Correct expression of the material derivative in continuum physics

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The material derivative is important in continuum physics. This Letter shows that the expression $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$, used in most literature and textbooks, is incorrect. The correct expression $\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(:)]$ is formulated.

In continuum physics, there are two ways of describing continuous media or flows, the Lagrangian description and the Eulerian description. In the Eulerian description, the material derivative with respect to time must be defined. For mass density $\rho(\boldsymbol{x},t)$, flow velocity $\boldsymbol{v}(\boldsymbol{x},t) = v_k \boldsymbol{e}_k$, and stress tensor $\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the material derivatives are given by:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_k}\frac{\partial x_k}{\partial t} = \frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho}{\partial x^1} + v_2 \frac{\partial \rho}{\partial x^2} + v_3 \frac{\partial \rho}{\partial x^3}, \quad (1)$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial x_k}\frac{\partial x_k}{\partial t} = \frac{\partial \boldsymbol{v}}{\partial t} + v_1 \frac{\partial \boldsymbol{v}}{\partial x^1} + v_2 \frac{\partial \boldsymbol{v}}{\partial x^2} + v_3 \frac{\partial \boldsymbol{v}}{\partial x^3}, \quad (2)$$

and

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + \frac{\partial \boldsymbol{\sigma}}{\partial x_k}\frac{\partial x_k}{\partial t} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + v_1 \frac{\partial \boldsymbol{\sigma}}{\partial x^1} + v_2 \frac{\partial \boldsymbol{\sigma}}{\partial x^2} + v_3 \frac{\partial \boldsymbol{\sigma}}{\partial x^3}, \quad (3)$$

respectively, where t is time, $\boldsymbol{x} = x_k \boldsymbol{e}_k$ is position, $\boldsymbol{e}_k$ is a base vector and $v_k = \frac{\partial x_k}{\partial t}$ is the flow velocity.

In most textbooks, handbooks and encyclopedia, such as Refs. 1–12, the above material derivatives are expressed as:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho, \quad (4)$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \quad (5)$$

and

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \quad (6)$$

where the gradient operator $\boldsymbol{\nabla} = \boldsymbol{e}_k \frac{\partial}{\partial x_k}$.

To simplify Eqs. (1)–(3) further, a differential operator is introduced as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}). \quad (7)$$

This operator is used by most fluid mechanics textbooks, including well-known graduate textbooks such as Landau & Lifshitz,[1,2] Prandtl,[3,4] Anderson,[5] Pope,[6] Cengel & Cimbala,[7] Kundu et al.[8] Woan,[9] wikipedia[10,11] and britannica[12].

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In the latest version of Landau & Lifshitz,[13] the material derivative of flow velocity is given in another form:

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}. \quad (8)$$

Besides the books mentioned above, there is a large amount of literature in fluid mechanics taking the same expressions as Refs. 1 and 13. This makes it difficult for readers to identify which expression for the material derivative is correct, causing great confusion to both students and scholars.

Although some authors, such as Lighthill,[14] Batchelor,[15] Frisch,[16] Nhan & Nam[17], Xie,[18] and Zhao[19] have used the correct expression, it seems that most fluid mechanics textbooks and academic literature have adopted an incorrect expression for the material derivative as in Ref. 1. Therefore, to revive the great influence of Landau in physics and fluid mechanics, we attempt to address this issue in this dedicated paper, where we revisit the material derivative to show why Eqs. (7) and (8) are incorrect, and derive a correct expression using standard tensor calculus.

In Landau & Lifshitz,[1] the material derivative is given as:

$$dx\frac{\partial \boldsymbol{v}}{\partial x} + dy\frac{\partial \boldsymbol{v}}{\partial y} + dz\frac{\partial \boldsymbol{v}}{\partial z} = (d\boldsymbol{r} \cdot \mathbf{grad})\boldsymbol{v}. \quad (9)$$

Although Landau's fluid mechanics is well known worldwide, the above expression is incorrect.

