Well-posedness and long-time behavior of solutions for two-dimensional Navier-Stokes equations with infinite delay and general hereditary memory

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Abstract

We address the dynamics of two-dimensional Navier-Stokes models with infinite delay and hereditary memory, whose kernels are a much larger class of functions than the one considered in the literature, on a bounded domain. We prove the existence and uniqueness of weak solutions by means of Faedo-Galerkin method. Moreover, we establish the existence of global attractor for the system with the existence of a bounded absorbing set and asymptotic compact property.

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1   Introduction

In this paper, we consider the two-dimensional Navier-Stokes model with past history and infinite delay effect in a bounded domain Ω with smooth boundary $\partial\Omega$ of the form

$$\begin{cases} \partial_t u - \nu \Delta u - \displaystyle\int_0^\infty \kappa(s)\Delta u(t-s)\mathrm{d}s \\ \qquad\qquad\qquad + (u\cdot\nabla)u + \nabla p = f + F(u_t), & (x,t)\in\Omega\times(0,+\infty), \\ \nabla\cdot u = 0, & (x,t)\in\Omega\times(0,+\infty), \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,+\infty), \\ u(x,\tau) = \phi(x,\tau), & (x,\tau)\in\Omega\times(-\infty,0], \end{cases} \tag{1.1}$$

where $u = u(x,t) = (u_1, u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, $f = f(x)$ is a given force field, ϕ is a given function, u_t is the function defined by the relation $u_t(\tau) = u(t+\tau)$, $\tau \in (-\infty, 0]$.

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The convolution term stands for memory effect whose kernel κ is a nonnegative summable function of total mass $\int_0^\infty \kappa(s)\mathrm{d}s = 1$ having the explicit form $\kappa(s) = \int_s^\infty \mu(r)dr$.

The Navier-Stokes equations reflect the basic mechanical law of viscous fluid flow, thus are very significant both in a purely mathematical sense and in the fluid applications including physics and biology. Since the first relative paper of Leray [24] published in 1933, the Navier-Stokes equations have been the object of numerous works. Caraballo and Han [12] considered the Navier-Stokes equations with delay effect which shows not only the present state but also its history in both autonomous and non-autonomous cases (see [15, 31, 39] and references therein). In 2001, Caraballo and Real [13] put forward the possibility of some kind of delay appearing in the Navier-Stokes equations in an open and bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with regular boundary Γ:

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(t) + g(t, u_t), \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

and they proved the existence and uniqueness of solutions. Subsequently, they established the existence of a pullback attractor when $N = 2$ in [14]. Concerning the Navier-Stokes equations with finite delay in an unbounded domain, Garrido-Atienza and Marín-Rubio [20] addressed the existence and uniqueness of solutions for both the evolutionary and the stationary cases. Especially, in the three-dimensional case, they only proved the existence of solutions. These results were extended in $\mathbb{R}^N$ with $2 \le N \le 4$ by Niche and Planas [37]. For unbounded delay, the authors investigated the well-posedness and asymptotic behavior of solutions [30, 35] and their references. In addition, Liu, Caraballo and Marín-Rubio [27] studied the existence and uniqueness of solutions for two-dimensional Navier-Stokes equations with infinite delay by using the phase space $BCL_{-\infty}(H)$ which will be defined below rather than $C_\gamma(H)$ carried out in [31] for a 2D Navier-Stokes model with infinite delay due to lacking the term $\Delta(\partial_t u)$. Recently, Anh and Thanh [4] discussed the 3D Navier-Stokes-Voigt equations with infinite delay of the form

$$\begin{cases} \partial_t u - \nu \Delta u - \alpha^2 \Delta(\partial_t u) + (u \cdot \nabla)u + \nabla p = f(t) + F(u_t), \\ \nabla \cdot u = 0, \end{cases}$$

where α is length-scale parameter describing the elasticity of the fluid. Some results on the existence and uniqueness of weak solutions and the existence of global attractors were proved. Other types of delay including constant, bounded variable delay as well as bounded distributed delay can be found in [10, 13] and references therein.

As for memory term, it plays an important role in the description of several phenomena such as non-Newtonian flows, soil mechanics and heat conduction theory and prescribes the effect of past history and provides a more realistic description (see [5] and references therein). So far, there are some authors having studied the asymptotic behavior of solutions

for the Navier-Stokes model incorporating memory effects. For example, Gatti, Giorgi and Pata [21] considered the following system

$$
\begin{cases}
\partial_t u - \omega \Delta u - (1-\omega) \int_0^\infty k_\epsilon(s)\Delta\eta \mathrm{d}s + (u \cdot \nabla)u + \nabla p = f, \\
\partial_t \eta^t = -\partial_s \eta^t + u, \\
\nabla \cdot u = 0, \ \ \nabla\eta^t = 0.
\end{cases}
$$

Here, $\omega \in (0,1)$ is a fixed parameter, the memory kernel is defined by

$$
k_\epsilon = \frac{1}{\epsilon^2}k\left(\frac{s}{\epsilon}\right), \qquad \epsilon \in (0,1].
$$

The authors proved the existence of global attractors and constructed a robust family of exponential attractors. Furthermore, Gal and Medjo [19] considered the three-dimensional Navier-Stokes-Voigt model with memory effect and proved the existence of uniform global attractors. In 2018, Di Plinio et al. [17] analyzed Navier-Stokes-Voigt model lacking kinematic viscosity with memory

$$
\begin{cases}
\partial_t u - \alpha\Delta(\partial_t u) - \int_0^\infty g(s)\Delta u(t-s)\mathrm{d}s + \beta u + (u \cdot \nabla)u + \nabla\pi = f, \\
\nabla \cdot u = 0,
\end{cases}
$$

which assumed the inequality

$$
\mu'(s) \leq -\delta\mu(s), \tag{1.3}
$$

holds for some $\delta > 0$ and achieved the well-posedness and asymptotic properties of solutions.

It is worth pointing out that the condition (1.3) is one of various types of memory conditions scarcely. In fact, the evolution equations, viscoelastic equations have been considered with following memory conditions to study the decay results of solutions. For example, Alabau-Boussouira, Cannarsa and Sforza [2] considered the evolution equations with the assumption

$$
\mu'(s) \leq -\delta\mu(s)^{1+\frac{1}{p}},
$$

for some $s \geq 0$ and $p > 2$. Subsequently, Messaoudi and Mustefa [33] studied the stability result for the memory type satisfying

$$
\mu'(t) \leq -\xi(t)\mu(t),
$$

where ξ is a nonincreasing differentiable function on $\mathbb{R}^+$. Moreover, Alabau-Boussouira [1] concerned the memory-dissipative evolution equations in a bounded domain with a positive function satisfying the form

$$
\mu'(s) \leq -\chi(\mu(s)), \tag{1.4}
$$

where χ is a nonnegative measurable function with $\chi(0) = \chi'(0) = 0$ and strictly increasing. In addition, Messaoudi and Mustafa [34] discussed the Timoshenko system where μ is a

positive nonincreasing function satisfying

$$\mu'(s) \leq -H(\mu(s))$$

and H is a regular and convex function. Moreover, Guesmia [22] considered an abstract linear dissipative integrodifferential equation with infinite memory satisfying

(H1) There exists an increasing strictly convex function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ satisfying

$$\Phi(0) = \Phi'(0) = 0 \text{ and } \lim_{t \to +\infty} \Phi'(t) = +\infty, \tag{1.5}$$

such that

$$\int_0^\infty \frac{\mu(s)}{\Phi^{-1}(-\mu'(s))} \mathrm{d}s + \sup_{s \in \mathbb{R}^+} \frac{\mu(s)}{\Phi^{-1}(-\mu'(s))} < +\infty. \tag{1.6}$$

For more similar arguments we refer the readers to [3, 8, 18, 25, 28, 32, 36].

To our best knowledge, there is no work about Navier-Stokes model (1.1) with a general memory condition (H1). Motivated by the literature above, we are interested in the well-posedness and asymptotic behavior of solutions for the two-dimensional Navier-Stokes equations with memory and infinite delay in a bounded domain. Here we establish the existence and uniqueness of solutions and global attractors on account of general memory condition (H1). Compared with [4], problem (1.1) becomes much harder since the trilinear term is out of control due to the absence of the voigt term $\Delta(\partial_t u)$. To overcome this difficulty, we combine the classical Galerkin approximation, energy method with two-dimensional trilinear estimates to obtain the existence and uniqueness of solutions for Navier-Stokes equations in two-dimensional case. For details, for the proof of the existence of solutions in this case, we can not get the convergence of auxiliary function J like [4] due to the presence of memory term $\|\eta^t\|_{1,\mu}^2$ here. For our purpose, we prove the strong convergence of delay term by continuous definition and integral mean value theorem directly. Additionally, to overcome the difficulty brought by the general memory condition (H1), we show the existence of global attractors for the system $S(t)$ by proving the existence of a bounded absorbing set through the use of a new lemma on the generalized differential inequality and the asymptotical compactness of $S(t)$. It should be note that, for the case of three-dimensions, the well-posedness of solutions is still open due to the difficulty of trilinear term estimates where we don't have 'better' exponents in the Sobolev embedding theorem.

The structure of this paper is as follows. In the next section, we recall some basic assumptions and the related lemmas. In Section 3, we prove the existence and uniqueness of weak solutions to problem (2.1) by using the Faedo-Galerkin method. Finally, the existence of a global attractor for the continuous semigroup generated by the weak solutions is studied in Section 4.

2 Notation and preliminaries

In this section, we follow the notation used in [5]. Denote

$$(u, v) := \int_\Omega \sum_{j=1}^2 u_j v_j \mathrm{d}x, \ \ u = (u_1, u_2), \ v = (v_1, v_2) \in (L^2(\Omega))^2,$$

$$((u.v)) := \int_\Omega \sum_{j=1}^2 \nabla u_j \nabla v_j \mathrm{d}x, \ \ u = (u_1, u_2), \ v = (v_1, v_2) \in (H_0^1(\Omega))^2,$$

and the associated norms $|u|^2 := (u, u)$, $\|u\|^2 := ((u, u))$.

