$f \in \mathcal{H}$ and $y \in X, f(y) = \langle f, k(y,.)\rangle_{\mathcal{H}}$. Conversely if $\mathcal{H}$ is an *RKHS* on X such that there is a unique positive definite kernel K on X such that $\{k(x,.), x \in X\} \subset \mathcal{H}$. In light of this bijective correspondence, positive definite kernels are usually called *Reproducing Kernels*.
It makes sense here to take an example of kernel functions using in machine learning. Other useful examples can be found in [5].

Example 2.1. The Brownian bridge kernel $\min(x, x')$ defined on R likewise is a positive kernel. Note that one-dimensional linear spline kernels with an infinite number of nodes [Vapnik et al., 1997] for $x, x' \in \mathbb{R}_0^+$ are given by

$$k(x, x') = \frac{\min(x, x')^3}{3} + \frac{\min(x, x')^2 |x - x'|}{2} + 1 + xx'.$$

Recall that two normed vector spaces V_1 and V_2 are said to be isometric if there is a bijective linear norm-preserving mapping between them. We call such V_1 and V_2 an identification of each other. We would like the dual space B^* of an *RKBS*, B, on X to be isometric to a Banach space of functions on X. As it explained in [16], we like an *RKBS*, B, to be reflexive in the sense that $(B^*)^* = B$.

A reproducing kernel Banach space (*RKBS*) on X is a reflexive Banach space of functions on X such that its topological dual B' is isometric to a Banach space of functions on X and the point evaluations are continuous linear functionals on both B and B'.

now we can introduce for a normed vector space V the following bilinear form on $V \times V^*$ by setting

$$\langle u, v^* \rangle_V := v^*(u), \qquad v \in V, v^* \in V^*.$$

It is called bilinear for the reason that for all $\alpha, \beta \in \mathbb{C}, u, v \in V$, and $u^*, v^* \in V^*$ there holds

$$\langle \alpha u + \beta v, u^* \rangle_V = \alpha \langle u, u^* \rangle_V + \beta \langle v, u^* \rangle_V$$

and

$$\langle u, \alpha u^* + \beta v^* \rangle_V = \alpha \langle u, u^* \rangle_V + \beta \langle u, v^* \rangle_V$$

note that if V is a reflexive Banach space then for any continuous linear functional T on V^* there exists a unique $u \in V$ such that

$$T(v^*) = \langle u, v^* \rangle_V, \qquad v^* \in V^*.$$

Lemma 2.2. *Suppose that B is an RKBS on X. Then there exists a unique function $K : X \times X \longrightarrow \mathbb{C}$ such that the following statements hold.*

(a) *For every $x \in X, K(.,x) \in B^*$ and*

$$f(x) = \langle f, K(.,x)\rangle_B, \qquad \textit{for all} \quad f \in B.$$

(b) *For every $x \in X, K(x,.) \in B$ and*

$$f^*(x) = \langle K(x,.), f^*\rangle_B, \qquad \textit{for all} \quad f^* \in B^*.$$

(c) *The linear span of $\{K(x,.) : x \in X\}$ is dense in B, namely,*

$$\overline{span}\{K(x,.) : x \in X\} = B.$$

(d) *The linear span of $\{K(.,x) : x \in X\}$ is dense in B^*, namely,*

$$\overline{span}\{K(.,x) : x \in X\} = B^*.$$

(e) *For all $x, y \in X$*

$$K(x,y) = \langle K(x,.), K(.,y)\rangle_B.$$

Proof. The proof is completely explained in theorem 2 of [16] . □

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The function K in previews lemma is called the *reproducing kernel* for the *RKBS*, B. Like in reproducing kernel Hilbert spaces and by Theorem 2 in [16], an *RKBS* has exactly one reproducing kernel. However, unlike *RKHS*, different *RKBS* may have the same reproducing kernel.

