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Article

New fixed point results via $(\theta,\psi)_{\mathcal{R}}$-weak contractions with an application

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Abstract: In this paper, inspired by Jleli and Samet [journal of inequalities and applications 38 (2014) 1–8] we introduce two new classes of auxiliary functions and utilize the same to define $(\theta,\psi)_{\mathcal{R}}$-weak contractions. Utilizing $(\theta,\psi)_{\mathcal{R}}$-weak contractions, we prove some fixed point theorems in the setting of relational metric spaces. We employ some examples to substantiate the utility of our newly proved results. Finally, we apply one of our newly proved results to ensure the existence and uniqueness of solution of a Volterra-type integral equation.

Keywords: Fixed point, θ-contraction, binary relation, integral equation.

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1. Introduction

The extensive utility of Banach contraction principle (cf.[3]) always inspires nonlinear analysts to improve this principle in various ways. For this kind of work one may recall Boyd and Wong [5], Matkowski [12], Ciric [6], Ran and Reurings [15], Jleli and Samet[9], Imdad et al.[8] and similar others. As Banach contraction principle is an existence and uniqueness result, it is very effectively utilized to establish the existence and uniqueness of solution of such problems arising in many branches of mathematics and outside mathematics. Due to its simplicity and applicability there has been many generalizations of this principle in various directions.

In 2014, Jleli and Samet [9] proved a result for a new type of contractions known as θ-contractions (or JS-contractions) as given below.

Definition 1. *Let $\theta:(0,\infty)\to(1,\infty)$ be a function satisfying the following conditions:*

(Θ_1) *θ is nondecreasing,*
(Θ_2) *for any $\{\alpha_n\}\subseteq(0,\infty)$, $\lim\limits_{n\to\infty}\alpha_n=0$ iff $\lim\limits_{n\to\infty}\theta(\alpha_n)=1$,*
(Θ_3) *$\exists l\in(0,\infty]$ and $0<r<1$ satisfying $\lim\limits_{\alpha\to 0^+}\frac{\theta(\alpha)-1}{\alpha^r}=l$,*
(Θ_4) *θ is continuous.*

Theorem 1. *[9] Let (E,d) be a complete generalized metric space and $f:E\to E$. Assume that there exist θ satisfying $\Theta_1,\Theta_2,\Theta_3$ and $\lambda\in(0,1)$ such that*

$$d(fa,fb)>0 \implies \theta(d(fa,fb))\leq(\theta(d(a,b)))^{\lambda},\ \textit{for all } a,b\in E. \tag{1}$$

Then f has a unique fixed point.

The mapping f in Theorem 1 is called θ-contraction or (JS-contraction).
In 2015, Hussain et al. [7] extended Theorem 1 for some new contraction mappings in which the authors used the condition (Θ_4) instead of (Θ_3). Imdad et al. [8] relaxed the condition (Θ_1) and called such mappings weak θ-contractions. Recently, there is yet another way to improve Banach contraction principle utilizing various types of binary relations. In 2004, Ran and Reurings [15] proved a fixed point result in metric space equipped with a partial order relation which was further generalized by Nieto and Rodríguez-López in [13,14]. In the same quest, in 2015, Alam and Imdad [1] generalized Banach contraction to a complete relational metric space.

The improvements accomplished in this paper are in fact four-fold which can be described as follows:

- to introduce the notion of $(\theta, \psi)_{\mathcal{R}}$-weak contraction;
- to prove our results in the setting of relational metric spaces;
- to adopt examples substantiating the utility of our proved results;
- to utilize our newly proved results to establish an existence and uniqueness result for the solution of a Volterra-type integral equation.

2. Preliminaries

In this manuscript, the set of all fixed points of $f : E \to E$ is denoted as $Fix(f)$.
Our main results involve relation theoretic concepts. So we recall some preliminaries of the same.

Definition 2. *[11] Let E be a nonempty set. A subset $\mathcal{R}$ of E^2 is said to be a binary relation on E. For $a, b \in E$ with $(a, b) \in \mathcal{R}$, we say that "a is related to b" or "a relates to b under $\mathcal{R}$". Sometimes, we write $a\mathcal{R}b$ instead of $(a, b) \in \mathcal{R}$. If $(a, b) \notin \mathcal{R}$, we say"$a$ is not related to b" or "a does not relate to b under $\mathcal{R}$." If $a\mathcal{R}b$, $b\mathcal{R}c$ implies $a\mathcal{R}c$, then $\mathcal{R}$ is called transitive.*

In this paper, $\mathcal{R}$ denotes for a nonempty binary relation defined on a nonempty set E. For brevity we only use "binary relation" instead of "nonempty binary relation."

Definition 3. *[1]*

- *A binary relation $\mathcal{R}$ on E is said to be f-closed if for any $a, b \in E$,*

$$(a, b) \in \mathcal{R} \Rightarrow (fa, fb) \in \mathcal{R}.$$

- *A sequence $\{a_n\} \subseteq E$ is called $\mathcal{R}$-preserving if $(a_n, a_{n+1}) \in \mathcal{R}\ \forall\ n \in \mathbb{N}$.*

Definition 4. *A sequence $\{a_n\} \subseteq E$ is called $\mathcal{R}$-preserving Cauchy sequence if it is a Cauchy sequence and $(a_n, a_{n+1}) \in \mathcal{R}\ \forall\ n \in \mathbb{N}$.*

Definition 5. *[1,2] Let (E, d) be a metric space and $\mathcal{R}$ a binary relation on E.*

- *$\mathcal{R}$ is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{a_n\}$ converging to a, $\exists\{a_{n_k}\} \subseteq \{a_n\}$ with $(a_{n_k}, a) \in \mathcal{R}$.*
- *(E, d) is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in E converges in E.*
- *A mapping $f : E \to E$ is called $\mathcal{R}$-continuous at $a \in E$ if for any $\mathcal{R}$-preserving sequence $\{a_n\}$ converging to a, we have $fa_n \to fa$. Moreover, f is $\mathcal{R}$-continuous on E if it is $\mathcal{R}$-continuous at each point of E.*

We need the following lemma in the sequel.