Subsequently, we consider the following usual abstract space

$$\mathcal{V} = \{u \in (C^\infty(\Omega))^2 : \nabla \cdot u = 0\}.$$

For simplicity of notation, we will denote by H the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$, and by V the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$. In what follows, we denote the dual space of H by H' and the dual space of V by V'. It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous. We will use $\| \cdot \|_*$ for the norm in V', and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'. Now we define $A : V \to V'$ by $(Au, v) = ((u, v))$, and it satisfies

$$Au = -P\Delta u, \qquad \text{for all } u \in D(A),$$

where P is the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^2$ onto the space V.

Let us denote trilinear form b on $V \times V \times V \to V'$ by

$$b(u, v, w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x, \qquad \forall \ u, v, w \in V.$$

We recall that

$$b(u, v, w) = -b(u, w, v), \qquad \text{for all } u, v, w \in V,$$

and consequently,

$$b(u, v, v) = 0, \qquad \text{for all } u, v \in V.$$

And the associated bilinear form $B : V \times V \to V'$ is given by $\langle B(u, v), w \rangle = b(u, v, w)$ for all $u, v, w \in V$.

As for prerequisites, the readers are expected to be familiar with the following lemma used throughout the paper.

Lemma 2.1. *[40] We have for all $u, v \in V$,*

$$|b(u, v, u)| \leq \|u\|_{(L^4(\Omega))^2}^2 \|v\| \leq 2^{-\frac{1}{2}} |u| \|u\| \|v\|.$$

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It is required that we choose a suitable phase space. There are several phase spaces which allow us to deal with infinite delays (see e.g. [31]). In this paper, inspired by recent works [27], we will use the following phase space

$$BCL_{-\infty}(H) = \{\varphi \in C((-\infty,,0];H) : \lim_{\tau \to -\infty} \varphi(\tau) \text{ exists in } H\},$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{BCL_{-\infty}(H)} = \sup_{\tau \in (-\infty,0]} |\varphi(\tau)|.$$

We introduce some notations and assumptions on the delay operator. Let X be a Banach space and consider a fixed $T > 0$. Given $u : (-\infty, T) \to X$, for each $t \in (0, T)$, we define the function u_t on $\tau \in (-\infty, 0]$. Referring to [4, 27], we now make the following assumptions:

(H2) $f \in H$;

(H3) Assume that $F : [0, T] \times BCL_{-\infty}(H) \to H$, then

    (1) for any $\xi \in BCL_{-\infty}(H)$, the mapping $[0, T] \ni t \mapsto F(t, \xi) \in (L^2(\Omega))^2$ is measurable,

    (2) $F(0) = 0$,

    (3) there exists a constant $L_F > 0$ such that, for any $t \in [0, T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,
$$|F(\xi) - F(\eta)| \le L_F \|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Remark 2.1. *It is noticed that the assumption* (H3)(2) *is not really restrictive. Indeed, if* $|F(0)| \in L^2(0, T)$, *we could redefine* $\hat{f}(t) = f(t) + F(0)$ *and* $\hat{F}(u_t) = F(u_t) - F(0)$, *which also fulfill the above assumptions. In addtion, conditions* (2) *and* (3) *imply that, for any* $\xi \in BCL_{-\infty}(H)$,
$$|F(t, \xi)| \le L_F \|\xi\|_{BCL_{-\infty}(H)}, \qquad \forall t \in [0, T].$$

Example 2.1. *We provide an example of the delay term* $F(u_t)$ *as in [31]. We take* $F : [0, T] \times BCL_{-\infty}(H) \to H$, *for all* $t \in [0, +\infty]$, $\xi \in BCL_{-\infty}(H)$. *We take* F *given by*

$$F(\xi) := \int_{-\infty}^{0} G(t, s, \xi(s))(x)\mathrm{d}s,$$

where the function $G : [0, T] \times (-\infty, 0] \times \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the following assumptions:*

    (1) $G(t, s, 0) = 0$ *for all* $(t, s) \in [0, T] \times (-\infty, 0)$;

    (2) *There exists a function* $\delta : (-\infty, 0) \to (0, \infty)$ *such that*

$$|G(t, s, u) - G(t,, s, v)|_{\mathbb{R}^2} \le \delta(s)|u - v|_{\mathbb{R}^2}, \qquad \forall u, v \in \mathbb{R}^2, \forall t \in [0, T], s \in (-\infty, 0).$$

We check that F satisfies the assumptions (H3)(3). It is well defined as a map with values in H:

$$
\begin{aligned}
|F(\xi) - F(\eta)|^2 &= \int_\Omega \left(\int_{-\infty}^0 |G(t,s,\xi(s)(x)) - G(t,s,\eta(s)(x))|_{\mathbb{R}^2} \right)^2 \mathrm{d}x \\
&\leq \int_\Omega \left(\int_{-\infty}^0 \delta(s)|\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2} \mathrm{d}s \right)^2 \mathrm{d}x \\
&\leq \int_\Omega \left(\int_{-\infty}^0 \delta(s)\mathrm{d}s \right) \left(\int_{-\infty}^0 \delta(s)|\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2}^2 \mathrm{d}s \right) \mathrm{d}x \\
&\leq \|\delta\|_{L^1(-\infty,0)} \int_{-\infty}^0 \delta(s) \int_\Omega |\xi(s)(x) - \eta(s)(x)|^2 \mathrm{d}x \mathrm{d}s \\
&\leq \|\delta\|_{L^1(-\infty,0)} \int_{-\infty}^0 \delta(s) \left(\sup_{s \leq 0} \int_\Omega |\xi(s)(x) - \eta(s)(x)|^2 \mathrm{d}x \right) \mathrm{d}s \\
&\leq \|\delta\|_{L^1(-\infty,0)} \|\xi - \eta\|_{BCL_{-\infty}}^2 .
\end{aligned}
$$

As in [16], to deal with the memory term, we introduce a new variable which reflects the past history of equation (1.1), that is,

$$
\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)\mathrm{d}r, \qquad s \geq 0,
$$

then

$$
\partial_t \eta^t(x,s) = u(x,t) - \partial_s \eta^t(x,s), \qquad s \geq 0.
$$

Using the fact that $\mu(s) = -\kappa'(s)$ and $\kappa(\infty) = 0$, and applying the Helmholtz-Leray projection to (1.1), we have

$$
\begin{cases}
\partial_t u + \nu A u + \displaystyle\int_0^{+\infty} \mu(s) A \eta^t(x,s)\mathrm{d}s \\
\qquad\qquad\qquad + B(u,u) = f + F(u_t), & (x,t) \in \Omega \times (0,+\infty), \\
\partial_t \eta^t(x,s) = -\partial_s \eta^t(x,s) + u(x,t), & (x,t) \in \Omega \times (0,+\infty), s \in (-\infty, 0], \\
u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\
\eta^t(x,s) = 0, & (x,t) \in \partial\Omega \times (0,\infty), s \in [0,\infty), \\
u(x,\tau) = \phi(x,\tau), & (x,\tau) \in \Omega \times (-\infty, 0], \\
\eta^0(x,s) = \eta_0(x,s) = \displaystyle\int_{-s}^0 \phi(x,r)dr, & (x,s) \in \Omega \times [0,\infty).
\end{cases} \tag{2.1}
$$

In consideration of memory term, let $L_\mu^2(\mathbb{R}^+, H)$ be the Hilbert space of the functions $\varphi : \mathbb{R}^+ \to H$ endowed with the inner product and norm respectively

$$
\langle \varphi_1, \varphi_2 \rangle_\mu = \int_0^\infty \mu(s) \langle \varphi_1, \varphi_2 \rangle \mathrm{d}s,
$$

$$
|\varphi|_\mu = \left(\int_0^\infty \mu(s)|\varphi|^2 \mathrm{d}s \right)^{\frac{1}{2}}.
$$

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Similarly, we make the Hilbert spaces $L^2_\mu(\mathbb{R}^+, V)$ and $L^2_\mu(\mathbb{R}^+, (H^2(\Omega))^2 \cap V)$ with the inner product

$$\langle \cdot, \cdot \rangle_{1,\mu} = \langle \nabla \cdot, \nabla \cdot \rangle_\mu,$$

$$\langle \cdot, \cdot \rangle_{2,\mu} = \langle \Delta \cdot, \Delta \cdot \rangle_\mu,$$

and endow with the norm $\|\varphi\|_{1,\mu}$ and $\|\varphi\|_{2,\mu}$ respectively.

In addition, we define the space $\mathcal{H}^* := BCL_{-\infty}(H) \times L^2_\mu(\mathbb{R}^+, V)$ with the following norm

$$\|(\psi, \varphi)\|^2_{\mathcal{H}^*} = \|\psi\|^2_{BCL_{-\infty}(H)} + \|\varphi\|^2_{1,\mu}.$$

3   Existence and uniqueness of weak solutions

In this section, we first give the definition of weak solutions to (2.1) and state the main result.

Definition 3.1. *Given functions $u_0 = \phi \in BCL_{-\infty}(H)$ and $\eta_0 = \int_{-s}^0 \phi(x,r)\mathrm{d}r \in L^2_\mu(\mathbb{R}^+, V)$. Functions $u \in C((-\infty, T]; H)$ and $\eta^t \in C([0,T]; L^2_\mu(\mathbb{R}^+, V))$ are called a weak solution of problem* (2.1) *for $v \in V$ and $\xi \in L^2_\mu(\mathbb{R}^+, V)$, if*

$$
\begin{aligned}
(\partial_t u, v) + \nu((u, v)) + \langle \eta^t, v \rangle_{1,\mu} + b(u, u, v) &= \langle f, v \rangle + (F(u_t), v), \\
\langle \partial_t \eta^t + \partial_s \eta^t, \xi \rangle_{1,\mu} &= \langle u, \xi \rangle_{1,\mu},
\end{aligned}
\tag{3.1}
$$

hold in the sense of $\mathcal{D}'(0, T)$, where

$$\partial_t u \in L^2(0, T; V^*)$$

and

$$\eta^t \in C([0, T]; L^2_\mu(\mathbb{R}^+, V)).$$

Theorem 3.1. *Assume that the assumptions* (H1) − (H3) *hold. Then for $u_0 \in BCL_{-\infty}(H)$, $\eta_0 \in L^2_\mu(\mathbb{R}^+, V)$ and $T > 0$, there exists a unique weak solution (u, η^t) of problem* (2.1).