Lemma 2.3. *Let W be a reflexive Banach space with dual space W^*. Suppose that there exists $\Phi : X \longrightarrow W$, and $\Phi^* : X \longrightarrow W^*$ such that*

$$\overline{span}\ \Phi(X) = W, \qquad and \qquad \overline{span}\ \Phi^*(X) = W^*.$$

Then $B := \{\langle u, \Phi(.)\rangle_W : u \in W\}$ with norm

$$\|\langle u, \Phi^*(.)\rangle_W\|_B := \|u\|_W$$

is an RKBS on X with the dual space $B^* := \{\langle \Phi(.), u^*\rangle_W : u^* \in W^*\}$ endowed with the norm

$$\|\langle \Phi(.), u^*\rangle_W\|_{B^*} := \|u^*\|_{W^*}$$

and the bilinear form

$$\langle\langle u, \Phi^*(.)\rangle_W, \langle \Phi(.), u^*\rangle_W\rangle_B := \langle u, u^*\rangle_W, \qquad u \in W, u^* \in W^*.$$

Moreover, the reproducing kernel K for B is

$$K(x, y) := \langle \Phi(x), \Phi^*(y)\rangle_W, \qquad x, y \in X.$$

Proof. The proof is completely explained in theorem 3 of [16]. □

Definition 2.4. The mapping $[., .] : X \times X \to K$ is called the semi-inner product ($s.i.p.$) if the following properties are satisfied:

(i) $[x + y, z] = [x, z] + [y, z]$ *for all* $x, y \in X$;
(ii) $[\lambda x, y] = \lambda[x, y]$ *for all* $x, y \in X$ *and* λ *a scalar in* K.;
(iii) $[x, x] \geq 0$ *for all* $x \in X$ *and* $[x, x] = 0$ *implies that* $x = 0$;
(iv) $|[x, y]|^2 \leq [x, x][y, y]$ *for all* $x, y \in X$;
(v) $[x, \lambda y] = \overline{\lambda}[x, y]$ *for all* $x, y \in X$ *and* λ *a scalar in* K.

Semi-inner product spaces have interesting properties which are out of our study. For more details, one can refer to [4, 10, 11].

Lemma 2.5. *A semi-inner-product $[., .]_V$ on a complex vector space V is an inner product if and only if*

$$[x, y + z]_V = [x, y]_V + [x, z]_V, \qquad for\ all \quad x, y, z \in V.$$

To see the details of proof, reader can refer to [16].
It was shown in [9] that a vector space V with a semi-inner-product is a normed space equipped with

$$\|x\|_V := [x, x]_V^{1/2}, \qquad x \in V.$$

Therefore, if a vector space V has a semi-inner-product, we always assume that its norm is induced by above equation and call V an $s.i.p.$ space. Conversely, every normed vector space V has a semi-inner product that induces its norm by this equation [6]. By the Cauchy-Schwartz inequality, if V is an $s.i.p.$ space then for each $x \in V$, $y \longrightarrow [y, x]_V$ is a continuous linear functional on V. We denote this linear functional by x. Following this definition, we have that

$$[x, y]_V = y(x) = \langle x, y\rangle_V, \qquad x, y \in V.$$

We call a normed vector space V a uniformly Frechet differentiable if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x - ty\|_V - \|x\|_V}{t} \qquad t \in \mathbb{R}$$

is approached uniformly on unit sphere on V.

Theorem 2.6. *Suppose B is an s.i.p. space. Then*

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(i) *(Riesz representation theorem) For each* $g \in B'$*, there exists a unique* $h \in B$ *such that* $g = h^*$*, i.e.,* $g(f) = [f, h]_B, f \in B$ *and* $\|g\|_{B'} = \|h\|_B$.

(ii) B' *is an s.i.p. space with respect to the semi inner product and semi norm defined by*

$$\langle h^*, f^* \rangle_{B'} := \langle f, h \rangle_B, f, h \in B \qquad \textit{and} \qquad \|h^*\|_{B'} := [h^*, h^*]_{B'}^{1/2}.$$

To see a complete proof of above theorem and more details, we refer the reader to [6].

