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Article

Note for the Riemann Hypothesis

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Abstract: Let $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log\log n}$ where log is the natural logarithm. Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order n. A trustworthy proof for the Riemann hypothesis has been considered as the Holy Grail of Mathematics by several authors. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. There are several statements equivalent to the famous Riemann hypothesis. We show if the inequality $R(N_{n+1}) < R(N_n)$ holds for n big enough, then the Riemann hypothesis is true. In this note, we prove that $R(N_{n+1}) < R(N_n)$ always holds for n big enough.

Keywords: Riemann hypothesis; prime numbers; Riemann zeta function; Chebyshev function

MSC: 11M26; 11A41; 11A25

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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. The hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, there have been several developments that have brought us closer to a proof of the Riemann hypothesis. There are many approaches to the Riemann hypothesis based on analytic number theory, algebraic geometry, non-commutative geometry, etc [1].

The Riemann zeta function $\zeta(s)$ is a function under the domain of complex numbers. It has zeros at the negative even integers: These are called the trivial zeros. The zeta function is also zero for other values of s, which are called nontrivial zeros. The Riemann hypothesis is concerned with the locations of these nontrivial zeros. Bernhard Riemann conjectured that the real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$.

The Riemann hypothesis's importance remains from its deep connection to the distribution of prime numbers, which are essential in many computational and theoretical aspects of mathematics. Understanding the distribution of prime numbers is crucial for developing efficient algorithms and improving our understanding of the fundamental structure of numbers. Besides, the Riemann hypothesis stands as a testament to the power and allure of mathematical inquiry. It challenges our understanding of the fundamental structure of numbers, inspiring mathematicians to push the boundaries of their field and seek ever deeper insights into the universe of mathematics.

2. Materials and Methods

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. We know the following inequalities:

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Proposition 1. *For $r \geq 0$ and $-1 \leq x < \frac{1}{r}$ [2, pp. 1]:*

$$(1+x)^r \leq \frac{1}{1-r\cdot x}.$$

Proposition 2. *For $x > -1$ [2, pp. 1]:*

$$\frac{x}{1+x} \leq \log(1+x) \leq x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [3].

Proposition 3. *We define [3, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the k^{th} prime number (Mathematicians also use the notation q_n to represent the n^{th} prime number). By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right)\\ &= \int_1^{\infty}\left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right)dx.\end{aligned}$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. In number theory, $\Psi(n) = n \cdot \prod_{q \mid n}\left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n.

Definition 1. *We say that* $\mathsf{Dedekind}(q_n)$ *holds provided that*

$$\prod_{q \leq q_n}\left(1 + \frac{1}{q}\right) \geq \frac{e^{\gamma}}{\zeta(2)} \cdot \log\theta(q_n).$$

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

We define $R(n) = \frac{\Psi(n)}{n\cdot\log\log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) \geq \frac{e^{\gamma}}{\zeta(2)}$ is satisfied.

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Proposition 4. *Unconditionally on Riemann hypothesis, we know that [4, Proposition 3 pp. 3]:*

$$\lim_{n\to\infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [5]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [6]:

$$f(x) = \Omega_R(g(x)) \text{ as } x \to \infty \text{ if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0;$$
$$f(x) = \Omega_L(g(x)) \text{ as } x \to \infty \text{ if } \liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_\pm(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as:

$$f(x) = \Omega_+(g(x)) \text{ if } \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [7, Theorem 3 pp. 376], [8, (5.5) pp. 111]:

$$f(x) = e^{\gamma} \cdot \log\theta(x) \cdot \prod_{q\leq x}\left(1 - \frac{1}{q}\right).$$

Finally, we have Nicolas' Theorem:

Proposition 5. *If the Riemann hypothesis is false then there exists a real b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$ [7, Theorem 3 (c) pp. 376], [8, Theorem 5.29 pp. 131]:*

$$\log f(x) = \Omega_\pm(x^{-b}).$$

Putting all together yields a proof for the Riemann hypothesis.

