Some Dirichlet forms on graphs as traces of one-dimensional diffusions

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Abstract

We compute explicitly traces of one-dimensional diffusion processes. The obtained trace forms can be regarded as Dirichlet forms on graphs. Then we discuss conditions ensuring the trace forms to be conservative. Finally, the obtained results are applied to the one-dimensional diffusion related to the Bessel's process of order ν.

1   Introduction

Throughout this paper we are concerned with computation of traces of one-dimensional diffusions generated by the Feller operator $\frac{d}{dm}\frac{d}{ds}$. We recall the known fact that such diffusions on an open interval I are characterized by a scale function s, i.e. a continuous strictly increasing function on I and speed measure m. Moreover, they are related in an appropriate way to Dirichlet forms with domains in $L^2(I, m)$ defined by

$$\mathcal{E}^{(s)}[u] := \int_I (\frac{du}{ds}(x))^2 \, ds(x),$$

on its domain $\operatorname{dom} \mathcal{E}^{(s)}$. Further details about the form are given in the next section. Given a diffusion of the above type, a positive measure μ with support $V \subset I$ and a linear operator $J : \operatorname{dom} \mathcal{E}^{(s)} \to L^2(V, \mu)$ we shall first compute the trace of $\mathcal{E}^{(s)}$ with respect to the measure μ by means of the method elaborated in [BBST19]. We shall demonstrate in particular that the obtained trace form in $L^2(F, \mu)$ is in fact a graph Dirichlet form if the measure μ is discrete.

Once the computation has been performed we shall turn our attention to study conservativeness property, i.e. conservation of total mass, for the trace Dirichlet form. We shall show that, for fixed $\mathcal{E}^{(s)}$, conservativeness depends strongly on the measure μ and its support V.

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The motivation rests on two facts: first to put the particular case for the Bessel's process analyzed in [BM20] in a general framework. Second, the significance of conservativeness property both in analysis and in probability. In fact in analysis conservativeness is equivalent to existence and uniqueness of solution of the heat equation with bounded initial data. Whereas in probability conservativeness implies that, almost surely, the related stochastic process starting at any point will have an infinite lifetime.

At this stage we mention that there is a huge literature concerned with conservative Dirichlet forms. Regarding the subject we refer the reader to [Stu94, AG12, MUW12] [Gim16, Gim17, KL12].

The paper is organized as follows. In section 2 we introduce some necessary definitions and notations concerning Dirichlet forms related to one-dimensional diffusions as well as Feller's classical properties of boundaries.

Section 3 is devoted to compute the trace of Dirichlet form on discrete sets as well as on composite of continuous and discrete sets. In Section 4 we study conservativeness property for traces of Dirichlet forms on discrete sets. In this respect we shall give necessary and sufficient conditions ensuring the trace form to be conservative. Thereby we extend [BM20, Theorem 3.7] to this general framework.

The obtained theoretical results will be applied to the one-dimensional diffusion related to the Bessel's process of order ν, in the last section.

2 Framework and basic notations

We start by introducing some notations.

Let $I := (r_1, r_2)$ where $-\infty \leq r_1 < r_2 \leq \infty$. Let us consider a continuous strictly increasing function $s : I \to \mathbb{R}$ (a scaling function for short). Thus s has the following representation

$$s(x) = \int_c^x \sigma(t)\, dt, \quad \forall\, x \in I,$$

where $\sigma > 0$ and $\sigma \in \mathcal{L}^1_{loc}(I)$. Obviously $ds(x) = \sigma(x)\, dx$.

Let us designate by $AC_{loc}(I)$ the space of locally absolutely continuous function on I and by $AC_s(I)$ the space of s-absolutely continuous functions on I, i.e. the set of functions $u : I \to \mathbb{R}$ such that there exists an absolutely continuous function ϕ with $u = \phi \circ s$.

Let us consider a speed measure m with full support I defined by

$$dm(x) = \rho(x)\, dx,$$

where $\rho > 0$ and $\rho \in \mathcal{L}^1_{loc}(I)$.