Proof. (i) **Existence**. We outline the proof of the existence with three steps.

Step 1: Galerkin scheme. First of all, let $\{\omega_j\}_{j=1}^\infty$ be a smooth orthonormal basis of H which is also orthogonal in V. Secondly, we select an orthonormal basis $\{\zeta\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+, V)$ with all ξ_j in $\mathcal{D}(\mathbb{R}^+, V)$, where $\mathcal{D}(I, X)$ is the space of infinitely differentiable X-valued functions with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution space on I with values in X^*, denoted by $\mathcal{D}(I, X^*)$. And one can take a complete set of normalized eigenfunctions for A in V such that $A\omega_j = \lambda_j \omega_j$. Given an integer n, denote by P_n and Q_n the projections on the subspaces $V_n = \text{span}\{\omega_1, \omega_2, \ldots, \omega_n\}$ and $L^2_\mu(\mathbb{R}^+, V) = \text{span}\{\zeta_1, \zeta_2 \ldots, \zeta_n\}$ respectively. Set

$$u_n(t) = \sum_{j=1}^n a_j(t)\omega_j \qquad \text{and} \qquad \eta_n^t(s) = \sum_{j=1}^n b_j(t)\zeta_j(s).$$

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Therefore, we seek a function (u_n, η_n^t) satisfying

$$
\begin{aligned}
\langle (\partial_t u_n, \eta_{n,t}^t), (\omega_k, \zeta_j) \rangle_{H \times L_\mu^2(\mathbb{R}^+, V)} &= \Big\langle \big(-\nu A u_n - \int_0^\infty \mu(s) A \eta_n^t(s) \mathrm{d}s \\
&\quad - B(u_n, u_n) + f + F(u_{nt}), u_n - \partial_s \eta_n^t \big), (\omega_k, \zeta_j) \Big\rangle_{H \times L_\mu^2(\mathbb{R}^+, V)},
\end{aligned}
$$

$$
(u_n, \eta_n^t)|_{t=0} = (P_n u_0, Q_n \eta_0),
$$

(3.2)

for a.e. $t \in [0, T]$ and every $k, j = 0, 1, \ldots, n$, where ω_0, ζ_0 are the zero vectors. Then, we take (ω_k, ζ_0) and (ω_0, ζ_k) in (3.2), and apply the divergence theorem to the term

$$
\Big\langle \int_0^\infty \Delta \eta_n^t \mathrm{d}s, \omega_k \Big\rangle,
$$

to get a system about the variable $a_k(t)$ and $b_k(t)$ of the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} a_k &= -\nu \lambda_k a_k - \sum_{j=1}^n b_j \langle \zeta_j, \omega_k \rangle_{1,\mu} - b \Big(\sum_{j=1}^n a_j \omega_j, \sum_{j=1}^n a_j \omega_j, \omega_k \Big) \\
&\quad + \langle f, \omega_k \rangle + (F(u_{nt}), \omega_k), \\
\frac{\mathrm{d}}{\mathrm{d}t} b_k &= \sum_{j=1}^n a_j \langle \omega_j, \zeta_k \rangle_{1,\mu} - \sum_{j=1}^n b_j \langle \zeta_j', \zeta_k \rangle_{1,\mu},
\end{aligned}
$$

(3.3)

with the initial conditions

$$
a_k(0) = \langle u_0, \omega_k \rangle_H,
$$

$$
b_k(0) = \langle \eta_0, \zeta_k \rangle_{1,\mu}.
$$

The above system of ordinary differential equations with infinite delay fulfills the conditions for the existence and uniqueness of local solutions in [23], so the appropriate solution (u_n, η_n^t) exists.

Step 2: A priori estimates. Multiplying the first equation of (3.3) by a_k and the second by b_k, then summing in k and adding the result, we get

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(|u_n|^2 + \|\eta_n^t\|_{1,\mu}^2 \right) = -\nu \|u_n\|^2 - \langle \partial_s \eta_n^t, \eta_n^t \rangle_{1,\mu} + \langle f, u_n \rangle + (F(u_{nt}), u_n).
$$

(3.4)

From the Cauchy's inequality and Remark 2.1, it follows that

$$
\langle f, u_n \rangle + (F(u_{nt}), u_n) \le \varepsilon \|u_n\|^2 + \frac{\|f\|_*^2}{4\varepsilon} + L_F \|u_{nt}\|_{BCL_{-\infty}(H)}^2,
$$

(3.5)

where we have used $|u_n| \le \|u_{nt}\|_{BCL_{-\infty}(H)}$. Applying integration by parts, we arrive at

$$
\langle \partial_s \eta_n^t, \eta_n^t \rangle_{1,\mu} = -\frac{1}{2} \int_0^\infty \mu'(s) \|\eta_n^t(s)\|^2 \mathrm{d}s \ge 0.
$$

(3.6)

Adding the (3.5) and (3.6) into (3.4) leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left(|u_n|^2 + \|\eta_n^t\|_{1,\mu}^2 \right) + 2(\nu - \varepsilon) \|u_n\|^2 \le \frac{\|f\|_*^2}{2\varepsilon} + 2 L_F \|u_{nt}\|_{BCL_{-\infty}(H)}^2, \qquad \forall\, t \in [0, T]. \quad (3.7)
$$

Choosing $\varepsilon > 0$ small enough such that $\nu - \varepsilon > 0$ and integrating (3.7) from 0 to t, we obtain

$$\left(|u_n|^2 + \|\eta_n^t\|_{1,\mu}^2\right) + 2(\nu - \varepsilon) \int_0^t \|u_n(r)\|^2 dr \leq |u_0|^2 + \|\eta^0\|_{1,\mu}^2 + \frac{\|f\|_*^2}{2\varepsilon} T$$

$$+ 2L_F \int_0^t \|u_{nr}\|_{BCL_{-\infty}(H)}^2 dr.$$

Thus,

$$\left(\|u_{nt}\|_{BCL_{-\infty}(H)} + \|\eta_n^t\|_{1,\mu}^2\right) + 2(\nu - \varepsilon) \int_0^t \|u_n(r)\|^2 dr$$

$$\leq (|u_0|^2 + \|\eta^0\|_{1,\mu}^2) + \frac{\|f\|_*^2}{2\varepsilon} T + 2L_F \int_0^t (\|u_{nr}\|_{BCL_{-\infty}(H)}^2 + \|\eta^t\|_{1,\mu}^2) dr.$$

By the Gronwall inequality, we have

$$\sup_{\tau \in [0,t]} |u_n(\tau)|^2 + \|\eta_n^t\|_{1,\mu}^2 \leq \left(|u_0|^2 + \|\eta^0\|_{1,\mu}^2 + \frac{\|f\|_*^2}{2\varepsilon} T\right) e^{2L_F t}, \quad (3.8)$$

which explains that there exists a constant C dependent of L_F, f, T, R such that

$$\|u_{nt}\|_{BCL_{-\infty}(H)} \leq C, \qquad \forall\, t \in [0, T],$$

where $R > 0$ and $\|\phi\|_{BCL_{-\infty}(H)} \leq R$. Especially, it's easy to imply that

$$\{u_n\} \text{ is bounded in } L^\infty(0, T; H).$$

Also, we claim that

$$\{\eta_n^t\} \text{ is bounded in } L^\infty(0, T; L_\mu^2(\mathbb{R}^+, V)), \quad (3.9)$$
$$\{u_n\} \text{ is bounded in } L^2(0, T; V).$$

Through the Banach-Alaoglu theorem, there exists a function (u, η^t) such that

$$\{u_n\} \rightharpoonup u \text{ weakly star in } L^\infty(0, T; H),$$
$$\{\eta_n^t\} \rightharpoonup \eta^t \text{ weakly star in } L^\infty(0, T; L_\mu^2(\mathbb{R}^+, V)), \quad (3.10)$$
$$\{u_n\} \rightharpoonup u \text{ weakly in } L^2(0, T; V).$$

Subsequently, we give the time derivative of the weak solution in $L^2(0, T; H)$. Assume that $\varphi \in V$. Taking the L^2 product of the Galerkin equation

$$\partial_t u_n + \nu A u_n + \int_0^{+\infty} \mu(s) A \eta_n^t(x, s) ds + B(u, u) = f + F(u_{nt}),$$

with φ, we obtain

$$\langle \partial_t u_n, \varphi \rangle + \langle \nu A u_n, \varphi \rangle + \left\langle \int_0^{+\infty} \mu(s) A \eta_n^t(x, s) ds, \varphi \right\rangle + \langle B(u, u), \varphi \rangle$$
$$= \langle f, \varphi \rangle + \langle F(u_{nt}), \varphi \rangle. \quad (3.11)$$

We estimate the norm $\|\partial_t u_n\|_{V^*}$ through the estimates of the right-hand side of the above identity. It is easy to get the estimates

$$|\langle \nu A u_n, \varphi\rangle| \le |\nu\langle \nabla u_n, \nabla\varphi\rangle| \le \nu\|u_n\|\|\varphi\|, \tag{3.12}$$

$$\left|\left\langle \int_0^{+\infty} \mu(s)A\eta_n^t(x,s)\mathrm{d}s, \varphi\right\rangle\right| \le \left|\left\langle \int_0^{+\infty}\mu(s)\nabla\eta_n^t(x,s)\mathrm{d}s, \nabla\varphi\right\rangle\right| \le \kappa(0)\|\eta_n^t\|_{1,\mu}\|\varphi\|, \tag{3.13}$$

$$\langle f, \varphi\rangle \le \|f\|_*\|\varphi\|. \tag{3.14}$$

For the nonlinear term, by Lemma 2.1, we obtain

$$\langle B(u,u), \varphi\rangle \le c|u_n|\|u_n\|\|\varphi\|. \tag{3.15}$$

Using the Poincaré inequality, we get

$$\langle F(u_{nt}), \varphi\rangle \le L_F \lambda_1^{-\frac{1}{2}}\|u_{nt}\|_{BCL_{-\infty}(H)}\|\varphi\|. \tag{3.16}$$

Plugging estimates $(3.12) - (3.16)$ into (3.11), we deduce the differential inequality

$$\|\partial_t u_n\|_{V^*} \le \nu\|u_n\| + c|u_n|\|u_n\| + \kappa(0)\|\eta^t\|_{1,\mu} + \|f\|_* + L_F\lambda_1^{-\frac{1}{2}}\|u_{nt}\|_{BCL_{-\infty}(H)}, \tag{3.17}$$

which shows that

$$\{\partial_t u_n\} \text{ is bounded in } L^2(0,T;V^*). \tag{3.18}$$

Also, we get

$$\int_0^T \int_\Omega (u_n \cdot \nabla)u_n \omega_j \psi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega (u \cdot \nabla)u \omega_j \psi \mathrm{d}x\mathrm{d}t,$$

through the argument in the Theorem 3.2 in [6], for any function ω_j in the basis and any continuously differentiable function ψ on $[0,T]$.