Lemma 1. *[4] Let (E, d) be a metric space and $\{a_n\} \subseteq E$ such that $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$. If $\{a_n\}$ is not Cauchy, then $\exists\, \epsilon > 0$ and two subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ of $\{a_n\}$ with $n_k \geq m_k > k$ such that*

$$\begin{aligned}\lim_{n\to\infty} d(a_{m_k}, a_{n_k}) &= \lim_{n\to\infty} d(a_{m_k}, a_{n_k+1}) = \lim_{n\to\infty} d(a_{m_k-1}, a_{n_k+1}) \\ &= \lim_{n\to\infty} d(a_{m_k+1}, a_{n_k+1}) = \epsilon.\end{aligned}$$

3. Main Results

Firstly, we introduce the following two classes of auxiliary functions which are relatively larger than the class of the auxiliary functions covered under Definition 1.

Definition 6. *Let Θ be the collection of all $\theta : (0, \infty) \to (1, \infty)$ which satisfy the following:*

(Θ_2) *for every sequence $\{a_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} \theta(a_n) = 1$,*
(Θ_3') *θ is lower semicontinuous.*

The following examples of the functions $\theta : (0, \infty) \to (1, \infty)$ belong to the class of Θ:

Example 1. $\theta(a) = \begin{cases} e^{\frac{a}{2}} \text{ if } a \leq 1 \\ e^{a} \text{ if } a > 1 \end{cases}$,

Example 2. $\theta(a) = \begin{cases} a+1 \text{ if } a \leq 1 \\ a+2 \text{ if } a > 1 \end{cases}$,

Example 3. $\theta(a) = \begin{cases} e^{\frac{a}{2}+\sin a} \text{ if } a \leq 2\pi \\ e^{\frac{a}{2}+\cos a} \text{ if } a > 2\pi \end{cases}$,

Example 4. $\theta(a) = \begin{cases} a^2+1 \text{ if } a \leq 2 \\ a^3 \text{ if } a > 2 \end{cases}$.

Next, we introduce yet another class of auxiliary functions:

Definition 7. *Let Ψ be the collection of all $\psi : (0, \infty) \to (1, \infty)$ which satisfy the following:*

(Ψ_2) *for every sequence $\{a_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} \psi(a_n) = 1$,*
(Ψ_3') *ψ is right upper semicontinuous.*

The following mappings $\psi : (0, \infty) \to (1, \infty)$ belong to the class Ψ:

Example 5. $\psi(a) = \begin{cases} e^{a}+1 \text{ if } a < 2 \\ a^4 \text{ if } a \geq 2 \end{cases}$,

Example 6. $\psi(a) = \begin{cases} \sqrt{a}+1 \text{ if } a < 1 \\ a+2 \text{ if } a \geq 1 \end{cases}$,

Example 7. $\psi(a) = \begin{cases} e^{\frac{a}{2}+a\cos a} \text{ if } a < \pi \\ e^{\frac{a}{2}+\sin a} \text{ if } a \geq \pi \end{cases}$,

Example 8. $\psi(a) = \begin{cases} a^2+1 \text{ if } a < 2 \\ e^2 \text{ if } a \geq 2 \end{cases}$.

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In what follows, we write $M(a,b) = \max\{d(a,b), d(a,fa), d(b,fb), \frac{d(a,fb)+d(b,fa)}{2}\}$.

Finally, we introduce the concept of $(\theta,\psi)_{\mathcal{R}}$-weak contractions as follows:

Definition 8. *Let (E,d) be a metric space, $\mathcal{R}$ a binary relation on E and $f : E \to E$. Then f is called $(\theta,\psi)_{\mathcal{R}}$-weak contraction if $\exists\, \lambda \in (0,1), \theta \in \Theta$ and $\psi \in \Psi$ with $\theta(t) \geq \psi(t)$ ($\forall\, t > 0$) such that*

$$\theta(d(fa,fb)) \leq (\psi(M(a,b)))^{\lambda}$$

$\forall\, a,b \in E$ such that $a\mathcal{R}b$ and $fa \neq fb$.

Now, we state and prove our first main result as follows:

Theorem 2. *Let (E,d) be a metric space endowed with a transitive binary relation $\mathcal{R}$ and $f : E \to E$. Assume that:*

(i) E is $\mathcal{R}$-complete,
(ii) $\exists\, a_0 \in E$ such that $a_0\mathcal{R}fa_0$,
(iii) $\mathcal{R}$ is f-closed,
(iv) f is $(\theta,\psi)_{\mathcal{R}}$-weak contraction and
(v) f is $\mathcal{R}$-continuous,

Then f has a fixed point.