We designate by

$$\mathcal{D}^{(s)} := \left\{ u \colon I \to \mathbb{R} :\ u \in AC_s(I), \quad \int_{r_1}^{r_2} (u'(x))^2\, \frac{dx}{\sigma(x)} < \infty \right\}.$$

2

Let us define a quadratic form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(I, \rho \, dx)$ by

$$\mathcal{D} := \mathcal{D}^{(s)} \cap L^2(I, \rho \, dx), \quad \mathcal{E}[u] := \int_{r_1}^{r_2} (u'(x))^2 \, \frac{dx}{\sigma(x)} \quad \text{for all } u \in \mathcal{D}. \tag{2.1}$$

From the classical reference book [FOT11, chap.1] on theory of Dirichlet forms we recall that a Dirichlet form $(\mathcal{Q}, \mathrm{dom}\,\mathcal{Q})$ is a densely defined, closed, symmetric and Markovian form on $L^2(I, m)$. Hence, the obtained quadratic form $\mathcal{E}$ is a Dirichlet form (see [CF12, p.63-64]).

Lemma 2.1. *Every function from $\mathcal{D}$ is continuous on I.*

Proof. Since every $u \in \mathcal{D}$ is s-absolutely continuous, it is a composition of continuous functions, hence continuous. $\square$

It is well known that $\mathcal{E}$ is a regular strongly local Dirichlet form in $L^2(I, \rho \, dx)$ (hence in particular closed and densely defined). Moreover, the positive self-adjoint operator associated with the form $\mathcal{E}$ via Kato's representation theorem, which we denote by L, is given by (see [Lin04] and [FOT11, Chap. 1])

$$D(L) = \Big\{ u \in \mathcal{D} : \ u' \in AC_{loc}(I), \ \mathcal{L}u = -\frac{1}{\rho\sigma}u'' + \frac{\sigma'}{\rho\sigma^2}u' \in L^2(I, \rho \, dx),$$
$$\text{with boundary conditions at } r_1 \text{ and } r_2 \Big\}$$
$$Lu = \mathcal{L}u \quad \text{for all } u \in D(L).$$

Remark 2.2. The second-order ordinary differential operator ([CF12, p.63-64])

$$\mathcal{L}u(x) = a(x)u''(x) + b(x)u'(x)$$

with real-valued functions $a > 0$ and b can be converted into Feller's canonical form $\frac{d}{dm}\frac{d}{ds}$ with

$$ds = \mathrm{e}^{-B(x)}dx, dm = \frac{\mathrm{e}^{B(x)}}{a(x)}dx, \quad B(x) = \int_{x_0}^{x} \frac{b(y)}{a(y)} \, dy.$$

Hence, by formal computation we get

$$-\int \mathcal{L}u.v \, dm = -\int v.d\frac{du}{ds} = \int \frac{du}{ds}\frac{dv}{ds} \, ds, \ \forall u, v \in \mathcal{C}_c^2(I).$$

Further, owing to Feller's canonical form, we can define the differential operator $\mathcal{L}$ on I by

$$\mathcal{L} := -\frac{d}{dm}\frac{d}{ds} = -\frac{1}{\rho}\frac{d}{dx}.\left(\frac{1}{\sigma}\frac{d}{dx}\right) = -\frac{1}{\rho\sigma}\frac{d^2}{dx^2} + \frac{\sigma'}{\rho\sigma^2}\frac{d}{dx}.$$

2.1 Feller's boundary classification

Let us introduce the following quantities

$$\Gamma_1 = \int_{r_1}^{c} m((c,x))ds(x), \quad \Sigma_1 = \int_{r_1}^{c} s((c,x))dm(x) \tag{2.2}$$

3

and

$$\Gamma_2 = \int_c^{r_2} m((x,c))ds(x), \quad \Sigma_2 = \int_c^{r_2} s((x,c))dm(x), \tag{2.3}$$

for some $r_1 < c < r_2$.