3. Results

The following inequality is a trivial result:

Lemma 1. *Let ϵ_1 be a positive integer between 0 and $e-1$ (i.e. $0 < \epsilon_1 < e-1$). Then,*

$$\log\left(1 - e^{-1}\cdot(\epsilon_1+1)\right) \geq -\frac{e^{-1}\cdot(\epsilon_1+1)}{1 - e^{-1}\cdot(\epsilon_1+1)}.$$

Proof. We can apply the Proposition 2 since $-e^{-1}\cdot(\epsilon_1+1) > -1$. Therefore, we only need to replace x by $-e^{-1}\cdot(\epsilon_1+1)$ in the following expression

$$\frac{x}{1+x} \leq \log(1+x).$$

□

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. Nevertheless, there exist some implications in case the Riemann hypothesis could be false. The following is a key Lemma.

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Lemma 2. *If the Riemann hypothesis is false, then there exist infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. Let's define a function called $g(x)$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

This function is based on some previously proven results (reference: [4, Theorem 4.2 pp. 5]). It involves several things: the constants γ and $\zeta(2)$, the Chebyshev function $\theta(x)$, and a product considering all prime numbers less than or equal to x.

We're interested in a specific condition, called $\mathsf{Dedekind}(q_n)$ (see Definition 1). This proof argues that $\mathsf{Dedekind}(q_n)$ could fail under the possibility that the Riemann hypothesis is false. That circumstance involves an infinitely many real numbers x_0 greater than or equal to 5. We claim that $\mathsf{Dedekind}(q_n)$ fails for infinitely many prime numbers q_n such that q_n refers to the largest prime number less than or equal to x_0. For this x_0, the value of $g(x_0)$ must be greater than 1 (or equivalently, $\log g(x_0) > 0$).

There's a previously established relationship between $g(x)$ and $f(x)$ [4, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

If the Riemann hypothesis (RH) is false, then there must be infinitely many numbers x for which $\log f(x) = \Omega_+(x^{-b})$ by Proposition 5. This result depends on another number b between 0 and $\frac{1}{2}$ (i.e. $0 < b < \frac{1}{2}$). Nicolas proved the general case $\log f(x) = \Omega_\pm(x^{-b})$, but we only need to use the notation Ω_+ under the domain of the real numbers. According to the Hardy and Littlewood definition, this would mean

$$\exists k > 0, \forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0) \colon \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, where we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies

$$\forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0) \colon \log f(y) \geq \frac{1}{\sqrt{y}}.$$

This inequality would mean that under a false RH, there are infinitely many widely spaced real numbers x where $\log f(x) \geq \frac{1}{\sqrt{x}}$. Here's how this connects back to our original function $g(x)$. Because of $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, hence if the false RH scenario holds, then there must be infinitely many such x_0 where $\log g(x_0) > 0$.

Finally, the proof establishes a link between these positive $\log g(x_0)$ values and the prime numbers. It shows that if the logarithm of $g(x_0)$ is positive for a specific $x_0 \geq 5$, then it must also be positive for the largest prime number q_n less than or equal to x_0. This connection arises from the properties of the terms used in the definition of $g(x)$ and the Chebyshev function. □

Lemma 3. *If $R(N_n)$ is strictly decreasing (i.e. $R(N_n) > R(N_{n+1})$) for n big enough then* $\mathsf{Dedekind}(q_n)$ *holds for n big enough.*

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Proof. Assume $R(N_n) > R(N_{n+1})$ for $n > n_0$ and that $\mathsf{Dedekind}(q_m)$ fails for $m > n_0$ that is

$$R(N_m) < \frac{e^\gamma}{\zeta(2)},$$

then for $n \geq m+1$ we have $R(N_{n+1}) < R(N_n) < \frac{e^\gamma}{\zeta(2)}$. This implies

$$\limsup_{n\to\infty} R(N_n) < \frac{e^\gamma}{\zeta(2)}$$

contradicting Proposition 4. □

This is the main insight.

Theorem 1. *The inequality $R(N_n) > R(N_{n+1})$ holds for n big enough.*

Proof. By Lemma 3, $\mathsf{Dedekind}(q_n)$ holds for n big enough if the following inequality is satisfied for a sufficiently large value of n:

$$R(N_{n+1}) < R(N_n).$$

This translates to:

$$\frac{\prod_{q\leq q_{n+1}}\left(1+\frac{1}{q}\right)}{\log\theta(q_{n+1})} < \frac{\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)}{\log\theta(q_n)}.$$