Proof. In view of (ii), there is $a_0 \in E$ such that $a_0\mathcal{R}fa_0$. Consider the sequence $\{a_n\}$ of Picard iterates of f based at a_0, i.e.,

$$a_n = f^n a_0 \; \forall\, n \in \mathbb{N}_0. \tag{2}$$

If $a_{n+1} = a_n$, for some $n \in \mathbb{N}_0$, then $fa_n = a_n$, i.e., a_n is a fixed point of f and there is nothing to prove. Now, assume that $a_n \neq a_{n+1}\; \forall\, n \in \mathbb{N}_0$. We claim that the sequence $\{a_n\}$ is $\mathcal{R}$-preserving. Due to condition (2) and (2), we have $a_0\mathcal{R}fa_0$ and $a_1 = fa_0$, hence, $a_0\mathcal{R}a_1$. Suppose $a_n\mathcal{R}a_{n+1}$, for some $n \in \mathbb{N}$. As $\mathcal{R}$ is f-closed, we have $fa_n\mathcal{R}fa_{n+1}$, i.e., $a_{n+1}\mathcal{R}a_{n+2}$. Hence, by mathematical induction, we conclude that $\{a_n\}$ is $\mathcal{R}$-preserving and $fa_n \neq fa_{n+1}\; \forall\, n \in \mathbb{N}_0$.
In view of the contraction condition (2), we have

$$\theta(d(a_n,a_{n+1})) = \theta(d(fa_{n-1},fa_n)) \leq (\psi(M(a_{n-1},a_n))^{\lambda},$$

where $M(a_{n-1},a_n)$

$$\begin{aligned}
&= \max\{d(a_{n-1},a_n), d(a_{n-1},fa_{n-1}), d(a_n,fa_n), \frac{d(a_n,fa_{n-1})+d(a_{n-1},fa_n)}{2}\} \\
&= \max\{d(a_{n-1},a_n), d(a_{n-1},a_n), d(a_n,a_{n+1}), \frac{d(a_n,a_n)+d(a_{n-1},a_{n+1})}{2}\} \\
&= \max\{d(a_{n-1},a_n), d(a_n,a_{n+1})\},
\end{aligned}$$

as $d(a_{n-1},a_{n+1}) \leq d(a_{n-1},a_n) + d(a_n,a_{n+1})$ and $\max\{x,y,\frac{z}{2}\} = \max\{x,y\}$ whenever $x,y,z \geq 0$ and $z \leq x+y$.
If $M(a_{n-1},a_n)$=$d(a_n,a_{n+1})$, then,

$$\theta(d(a_n,a_{n+1})) \leq \psi((d(a_n,a_{n+1})))^{\lambda} < \psi(d(a_n,a_{n+1})) \leq \theta(d(a_n,a_{n+1}))$$

a contradiction. Hence,

$$M(a_{n-1},a_n) = d(a_{n-1},a_n),$$

so that

$$\theta(d(a_n, a_{n+1})) \leq (\psi(d(a_{n-1}, a_n)))^\lambda \leq (\theta(d(a_{n-1}, a_n)))^\lambda,$$

as $\theta(t) \geq \psi(t)\ \forall\ t > 0$. Finally, we have

$$1 < \theta(d(a_n, a_{n+1})) \leq (\theta(d(a_{n-1}, a_n)))^\lambda \leq \cdots \leq (\theta(d(a_0, a_1))^{\lambda^n}.$$

Now, letting $n \to \infty$, we obtain

$$1 \leq \lim_{n\to\infty} \theta(d(a_n, a_{n+1})) \leq 1, \text{so that} \lim_{n\to\infty} \theta(d(a_n, a_{n+1})) = 1.$$

Now, making use of (Θ_2) we get

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0.$$

Now, we proceed to prove that $\{a_n\}$ is Cauchy. Let on contrary that $\{a_n\}$ is not Cauchy. From Lemma 1, one can infer that there exist an $\epsilon > 0$ and $\{m_k\}, \{n_k\} \subseteq \mathbb{N}$ with $n_k \geq m_k > k$ such that

$$\{d(a_{m_k}, a_{n_k})\}, \{d(a_{m_k}, a_{n_k+1})\}, \{d(a_{m_k-1}, a_{n_k})\}, \{d(a_{m_k-1}, a_{n_k+1})\}, \{d(a_{m_k+1}, a_{n_k+1})\} \tag{3}$$

tend to ϵ when $k \to \infty$.

As $\mathcal{R}$ is transitive, so $a_{m_k+1}\mathcal{R}a_{n_k+1}$, for all $k \in \mathbb{N}$. Also, for sufficiently large k_0, $fa_{m_k} \neq fa_{n_k}$, for all $k \geq k_0$ (as $d(a_{m_k+1}, a_{n_k+1}) \to \epsilon$). Therefore, we have

$$\theta(d(a_{n_k+1}, a_{m_k+1})) = \theta(d(fa_{n_k}, fa_{m_k})) \leq (\psi(M(a_{n_k}, a_{m_k}))^\lambda, \tag{4}$$

so that

$$\liminf_{k\to\infty} \theta(d(a_{n_k+1}, a_{m_k+1})) \leq \liminf_{k\to\infty} (\psi(M(a_{n_k}, a_{m_k})))^\lambda \leq \limsup_{k\to\infty} (\psi(M(a_{n_k}, a_{m_k})))^\lambda,$$

wherein $M(a_{n_k}, a_{m_k})$

$$\begin{aligned} &= \max\{d(a_{n_k}, a_{m_k}), d(a_{n_k}, fa_{n_k}), d(a_{m_k}, fa_{m_k}), \frac{d(a_{n_k}, fa_{m_k}) + d(a_{m_k}, fa_{n_k})}{2}\} \\ &= \max\{d(a_{n_k}, a_{m_k}), d(a_{n_k}, a_{n_k+1}), d(a_{m_k}, a_{m_k+1}), \frac{d(a_{n_k}, a_{m_k+1}) + d(a_{m_k}, a_{n_k+1})}{2}\}. \end{aligned}$$