It is well known ([JYC09]) that the boundaries r_1 and r_2 of I can be classified w.r.t the Feller operator $\frac{d}{dm}\frac{d}{ds}$ into four classes as follows (we refer the reader to [Ito06, p.151-152] or [Man69, p.24-25])

(a) r_i a regular boundary if $\Gamma_i < \infty,\ \Sigma_i < \infty$,

(b) r_i an exit boundary if $\Gamma_i < \infty,\ \Sigma_i = \infty$,

(c) r_i an entrance boundary if $\Gamma_i = \infty,\ \Sigma_i < \infty$,

(d) r_i a natural boundary if $\Gamma_i = \infty,\ \Sigma_i = \infty$.

Definition 2.3. We say that the boundary r_1 (resp. r_2) is approachable whenever

$$s(r_1) > -\infty \quad (\text{resp. } s(r_2) < \infty).$$

According to the inequality (3.2), if r_1, (resp. r_2) is approachable, then for any element from $\mathcal{D}^{(s)}$ we have $u(r_1) = \lim_{x\downarrow r_1, x\in I} u(x) < \infty$, (resp. $u(r_2) = \lim_{x\uparrow r_2, x\in I} u(x) < \infty$) and $u \in \mathcal{C}([r_1, r_2))$, (resp. $u \in \mathcal{C}((r_1, r_2])$).
One has in particular that space $\mathcal{D}^{(s)}$ is a uniformly dense sub-algebra of $\mathcal{C}([r_1, r_2])$ if both r_1 and r_2 are approachable (for more details we refer to [CF12, Chap. II]).

Definition 2.4. We call that boundary r_1 (resp. r_2) is called regular whenever

it is approachable and $m((r_1, c)) < \infty,$ (resp. $m((c, r_2)) < \infty$) $\forall\, c \in (r_1, r_2)$.

Remark 2.5. Regularity condition of boundaries $r_i,\ i = 1, 2$ (see [RW00, chpa.5] for more details) is very like the concept of irreducibility which has the following probabilistic terminology

$$\mathbb{P}_x(H_y < \infty) > 0,\ \forall x, y \in I,$$

where H_y is the hitting time of $\{y\}$ relative to a diffusion process. Likewise, ([RW00, chap.5]) it allows us to characterise one-dimensional diffusion essentially by a scaling function s and a speed measure m.

2.2 Extended Dirichlet space

Let us now introduce the extended Dirichlet space of $\mathcal{E}$ ([CF12, chap.1]), which we denote by $\mathcal{D}_e$.

Definition 2.6. Let $(\mathcal{E}, \mathcal{D})$ be a closed symmetric form on $L^2(I, m)$. Denote by $\mathcal{D}_e$ the totality of m-equivalence classes of all m-measurable functions f on I such that $|f| < \infty\ [m]$ and there exists an $\mathcal{E}$-Cauchy sequence $\{f_n, n \geq 1\} \subset \mathcal{D}$ such that $\lim_{n\to\infty} f_n = f,\ m - a.e.$ on I. $\{f_n\} \subset \mathcal{D}$ in the above is called an approximating sequence of $f \in \mathcal{D}_e$. We call the space $\mathcal{D}_e$ the extended space attached to $(\mathcal{E}, \mathcal{D})$. When the latter is a Dirichlet form on $L^2(I, m)$, the space $\mathcal{D}_e$ will be called its extended Dirichlet space.

To determine the extended Dirichlet form in our case we shall using the following proposition which is mentioned by [CF12, p.66].

Proposition 2.7. *Assume that both r_1 and r_2 are approachable but non-regular. If we let*

$$\mathcal{D}_0^{(s)} := \{u \in \mathcal{D}^{(s)} \; : \; u(r_1) = 0 = u(r_2)\}, \tag{2.4}$$

then

$$\mathcal{D} \subset \mathcal{D}_e = \mathcal{D}_0^{(s)}, \tag{2.5}$$

and $(\mathcal{E}, \mathcal{D})$ is a regular, strongly local, transient, and irreducible Dirichlet form on $L^2(I, m)$.

It is well known that $\mathcal{E}$ is transient if and only if (see [CF12, Theorem 2.2.11]) either r_1 or r_2 is approachable and non-regular. Otherwise it is recurrent .