Example 2.7. Let (X, A, μ) be a measure space and $B := L_p(X, \mu)$ for some $p \in (1, +\infty)$. It is an $s.i.p.$ space with dual $B' := L_q(X, \mu)$ where $q = \dfrac{p}{p-1}$. For each $f \in B$, its dual element in B' is

$$f^* = \frac{\overline{f}|f|^{p-2}}{\|f\|_{L_p(X,\mu)}^{p-2}}$$

Consequently, the semi-inner product on B is

$$[f, g]_B = g^*(f) = \frac{\int_X f\overline{g}|g|^{p-2}\, d\mu}{\|g\|_{L_p(X,\mu)}^{p-2}} \tag{1}$$

Recall that a normed vector space V is uniformly convex if for all $\epsilon\rangle 0$ there exists a $\delta\rangle 0$ such that $\|x+y\|_V \leq 2-\delta$ for all x, y in unit sphere on V with $\|x-y\|_V \geq \epsilon$.

Definition 2.8. Let X be a set. We call a uniformly convex and uniformly Frechet differentiable $RKBS$ on X an $s.i.p.$ reproducing kernel Banach space ($s.i.p.RKBS$).

We can also have other view to a $RKBS$. Although our next definition of an $RKBS$ is coincides with our last definition, but it may be more useful.

Definition 2.9. Let Ω and Ω' be locally compact Hausdorff spaces equipped with finite Borel measures μ and μ', respectively. Let $\mathcal{B}$ be a Banach space composed of functions $f \in L_0(\Omega)$, whose dual space B' is isometrically equivalent to a normed space F consisting of functions $g \in L_0(\Omega')$, and that $K \in L_0(\Omega \times \Omega')$ is a kernel. We call $\mathcal{B}$ a right-sided reproducing kernel Banach space and K its right-sided reproducing kernel if

(i) $K(x, .) \in B'$, *for each* $x \in \Omega$,

(ii) $\langle f, K(x, .)\rangle_B = f(x)$, *for all* $f \in B, x \in \Omega$.

If the Banach space $\mathcal{B}$ reproduces from the other side, that is,

(iii) $K(., y) \in B$, *for each* $y \in \Omega'$,

(iv) $\langle K(., y), g\rangle_B = g(y)$, *for all* $g \in B', y \in \Omega'$,

then $\mathcal{B}$ is called a left-sided reproducing kernel Banach space and K its left-sided reproducing kernel. If two-sided reproducing properties $(i-iv)$ as above are satisfied, we say that $\mathcal{B}$ is a two-sided reproducing kernel Banach space with the two-sided reproducing kernel K.

Observing the conjugated structures, the adjoint kernel K' of the reproducing kernel K is defined by

$$K'(y, x) := K(x, y), \qquad \textit{for all} \quad x \in \Omega, y \in \Omega'.$$

The features of the two-sided reproducing kernel K and its adjoint kernel K' can be represented by the dual-bilinear products in the following manner

$$K(x, y) = \langle K'(y, .), K(x, .)\rangle_B = \langle K(., y), K'(., x)\rangle_B = K'(y, x),$$

for all $x \in \Omega$ and all $y \in \Omega'$. This means that both $x \longmapsto K(x, .), y \longmapsto K(., y)$ and $x \longmapsto K'(., x), y \longmapsto K'(y, .)$ can be viewed as the feature maps of $RKBS$s.

Clearly, the classical reproducing kernels of $RKHS$s can be seen as a two-sided reproducing kernels of the two-sided $RKBS$. It is well-known that the reproducing kernels of $RKHS$s are always positive definite while the reproducing kernels of $RKBS$s may be neither symmetric nor positive definite.

3. Weighted Reproducing Property

In this section, we introduce weighted reproducing kernels. The main propose of introducing this concept is to learn in a space with different weighted of hypothesis. We just label two weights to the kernels in the text but it can be extended in to finite ones respected to the complexity of learning subjects.

In the following, we first define relative reproducing kernels on Banach spaces.

Definition 3.1. Let Ω and Ω' be locally compact Hausdorff spaces equipped with finite Borel measures μ and μ', respectively. Let $\mathcal{B}$ be a Banach space composed of functions $f \in L_0(\Omega)$, whose dual space $\mathcal{B}'$ is isometrically equivalent to a normed space F consisting of functions $g \in L_0(\Omega')$, and that $M_{x,y} \in L_0(\Omega \times \Omega')$ is a kernel. We call $\mathcal{B}$ a right-sided relative reproducing kernel Banach space ($RRKBS$) and $M_{x,y}$ its right-sided relative reproducing kernel if

(i') $M_{x,y} \in \mathcal{B}'$, *for each* $x \in \Omega$,
(ii') $\langle f, M_{x,y}\rangle_B = f(x) - f(y)$, *for all* $f \in B, x, y \in \Omega$.