Observe that

$$d(a_{n_k}, a_{m_k+1}) + d(a_{m_k}, a_{n_k+1}) \leq d(a_{n_k}, a_{m_k}) + d(a_{m_k}, a_{m_k+1}) + d(a_{m_k}, a_{n_k}) + d(a_{n_k}, a_{n_k+1}).$$

Now, as $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$, and $d(a_{n_k}, a_{m_k}) \to \epsilon$, so due to Lemma 1 we have

$$\lim_{k\to\infty} M(a_{m_k}, a_{n_k}) = \epsilon \tag{5}$$

Now, using (3), (4) and (5), we get

$$\theta(\epsilon) \leq \liminf_{k\to\infty} \theta(d(a_{n_k+1}, a_{m_k+1})) \leq \limsup_{k\to\infty} (\psi(M(a_{n_k}, a_{m_k})))^\lambda \leq (\psi(\epsilon))^\lambda < \psi(\epsilon) \leq \theta(\epsilon),$$

a contradiction. Thus, $\{a_n\}$ is Cauchy. As $\{a_n\}$ is $\mathcal{R}$-preserving Cauchy in E which is $\mathcal{R}$-complete, therefore, there is some $a \in E$ such that

$$\lim_{n\to\infty} a_n = a.$$

As f is also $\mathcal{R}$-continuous, we obtain

$$fa_n \to fa \text{ and, as } n \to \infty,$$

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i.e., $a_{n+1} \to fa$ as $n \to \infty$. Now, using the uniqueness of the limit, we have $fa = a$. Hence f has a fixed point in E. This ends the proof. $\square$

Next, we adopt the following example which exhibit the utility of Theorem 2.

Example 9. *Let $E = \mathbb{R}^+$ and d be the usual metric defined by $d(a,b) = |a-b|$, $\forall\, a,b \in E$. Define $\{\alpha_n\}$ as*

$$\alpha_n = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for all } n \in \mathbb{N},$$

whose first few terms are 6,30,90,210 and so on. Define $\mathcal{R}$ on E as

$$\mathcal{R} = \{(1,1),(1,\alpha_i),(\alpha_i,\alpha_j) : i,j \in \mathbb{N}, i < j\}.$$

Define f on E by

$$f(x) = \begin{cases} x, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } 1 \leq x < 6, \\ 1 + \frac{5}{24}(x-6), \text{ if } \alpha_1 \leq x < \alpha_2, \\ \alpha_{i-1} + \frac{\alpha_i - \alpha_{i-1}}{\alpha_{i+1}-\alpha_i}(x-\alpha_i), \text{ if } \alpha_i \leq x < \alpha_{i+1}. \end{cases}$$

The mapping f is continuous (see Figure 1). To show that $\mathcal{R}$ is f-closed, consider the following three different cases.
Case I: let $(a,b) = (1,1) \in \mathcal{R}$, then $(fa,fb) = (1,1) \in \mathcal{R}$.
Case II: when $(a,b) = (1,\alpha_i) \in \mathcal{R}$, then $(fa,fb) = (1,1)$ or $(1,\alpha_i)$, i.e., $fa\mathcal{R}fb$.
Case III: when $(a,b) = (\alpha_i,\alpha_j), i < j$; then $(fa,fb) = (1,\alpha_{j-1})$ or $(\alpha_{i-1},\alpha_{j-1})$ and hence $fa\mathcal{R}fb$. Thus, $\mathcal{R}$ is f-closed.

Now, to show that f is a $(\theta,\psi)_{\mathcal{R}}$-weak contraction mapping, we define $\theta,\psi : \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$\theta(t) = e^{e^{\gamma_1(t)}}, \psi(t) = e^{e^{\gamma_2(t)}},$$

where $\gamma_1,\gamma_2 : \mathbb{R}^+ \to \mathbb{R}$ are given by

$$\gamma_1(t) = \begin{cases} \frac{-1}{t} + \frac{t}{2}, t \leq 10, \\ \frac{-1}{t} + t, t > 10, \end{cases} \quad \text{and } \gamma_2(t) = \begin{cases} \frac{-1}{t} + \frac{t}{3}, t < 25, \\ \frac{-4}{t} + t, t \geq 25. \end{cases}$$

We have to show that there exists some $\lambda \in (0,1)$ such that

$$\begin{aligned} \theta(d(fa,fb)) &\leq (\psi(M(a,b)))^{\lambda} \\ \text{i.e., } e^{e^{\gamma_1(d(fa,fb))}} &\leq (e^{e^{\gamma_2(M(a,b))}})^{\lambda} = e^{\lambda e^{\gamma_2(M(a,b))}} \\ \text{or, } e^{\gamma_1(d(fa,fb))} &\leq \lambda e^{\gamma_2(M(a,b))} \\ \text{or, } \gamma_1(d(fa,fb)) &\leq \ln(\lambda) + \gamma_2(M(a,b)) \end{aligned} \tag{6}$$

We don't need to consider the cases $(1,1),(1,\alpha_1) \in \mathcal{R}$ as $fa = fb$. Now, we distinguish the following four cases.
Case I: if $a = 1$ and $b = \alpha_2 = 30$, then $d(a,b) = 29, d(fa,fb) = 5$,

$$\begin{aligned} \text{where } M(a,b) &= \max\{d(a,b),d(a,fa),d(b,fb),\frac{d(a,fb)+d(b,fa)}{2}\} \\ &= \max\{29,0,24,\frac{5+29}{2}\} = 29, \end{aligned}$$