By virtue of classical Feller's test of non-explosion, $\mathcal{E}$ is conservative if and only if (see [CF12, p. 126] and the discussion made there)

$$\int_{(r_1,c)} m((x,c)) \, ds(x) = \int_{(c,r_2)} m((c,x)) \, ds(x) = \infty, \; \forall \; r_1 < c < r_2. \tag{2.6}$$

In this case, the boundaries r_1 and r_2 are non-exit points.

Remark 2.8. We have the following discussion about the boundary conditions at r_1 and r_2 on $D(L)$:

   (i) If r_1 (resp. r_2) is an exit endpoint then we have the boundary condition at r_1 (resp. r_2)

$$\lim_{x \to r_1} u(x) = 0 \; (\text{resp. } \lim_{x \to r_2} u(x) = 0).$$

   (ii) If r_1 (resp. r_2) is an entrance endpoint then we have the boundary condition at r_1 (resp. r_2)

$$\lim_{x \to r_1} \frac{1}{\sigma(x)} \frac{du(x)}{dx} = 0 \; (\text{resp. } \lim_{x \to r_2} \frac{1}{\sigma(x)} \frac{du(x)}{dx} = 0).$$

   (iii) If r_1, (resp. r_2) is a natural endpoint then there is no boundary condition needed.

3 Computation of the trace of transient form $\check{\mathcal{E}}$

Let $V = \{x_k, \; k \in \mathbb{N} \} \subset (r_1, r_2)$ be a finite or countable set, where $(x_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence, i.e., $x_k < x_{k+1}$ for all $k \in \mathbb{N}$. Assume that

$$d := \sup_{k \in \mathbb{N}} (x_{k+1} - x_k) < \infty. \tag{3.1}$$

In addition, let $x_\infty = \lim_{k \to \infty} x_k$ which can be finite or not. Next, we will investigate the following two cases for a transient Dirichlet form $\mathcal{E}$:

   (a) V has no accumulation point.

(b) V has x_∞ as an accumulation point.

We have the following definition of capacity which is a set function associated to Dirichlet form and it plays an important role to measure the size of sets adapted to the form.

Definition 3.1. We define the 1-capacity Cap associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ by

$$Cap(U) := \inf\{\mathcal{E}_1[u] \ : \ u \in \mathcal{D}, u \geq 1, m - ae. \text{ in } U\}$$

for an open set $U \subset I$, and

$$Cap(A) := \inf\{Cap(U) \ : \ U \subset I \text{ open}, \ A \subset U\}$$

for a Borel set $A \subset I$, where

$$\mathcal{E}_1[u] := \mathcal{E}[u] + \int_I u^2(x) \ \rho(x) \ dx, \text{ for } u, v \in \mathcal{D}.$$

The following lemma show that a diffusion process associated with a Dirichlet form $\mathcal{E}$ enjoys a strong irreducibility property which means that any two point of I are connected for a diffusion process (we refer to [Fuk14]).

Lemma 3.2. *For all $x \in V$ we have $Cap(\{x\}) > 0$.*

Proof. An elementary identity leads to

$$u(\xi) - u(y) = \int_y^\xi \frac{du}{ds}(x) \ ds(x), \ \forall \ r_1 < \xi < y < r_2.$$

By Hölder inequality, we get

$$\left(u(\xi) - u(y)\right)^2 \leq s([y, \xi]).\mathcal{E}[u], \ \forall \ r_1 < \xi < y < r_2. \tag{3.2}$$

Then we get by integrating the both hand side on each compact set $K \subset I$ that there is a positive constant C_K such that

$$\sup_{x \in K} u(x)^2 \leq C_K \mathcal{E}_1[u], \ \forall u \in \mathcal{D}. \tag{3.3}$$

Hence $Cap(\{x\}) \geq \frac{1}{C_K} > 0$, for every $x \in V$, i.e. each point of I has a positive capacity relative to the Dirichlet form $(\mathcal{E}, \mathcal{D})$. $\qquad\square$

We shall start by the first case (a) which said that V has no accumulation point, i.e. $x_\infty = \infty$. We suppose in this case that interval $(r_1, r_2) = (x_1, \infty)$.