Proof:

Since $\boldsymbol{v} \cdot \mathbf{grad} = \boldsymbol{v} \cdot \boldsymbol{\nabla} = (v_i \boldsymbol{e}_i) \cdot (\partial_j \boldsymbol{e}_j) = v_{i,j} \boldsymbol{e}_i \cdot \boldsymbol{e}_j = v_{i,j} \delta_{ij} = \mathrm{div}\boldsymbol{v}$, and $(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (v_{j,i} \boldsymbol{e}_i \cdot \boldsymbol{e}_j)(v_k \boldsymbol{e}_k) = \delta_{ij} v_{j,i} v_k \boldsymbol{e}_k = v_{i,i} v_k \boldsymbol{e}_k = (\mathrm{div}\boldsymbol{v})\boldsymbol{v}$, where the divergence is $\mathrm{div}\boldsymbol{v} = \frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3}$, so $(\mathrm{div}\boldsymbol{v})\boldsymbol{v} = (\frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3})\boldsymbol{v}$. Thus:

$$(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} \neq v_1 \frac{\partial \boldsymbol{v}}{\partial x^1} + v_2 \frac{\partial \boldsymbol{v}}{\partial x^2} + v_3 \frac{\partial \boldsymbol{v}}{\partial x^3}. \quad (10)$$

In the latest version of Landau & Lifshitz,[13] for reasons which are unclear, the material derivative expression has been changed to $\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}$; this is also incorrect.

Proof:

Since $\boldsymbol{v}\boldsymbol{\nabla} = (v_i \boldsymbol{e}_i)(\partial_j \boldsymbol{e}_j) = v_{i,j} \boldsymbol{e}_i \boldsymbol{e}_j$,

$$(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} = (v_{j,i} \boldsymbol{e}_i \boldsymbol{e}_j)(v_k \boldsymbol{e}_k) = (v_{j,i} \boldsymbol{e}_i \boldsymbol{e}_j)(v_k \boldsymbol{e}_k),$$

which indicates that $v\nabla$ is a 3rd-order tensor rather than a vector. Thus:

$$(v\nabla)v \neq v_1\frac{\partial v}{\partial x^1} + v_2\frac{\partial v}{\partial x^2} + v_3\frac{\partial v}{\partial x^3}. \qquad (11)$$

To obtain the correct formulation of the material derivative, we perform some basic tensor calculus[20]. The del operator is a vector differential operator, and defined by $\nabla = e_i\frac{\partial}{\partial x_i}$. An important note concerning the del operator ∇ is in order. Two types of gradients are used in continuum physics: left and right gradients. The left gradient is the usual gradient and the right gradient is the transpose of the forward gradient operator.

To see the difference between the left and right of gradients, consider a vector function $A = A_i(x)e_i$. The left gradient of a vector A is $\nabla \otimes A \equiv e_j\frac{\partial}{\partial x_j} \otimes (A_ie_i) = \frac{\partial A_i}{\partial x_j}e_j \otimes e_i = A_{i,j}e_j \otimes e_i$, or written as $\nabla A \equiv e_j\frac{\partial}{\partial x_j}(A_ie_i) = \frac{\partial A_i}{\partial x_j}e_je_i = A_{i,j}e_je_i$. The right gradient of a vector A is $A \otimes \nabla \equiv (A_ie_i) \otimes e_j\frac{\partial}{\partial x_j} = \frac{\partial A_i}{\partial x_j}e_i \otimes e_j = A_{i,j}e_i \otimes e_j$, or written as $A\nabla \equiv (A_ie_i)e_j\frac{\partial}{\partial x_j} = \frac{\partial A_i}{\partial x_j}e_ie_j = A_{i,j}e_ie_j$, where $A_{i,j} = \frac{\partial A_i}{x_j}$.

The gradient of a scalar function is a vector, the divergence of a vector-valued function is a scalar $\nabla \cdot A$, and the gradient of a vector-valued function is a second-order tensor ∇A. Although the del operator has some of the properties of a vector, it does not have them all, because it is an operator. For instance, $\nabla \cdot A$ is a scalar (called the divergence of A) whereas $A \cdot \nabla$ is a scalar differential operator, where A is a vector. Thus the del operator ∇ does not commute in this sense.

It worth to emphasise that the right gradient of a vector $A\nabla$ is a more natural one, is often used in defining the deformation gradient tensor, displacement gradient tensor, and velocity gradient tensor. It is clear that $A\nabla = (\nabla A)^T$.