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so that

$$\begin{aligned} 4+\gamma_1(d(fa,fb)) &= 4+\gamma_1(5) = 10-\frac{1}{5}+\frac{5}{2} \\ &\leq -\frac{4}{29}+29 = \gamma_2(29) \\ \text{i.e., } 4+\gamma_1(d(fa,fb)) &= \gamma_2(M(a,b)). \end{aligned}$$

Case II: if $a = 1$ *and* $b = \alpha_i, i > 2$; *then* $d(a,b) \geq 89, d(fa,fb) \geq 29$

$$\begin{aligned} \text{wherein } M(a,b) &= \max\{d(a,b), d(a,fa), d(b,fb), \frac{d(a,fb)+d(b,fa)}{2}\} \\ &\geq \max\{89, 0, 60, \frac{29+89}{2}\} = 89, \end{aligned}$$

so that

$$\begin{aligned} 4(d(fa,fb)) - M(a,b) &< 4(d(fa,fb)) \\ &< M(a,b)d(fa,fb) \\ &< M(a,b)(d(fa,fb))(M(a,b)-d(fa,fb)-4) \\ \text{yielding thereby } \frac{4}{M(a,b)} - \frac{1}{d(fa,fb)} &< M(a,b)-d(fa,fb)-4 \\ \text{or, } 4+d(fa,fb)-\frac{1}{d(fa,fb)} &< -\frac{4}{M(a,b)}+M(a,b) \\ \text{i.e., } 4+\gamma_1(d(fa,fb)) &\leq \gamma_2(M(a,b)). \end{aligned}$$

Case III: if $a = \alpha_1 = 6$ *and* $b = \alpha_2 = 30$; *then* $d(a,b) = 24, d(fa,fb) = (1,6) = 5$

$$\begin{aligned} \text{where } M(a,b) &= \max\{d(a,b), d(a,fa), d(b,fb), \frac{d(a,fb)+d(b,fa)}{2}\} \\ &= \max\{24, 5, 24, \frac{0+29}{2}\} = 24, \end{aligned}$$

so that

$$\begin{aligned} 4+\gamma_1(d(fa,fb)) &= 4+\gamma_1(5) \\ &= 4-\frac{1}{5}+\frac{5}{2} \\ &\leq -\frac{1}{24}+\frac{24}{3} \\ &= -\frac{1}{M(a,b)}+\frac{M(a,b)}{3} \\ \text{i.e., } 4+\gamma_1(d(fa,fb)) &= \gamma_2(M(a,b)). \end{aligned}$$

Case IV: if $a = \alpha_i$ *and* $b = \alpha_j, (i,j) \neq (1,2)$; *then* $d(a,b) \geq 60, d(fa,fb) \geq 24$

$$\begin{aligned} \text{where } M(a,b) &= \max\{d(a,b), d(a,fa), d(b,fb), \frac{d(a,fb)+d(b,fa)}{2}\} \\ &\geq \max\{24, 5, 24, \frac{0+29}{2}\} = 24, \end{aligned}$$

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so that

$$\begin{aligned} 4(d(fa,fb)) - M(a,b) &< 4(d(fa,fb)) \\ &< M(a,b)d(fa,fb) \\ &< M(a,b)(d(fa,fb))(M(a,b) - d(fa,fb) - 4) \\ \text{yielding thereby } \frac{4}{M(a,b)} - \frac{1}{d(fa,fb)} &< M(a,b) - d(fa,fb) - 4 \\ \text{or, } 4 + d(fa,fb) - \frac{1}{d(fa,fb)} &< \frac{4}{M(a,b)} + M(a,b) \\ \text{i.e., } 4 + \gamma_1(d(fa,fb)) &\leq \gamma_2(M(a,b). \end{aligned}$$

Therefore, in all four cases, we have

$$\gamma_1(d(fa,fb)) \leq -4 + \gamma_2(M(a,b)\ \forall\, a,b \in E \text{ whenever } a\mathcal{R}b \text{ and } d(fa,fb) > 0$$

Thus the condition (6) is satisfied if we take $\lambda = \frac{1}{e^4} \in (0,1)$. *Therefore, we have furnished a* $\lambda \in (0,1)$ *such that* $\theta(d(fa,fb)) \leq (\psi(M(a,b)))^\lambda$, *i.e.,* f *is a* $(\theta,\psi)_\mathcal{R}$*-weak contraction mapping on* X. *Observe that, the remaining assumptions of Theorem 2 are also fulfilled. Thus* f *possesses a fixed point in* E. *Observe that* f *has infinitely many fixed points, in fact,* $Fix(f) = [0,1]$ *(see Figure 1).*

The pre-existing results in this direction, say, the results of Jleli and Samet[9], Hussain et al. [7] and Imdad et al.[8] can not be applied in this example as these results require the contraction condition to hold on the whole space. But here in this example the contraction condition holds for those $a,b \in E$ *which are related under the binary relation* $\mathcal{R}$.