6

3.1 V has no accumulation point

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that $a_k > 0$ for all $k \in \mathbb{N}$. Let us consider the atomic measure defined as follows

$$\mu = \sum_{k\in\mathbb{N}} a_k \delta_{x_k}.$$

We define now the Hilbert space $\ell^2(V,\mu)$ equipped with the product given by

$$(u,v) := \sum_{k\in\mathbb{N}} u(x_k)v(x_k)\, a_k \quad \text{and} \quad \|u\| := \sqrt{(u,u)}.$$

Let $\check{\mathcal{E}}$ be the trace of $\mathcal{E}$ on the discrete set V (see [FOT11, BBST19]). We shall adopt the method elaborated in [BBST19] to compute explicitly $\check{\mathcal{E}}$.

Let J be the restriction operator defined from $D(J) \subseteq (\mathcal{E},\mathcal{D}_e)$ to $\ell^2(V,\mu)$ by

$$D(J) := \Big\{ u \in \mathcal{D}_e : \sum_{k\in\mathbb{N}} a_k u(x_k)^2 < \infty \Big\},$$

$$Ju := u|_V \quad \text{for all } u \in D(J).$$

We quote that since functions with finite support are dense in $\ell^2(V,\mu)$, the operator J has dense range. Obviously

$$\ker J = \big\{ u \in D(J) : \ u(x_k) = 0 \quad \text{for all } k \in \mathbb{N} \big\}.$$

We shall first determine the extended domain $\mathcal{D}_e$ of the trace form $\check{\mathcal{E}}$ according to Proposition 2.7.

$$\mathcal{D}_e = \{ u\colon I \to \mathbb{R} : \ u \in AC_s(I), \ \int_{x_1}^{\infty} (u'(x))^2\, \frac{dx}{\sigma(x)} < \infty \ \text{s.t.} \ \ u(x_1) = 0 = \lim_{x\to\infty} u(x) \}.$$

Accordingly we can compute $\check{\mathcal{E}}$ following [BBST19, Prop.3.1]. To that end we designate by P the orthogonal projection in the Dirichlet space $(\mathcal{E},\mathcal{D}_e)$ onto the $\mathcal{E}$-orthogonal complement of $\ker J$. Clearly

$$(\ker J)^{\perp_\mathcal{E}} = \{ u \in \mathcal{D}_e \ : \ \mathcal{E}(u,v) = 0, \quad \text{for all } v \in \ker J \}.$$

We can define the quadratic from $\check{\mathcal{E}}$ as follows

$$\check{\mathcal{E}}[Ju] = \mathcal{E}[Pu], \ \text{for all } u \in D(J).$$

Since J is closed in $(\mathcal{D}_e,\mathcal{E})$, then, from [BBST19, Prop.3.1], the form $\check{\mathcal{E}}$ is closed in $\ell^2(V,\mu)$.

Lemma 3.3. *Let $u \in \mathcal{D}$. Then Pu is the unique solution in $\mathcal{D}_e$ of the following Sturm-Liouville problem*

$$-\frac{1}{\sigma}(Pu)'' + \frac{\sigma'}{\sigma^2}(Pu)' = 0 \quad in \ \ (x_1,\infty)\setminus V, \tag{3.4}$$

$$Pu = u \quad on \ V.$$

7

Proof. Let $u \in D(J)$. Since P is the $\mathcal{E}$-orthogonal projection from $\mathcal{D}_e$ onto $(Ker J)^{\perp}$, then we obtain

$$\mathcal{E}(Pu, v) = 0, \quad \forall v \in \mathcal{C}_c^{\infty}((x_1, \infty) \setminus V),$$

which is equivalent to

$$-\frac{1}{\sigma}(Pu)'' + \frac{\sigma'}{\sigma^2}(Pu)' = 0 \text{ in the sense of distribution in } (x_1, \infty) \setminus V. \qquad (3.5)$$

