K(5) \\ w_1+w_3+w_4=K(5) \end{cases}$$

Under the constraint $\|\vec{V}_i\|\leq 1, i=1,\dots,5$.

By symmetry, we can assume the following

$$\alpha_5=\alpha_4=\alpha_3=\alpha_2=\alpha$$

$$\gamma_5=\gamma_2=\gamma$$

$$\lambda_3=\lambda_2=\lambda$$

$$w_4=w_3=w$$

$$\beta_1=\gamma_1=\lambda_1=w_1$$

As $\|\vec{V}_1\|_2\leq 1$, we can put

$$\beta_1=\gamma_1=\lambda_1=w_1=\frac{1}{2}$$

The system will be summarized as follow:

$$\begin{cases} 4\alpha=K(5) \\ 2\beta=K(5)-\frac{1}{2} \\ 2\gamma=K(5)-\frac{1}{2} \\ 2\lambda=K(5)-\frac{1}{2} \\ 2w=K(5)-\frac{1}{2} \end{cases}$$

Under the constraints

$$\begin{aligned} 1 &= \alpha^2+\beta^2+\gamma^2 \\ &= \alpha^2+\beta^2+w^2 \\ &= \alpha^2+\lambda^2+w^2 \\ &= \alpha^2+\lambda^2+\gamma^2 \end{aligned}$$

The system be equivalent to quadratic equation

$$\left(\frac{K(5)}{4}\right)^2+2\left(\frac{K(5)}{2}-\frac{1}{4}\right)^2=1.$$

So we can conclude that the value of $K(5)$ is lower bounded by $\frac{4+\sqrt{142}}{9}$ i.e.

$$K(5) \geq \frac{4+\sqrt{142}}{9}.$$

To see the importance of the way of distributing the rows among the axes is very important, we try to make, as an example, another configuration as follows:

$$\begin{cases} (r_7, r_8, r_9, r_{10}) & for\ \alpha - Axe \\ (r_2, r_3, r_4) & for\ \beta - Axe \\ (r_{12}, r_{14}, r_{16}) & for\ \gamma - Axe \\ (r_5, r_{13}, r_1) & for\ \lambda - Axe \\ (r_6, r_{11}, r_{15}) & for\ w - Axe \end{cases}$$

The five vectors coordinate will be summarized under the below matrix

$$[\overrightarrow{V_1}, \quad \overrightarrow{V_2}, \quad \overrightarrow{V_3}, \quad \vec{V}_4, \ \vec{V}_5\ ] = \begin{bmatrix} 0 & -\alpha_2 & -\alpha_3 & \alpha_4 & 0 \\ 0 & 0 & \beta_3 & \beta_4 & \beta_5 \\ -\gamma_1 & 0 & 0 & -\gamma_4 & \gamma_5 \\ \lambda_1 & \lambda_2 & 0 & 0 & \lambda_5 \\ -w_1 & w_2 & 0 & w_4 & 0 \end{bmatrix}$$

The system that needs to be solved is formulated as follows:

$$\begin{gathered} \alpha_2 + \alpha_3 + \alpha_4 = K(5) \\ \beta_3 + \beta_4 + \beta_5 = K(5) \\ \gamma_1 + \gamma_4 + \gamma_5 = K(5) \\ \lambda_1 + \lambda_2 + \lambda_5 = K(5) \\ w_1 + w_2 + w_4 = K(5) \end{gathered}$$

Under the constraints: $\left\|\vec{V}_i\right\|_2 \leq 1$.

By using this type of distribution, the symmetry of the matrix $[\vec{V}_1, \dots, \vec{V}_5]$ is not preserve, which it makes the system hard to be solve analytically and number of zero coordinate in the set of vectors $\vec{V_\iota}$ has been increased from 9 times to 10 times.

Therefore, the system that needs to be optimized is as follows:

$$\begin{aligned} \text{Max}\, K(5) &= \alpha_2 + \alpha_3 + \alpha_4 \\ &= \beta_3 + \beta_4 + \beta_5 \\ &= \gamma_1 + \gamma_4 + \gamma_5 \\ &= \lambda_1 + \lambda_2 + \lambda_5 \\ &= w_1 + w_2 + w_4 \end{aligned}$$

Under the constraints: $\left\|\vec{V}_i\right\|_2 \leq 1$.

The value of $K(5)$ is very sensitive to the distribution choices, please refer the below Table 2 for different choices.