Nevertheless, it is not sufficient to pass to the limit in the delay term $F(u_t)$ because of its discontinuity, we need some kind of strong convergence.

Step 3: Approximation in $BCL_{-\infty}(H)$ of the initial datum.

In this part, we establish

$$u_{nt} \to u_t \text{ in } [0,T] \times BCL_{-\infty}(H) \quad \forall\, t \in (-\infty, T], \tag{3.19}$$

by proving the following:

$$P_n\phi \to \phi \quad \text{in } [0,T] \times BCL_{-\infty}(H),$$
$$u_n \to u \quad \text{in } C([0,T];H). \tag{3.20}$$

Let us check the first convergence of (3.20). Indeed, if not, then there exists $\varepsilon > 0$ and a subsequence (relabeled the same), such that

$$|P_n\phi(\theta_n) - \phi(\theta_n)| > \varepsilon. \tag{3.21}$$

One can assume that $\theta_n \to -\infty$, otherwise if $\theta_n \to \theta$, then $P_n\phi(\theta_n) \to \phi(\theta_n)$, since $|P_n\phi(\theta_n) - \phi(\theta)| \leq |P_n\phi(\theta_n) - P_n\phi(\theta)| + |P_n\phi(\theta) - \phi(\theta)| \to 0$ as $n \to +\infty$. However, with $\theta_n \to -\infty$ as $n \to +\infty$, if we denote $x = \lim\limits_{\theta\to-\infty} \phi(\theta)$, we get

$$|P_n\phi(\theta_n) - \phi(\theta_n)| \leq |P_n(\phi(\theta_n)) - P_n x| + |P_n x - x| + |x - \phi(\theta_n)| \to 0,$$

which is a contradiction with (3.21), so the first formula of (3.20) holds. Now, we prove the strong convergence of $\{u_n\}$ to u in $C([0,T];H)$ via embedding theorem. From Step 2, we have

$$u_n(t) \to u(t) \qquad \text{in } H \text{ for a.e. } t \in (0,T). \tag{3.22}$$

Observing that

$$u_n(t) - u_n(s) = \int_s^t (u_n)'(r)\mathrm{d}r \quad \text{in } V' \quad \forall\, s,t \in [0,T],$$

and (3.18), we have that $\{u_n\}$ is equi-continuous on $[0,T]$ with values in V'. And from the compactness of V into V', (3.9) and the equi-continuity in V', using the Arzela-Ascoli theorem, we obtain

$$u_n(t) \to u(t) \qquad \text{in } C([0,T];V'). \tag{3.23}$$

On account of (3.9), we obtain that for any sequence $\{t_n\} \subset [0,T]$ with $t_n \to t$,

$$u_n(t_n) \rightharpoonup u(t) \qquad \text{weakly in } H, \tag{3.24}$$

where we have used (3.23) to identify the weak limit $u(t)$. At present, we prove the second convergence of (3.20) by a contrary argument. If it is not so, then considering $u \in C([0,T];H)$, there would exist $\varepsilon > 0$, a value $t_0 \in [0,T]$ and subsequences (relabeled the same) u_n and $t_n \subset [0,T]$ with $\lim\limits_{n\to+\infty} t_n = t_0$ such that

$$|u_n(t_n) - u(t_0)| \geq \varepsilon, \quad \forall\, n. \tag{3.25}$$

On the one hand, Since $u(\cdot)$ is continuous at t_0, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|u(t_n) - u(t_0)|^2 \leq \frac{\varepsilon}{20}, \qquad \forall\, n \geq n_\delta. \tag{3.26}$$

On the other hand, using the fact that

$$u_n(t) \to u(t) \qquad \text{in } L^2(0,T;H),$$

we are easy to get

$$\int_{t_0-\delta}^{t_0+\delta} |u_n(s) - u(s)|^2 \mathrm{d}s < \varepsilon,$$

then there is a $t' \in (t_0 - \delta, t_0 + \delta)$ such that

$$|u_n(t') - u(t')|^2 < \frac{\varepsilon}{4}, \tag{3.27}$$

with the integral mean value theorem. In addition, using the fact that $u(t)$ is continuous and choosing $|t_n - t'| < |t_n - t_0| + |t_0 - t'| < 2\delta$, we can obtain

$$|u_n(t_n) - u(t_0)|^2 \leq 2(|u_n(t_n) - u(t_n)|^2 + |u(t_n) - u(t_0)|^2)$$
$$\leq 2[2(|u_n(t_n) - u_n(t')|^2 + |u_n(t') - u(t')|^2 + |u(t') - u(t_n)|^2) \qquad (3.28)$$
$$+ |u(t_n) - u(t_0)|^2].$$

Combining (3.26) and (3.27), we deduce that

$$|u_n(t_n) - u(t_0)|^2 < \varepsilon,$$

which contradicts (3.25). It impies obviously that

$$|u_n(t_n) - u(t_0)|^2 \to 0.$$

Hence, we conclude

$$F(u_{nt}) \to F(u_t) \qquad \text{in } L^2(0, T; H).$$

At last, we will show that the limit (u, η^t) is a weak solution of (2.1). Choosing the arbitrary test function

$$\psi = (\varphi, \xi) \in \mathcal{D}([0, T], V \cap L^\infty(\Omega)) \times \mathcal{D}([0, T], \mathcal{D}(\mathbb{R}^+, V))$$

of the form

$$\varphi(t) = \sum_{j=1}^n a_j(t)\omega_j \qquad \text{and} \qquad \xi = \sum_{j=1}^n b_j(t)\zeta_j,$$

where n is a fixed integer, and $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ are given functions in $\mathcal{D}((0, T))$. We replace (ω_k, ζ_j) with (φ, ξ) in (3.2). Then integrating the resulting equation over $(0, T)$ and passing to the limits, we obtain

$$\int_0^T [(\partial_t u_n(t), \varphi) + \langle \partial_t \eta^t, \xi \rangle_{1,\mu}] dt$$
$$= -\nu \int_0^T [((u(t), \varphi)) + \langle \eta^t, \varphi \rangle] dt - \int_0^T b(u(t), u(t), \varphi) dt \qquad (3.29)$$
$$+ \int_0^T [\langle f, \varphi \rangle + (F(u_t), \varphi)] dt + \int_0^T [-\langle \partial_s \eta^t, \xi \rangle_{1,\mu} + \langle u, \xi \rangle_{1,\mu}] dt.$$

Using a density argument, we have that (u, η^t) satisfies the equation in the weak sense. Moreover, we check that (u, η^t) also fulfill the initial datum $(u, \eta^t)(0) = (u_0, \eta^0)$. Therefore, it implies (u, η^t) is a weak solution of problem (2.1).

(ii) **Uniqueness**. We assume that (u_1, η_1^t) and (u_2, η_2^t) are different solutions of (2.1). Let $(u_3, \eta_3^t) = (u_1, \eta_1^t) - (u_2, \eta_2^t)$, then

$$\partial_t u_3 + \nu A u_3 + \int_0^\infty \mu(s) A \eta_3^t(x, s) ds + B(u_1, u_1)$$
$$- B(u_2, u_{2)} = F(u_{1t}) - F(u_{2t}). \qquad (3.30)$$

Multiplying the equation (3.30) by u_3, and using the fact that

$$b(u_1,u_1,u_3) - b(u_2,u_2,u_3) = b(u_3,u_1,u_3),$$

we can get

$$\frac{1}{2}\frac{d}{dt}|u_3|^2 + \nu\|u_3\|^2 + b(u_3,u_1,u_3) + \int_0^\infty \mu(s)\int_\Omega \nabla\eta_3^t \nabla u_3 dxds \tag{3.31}$$
$$=(F(u_{1t}) - F(u_{2t}), u_3).$$

Briefly, we get

$$|(F(u_{1t}) - F(u_{2t}), u_3)| \le L_F\|u_{3t}\|^2_{BCL_{-\infty}(H)}.$$

Integrating by parts, we obtain

$$\int_0^\infty \mu(s)\int_\Omega \nabla\eta_3^t \nabla u_3 dx = \int_0^\infty \mu(s)\int_\Omega \nabla\eta_3^t \nabla\partial_t\eta_3^t dx + \int_0^\infty \mu(s)\int_\Omega \nabla\partial_s\eta_3^t \nabla\eta_3^t dxds$$
$$= \frac{1}{2}\frac{d}{dt}\|\eta_3^t\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)\int_\Omega |\nabla\eta_3^t|^2 dxds.$$

In particular,

$$|b(u_3,u_1,u_3)| \le 2^{-\frac{1}{2}}|u_3|\|u_3\|\|u_1\| \le \nu\|u_3\|^2 + \frac{1}{8\nu}\|u_1\|^2|u_3|^2.$$

Therefore,

$$\frac{d}{dt}(|u_3(t)|^2 + \|\eta_3^t\|^2_{1,\mu}) \le \frac{1}{8\nu}\|u_1\|^2|u_3|^2 + 2L_F\|u_{3t}\|^2_{BCL_{-\infty}(H)}$$
$$\le \left(\frac{1}{8\nu}\|u_1\|^2 + 2L_F\right)\left(\|u_{3t}\|^2_{BCL_{-\infty}(H)} + \|\eta_3^t\|^2\right).$$

Whence applying the Gronwall inequality, we get

$$\sup_{\tau\in[0,t]}|u_3(\tau)|^2 + \|\eta_3^t\|^2_{1,\mu} \le (|u_3(\tau)|^2 + \|\eta_3^0\|^2_{1,\mu})\exp\left(\left(\frac{1}{8\nu} + 2L_F\right)\left(\int_0^t(1 + \|u_1(r)\|^2)dr\right)\right).$$

$\square$

Remark 3.1. *The argument presented in Theorem 3.1 also shows that the existence of u_t is global in time. In fact, one can clarify that $(u_t,\eta^t) \in \mathcal{H}^*$.*

4   Existence of global attractor

For each $(u_0,\eta^0) \in \mathcal{H}^*$, satisfying $u_0 = \phi$ and $\eta^0 = \int_{-s}^0 \phi(x,r)dr$, we denote a semigroup $S(t) : \mathcal{H}^* \to \mathcal{H}^*$ by the formula

$$S(t)(u_0,\eta^0) = (u(t+\tau),\eta^t), \qquad \tau\in(-\infty,0],$$

where $(u(t),\eta^t)$ is the unique global weak solution of (2.1). In this section, we will prove the existence of a compact global attractor for the semigroup $S(t)$.