Multiplying the latter equation by a positive term $\frac{1}{\rho}$. Pu be a solution with smooths coefficients on $(x_1, \infty) \setminus V$. Moreover since J is a closed operator then $Ker J$ is also closed, hence $u - Pu \in Ker J$ and $Pu \in D(J)$, then $Pu \in \mathcal{D}_e$ and hence Pu is s-absolutely continuous in $(x_1, \infty) \setminus V$ and the equation (3.5) is fulfilled pointwise on $(x_1, \infty) \setminus V$. Pu is the $\mathcal{E}$-orthogonal projection from $\mathcal{D}_e$ onto $(Ker J)^{\perp}$, hence $u = Pu$ everywhere on V. The converse is obvious. $\qquad \square$

Let us now compute explicitly the $\mathcal{E}$-orthogonal projection Pu the solution of the boundary value problem (3.4).

Lemma 3.4. *Let $u \in \mathcal{D}$. Pu can be expressed in the following way*

$$Pu(x) = -\frac{\int_c^{x_k} \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big) + u(x_k) + \frac{\int_c^x \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big), \quad (3.6)$$

for all $x \in [x_k, x_{k+1}]$ and $k \in \mathbb{N}$, where c is a constant in (x_1, ∞) such that $s(c) = 0$.

Proof. In fact the differential equation (3.4) has the solution for all $k \in \mathbb{N}$ as follows

$$Pu(x) = C_1 + C_2 \int_c^x \sigma(\tau)\, d\tau, \ \forall \, c > x_1, \text{ in } [x_k, x_{k+1}], \qquad (3.7)$$

where C_1 and C_2 are two real constants to be determined according to the boundary conditions.

Then, we have

$$Pu(x_k) = C_1 + C_2 \int_c^{x_k} \sigma(\tau) d\tau = u(x_k),$$

$$Pu(x_{k+1}) = C_1 + C_2 \int_c^{x_{k+1}} \sigma(\tau) d\tau = u(x_{k+1}), \ \forall r_1 < c < r_2,$$

which leads to get the following expression

$$
\begin{aligned}
Pu(x) &= -\frac{\int_c^{x_k} \sigma(\tau)\, d\tau}{\int_{x_k}^{x_{k+1}} \sigma(\tau)\, d\tau}\big(u(x_{k+1}) - u(x_k)\big) + u(x_k) + \frac{\int_c^x \sigma(\tau)\, d\tau}{\int_{x_k}^{x_{k+1}} \sigma(\tau)\, d\tau}\big(u(x_{k+1}) - u(x_k)\big) \\
&= -\frac{\int_c^{x_k} \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big) + u(x_k) + \frac{\int_c^x \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big),
\end{aligned}
$$

for all $x \in [x_k, x_{k+1}]$, and $c \in (x_1, \infty)$. $\qquad \square$

Lemma 3.5. *For every $u \in \mathcal{D}_e$. It holds*

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \left[(Pu)'(x_{k+1}^-)u(x_{k+1})\frac{1}{\sigma(x_{k+1})} - (Pu)'(x_k^+)u(x_k)\frac{1}{\sigma(x_k)} \right]$$

$$= \sum_{k=1}^{\infty} \left(\int_{x_k}^{x_{k+1}} \sigma(\tau)d\tau \right)^{-1} \left(u(x_{k+1}) - u(x_k) \right)^2, \tag{3.8}$$

where $(Pu)'(x_k^+)$ and $(Pu)'(x_{k+1}^-)$ are the right derivative at x_k and the left derivative at x_{k+1} respectively.