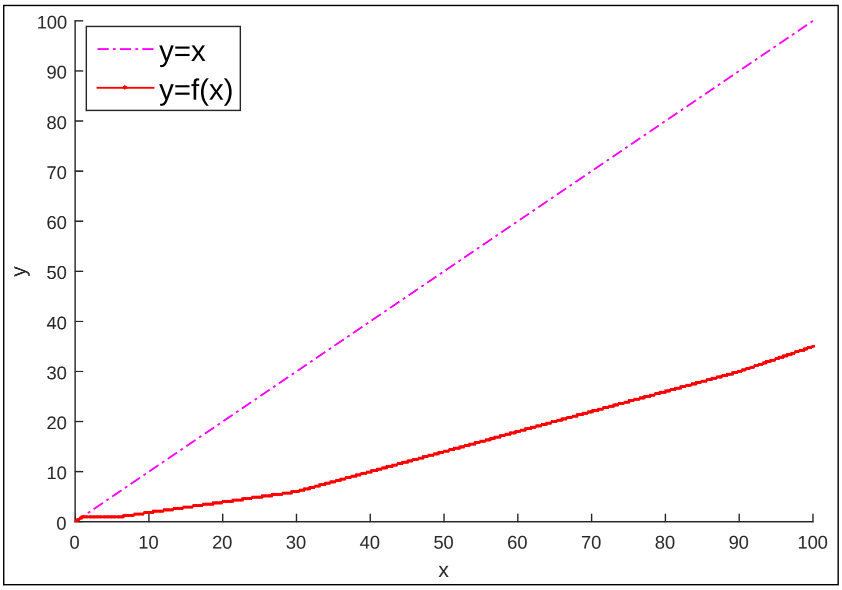

Figure 1: Graph of $y = x$ and $y = fx$ in Example 9.

Now, we prove an analogous of Theorem 2 using the d-self-closedness property.

Theorem 3. *The conclusion of Theorem 2 holds true if the assumption (e) is replaced by:*

(e') $\mathcal{R}$ *is* d*-self-closed.*

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Proof. On the lines of the proof of Theorem 2, we can show that $\{a_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence which converging to $a \in E$. Our aim is to show that $a = fa$. Suppose on contrary that $d(a, fa) > 0$. In view of the condition (e'), $\exists\{a_{n_k}\} \subseteq \{a_n\}$ with $a_{n_k}\mathcal{R}a \;\forall\, k \in \mathbb{N}$. Now, as $a_n \to a$ and $a \neq fa$, for sufficiently large k_0, we have $fa_{n_k} \neq fa$ for all $k \geq k_0$. Hence, we have (for all $k \geq k_0$)

$$\theta(d(a_{n_k+1}, fa)) \leq (\psi(M(a_{n_k+1}, a))^{\lambda}, \tag{7}$$

where $M(a_{n_k+1}, a) =$

$$\max\{d(a_{n_k+1}, a), d(a_{n_k+1}, fa_{n_k+1}), d(a, fa), \frac{d(a, fa_{n_k+1}) + d(a_{n_k+1}, fa)}{2}\}.$$

As

$$\frac{d(a, fa_{n_k+1}) + d(a_{n_k+1}, fa)}{2} \leq \frac{d(a, fa_{n_k+1}) + d(a_{n_k+1}, a) + d(a, fa)}{2}$$

and the sequence $\{a_n\}$ converges to a with $d(a, fa) > 0$, we have (for all $k \geq k_0$)

$$\begin{gathered} d(a_{n_k+1}, a) < d(a, fa), d(a_{n_k+1}, fa_{n_k+1}) = d(a_{n_k+1}, a_{n_k+2}) < d(a, fa), \\ d(a, fa_{n_k+1}) + d(a_{n_k+1}, a) = d(a, a_{n_k+2}) + d(a_{n_k+1}, a) < \tfrac{d(a,fa)}{2}, \end{gathered}$$

so that (for all $k \geq k_0$)

$$M(a_{n_k+1}, a) = d(a, fa).$$

Hence, (7) reduces to

$$\theta(d(a_{n_k+1}, fa)) \leq (\psi(d(a, fa)))^{\lambda}, \text{ for all } k \geq k_0,$$

implying thereby

$$\liminf_{n\to\infty} \theta(d(a_{n_k+1}, fa)) \leq \liminf_{n\to\infty}((\psi(d(a, fa)))^{\lambda} = (\psi(d(a, fa)))^{\lambda}.$$

As θ is lower semicontinuous, $\liminf_{n\to\infty} \theta(d(a_{n_k+1}, fa)) \geq \theta(d(a, fa))$, which gives rise to the following:

$$\theta(d(a, fa)) \leq (\psi(d(a, fa)))^{\lambda} < \psi(d(a, fa)) \leq \theta(d(a, fa)).$$

This is a contradiction. Hence, the assumption $d(a, fa) \neq 0$ is wrong. Therefore, we must have $d(a, fa) = 0$, i.e., a is a fixed point of f. This completes the proof. □

Next, we prove the following corresponding uniqueness result.

Theorem 4. *If we assume in addition to the assumptions of Theorem 2 (or Theorem 3), that $Fix(f)$ is complete or $Fix(f)$ is $\mathcal{R}^S$-connected, then f has a unique fixed point in E.*

Proof. In view of Theorem 2 (or 3), we have $Fix(f)$ is nonempty. Assume that $Fix(f)$ is complete and let a, b be two different points in $Fix(f)$. So $a\mathcal{R}b$ or $b\mathcal{R}a$, i.e., $[a, b] \in \mathcal{R}$ or $a\mathcal{R}^S b$. As $a = fa, b = fb$ and $a \neq b$, we have $d(fa, fb) > 0$. Hence, using the contraction condition (d), we get

$$\theta(d(a, b)) = \theta(d(fa, fb)) \leq (\theta(d(a, b)))^{\lambda} \text{ for some } \lambda \in (0, 1).$$

This is a contradiction as $\theta(d(a, b)) > 0$. Hence, our assumption that $a \neq b$ is wrong. Therefore, f has a unique fixed point in E.