Rows	alpha	beta	Gamma	Lambda	w	Index vector and coorinate =0	Dimension			
1	1	1	1	1	1	b				
2	-1	1	1	1	1	g	dim=2			
3	1	-1	1	1	1	b		dim=3		
4	-1	-1	1	1	1	l	dim=2			
5	1	1	-1	1	1	b l			dim=4	
6	-1	1	-1	1	1	g w	dim=2			
7	1	-1	-1	1	1	b		dim=3		
8	-1	-1	-1	1	1	w	dim=2			
9	1	1	1	-1	1	w				dim=5
10	-1	1	1	-1	1	g	dim=2			
11	1	-1	1	-1	1	w		dim=3		
12	-1	-1	1	-1	1	l	dim=2		dim=4	
13	1	1	-1	-1	1	l				
14	-1	1	-1	-1	1	g	dim=2			
15	1	-1	-1	-1	1	a		dim=3		
16	-1	-1	-1	-1	1	a	dim=2			

Table 2 How to gather rows of the matrix in order to eliminate a certain axes-coordinates.

Evaluation of $K(n)$

In dimension n, it is very crucial to find a best way to distribute all possible combinations of the vectors $\vec{\varepsilon} = (1, \varepsilon_1, \dots, \varepsilon_{n-1})$ among the n axes. We assume that we have $\left\lceil \frac{2^{n-1}}{n} \right\rceil$ different combinations of the vector of $\vec{\varepsilon}$ for which

$$\left\| \vec{V}_n + \sum_{i=1}^{n-1} \varepsilon_i \vec{V}_i \right\|_\infty = \left| \sum \varepsilon_i x_i + x_n \right| = K(n),$$

where x_i is the coordinate of vector $\vec{V}_i$ corresponding to x-Axe.

The $\left\lceil \frac{2^{n-1}}{n} \right\rceil$ vectors that have been assign to one axes has a dimension of order $\left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$, and as consequence, it implies that each vector $\vec{V}_i$ has at most $\left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$ null coordinates.

To evaluate the constant $K(n)$, it is enough to solve the below optimization equation

$$K(n) = \max_{x_i} \left(\sum_{i=1}^{n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil} x_i \right).$$

By imposing the symmetry conditions by choosing a good way of distribution, the non-null coordinate in each axes as constant values, i.e., $x_i = x$.

Let B a suset $\{1, \dots, n\}$ of cardinality around $n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$ and from the condition that $\|\vec{V}_n\|_2 \leq 1$, the upper bound of x can be found as following

$$\sum_{j\in B}\left(x_i^j\right)^2=\sum_{j\in B}(x)^2\leq 1\ \Rightarrow x\leq\frac{\sqrt{n-\left\lceil log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil}}{n-\left\lceil log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil}.$$

The lower bound of Kmolos conjecture can be calculated as follows:

$$\begin{aligned} K(n) &\geq \sqrt{n-\left\lceil log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil} \\ &\geq \sqrt{\log(n)+1}. \end{aligned}$$

Under our lemma, if it exists an natural n such that $n = 2^k$, then the symmetry conditions can be used always in order to conclude that

$$K(n)=\sqrt{n-\left\lceil log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil}=\sqrt{\log_2(n)+1}.$$

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References

1. A. Dvoretzky, Some results on convex bodies and Banach spaces, Proceedings of the Symposium on Linear Spaces, J erusalem 1961, pp. 123-160.
2. A.Szankowski, On Dvoretzky's theorem on almost spherical sections of convex bodies, Israel J. Math. 17, (1974), 325-338.].
3. B. Chazelle. The discrepancy Method Cambridge University, Press, 1991.
4. D. Hajela, On a Conjecture of Komlos about Signed Sums of Vectors inside the Sphere . Europ. J. Combinatorics (1988) 9, 33-37.
5. J.Becka and T.Fiala. Integer-making theorem. Discrete Applied Mathematics, 3(1):1-8,1981.

6. J. Beck and V.T. Sos. Discrepancy theory. In Handbook of combinatorics (vol.2), page 1446. MIT Press, 1996
7. J. Matousek, Geometric Discrepancy. “ An Illustrated Guide , Springer Verlag 2010
8. J. Spencer. Ten lectures non the probabilistic method: second Edition. SIAM, 1994J.
9. J. Spencer, Six standard deviations suffice, Trans. Amer. Math. Soc. 289 (1985), 679-706
10. T. Figiel, Some remarks on Dvoretzky's theorem on almost spherical sections of convex bodies, Colloq. Math. 24 (1972), 241-252.
11. W. Banaszczyk . Balancing vectors and Gaussian measures of n-dimensional convex bodies, Random structures and algorithms 12(4) :351-360,1998