Proof. Let $u \in \mathcal{D}_e$. A straightforward computation leads to

$$\check{\mathcal{E}}[Ju]$$
$$= \int_{r_1}^{r_2} ((Pu)'(x))^2 \frac{dx}{\sigma(x)} = \int_{x_1}^{\infty} ((Pu)'(x))^2 \frac{dx}{\sigma(x)}$$
$$= \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} \left(-\frac{(Pu)''(x)}{\sigma(x)} + \frac{\sigma'(x)(Pu)'(x)}{\sigma(x)^2} \right) (Pu)(x)dx + \sum_{k=1}^{\infty} \frac{(Pu)'(x)(Pu)(x)}{\sigma(x)}\bigg|_{x_k}^{x_{k+1}}$$
$$= \sum_{k=1}^{\infty} \left[(Pu)'(x_{k+1}^-)u(x_{k+1})\frac{1}{\sigma(x_{k+1})} - (Pu)'(x_k^+)u(x_k)\frac{1}{\sigma(x_k)} \right]. \tag{3.9}$$

From the expression of Pu we can compute its derivative $(Pu)'$ for all $k \in \mathbb{N}$,

$$(Pu)'(x) = \frac{\sigma(x)}{s([x_k, x_{k+1}])} \left(u(x_{k+1}) - u(x_k) \right), \quad \text{in } [x_k, x_{k+1}].$$

Finally we obtain the trace form $\check{\mathcal{E}}$ of pure jump type

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s([x_k, x_{k+1}])} \left(u(x_{k+1}) - u(x_k) \right)^2$$

$$= \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \left(u(x_{k+1}) - u(x_k) \right)^2. \tag{3.10}$$

$\square$

In the sequel we shall recall some definition of weighted graphs to construct the discrete Dirichlet forms.

Definition 3.6. Let (V, b, c, μ) be a weighted graph consists of a countable set V defined as before, a measure $\mu : V \longrightarrow (0, \infty)$ and weight function $b : V \times V \longrightarrow [0, \infty)$ with $b(x, x) = 0$ for all $x \in V$ satisfying the following two properties

    (a) $b(x, y) = b(y, x)$ for all $x, y \in V$.

    (b) $\sum_{y \in V; y \sim x} b(x, y) < \infty$ for all $x \in V$.

We define a function $c : V \longrightarrow (0, \infty)$ which can be interpreted as a killing term or as a potential. We say that two vertices $x, y \in V$ are neighbors or connected by an edge if $b(x, y) > 0$ and we write $x \sim y$.

In order to describe the trace form $\check{\mathcal{E}}$ we have to introduce the next form
For every $u \in \ell^2(V, \mu)$. Set

$$Q[u] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2. \tag{3.11}$$

Then, $\check{\mathcal{E}} = Q_{|_{ranJ}}$.

Theorem 3.7. *Suppose that $\mu(V) = \infty$. Then*

$$D(\check{\mathcal{E}}) := \{ u \in \ell^2(V, \mu) : \ Q[u] < \infty \}, \ \check{\mathcal{E}}[u] = Q[u], \ \textit{for all } u \in D(\check{\mathcal{E}}).$$

Proof. Let us rewrite the trace form $\check{\mathcal{E}}$ as follows

$$
\begin{aligned}
\check{\mathcal{E}}[Ju] &= \sum_{x_k \in V} \sum_{x_j \sim x_k} b(x_k, x_j) \big(u(x_k)) - u(x_j) \big)^2 \\
&= \sum_{k \in \mathbb{N}} b(x_{k+1}, x_k) \big(u(x_{k+1})) - u(x_k) \big)^2 \\
&= \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2,
\end{aligned} \tag{3.12}
$$

where $b(x_{k+1}, x_k) = \frac{1}{s(x_{k+1}) - s(x_k)} > 0$ if $x_{k+1} \sim x_k$ (i.e. $\exists r > 0 : |x_{k+1} - x_k| = r$) and $b(x_{k+1}, x_k) = 0$ otherwise. Moreover, if $\mu(V) = \sum_{k \in \mathbb{N}} a_k = \infty$ and $b(x_k, x_{k+1}) > 0$ for all $k \in \mathbb{N}$, the condition (A) from [KL12] is fulfilled which yields the assertion. $\square$