Now, if $Fix(f)$ is $\mathcal{R}^S$-connected and $a, b \in Fix(f)$, then there exist $a_1, a_2..., a_{n-1}$ in $Fix(f)$ such that $a_i\mathcal{R}^S a_{i+1} \;\forall\, i : 0 \leq i \leq n-1$ where $a_0 = a$ & $a_n = b$. Now, as $a_i\mathcal{R}^S a_{i+1}$ and $a_i, a_{i+1} \in Fix(f) \;\forall\, i : 0 \leq i \leq n-1$; from the earlier part of the theorem we have $a_i = a_{i+1} \;\forall\, i$. Therefore, $a = b$, *i.e.*, the fixed point of f is unique. This concludes the proof. □

Now, to substantiate the utility of Theorems 3 and 4, we furnish the following example:

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Example 10. *Let $E = (-1, 3]$ and d be the usual metric on E. Define $\mathcal{R}$ on E as:*

$$\mathcal{R} = \{(0,0), (0,1), (1,0), (1,1), (0,3), (1,3)\}.$$

Then $\mathcal{R}$ is transitive. Now, define $f : E \to E$ by

$$f(t) = \begin{cases} 0, \ if \ -1 < t \leq 1, \\ 1, \ if \ 1 < t \leq 3. \end{cases}$$

We observe that the folowing holds:

- *E is $\mathcal{R}$-complete, as for any $\mathcal{R}$-preserving Cauchy sequence $\{a_n\}$ in E, $\exists\ K \in \mathbb{N}$ such that $a_n = 1$, $\forall\ n \geq K$ or $a_n = 0$, $\forall\ n \geq K$, i.e., $\{a_n\}$ converges to 0 or 1;*
- *$0 \in E$ and $0\mathcal{R}f0$;*
- *$\mathcal{R}$ is f-closed;*

Now, we show that f is a $(\theta, \psi)_{\mathcal{R}}$-weak contraction. Take $\theta \in \Theta, \psi \in \Psi$ as the following:

$$\theta(t) = e^{e^{\gamma_1(t)}}, \psi(t) = e^{e^{\gamma_2(t)}},$$

where $\gamma_1, \gamma_2 : \mathbb{R}^+ \to \mathbb{R}$ are given by

$$\gamma_1(t) = \begin{cases} \frac{-1}{t} + \frac{t}{2}, t \leq 10, \\ \frac{-1}{t} + t, t > 10, \end{cases} \quad and \ \gamma_2(t) = \begin{cases} \frac{-1}{t} + \frac{t}{3}, t < 25, \\ \frac{-4}{t} + t, t \geq 25. \end{cases}$$

We have to show that there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} \theta(d(fa, fb)) &\leq (\psi(M(a,b)))^{\lambda} \\ i.e., e^{e^{\gamma_1(d(fa,fb))}} &\leq (e^{e^{\gamma_2(M(a,b))}})^{\lambda} = e^{ke^{\gamma_2(M(a,b))}} \\ or, e^{\gamma_1(d(fa,fb))} &\leq ke^{\gamma_2(M(a,b))} \\ or, \gamma_1(d(fa, fb)) &\leq \ln(\lambda) + \gamma_2(M(a,b)). \end{aligned}$$

Observe that $d(fa, fb) > 0$ implies that $(a, b) \in \{(0,3), (1,3)\}$. Therefore, we consider only the two cases $(0,3)$ and $(1,3)$.
Case I: let $(a, b) = (0, 3)$. Then

$$d(fa, fb) = d(0,1) = 1 < 3 = d(0,3) = d(a,b) \leq M(a,b).$$

Observe that,

$$\frac{1}{2} + \gamma_1(1) = 0 \leq \frac{2}{3} = -\frac{1}{3} + 1 = \gamma_2(3).$$

Case II: if $(a, b) = (1, 3)$, then

$$d(fa, fb) = d(0,1) = 1 < 2 = d(1,3) = d(a,b) \leq M(a,b).$$

As

$$\frac{1}{2} + \gamma_1(1) = 0 \leq \frac{1}{6} = -\frac{1}{2} + \frac{2}{3} = \gamma_2(2),$$

Therefore, for $\lambda = e^{-\frac{1}{2}} \in (0,1)$, we have

$$\gamma_1(d(fa, fb)) \leq \ln(\lambda) + \gamma_2(M(a,b)),$$

or

$$\theta(d(fa, fb)) \leq (\psi(M(a,b)))^{\lambda},$$

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for all $a, b \in E$ with $a\mathcal{R}b$ and $fa \neq fb$. Hence, the condition (d) is satisfied.

Next, in order to verify (e'), observe that for any $\mathcal{R}$-preserving sequence $\{a_n\}$ converging to some $a \in E$, there is some $K \in \mathbb{N}$ such that either $a_n = 0\ \forall\, n \geq K$ or $a_n = 1\ \forall\, n \geq K$. Hence, $\{a_{K+i}\}_{i \in \mathbb{N}_0}$ is a subsequence of $\{a_n\}$ such that $a_{K+i}\mathcal{R}a$ for each $i \in \mathbb{N}_0$.
Therefore, all the assumptions of Theorem 3 are satisfied. Hence, f has a fiaed point in E. We can see that 0 is the fixed point of f. Also $Fix(f) = \{0\}$, which is complete as $(0, 0) \in \mathcal{R}$. Thus, Theorem 4 ensures the uniqueness of the fixed point of f.

4. Applications

In this section, we apply Theorem 4 to find the existence and uniqueness of solution for the following integral equation:

$$a(s) = \int_0^s G(s, v, a(v))dv + g(s), s \in [0, 1], \tag{8}$$

where G is a continuous function from $[0, 1] \times [0, 1] \times [0, 1]$ to [0,1] and g is a continuous function from [0,1] to [0,1].