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4.1 Continuity of the semigroup in $\mathcal{H}^*$

Proposition 4.1. *Assume that conditions* (H1) − (H3) *hold. For* $(u_0, \eta^0) \in \mathcal{H}^*$, *the semigroup* $S(t)$ *is continuous on* $\mathcal{H}^*$.

Proof. Denoting (u_i, η_i^t) for $i = 1, 2$, the corresponding solutions to initial datum $(u_{0i}, \eta_i^0) \in \mathcal{H}^*$. Considering the equations satisfied by u_i for $i = 1, 2$, acting on the element $u_1 - u_2$, and taking the difference, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_1 - u_2|^2 + \|\eta_1^t - \eta_2^t\|_{1,\mu}\right) + \nu\|u_1 - u_2\|^2 - \frac{1}{2}\int_0^\infty \mu'(s)|\nabla(\eta_1^t - \eta_2^t)|^2\mathrm{d}s$$
$$+ b(u_1, u_1, u_1 - u_2) - b(u_2, u_2, u_1 - u_2) = (F(u_{1t}) - F(u_{2t}), u_1 - u_2).$$

As in the proof of Theorem 3.1, using the Ladyzhenskaya's inequality, we get

$$|b(u_1, u_1, u_1 - u_2) - b(u_2, u_2, u_1 - u_2)| = |b(u_1 - u_2, u_1, u_1 - u_2)|$$
$$\leq C|u_1 - u_2|\|u_1 - u_2\|\|u_1\|.$$

And by the assumptions on delay term, we yields

$$|u_{1t} - u_{2t}|_{BCL_{-\infty}(H)}$$
$$= \sup_{\tau \in (-\infty, 0]} |u_1(t + \tau) - u_2(t + \tau)|$$
$$= \max\left\{\sup_{\tau \in (-\infty, -t]} |\phi_1(t + \tau) - \phi_2(t + \tau)|; \sup_{\tau \in [-t, 0]} |u_1(t + \tau) - u_2(t + \tau)|\right\}$$
$$\leq \max\left\{\|\phi_1 - \phi_2\|_{BCL_{-\infty}(H)}; \max_{\tau \in [0,t]} |u_1(\tau) - u_2(\tau)|\right\},$$

where $\phi_1, \phi_2 \in BCL_{-\infty}(H)$ are the initial datum of u. Therefore,

$$|u_1 - u_2|^2 + \|\eta_1^t - \eta_2^t\|_{1,\mu} + 2\nu\int_0^t \|u_1 - u_2\|^2\mathrm{d}r = |u_1(0) - u_2(0)|^2 + \|\eta_1^0 - \eta_2^0\|_{1,\mu}^2$$
$$+ 2\int_0^t b(u_1 - u_2, u_1, u_1 - u_2)\mathrm{d}r + 2\int_0^t (F(u_{1t}) - F(u_{1t}), u_1 - u_2)\mathrm{d}r,$$

then, we conclude that
$$|u_1 - u_2|^2 + \|\eta_1^t - \eta_2^t\|_{1,\mu}$$
$$\leq |\phi_1(0) - \phi_1(0)|^2 + \left\|\int_{-s}^0 \nabla\phi_1(x, r)\mathrm{d}r - \int_{-s}^0 \nabla\phi_2(x, r)\mathrm{d}r\right\|_{1,\mu}^2 + \frac{C}{\nu}\int_0^t \|u_1\|^2|u_1 - u_2|^2\mathrm{d}r$$

$$+ 2L_F\|\phi_1 - \phi_2\|_{BCL_{-\infty}(H)}\int_0^t |u_1(r) - u_2(r)|\mathrm{d}r$$

$$+ 2L_F\int_0^t |u_1(r) - u_2(r)|\max_{\tau \in [0,r]} |u_1(\tau) - u_2(\tau)|\mathrm{d}r$$

$$\leq |\phi_1(0) - \phi_1(0)|^2 + \left\|\int_{-s}^0 \nabla\phi_1(x, r)\mathrm{d}r - \int_{-s}^0 \nabla\phi_2(x, r)\mathrm{d}r\right\|_{1,\mu}^2$$

$$+ \frac{C}{\nu}\int_0^t \max_{\tau \in [0,r]} \|u_1\|^2|u_1(\tau) - u_2(\tau)|^2\mathrm{d}r + L_F\|\phi_1 - \phi_2\|_{BCL_{-\infty}(H)}^2$$

$$+ L_F\int_0^t |u_1 - u_2|^2\mathrm{d}r + 2L_F\int_0^t \max_{\tau \in [0,r]} |u_1(\tau) - u_2(\tau)|^2\mathrm{d}r.$$

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Substituting t by $r \in [0,t]$, then

$$\max_{r \in [0,t]} |u_1 - u_2|^2 + \|\eta_1^t - \eta_2^t\|_{1,\mu}$$

$$\leq |\phi_1(0) - \phi_1(0)|^2 + \left\| \int_{-s}^0 \nabla \phi_1(x,r) \mathrm{d}r - \int_{-s}^0 \nabla \phi_2(x,r) \mathrm{d}r \right\|_{1,\mu}^2 + L_F \|\phi_1 - \phi_2\|_{BCL_{-\infty}(H)}$$

$$+ \int_0^t \left(\frac{C}{\nu} \|u_1\|^2 + 3L_F \right) \max_{\tau \in [0,r]} |u_1(\tau) - u_2(\tau)|^2 \mathrm{d}r$$

$$\leq |\phi_1(0) - \phi_1(0)|^2 + \left\| \int_{-s}^0 \nabla \phi_1(x,r) \mathrm{d}r - \int_{-s}^0 \nabla \phi_2(x,r) \mathrm{d}r \right\|_{1,\mu}^2 + L_F \|\phi_1 - \phi_2\|_{BCL_{-\infty}(H)}$$

$$+ \int_0^t \left(\frac{C}{\nu} \|u_1\|^2 + 3L_F \right) \max_{\tau \in [0,r]} \left\{ |u_1(\tau) - u_2(\tau)|^2 + \|\eta_1^\tau - \eta_2^\tau\|_{1,\mu}^2 \right\} \mathrm{d}r.$$

This complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.2   Existence of a absorbing set for the semigroup

In this subsection, we state and prove the existence of a bounded absorbing set. Before proving the result, let us first introduce the "modified" energy

$$\hat{E}(t) = \frac{1}{2}(\|u_t\|_{BCL_{-\infty}(H)}^2 + \|\eta^t\|_{1,\mu}^2), \tag{4.1}$$

which has the following relation to the energy associated to the problem (2.1)

$$E(t) = \frac{1}{2}(|u(t)|^2 + \|\eta^t\|_{1,\mu}^2) \leq \hat{E}(t).$$

A direct differentiation to $(2.1)_1$ leads to

$$E'(t) = -\nu \|u(t)\|^2 + \frac{1}{2} \int_0^\infty \mu'(s) \|\eta^t\|^2 \mathrm{d}s + \int_\Omega f u \mathrm{d}x + \int_\Omega F(u_t) u \mathrm{d}x, \tag{4.2}$$

and then using the Young's inequality, we get

$$\int_\Omega f u \mathrm{d}x \leq \nu \|u(t)\|^2 + c_1 \|f\|_*^2,$$

in conclusion,

$$\hat{E}'(t) \leq \frac{1}{2} \int_0^\infty \mu'(s) \|\eta^t\|^2 \mathrm{d}s + c_1 \|f\|_*^2 + \|u_t\|_{BCL_{-\infty}(H)}^2. \tag{4.3}$$

Lemma 4.1. *Assume that* (H1) − (H3) *are satisfied, then there exists a bounded absorbing set for the semigroup* $S(t)$.

The proof of Lemma 4.1 is based on three lemmas.

Lemma 4.2. *Under the assumptions* (H1) − (H3), *the functional*

$$K_1(t) = -\int_\Omega u \int_0^\infty \mu(s) \eta^t \mathrm{d}s \mathrm{d}x,$$

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satisfies, for all $t \in \mathbb{R}^+$,

$$K_1'(t) \leq -\left(\kappa(0) - c_2 - c_{11}\right)\|u_t\|^2_{BCL_{-\infty}(H)} - c_3 \int_0^\infty \mu'(s)\|\eta^t\|^2 ds + (c_4\nu + c_6)\|u(t)\|^2$$
$$+ (c_5 + c_6 + c_8|u(t)|^2 + c_{10} + c_{12})\|\eta^t\|^2_{1,\mu} + c_9\|f\|^2_*,$$

(4.4)

where $c_i, i = 1, 2, ..., 12$ are suitable constants.