1. Mass density $\rho = \rho(x,t)$ is a scalar-valued function of x and t and its differential is $d\rho = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x} \cdot \frac{\partial x}{\partial t}dt$. For a scalar $\frac{\partial \rho}{\partial x} = \rho\nabla = \nabla\rho$, so $d\rho = \frac{\partial \rho}{\partial t}dt + (\rho\nabla) \cdot vdt = \frac{\partial \rho}{\partial t}dt + v \cdot (\nabla\rho)dt$, or, dividing both sides by dt,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\rho\nabla) \cdot v = \frac{\partial \rho}{\partial t} + v \cdot (\nabla\rho). \qquad (12)$$

2. Flow velocity $v = v(x,t)$ is a vector-valued function of x and t and its differential is $dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial t}dt$. For a vector $\frac{\partial v}{\partial x} = v \otimes \nabla = v\nabla = (\nabla v)^T$. Strictly speaking, the right gradient of v should be written as $v \otimes \nabla$; here, in order to agree as far as possible with conventional presentation, we write $v \otimes \nabla = v\nabla$ and, similarly for the left gradient, $\nabla \otimes v = \nabla v$. Hence $(v\nabla) \cdot v = v \cdot (\nabla v)$ and $dv = \frac{\partial v}{\partial t}dt + (v\nabla) \cdot vdt = \frac{\partial \rho}{\partial t}dt + v \cdot (\nabla v)dt$, or, dividing both sides by dt,

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v\nabla) \cdot v = \frac{\partial v}{\partial t} + v \cdot (\nabla v). \qquad (13)$$

3. The 2nd order stress tensor $\sigma = \sigma(x,t)$ is a tensor-valued function of x and t and its differential is $d\sigma = \frac{\partial \sigma}{\partial t}dt + \frac{\partial \sigma}{\partial x} \cdot \frac{\partial x}{\partial t}dt$. For an arbitrary tensor $\frac{\partial \sigma}{\partial x} = \sigma\nabla \neq (\nabla\sigma)^T$, hence

$d\sigma = \frac{\partial \sigma}{\partial t}dt + (\sigma\nabla) \cdot vdt$, or dividing both sides by dt leads to $\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial t} + (\sigma\nabla) \cdot v$. Since $(\sigma\nabla) \cdot v = v \cdot (\nabla\sigma)$,

$$\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial t} + v \cdot (\nabla\sigma). \qquad (14)$$

If an operator must be introduced for the material derivative, it should be in the following form:

$$\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + v \cdot [\nabla(:)]. \qquad (15)$$

For a scalar function $f(x,t)$, this leads to $\frac{df}{dt} = \frac{\partial f}{\partial t} + v \cdot (\nabla f)$. If f is considered as a distributed function of gas molecules in their phase space, this derivative is as in Eq. (3.2) in Physical Kinetics by Lifshitz and Pitaevskii.[2]

For a vector function $u(x,t)$, Eq. (15) yields $\frac{du}{dt} = \frac{\partial u}{\partial t} + v \cdot (\nabla u)$. For a tensor function $A(x,t)$, Eq. (15) yields $\frac{dA}{dt} = \frac{\partial A}{\partial t} + v \cdot (\nabla A)$.

Using the material derivative operator $\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + v \cdot [\nabla(:)]$, the Navier-Stokes momentum equation is given by $\frac{\partial v}{\partial t} + v \cdot (\nabla v) = \nu\nabla^2 v - \frac{1}{\rho}\nabla p$, where p is flow pressure and ν is kinematic viscosity.

Assuming $\det \nabla v \neq 0$, the Navier-Stokes momentum equation can be rewritten as:

$$\begin{aligned} v &= \left(\nu\nabla^2 v - \frac{1}{\rho}\nabla p - \frac{\partial v}{\partial t}\right) \cdot (\nabla v)^{-1} \\ &= (v\nabla)^{-1} \cdot \left(\nu\nabla^2 v - \frac{1}{\rho}\nabla p - \frac{\partial v}{\partial t}\right). \end{aligned} \qquad (16)$$

If the incorrect material derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)$ is used in the Navier-Stokes momentum equation, then Eq. (16) cannot be obtained.

The mistake in the definition $\frac{d}{dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)$ is mainly the incorrect representation of $v \cdot \nabla$; ∇ is a gradient operator rather than a vector, so that:

$$\begin{aligned} v \cdot \nabla &= (v_1e_1 + v_2e_2 + v_3e_3) \cdot \left(e_1\frac{\partial}{\partial x^1} + e_2\frac{\partial}{\partial x^2} + e_3\frac{\partial}{\partial x^3}\right) \\ &= \frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} \neq v_1\frac{\partial}{\partial x^1} + v_2\frac{\partial}{\partial x^2} + v_3\frac{\partial}{\partial x^3}. \end{aligned}$$

In conclusion, the material derivative operator is correctly defined as: $\frac{d(:)}{dt} = \frac{\partial}{\partial t}(:) + v \cdot [\nabla(:)]$.

Availability of data: There are no data in this theoretical article.

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