Consider the Banach space $(E, \|.\|)$ of all continuous functions $a : [0, 1] \to [0, 1]$ equipped with the norm

$$\|a\| = \max_{t \in [0,1]} |a(t)|.$$

Define a metric d on E by

$$d(a, b) = \max_{t \in [0,1]} |a(t) - b(t)|.$$

Then (E, d) is a complete metric space.

Now, we state and prove our first result in this section as follows:

Theorem 5. *If G is nondecreasing in the third variable and $M > 0$ such that*

$$|G(s, v, a(v) - G(s, v, b(v))| \leq |a(v) - b(v)|e^{-\frac{1}{1+M(a,b)} - M}$$

for all $s, v \in [0, 1]$ & $a, b \in E$ such that $a \leq b$, then the existence of a lower solution of the integral equation (8) ensures the existence of a unique solution of the same.

Proof. Define a self mapping $f : E \to E$ by

$$(fa)(s) = \int_0^s G(s, v, a(v))dv + g(s)\ \forall\, a \in E.$$

Define $\mathcal{R}$ on E by

$$\mathcal{R} = \{(a, b) \in E \times E : a(s) \leq b(s)\ \forall\, s \in [0, 1]\}.$$

For any $(a, b) \in \mathcal{R}$, we have

$$\begin{aligned}(fa)(s) &= \int_0^s G(s, v, a(v))dv + g(s) \\ &\leq \int_0^s G(s, v, b(v))dv + g(s) \\ &= (fb)(s).\end{aligned}$$

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Therefore $\mathcal{R}$ is f-closed. In the hypotheses, we have assumed the existence of a lower solution of (8), i.e., there is some $a_0 \in E$ such that the following holds:

$$a_0(s) \leq \int_0^s k(s,v,a_0(v))dv + g(s).$$

Therefore, $a_0 \in E$ is such that $a_0(s) \leq (fa_0)(s)$ for each $s \in [0,1]$. Hence $a_0\mathcal{R}fa_0$. Now, for any $a,b \in E$ such that $a\mathcal{R}b$, we have

$$\begin{aligned}
|f(a(s)) - f(b(s))| &= \left|\int_0^s (G(s,v,a(v)) - G(s,y,b(v))dv\right| \\
&\leq \int_0^s |(G(s,v,a(v)) - G(s,y,b(v))|dv \\
&\leq \int_0^s |a(v) - b(v)|e^{-\frac{1}{1+M(a,b)}-M}dv \\
&\leq e^{-\frac{1}{1+M(a,b)}-M} \sup_{v\in[0,1]} |a(v) - b(v)| \int_0^s dv \\
&= d(a,b)e^{-\frac{1}{1+M(a,b)}-M} \int_0^s dv \\
&= d(a,b)e^{-\frac{1}{1+M(a,b)}-M}s.
\end{aligned}$$

Taking maximum over both the sides, we get

$$\begin{aligned}
d(fa,fb) &\leq d(a,b)e^{-\frac{1}{1+M(a,b)}-M} \\
&\leq M(a,b)e^{-\frac{1}{1+M(a,b)}-M} \\
\implies e^{d(fa,fb)} &\leq e^{M(a,b)e^{-\frac{1}{1+M(a,b)}}e^{-M}} \\
&= \left(e^{M(a,b)e^{-\frac{1}{1+M(a,b)}}}\right)^{e^{-M}}.
\end{aligned}$$

Consider $\theta(t) = e^t, \psi(t) = e^{te^{-\frac{1}{1+t}}}, \lambda = e^{-M} \in (0,1)$; then f becomes a $(\theta,\psi)_{\mathcal{R}}$-weak contraction. Now, for any $\mathcal{R}$-preserving sequence $\{a_n\}$ in E converging to $a \in E$, we have

$$a_0(t) \leq a_1(t) \leq a_2(t) \leq \cdots \leq a_n(t) \leq a_{n+1}(t) \leq \ldots, \ \forall\, t \in [0,1].$$

Thus, $a_n(t) \leq a(t)$ for all $t \in [0,1]$. Therefore, $\mathcal{R}$ is d-self closed in E. Hence, using Theorem 2, we conclude that f has a fixed point, i.e., there is some $a \in E$ such that

$$a(s) = \int_0^s G(s,v,a(v))dv + g(s).$$

Now, for any $a,b \in \ Fix(f); c = \max\{a,b\} \in E$ and $c \in Fix(f)$. Also $a \leq c, b \leq c$, i.e., $a\mathcal{R}c$ and $b\mathcal{R}c$. Therefore, $Fix(f)$ is $\mathcal{R}^S$ connected. Hence, Theorem 4 ensures the uniqueness of the solution of the integral equation (8). This accomplishes the proof. $\square$

Next, we provide the following theorem in the presence of an upper solution.

Theorem 6. *If G is nondecreasing in the third variable and $M > 0$ such that*

$$|G(s,v,a(v) - G(s,v,b(v))| \leq |a(v) - b(v)|e^{-\frac{1}{1+M(a,b)}-M}$$

for all $s,v \in [0,1]$ & $a,b \in E$ such that $a \leq b$, then the existence of an upper solution of the integral equation (8) ensures the existence of a unique solution of the same.

Proof. In this case, we define the binary relation $\mathcal{R}$ as follows:

$$\mathcal{R} = \{(a,b) \in E \times E : a(s) \geq b(s) \; \forall \, s \in [0,1]\}.$$

Now, following the same steps as of Theorem 5 one can see that all the hypotheses of Theorem 4 hold true. Therefore, the existence and uniqueness of the solution of the integral equation (8) is ensured (due to Theorem 4). □

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