Proof. Let us now multiply $(2.1)_1$ by $\int_0^\infty \mu(s)\eta^t ds$. We get

$$\left\langle \partial_t u, \int_0^\infty \mu(s)\eta^t ds \right\rangle + \left\langle \nu A u, \int_0^\infty \mu(s)\eta^t ds \right\rangle + \left\langle \int_0^\infty \mu(s)A\eta^t ds, \int_0^\infty \mu(s)\eta^t ds \right\rangle$$
$$+ b\left(u, u, \int_0^\infty \mu(s)\eta^t ds\right) = \left\langle f, \int_0^\infty \mu(s)\eta^t ds \right\rangle + \left\langle F(u_t), \int_0^\infty \mu(s)\eta^t ds \right\rangle.$$

By $(2.1)_2$, we find

$$\left\langle \partial_t u, \int_0^\infty \mu(s)\eta^t ds \right\rangle = \partial_t \left\langle u, \int_0^\infty \mu(s)\eta^t ds \right\rangle - \left\langle u, \int_0^\infty \mu(s)\partial_t \eta^t ds \right\rangle$$
$$= -K_1'(t) - \kappa(0)|u(t)|^2 - \left\langle u, \int_0^\infty \mu'(s)\eta^t ds \right\rangle,$$

therefore,

$$K_1'(t) = -\kappa(0)|u(t)|^2 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

(4.5)

where

$$I_1 = -\left\langle u, \int_0^\infty \mu'(s)\eta^t ds \right\rangle, \qquad I_2 = \left\langle \nu A u, \int_0^\infty \mu(s)\eta^t ds \right\rangle,$$
$$I_3 = \left\langle \int_0^\infty \mu(s)A\eta^t ds, \int_0^\infty \mu(s)\eta^t ds \right\rangle, \quad I_4 = b\left(u, u, \int_0^\infty \mu(s)\eta^t ds\right),$$
$$I_5 = -\left\langle f, \int_0^\infty \mu(s)\eta^t ds \right\rangle \qquad I_6 = -\left\langle F(u_t), \int_0^\infty \mu(s)\eta^t ds \right\rangle.$$

can be estimated in this way

$$I_1 \leq \left|\left\langle u, \int_0^\infty \mu'(s)\eta^t ds \right\rangle\right| \leq c_2|u(t)|^2 - c_3 \int_0^\infty \mu'(s)\|\eta^t\|^2 ds,$$

$$I_2 = \left\langle \nu A u, \int_0^\infty \mu(s)\eta^t ds \right\rangle \leq c_4\nu\|u(t)\|^2 + c_5\|\eta^t\|^2_{1,\mu},$$

$$I_3 = \left\langle \int_0^\infty \mu(s)A\eta^t ds, \int_0^\infty \mu(s)\eta^t ds \right\rangle \leq c_6\|\eta^t\|^2_{1,\mu},$$

$$I_4 \leq \left|b\left(u, \int_0^\infty \mu(s)\eta^t ds, u\right)\right| \leq |u(t)|\|u(t)\|\|\eta^t\|_{1,\mu} \leq c_7\|u(t)\|^2 + c_8|u(t)|^2\|\eta^t\|^2_{1,\mu},$$

$$I_5 \leq \left|\left\langle f, \int_0^\infty \mu(s)\eta^t ds \right\rangle\right| \leq c_9\|f\|^2_* + c_{10}\|\eta^t\|^2_{1,\mu},$$

$$I_6 \leq \left|\left\langle F(u_t), \int_0^\infty \mu(s)\eta^t ds \right\rangle\right| \leq c_{11}\|u_t\|^2_{BCL_{-\infty}(H)} + c_{12}\|\eta^t\|^2_{1,\mu},$$

by means of the Young's inequality and Lemma 3.2 in [22]. Taking above estimates into account, we are easy to get (4.4). □

Lemma 4.3. *Under the assumptions* (H1) − (H3), *the functional*

$$K_2(t) = \frac{1}{2}(u, u),$$

satisfies, for all $t \in \mathbb{R}^+$,

$$K_2'(t) \le -(\nu - c_{15})\|u(t)\|^2 + (c_{13} + c_{17})\|u_t\|_{BCL_{-\infty}(H)}^2 + c_{14}\|\eta^t\|_{1,\mu} + c_{16}\|f\|_*^2, \qquad (4.6)$$

where $c_i, i = 13, 14, ..., 16$ *are suitable constants.*

Proof. Multiplying $(2.1)_1$ by u and using the Young's inequality, then we can obtain the desired result. □

For positive constants M_1, M_2, we define a Lyapunov functional

$$L_1(t) = M_1 \hat{E}(t) + M_2 K_1(t) + K_2(t).$$

Combining the (4.1), Lemma 4.2 and Lemma 4.3, we arrive at

$$\begin{aligned}
L_1'(t) \le &- [M_2(\kappa(0) - c_2 - c_{10}) - c_{13} - c_{17}]\|u_t\|_{BCL_{-\infty}(H)}^2 \\
&+ \left[\frac{M_1}{2} - M_2 c_3\right] \int_0^\infty \mu'(s)\|\eta^t\|_{1,\mu}^2 \mathrm{d}s + [M_2(c_4\nu + c_6) - \nu + c_{15}]\|u(t)\|^2 \\
&+ [M_2(c_5 + c_6 + c_7|u(t)|^2 + c_{10} + c_{12}) + c_{14}]\|\eta^t\|_{1,\mu}^2 + [M_1 c_1 + M_2 c_9 + c_{16}]\|f\|_*^2.
\end{aligned}$$

Then we choose M_1 large enough such that

$$\frac{M_1}{2} - M_2 c_3 > 0,$$

and choose M_2 satisfying

$$0 < \frac{c_{13} + c_{17}}{\kappa(0) - c_2 - c_{10}} < M_2 \le \frac{\nu - c_{15}}{c_4\nu + c_6},$$

where

$$\frac{c_{13} + c_{17}}{\kappa(0) - c_2 - c_{10}} < \frac{\nu - c_{15}}{c_4\nu + c_6}.$$

Thus, there exist two positive constants m and c such that

$$L_1' \le -m\hat{E}(t) + c\|\eta^t\|_{1,\mu}^2 + c\|f\|_*^2. \qquad (4.7)$$

In addition (see [9]), we can choose M_1 large enough so that

$$L_1(t) \sim \hat{E}(t),$$

i.e., $L_1(t)$ is equivalent to $\hat{E}(t)$. Here, we introduce an approach to estimate $\hat{E}(t)$ and refer the readers to [22] for more details.

Lemma 4.4. *Under the assumptions* (H1) − (H3), *there exists a constant* $\beta_1 > 0$ *such that for every* $\delta_0 >$ *and* $t \in \mathbb{R}^+$,

$$\Phi'(\delta_0 \hat{E}(t))\|\eta^t\|_{1,\mu}^2 \leq -\beta_1(\hat{E}(t) - c_1\|f\|_*^2 + \|u_t\|_{BCL_{-\infty}(H)}^2) + \beta_1\delta_0\hat{E}(t)\Phi'(\delta_0\hat{E}(t)). \quad (4.8)$$

Proof. From the assumption (H1), without loss of generality, we can assume that $\mu'(s) < 0$. Let Φ^* be the convex conjugate of Φ in the sense of Young (see [7]), so for all $t \in \mathbb{R}^+$,

$$\Phi^*(t) = t(\Phi')^{-1}(t) - \Phi((\Phi')^{-1}(t)) \leq t(\Phi')^{-1}(t)$$

Let $\delta_0, \tau_1, \tau_2 > 0$. Using the general Young's inequality: $t_1 t_2 \leq \Phi(t_1) + \Phi^*(t_2)$ for

$$t_1 = \Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2), \qquad t_2 = \frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)},$$

we deduce that for all $t \in \mathbb{R}^+$,

$$\begin{aligned}
\int_0^\infty \mu(s)\|\eta^t\|^2 ds &= \frac{1}{\tau_1\Phi'(\delta_0\hat{E}(t))}\int_0^\infty \Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)}ds \\
&\leq -\frac{\tau_2}{\tau_1\Phi'(\delta_0\hat{E}(t))}\int_0^\infty \mu'(s)\|\eta^t\|^2 ds \\
&\quad + \frac{1}{\tau_1\Phi'(\delta_0\hat{E}(t))}\int_0^\infty \Phi^*\left(\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)}\right)ds \\
&\leq -\frac{2\tau_2(\hat{E}(t) - c_1\|f\|_*^2 - \|u_t\|_{BCL_{-\infty}(H)}^2)}{\tau_1\Phi'(\delta_0\hat{E}(t))} \\
&\quad + \int_0^\infty \frac{\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)} \times (\Phi')^{-1}\left(\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)}\right)ds,
\end{aligned}$$

$$\quad (4.9)$$

where we used (4.3) and (4.2). Observe that Φ^{-1} is concave and $\Phi^{-1}(0) = 0$. Thus, for any $0 \leq s_1 < s_2$, the function $K(s) = \frac{s}{\Phi^{-1}(s)}$ satisfies

$$K(s_1) = \frac{s_1}{\Phi^{-1}\left(\frac{s_1}{s_2}s_2 + (1 - \frac{s_1}{s_2})0\right)} \leq \frac{s_1}{\frac{s_1}{s_2}\Phi^{-1}(s_2) + (1 - \frac{s_1}{s_2})\Phi^{-1}(0)} = \frac{s_2}{\Phi^{-1}(s_2)} = K(s_2).$$

Notice that $\int_0^\infty \mu(s)\|\eta^t\|^2 ds$ is bounded. Then we can find a suitable constant N_1 such that

$$\|\eta^t\|^2 \leq N_1. \quad (4.10)$$

On the other hand, using the fact that $(\Phi')^{-1}$ is nondecreasing and (4.10), we get

$$\begin{aligned}
(\Phi')^{-1}\left(\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)\|\eta^t\|^2}{\Phi^{-1}(-\tau_2\mu'(s)\|\eta^t\|^2)}\right) &= (\Phi')^{-1}\left(\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)}{-\tau_2\mu'(s)}K(-\tau_2\mu'(s)\|\eta^t\|^2)\right) \\
&\leq (\Phi')^{-1}\left(\frac{\tau_1\Phi'(\delta_0\hat{E}(t))\mu(s)}{-\tau_2\mu'(s)}K(-\tau_2 N_1\mu'(s))\right) \\
&= (\Phi')^{-1}\left(\frac{\tau_1 N_1\Phi'(\delta_0\hat{E}(t))\mu(s)}{\Phi^{-1}(-\tau_2 N_1\mu'(s))}\right).
\end{aligned}$$