Applying logarithms to both sides and expanding the terms, we get:

$$\log\log\theta(q_{n+1}) > \log\log\theta(q_n) + \sum_{q_n<q\leq q_{n+1}}\log\left(1+\frac{1}{q}\right).$$

Dividing both sides by $\log\log\theta(q_{n+1})$ (since q_{n+1} is large enough to ensure $\log\log\theta(q_{n+1}) > 0$), we have:

$$1 > \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})} + \frac{\sum_{q_n<q\leq q_{n+1}}\log\left(1+\frac{1}{q}\right)}{\log\log\theta(q_{n+1})}.$$

Taking exponentials of both sides yields:

$$e > \exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)\cdot\left(\prod_{q_n<q\leq q_{n+1}}\left(1+\frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n+1})}}.$$

For a sufficiently large prime q_{n+1}, we can leverage the property $e = x^{\frac{1}{\log x}}$ for $x > 0$ to obtain:

$$e = (\log\theta(q_{n+1}))^{\frac{1}{\log\log\theta(q_{n+1})}}.$$

Therefore, it suffices to show that:

$$\log\theta(q_{n+1}) > \prod_{q_n<q\leq q_{n+1}}\left(1+\frac{1}{q}\right).$$

This simplifies to:

$$\log\theta(q_{n+1}) > 1 + \frac{1}{q_{n+1}}$$

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which is trivially true for n big enough. That would mean

$$e \cdot (1 - \epsilon_2) = \left(\prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) \right)^{\frac{1}{\log \log \theta(q_{n+1})}}$$

for some positive integer ϵ_2 between 0 and 1 (i.e. $0 < \epsilon_2 < 1$). Besides, we have:

$$1 + \epsilon_1 = \exp\left(\frac{\log \log \theta(q_n)}{\log \log \theta(q_{n+1})} \right)$$

where ϵ_1 is a positive integer between 0 and $e - 1$ (i.e. $0 < \epsilon_1 < e - 1$). Our goal is to prove:

$$e > (1 + \epsilon_1) \cdot e \cdot (1 - \epsilon_2),$$

which simplifies to:

$$\epsilon_2 > \frac{\epsilon_1}{\epsilon_1 + 1}.$$

We can also see that:

$$1 - e^{-1} \cdot \left(\prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) \right)^{\frac{1}{\log \log \theta(q_{n+1})}} = \epsilon_2.$$

Using Proposition 1 and the fact that $-1 \leq \left(\prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) - 1 \right) < \log \log \theta(q_{n+1})$ (due to a sufficiently large q_{n+1}), we obtain

$$\begin{aligned} \left(\prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) \right)^{\frac{1}{\log \log \theta(q_{n+1})}} &= \left(1 + \prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) - 1 \right)^{\frac{1}{\log \log \theta(q_{n+1})}} \\ &\leq \frac{1}{1 - \frac{\left(\prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right) - 1 \right)}{\log \log \theta(q_{n+1})}} \\ &= \frac{\log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) + 1 - \prod_{q_n < q \leq q_{n+1}} \left(1 + \frac{1}{q}\right)} \\ &= \frac{\log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}}. \end{aligned}$$

So, we arrive at:

$$1 - \frac{e^{-1} \cdot \log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}} \leq \epsilon_2.$$

Combining steps, this follow as

$$1 - \frac{e^{-1} \cdot \log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}} > \frac{\epsilon_1}{\epsilon_1 + 1}.$$

After simple distribution, we make

$$\frac{\epsilon_1 + 1}{\epsilon_1} - \frac{e^{-1} \cdot \frac{\epsilon_1 + 1}{\epsilon_1} \cdot \log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}} > 1$$

and

$$1 > \frac{e^{-1} \cdot (\epsilon_1 + 1) \cdot \log \log \theta(q_{n+1})}{\log \log \theta(q_{n+1}) - \frac{1}{q_{n+1}}}$$