If the Banach space $\mathcal{B}$ reproduces from the other side, that is,

(iii') $M_{x,y} \in B$, *for each* $y \in \Omega'$,
(iv') $\langle M_{x,y}, g\rangle_B = g(y) - g(x)$, *for all* $g \in \mathcal{B}', x, y \in \Omega'$,

then $\mathcal{B}$ is called a left-sided relative reproducing kernel Banach space and $M_{x,y}$ its left-sided reproducing kernel. If two-sided relative reproducing properties $(i' - iv')$ as above are satisfied, we say that $\mathcal{B}$ is a two-sided relative reproducing kernel Banach space with the two-sided relative reproducing kernel $M_{x,y}$.

This definition is extending of relative reproducing kernel Hilbert space which is defined by *Daniel Alpay et.al* in [1]. We generalized this definition to multiple form and prove some theorems for $RRKBS$s and its generalized form in next section.

Lemma 3.2. *For an s.i.p.$RRKBS$, the function $M_{x,y}$ in definition 3 is unique and satisfies*

$$M_{x,y}(t) + M_{y,z}(t) = M_{x,z}(t), \qquad \textit{for all} \quad x, y, z, t \in \Omega. \tag{2}$$

Proof. Uniqueness is obvious by Riesz Representation Theorem. By definition, for every f we have

$$\langle f, M_{x,y} + M_{y,z}\rangle = \langle f, M_{x,y}\rangle + \langle f, M_{y,z}\rangle$$
$$= f(y) - f(x) + f(z) - f(y) = f(z) - f(y) = \langle f, M_{x,z}\rangle.$$

□

Lemma 3.3. *A Banach space B of functions on the set Ω is an s.i.p.$RRKBS$ if and only if there exists a function $h_x : \Omega \longmapsto B$ and an everywhere defined possibly unbounded linear functional $C : B \longmapsto \mathbb{C}$ such that*

$$F(x) = \langle F, h_x\rangle_B + C(F), \qquad \textit{for all} \quad F \in B, x \in \Omega$$

Proof. One direction is obvious by applying definition 3. Conversely take any $x_0 \in \Omega$. Then, definition 3 with $y = x_0$ implies above equation with $h_x = M_{x,x_0}$ and $C(F) = F(x_0)$. □

Definition 3.4. Let Ω and Ω' be locally compact Hausdorff spaces equipped with finite Borel measures μ and μ', respectively. Let $\mathcal{B}$ be a Banach space composed of functions $f \in L_0(\Omega)$, whose dual space $\mathcal{B}'$ is isometrically equivalent to a normed space F consisting of functions $g \in L_0(\Omega')$, and that $M^{\alpha_1,\alpha_2}_{x_1,x_2} \in L_0(\Omega \times \Omega')$ is a kernel. We call $\mathcal{B}$ a right-sided Generalized reproducing kernel Banach space $(WRKBS)$ and $M^{\alpha_1,\alpha_2}_{x_1,x_2}$ its right-sided Generalized reproducing kernel if

(i'') $M^{\alpha_1,\alpha_2}_{x_1,x_2} \in \mathcal{B}'$, *for each* $x_1, x_2 \in \Omega$,

(ii'') $\langle f, M^{\alpha_1,\alpha_2}_{x_1,x_2}\rangle_B = \sum_{i=1}^{2} \alpha_i f(x_i)$, *for all* $f \in B, x_1, x_2 \in \Omega$.

If the Banach space $\mathcal{B}$ reproduces from the other side, that is,

(iii'') $M^{\alpha_1,\alpha_2}_{x_1,x_2} \in B$, *for each* $x_1, x_2 \in \Omega'$,

(iv'') $\langle M^{\alpha_1,\alpha_2}_{x_1,x_2}, g\rangle_B = \sum_{i=1}^{2} \alpha_i g(x_i)$, *for all* $g \in \mathcal{B}', x_1, x_2 \in \Omega'$.

then $\mathcal{B}$ is called a left-sided generalized reproducing kernel Banach space and $M^{\alpha_1,\alpha_2}_{x_1,x_2}$ its left-sided reproducing kernel. If two-sided generalized reproducing properties $(i''-iv'')$ as above are satisfied, we say that $\mathcal{B}$ is a two-sided generalized reproducing kernel Banach space with the two-sided generalized reproducing kernel $M^{\alpha_1,\alpha_2}_{x_1,x_2}$.