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Therefore, (4.9) can be rewritten as

$$
\int_0^\infty \mu(s)\|\eta^t\|^2 \mathrm{d}s \le -\frac{2\tau_2(\hat{E}'(t) - c_1\|f\|_*^2 - \|u_t\|^2_{BCL_{-\infty}(H)})}{\tau_1 \Phi'(\delta_0 \hat{E}(t))}
$$
$$
+ N_1 \int_0^\infty \frac{\mu(s)}{\Phi^{-1}(-\tau_2 N_1 \mu'(s))} (\Phi')^{-1}\left(\frac{\tau_1 N_1 \Phi'(\delta_0 \hat{E}(t))\mu(s)}{\Phi^{-1}(-\tau_2 N_1 \mu'(s))}\right) \mathrm{d}s. \tag{4.11}
$$

From Assumption (H1), it is evident to show that for some positive constants N_2 and N_3,

$$
\sup_{s\in\mathbb{R}^+} \frac{\mu(s)}{\Phi^{-1}(-\mu'(s))} := N_2 < +\infty \quad \text{and} \quad \int_0^\infty \frac{\mu(s)}{\Phi^{-1}(-\mu'(s))} \mathrm{d}s := N_3 < +\infty.
$$

Then, choose $\tau_1 = \frac{1}{N_1 N_2}$, $\tau_2 = \frac{1}{N_1}$ in (4.11). This yields

$$
\int_0^\infty \mu(s)\|\eta^t\|^2 \mathrm{d}s \le -\frac{2N_2(\hat{E}'(t) - c_1\|f\|_*^2 - \|u_t\|^2_{BCL_{-\infty}(H)})}{\Phi'(\delta_0 \hat{E}(t))} + N_1 N_3 \delta_0 \hat{E}(t),
$$

which gives (4.8) with $\beta_1 = \max\{2N_2, N_1 N_3\}$. $\qquad\qquad\square$

Proof of Lemma 4.1. Multiplying (4.7) by $\Phi'(\delta_0 \hat{E}(t))$, we obtain

$$
\Phi'(\delta_0 \hat{E}(t))L_1'(t) \le -m\Phi'(\delta_0 \hat{E}(t))\hat{E}(t) + c\Phi'(\delta_0 \hat{E}(t))\|\eta^t\|^2_{1,\mu} + c\Phi'(\delta_0 \hat{E}(t))\|f\|_*^2
$$
$$
\le -m\Phi'(\delta_0 \hat{E}(t))\hat{E}(t) - c\beta_1(\hat{E}'(t) - c_1\|f\|_*^2 - \|u_t\|^2_{BCL_{-\infty}(H)})
$$
$$
+ c\beta_1 \delta_0 \hat{E}(t)\Phi'(\delta_0 \hat{E}(t)) + c\Phi'(\delta_0 \hat{E}(t))\|f\|_*^2
$$
$$
\le -(m - c\beta_1\delta_0)\hat{E}(t)\Phi'(\delta_0 \hat{E}(t)) - c\beta_1(\hat{E}'(t) - c_1\|f\|_*^2 - \|u_t\|^2_{BCL_{-\infty}(H)})
$$
$$
+ c\Phi'(\delta_0 \hat{E}(t))\|f\|_*^2
$$
$$
\le -(m - c\beta_1\delta_0)\hat{E}(t)\Phi'(\delta_0 \hat{E}(t)) - c\beta_1\hat{E}'(t) + c.
$$

where we have used the fact that Φ' is bounded. Choosing δ_0 small enough such that $\beta_2 := m - c\beta_1\delta_0 > 0$ and introducing the functional

$$
L_2(t) := \Phi'(\delta_0 \hat{E}(t))L_1(t) + c\beta_1 \hat{E}(t),
$$

we are led to the following differential inequality (note that $\Phi'(\delta_0 \hat{E}(t))$ is nonincreasing.)

$$
L_2'(t) \le \Phi'(\delta_0 \hat{E}(t))L_1'(t) + c\beta_1 \hat{E}'(t) \le -\beta_2 \Phi'(\delta_0 \hat{E}(t))\hat{E}(t) + c \le -\beta_2 \Phi'(\delta_0 L_2(t))L_2(t) + c,
$$

due to $L_2(t) \sim \hat{E}(t)$. Finally, applying the Lemma 4.5 introduced below, we get

$$
L_2(t) \le H_1^{-1}(\beta_2 t) + H_2^{-1}\left(\frac{c}{\beta_2}\right),
$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} \mathrm{d}s$ and $H_2(t) = L_2(t)\Phi'(\delta_0 L_2(t))$. The equivalence $L_2(t) \sim \hat{E}(t)$ give the fact that there exists a $\rho_0 > 0$ so that

$$
\|(u_t, \eta^t)\|_{\mathcal{H}^*} \le \rho_0,
$$

for all $(u_0, \eta^0) \in \mathcal{B}$ and $t \geq t_0$, where

$$\mathcal{B} = \left\{ v \in \mathcal{H}^* : \|v\|_{\mathcal{H}^*} \leq \sqrt{H_1^{-1}(\beta_2 t_0) + H_2^{-1}\left(\frac{c}{\beta_2}\right)} \right\},$$

is the bounded absorbing set.   □

Lemma 4.5. *[29] Let $y(t)$ be a positive absolutely continuous function on $(0, +\infty)$ satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \leq -\beta H_2(y(t)) + \gamma,$$

for $\beta > 0$, $\gamma > 0$. Then for any $t > 0$,

$$y(t) \leq H_1^{-1}(\beta) + H_2^{-1}\left(\frac{\gamma}{\beta}\right),$$

where H_1^{-1}, H_2^{-1} are defined as follow

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} \mathrm{d}s, \tag{4.12}$$

and

$$H_2(t) = \begin{cases} t & \text{if } \Phi \text{ is linear on } [0, \epsilon_1], \\ t\Phi'(\epsilon_0 t) & \text{if } \Phi'(0) = 0 \text{ and } \Phi'' > 0 \text{ on } [0, \epsilon_1]. \end{cases} \tag{4.13}$$

and H_1^{-1}, H_2^{-1} represent the inverse function of H_1, H_2 respectively.

4.3   Asymptotical compact of the semigroup

To show that the semigroup $S(t)$ exists a global attractor, it remains to prove that $S(t)$ is asymptotically compact. It is necessary to consider the difficulty caused by the lack of the compactness of $L^2(\mathbb{R}^+, (H^2(\Omega))^2 \cap V) \hookrightarrow L^2(\mathbb{R}^+, V)$. Now we recall some preliminary results what we need.

4.3.1   Decomposition of the equation

It is convenient to make asymptotic estimates when we decompose the solution $S(t)(u_0, \eta^0) = (u_t, \eta^t)$ of problem (2.1) as

$$S(t)(u_0, \eta^0) = S_1(t)(u_0, \eta^0) + S_2(t)(u_0, \eta^0).$$

where $S_1(t)(u_0, \eta^0) = (v_t^\varepsilon, \zeta^{t\varepsilon})$ and $S_2(t)(u_0, \eta^0) = (w^\varepsilon, \xi^{t\varepsilon})$ with

$$u = v^\varepsilon + w^\varepsilon, \quad \eta^t = \zeta^{t\varepsilon} + \xi^{t\varepsilon},$$

$$z_1 = (v^\varepsilon, \zeta^{t\varepsilon}), \quad z_2 = (w^\varepsilon, \xi^{t\varepsilon}),$$

where $(v^\varepsilon, \zeta^{t\varepsilon})$ and $(w^\varepsilon, \xi^{t\varepsilon})$ are respectively the unique solutions of the problems

$$
\begin{cases}
\partial_t v^\varepsilon + \nu A v^\varepsilon + \displaystyle\int_0^\infty \mu(s) A \zeta^{t\varepsilon} \mathrm{d}s + B(u, v^\varepsilon) = F(u_t) - F(w_t^\varepsilon) \\[2mm]
\nabla \cdot v^\varepsilon = 0, \\[2mm]
\partial_t \zeta^{t\varepsilon}(x, s) = -\partial_s \zeta^{t\varepsilon}(x, s) + v^\varepsilon(x, t), \\[2mm]
v^\varepsilon(x, t)|_{\partial\Omega} = 0, \qquad v^\varepsilon(x, \tau) = \phi(x, \tau)|_{\tau \in (-\infty, 0]}, \\[2mm]
\zeta^{t\varepsilon}(x, s)|_{\partial\Omega} = 0, \qquad \zeta^0(x, s) = \displaystyle\int_{-s}^0 \phi(x, r) \mathrm{d}r,
\end{cases}
\tag{4.14}
$$

and

$$
\begin{cases}
\partial_t w^\varepsilon + \nu A w^\varepsilon + \displaystyle\int_0^\infty \mu(s) A \xi^{t\varepsilon} \mathrm{d}s + B(u, w^\varepsilon) = F(w_t^\varepsilon) + f \\[2mm]
\nabla \cdot w^\varepsilon = 0, \\[2mm]
\partial_t \xi^{t\varepsilon}(x, s) = -\partial_s \xi^{t\varepsilon}(x, s) + w^\varepsilon(x, t), \\[2mm]
w^\varepsilon(x, t)|_{\partial\Omega} = 0, \qquad w^\varepsilon(x, \tau) = 0, \\[2mm]
\xi^{t\varepsilon}(x, s)|_{\partial\Omega} = 0, \qquad \xi^0(x, s) = 0.
\end{cases}
\tag{4.15}
$$

Using the similar argument to the proof of Theorem 3.1, we can establish the existence and uniqueness of solutions to problem (4.14) and (4.15). Furthermore, as for problem (4.15), we can show the solution $(w^\varepsilon, \zeta^{t\varepsilon})$ is a strong solution for the reason that $f^\varepsilon \in H$ and the initial values are zero. Now, we make the a priori estimates to the above problems.

4.3.2   The first a priori estimates

We show the decay estimates for the solution of problem (4.14).