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where
$$\log\log\theta(q_{n+1}) - \frac{1}{q_{n+1}} > e^{-1}\cdot(\epsilon_1+1)\cdot\log\log\theta(q_{n+1}).$$
Using further manipulations, we arrive at:
$$-\frac{1}{q_{n+1}} > \left(e^{-1}\cdot(\epsilon_1+1)-1\right)\cdot\log\log\theta(q_{n+1}).$$
and
$$1 < q_{n+1}\cdot\left(1-e^{-1}\cdot(\epsilon_1+1)\right)\cdot\log\log\theta(q_{n+1})$$
which is
$$0 < \log q_{n+1} + \log\left(1-e^{-1}\cdot(\epsilon_1+1)\right) + \log\log\log\theta(q_{n+1})$$
after of applying the logarithm to both sides. That could be rewritten as
$$0 < -\frac{e^{-1}\cdot(\epsilon_1+1)}{1-e^{-1}\cdot(\epsilon_1+1)} + \log q_{n+1} + \log\log\log\theta(q_{n+1})$$
by Lemma 1. That is equivalent to
$$\frac{1}{e\cdot(\epsilon_1+1)^{-1}-1} < \log q_{n+1} + \log\log\log\theta(q_{n+1})$$
since
$$\frac{e^{-1}\cdot(\epsilon_1+1)}{1-e^{-1}\cdot(\epsilon_1+1)} = \frac{1}{e\cdot(\epsilon_1+1)^{-1}-1}$$
after multiplying the fraction (so above as below) by $e\cdot(\epsilon_1+1)^{-1}$. The inequality
$$\frac{1}{e\cdot(\epsilon_1+1)^{-1}-1} < \log q_{n+1} + \log\log\log\theta(q_{n+1})$$
is the same as
$$\frac{1}{\exp\left(1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)-1} < \log q_{n+1} + \log\log\log\theta(q_{n+1})$$
because of
$$\epsilon_1 = \exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)-1.$$
We can further deduce that
$$\frac{1}{\exp\left(1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)-1} < \log q_{n+1} + \log\log\log\theta(q_{n+1})$$
holds whenever
$$\log q_{n+1} + \log\log\log\theta(q_{n+1}) < \exp\left(1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)\cdot(\log q_{n+1} + \log\log\log\theta(q_{n+1}))$$
also holds. Finally, we can infer that
$$\log q_{n+1} + \log\log\log\theta(q_{n+1}) < \exp\left(1-\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right)\cdot(\log q_{n+1} + \log\log\log\theta(q_{n+1}))$$

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trivially holds by the fact that

$$\exp\left(1 - \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n+1})}\right) > 1$$

under the supposition that n is big enough. □

This is the main theorem.

Theorem 2. *The Riemann hypothesis is true.*

Proof. In virtue of Lemmas 2 and 3, the Riemann hypothesis is true if the inequality

$$R(N_{n+1}) < R(N_n)$$

holds for n big enough. Consequently, the Riemann hypothesis is true by Theorem 1. □

4. Discussion

In number theory, the difference between consecutive prime numbers is called a *prime gap*. The n^{th} prime gap refers specifically to the difference between the prime numbers at positions n and $(n+1)$ in the sequence of primes.

In 1936, Harald Cramér, a Swedish mathematician, proposed a conjecture about the size of prime gaps. Cramér conjecture states that the difference between consecutive prime gaps grows no faster than the square of the logarithm of the larger prime gap (i.e. $q_{n+1} - q_n = O((\log q_n)^2)$). Here, the big O notation represents an upper bound on the order of magnitude of a function.

However, there's growing evidence that Cramér conjecture might be incorrect [9]. Recent research suggests the conjecture may be violated for infinitely many prime gaps. This conclusion is based on results from a yet-to-be-peer-reviewed paper [10, Proposition 4 pp. 5; Proposition 7 pp. 7]. While Theorem 1 in this work seems to disprove the conjecture, its validity depends on the full publication of [10] through peer review.

5. Conclusion

The Riemann hypothesis holds immense significance not only for number theory, but also for fields as diverse as cryptography and particle physics. A proof wouldn't just offer deep insights into the nature and distribution of prime numbers, the fundamental building blocks of integers. It would fundamentally reshape various mathematical landscapes, sparking entirely new lines of inquiry. For example, a proven Riemann hypothesis could lead to more efficient methods of prime number generation, which are crucial for securing online communication in cryptography. Furthermore, its implications might extend beyond pure mathematics, potentially influencing our understanding of the distribution of energy levels in complex systems studied in particle physics. In essence, a resolution to the Riemann hypothesis could be a catalyst for groundbreaking discoveries across a wide range of scientific disciplines.

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Short Biography of Authors

Frank Vega is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article "Robin's criterion on divisibility" makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.