Lemma 3.5. *For an s.i.p.$WRKBS$, the function $M^{\alpha_1,\alpha_2}_{x_1,x_2}$ in definition 3 is unique and satisfies:*

$$M^{\alpha_1,\alpha_2}_{x_1,x_2} - M^{\beta_1,\beta_2}_{y_1,y_2} = M^{\alpha_1,\beta_1}_{x_1,y_1} + M^{\alpha_2,\beta_2}_{x_2,y_2} \tag{3}$$

Proof. Uniqueness is obvious by Riesz Representation Theorem. By definition 3 we have

$$\begin{aligned}\langle f, M^{\alpha_1,\alpha_2}_{x_1,x_2} - M^{\beta_1,\beta_2}_{y_1,y_2}\rangle &= \langle f, M^{\alpha_1,\alpha_2}_{x_1,x_2}\rangle - \langle f, M^{\beta_1,\beta_2}_{y_1,y_2}\rangle \\ &= \sum_{i=1}^{2} \alpha_i f(x_i) - \sum_{i=1}^{2} \beta_i f(y_i) \\ &= (\alpha_1 f(x_1) - \beta_1 f(y_1)) + (\alpha_2 f(x_2) - \beta_2 f(y_2)) \\ &= \langle f, M^{\alpha_1,\beta_1}_{x_1,y_1}\rangle + \langle f, M^{\alpha_2,\beta_2}_{x_2,y_2}\rangle\end{aligned}$$

Since above equations hold for every f and every set of points $\{x_i\}_{i=1}^{2}$ and scalers $\{\alpha_i\}_{i=1}^{2}$ and $\{\beta_j\}_{j=1}^{2}$, proof is complete. □

Lemma 3.6. *Let $\mathcal{B}$ be a s.i.p.$WRKBS$ of degree 3. Then for every $\alpha \in \mathbb{C}$ the generalized reproducing kernel satisfies:*

$$M^{1,1,1}_{x,y,z} = M^{1,\alpha}_{x,y} + M^{1-\alpha,1}_{y,z} \tag{4}$$

Proof. By definition 3 for every $F \in B$ we have

$$\langle F, M^{1,\alpha}_{x,y} + M^{1-\alpha,1}_{y,z}\rangle_B = F(x) + \alpha F(y) + (1-\alpha)F(y) + F(z) = \langle F, M^{1,1,1}_{x,y,z}\rangle_B$$

□

Theorem 3.7. *A Banach space $\mathcal{B}$ of functions on the set Ω is an s.i.p.$WRKBS$ if and only if there exists a function $h_x : \Omega \longmapsto B$, a real number α_x and an everywhere defined possibly unbounded linear functional $C : B \longmapsto \mathbb{C}$ such that*

$$F(x) = \langle F, \alpha^{-1} h_x\rangle_B + C(F), \qquad \textit{for all} \quad F \in B, x \in \Omega. \tag{5}$$

Proof. One direction is clear by definition. Conversely fix $y_1 \in \Omega$. Then, applying definition 3 with x, y_1 implies the claimed equation with

$$h_x = M^{\alpha_1,\alpha_2}_{x,y_1} \qquad \textit{and} \qquad C(F) = F(y_1).$$

□

Corollary 3.8. *Let $\mathcal{B}$ be a s.i.p.$WRKBS$ of functions defined on the set Ω, and assume that for $n-1$ points $x_2, x_3, \ldots, x_n \in \Omega$ the evaluation $F \longmapsto F(x_2) + F(x_3) + \cdots + F(x_n)$ is bounded. Then, $\mathcal{B}$ is an $WRKBS$.*

Proof. It suffices to write for $x \in \Omega$,

$$F(x) = (F(x) - F(x_2) - F(x_3) - \cdots - F(x_n)) + F(x_2) + F(x_3) + \cdots + F(x_n).$$

$\square$

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