Lemma 4.6. *Under the conditions* $(H1) - (H3)$*, for any* $\varepsilon > 0$*, the solution of problem* (4.14) *has the following estimate: there exists a constant* $d_0 > 0$*, such that for every* $t \geq 0$*,*

$$
\|S_1(t)(u_0, \eta^0)\|_{\mathcal{H}^*}^2 \leq Q(\|(u_0, \eta^0)\|_{\mathcal{H}^*}) e^{-d_0 t}.
$$

Proof. Multiplying the first equation of (4.14) by v^ε and integrating over Ω, we get

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (|v^\varepsilon|^2 + \|\zeta^{t\varepsilon}\|_{1,\mu}^2) + \nu \|v^\varepsilon\|^2 + \int_0^\infty \mu(s) \frac{\mathrm{d}}{\mathrm{d}s} \|\zeta^{t\varepsilon}\|_{1,\mu}^2 \mathrm{d}s \\
= (F(u_t) - F(w_t^\varepsilon), v^\varepsilon).
\end{aligned}
\tag{4.16}
$$

The right hand of (4.16) cen be eatimated in this way

$$
(F(u_t) - F(w_t^\varepsilon), v^\varepsilon) \leq L_F \|v_t^\varepsilon\|_{BCL_{-\infty}(H)}^2.
$$

Hence, by Poincaré inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}(|v^\varepsilon|^2 + \|\zeta^{t\varepsilon}\|_{1,\mu}^2) + \nu\lambda_1 |v^\varepsilon|^2 + 2\int_0^\infty \mu(s) \frac{\mathrm{d}}{\mathrm{d}s} \|\zeta^{t\varepsilon}\|_{1,\mu}^2 \mathrm{d}s \leq 2L_F \|v_t^\varepsilon\|_{BCL_{-\infty}(H)}^2.
$$

Similarly to the proof in Lemma 4.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(|v^\varepsilon(t)|^2 + \|\zeta^{t\varepsilon}\|_{1,\mu}^2) \leq -c(|v^\varepsilon|^2 + \|\zeta^{t\varepsilon}\|_{1,\mu}^2)\Phi'(\delta_1(|v^\varepsilon(t)|^2 + \|\zeta^{t\varepsilon}\|_{1,\mu}^2)), \qquad (4.17)$$

which implies that

$$\|S_1(t)(u_0,\eta^0)\|_{\mathcal{H}^*}^2 \leq Q(\|(u_0,\eta^0)\|_{\mathcal{H}^*})e^{-d_0t},$$

for some $d_0 > 0$. □

4.3.3   The second a priori estimates

Since the presence of the convolution integral in which the embedding $L_\mu^2(\mathbb{R}^+, (H^2(\Omega))^2 \cap V) \hookrightarrow L_\mu^2(\mathbb{R}^+, V)$ is not compact, we apply the compactness theorem and the construction of the compact subspaces in [5] to overcome the difficulty.

Lemma 4.7. *Under the assumptions* (H1) − (H3), *for any* $\varepsilon > 0$ *and the initial data* $(u_0,\eta^0) \in \mathcal{H}^*$, *there exists* $M > 0$ *such that for any given* $T > 0$,

$$\|S_2(t)(u_0,\eta^0)\|_{V \times L_\mu^2(\mathbb{R}^+, (H^2(\Omega))^2 \cap V)}^2 \leq M, \ \forall\, t \geq T,$$

where M *depends on* $|u|^2\|u\|^2, \|f\|_*^2$ *and* $\|z_0\|_{\mathcal{H}^*}^2$.

Proof. Multiplying the first equation of (4.15) by Aw^ε and using the Cauchy inequality, then we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w^\varepsilon\|^2 + \|\xi^{t\varepsilon}\|_{1,\mu}^2) + \frac{\nu}{2}|Aw^\varepsilon|^2 + \int_0^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\xi^{t\varepsilon}\|_{2,\mu}^2\mathrm{d}s + b(u,w^\varepsilon,Aw^\varepsilon)$$
$$\leq \frac{1}{\nu}\|f\|_*^2 + \frac{L_F}{\nu}\|w_t^\varepsilon\|_{BCL_{-\infty}(H)}^2. \qquad (4.18)$$

Recall that

$$b(u,w^\varepsilon,Aw^\varepsilon) \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|w^\varepsilon\|^{\frac{1}{2}}|Aw^\varepsilon|^{\frac{3}{2}} \leq \frac{\nu}{4}|Aw^\varepsilon|^2 + \left(\frac{C}{\nu}|u|^2\|u\|^2\right)\|w^\varepsilon\|^2,$$

hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w^\varepsilon\|^2 + \|\xi^{t\varepsilon}\|_{1,\mu}^2) + \frac{\nu}{4}|Aw^\varepsilon|^2 \leq \left(\frac{C}{\nu}|u|^2\|u\|^2\right)\|w^\varepsilon\|^2 + \frac{1}{\nu}\|f\|_*^2 + \frac{L_F}{2\nu}\|w_t^\varepsilon\|_{BCL_{-\infty}(H)}^2,$$

and using Poincaré inequality then we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w^\varepsilon\|^2 + \|\xi^{t\varepsilon}\|_{1,\mu}^2) + \frac{\nu}{4}|Aw^\varepsilon|^2 \leq \left(\frac{C}{\nu}|u|^2\|u\|^2\right)\|w^\varepsilon\|^2 + \frac{1}{\nu}\|f\|_*^2 + \frac{L_F}{2\nu\lambda_1}\|w_t^\varepsilon\|_{BCL_{-\infty}(H)}^2.$$

By Gronwall inequality, we can find a $T > 0$ large enough such that

$$\|S_2(t)(u_0,\eta^0)\|_{V \times L_{1,\mu}^2(\mathbb{R}^+, H^2(\Omega)^2 \cap V)}^2 \leq M, \qquad for\ all\ t \geq T.$$

□

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Besides, for any $\xi_0 \in L^2_\mu(\mathbb{R}^+, V)$, the Cauchy problem (see [11], [38])

$$
\begin{cases}
\partial_t \xi^t = -\partial_s \xi^t + w, & t > 0, \\
\xi^0 = \xi_0,
\end{cases}
$$

has a unique solution $\xi^t \in C(\mathbb{R}^+, L^2_\mu(\mathbb{R}^+, V))$ and

$$
\xi^t(s) =
\begin{cases}
\displaystyle\int_0^s w(t-r)\mathrm{d}r, & 0 < s \le t, \\
\displaystyle\int_0^t w(t-r)\mathrm{d}r, & s > t.
\end{cases}
\tag{4.19}
$$

Then, we have

$$
\xi^{t\varepsilon}(s) =
\begin{cases}
\displaystyle\int_0^s w^\varepsilon(t-r)\mathrm{d}r, & 0 < s \le t, \\
\displaystyle\int_0^t w^\varepsilon(t-r)\mathrm{d}r, & s > t.
\end{cases}
\tag{4.20}
$$

Let $\mathcal{B}_0$ be the bounded absorbing set obtained in Lemma 4.1. We have the following results.

Lemma 4.8. *Assume that the assumptions* $(H1) - (H3)$ *are satisfied, set*

$$
\mathcal{K}^\varepsilon_T = RS_2(T)B_0,
$$

for every given $T > 0$ large enough, where $R : \mathcal{H}^* \to L^2_\mu(\mathbb{R}^+, V))$ is the canonical projection operator. Then there exists a positive constant $N = N(\|B_0\|_{\mathcal{H}^*})$ such that

(1) $\mathcal{K}^\varepsilon_T$ *is bounded in* $L^2_\mu(\mathbb{R}^+, (H^2(\Omega))^2 \cap V) \cap H^1_\mu(\mathbb{R}^+, H)$;

(2) $\displaystyle\sup_{\xi \in \mathcal{K}^\varepsilon_T} \|\xi(s)\|^2_V \le N,$

where $\{S_2(t)\}_{t \ge 0}$ is the solution semigroup of (4.15).

Proof. By (4.20), we have

$$
\partial_s \xi^{t\varepsilon} =
\begin{cases}
w^\varepsilon(t-s), & 0 < s \le t, \\
0, & s > t.
\end{cases}
$$

Combining with the Lemma 4.7, we conclude that (1) holds. Afterwards, from (4.20) we can deduce that

$$
\|\xi^{T\varepsilon}\|^2_V \le
\begin{cases}
\displaystyle\int_0^s \|w^\varepsilon(T-r)\|^2_V \mathrm{d}r \le \int_0^T \|w^\varepsilon(T-r)\|^2_V \mathrm{d}r, & 0 < s \le T, \\
\displaystyle\int_0^T \|w^\varepsilon(T-r)\|^2_V \mathrm{d}r, & s > T.
\end{cases}
$$

On account of Lemma 4.7 again, we have that (2) holds. $\qquad\square$

Moreover, due to $(H^2(\Omega))^2 \cap V \hookrightarrow V$ compactly, we have that $\mathcal{K}^\varepsilon_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, V)$ thanks to the following lemma.

Lemma 4.9. *([26, 38]) Suppose that $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function and satisfies the following condition: if there exists a $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$ for all $s \geq s_0$. Let A_0, A_1, A_2 be Banach spaces, where A_0, A_2 are reflexive and satisfy*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2,$$

and the embedding $A_0 \hookrightarrow A_1$ is compact. Let $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+, A_1)$ satisfy

  (i) *$\mathcal{C}$ is bounded in $L^2_\mu(\mathbb{R}^+, A_0) \cap H^1_\mu(\mathbb{R}^+, A_2)$;*

  (ii) *$\sup\limits_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{A_1} \leq h(s)$ for all $s \in \mathbb{R}^+$, where $h \in L^2_\mu(\mathbb{R}^+)$.*

Then $\mathcal{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+, A_1)$.

The semigroup $S(t)$ has a bounded absorbing set in $\mathcal{H}^*$ and $S(t)$ is asymptotically compact in $\mathcal{H}^*$, thus we can get the main result as follows.

Theorem 4.1. *Under the assumptions* (H1) − (H3), *$S(t)$ has a global attractor $\mathcal{A}_0$ in $\mathcal{H}^*$.*

Proof. From Proposition 4.1, Lemma 4.1 and the fact that $S(t)$ is asymptotically compact in $\mathcal{H}^*$, we are easy to complete the proof. □

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interests.

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