Then, the associated self-adjoint discrete operator $\check{L}$ is given by

$$
\begin{aligned}
D(\check{L}) &= \big\{ u \in \ell^2(V, \mu) : \tilde{L}u \in \ell^2(V, \mu) \big\} \\
\check{L}u &= \tilde{L}u \quad \text{for all } u \in D(\check{L}),
\end{aligned}
$$

where for all $k \in \mathbb{N}$. We get

$$
\tilde{L}u(x_k) = \tag{3.13}
$$
$$
- \frac{u(x_{k+1})}{a_k(s(x_{k+1}) - s(x_k))} + \frac{u(x_k)(s(x_{k+1}) - s(x_{k-1}))}{a_k(s(x_{k+1}) - s(x_k))(s(x_k) - s(x_{k-1}))} - \frac{u(x_{k-1})}{a_k(s(x_k) - s(x_{k-1}))}.
$$

Remark 3.8. Assume $V = \mathbb{Z}$ and $a_k = 1$ for all $k \in \mathbb{Z}$. Then the expression (3.13) can be regarded as a discrete Jacobi operator which has the following form

$$\mathcal{J}u(k) := A(k)u(k + 1) + B(k)u(k) + A(k - 1)u(k - 1), \ \forall k \in \mathbb{N}. \tag{3.14}$$

3.2 V has an accumulation point

Now we consider the second case (b) where the sequence $(x_k)_{k \in \mathbb{N}}$ of the set V is convergent and it has x_∞ as an accumulation point .

We keep the same definitions as in the third section. We consider again the case where $\mathcal{E}$ is transient Dirichlet form. For $u \in \mathcal{D}$ Pu is the unique solution in $\mathcal{D}_e$ of the differential equation with boundary condition

$$-\frac{1}{\sigma}(Pu)'' + \frac{\sigma'}{\sigma^2}(Pu)' = 0 \quad \text{in} \quad \bigcup_{k=1}^{\infty}(x_k, x_{k+1}) \cup (x_\infty, \infty), \tag{3.15}$$

$$Pu = u \quad \text{on } V := \{x_1, ..., x_\infty\},$$

which can be expressed as follows

$$Pu(x) = -\frac{\int_c^{x_k} \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big) + u(x_k) + \frac{\int_c^x \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big),$$

for all $x \in [x_k, x_{k+1}]$, $k \in \mathbb{N}$,

and

$$Pu(x) = \frac{\int_c^{x_\infty} \sigma(\tau)\, d\tau}{s([x_\infty, \infty))}u(x_\infty) + u(x_\infty) - \frac{\int_c^x \sigma(\tau)\, d\tau}{s([x_\infty, \infty))}u(x_\infty),$$

for all $x \in (x_\infty, \infty)$ and for all fixed arbitrary $c \in (x_1, \infty)$.

We can compute now the trace $\check{\mathcal{E}}$ which is decomposed into the sum of a non-local and a killing Dirichlet form.

Lemma 3.9. *For every $u \in \mathcal{D}_e$, it holds*

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)}\big(u(x_{k+1}) - u(x_k)\big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))}. \tag{3.16}$$

Proof. Since the end-point ∞ is an approachable boundary, i.e., $s(\infty) < \infty$, then we have $\lim_{x \to \infty} Pu(x) = \lim_{x \to \infty} u(x) = 0$ and $s([x_\infty, \infty)) < \infty$. Owing to this argument we can obtain the following explicit computation of $\check{\mathcal{E}}$.

Let $u \in \mathcal{D}_e$. We get

$$\check{\mathcal{E}}[Ju] = \int_{r_1}^{r_2}((Pu)'(x))^2\frac{dx}{\sigma(x)} = \int_{x_1}^{\infty}((Pu)'(x))^2\frac{dx}{\sigma(x)}$$

$$= \sum_{k=1}^{\infty}\int_{x_k}^{x_{k+1}}((Pu)'(x))^2\frac{dx}{\sigma(x)} + \int_{x_\infty}^{\infty}((Pu)'(x))^2\frac{dx}{\sigma(x)}$$

$$= \sum_{k=1}^{\infty}\int_{x_k}^{x_{k+1}}\left(-\frac{(Pu)''(x)}{\sigma(x)} + \frac{\sigma'(x)(Pu)'(x)}{\sigma(x)^2}\right)(Pu)(x)dx + \sum_{k=1}^{\infty}\frac{(Pu)'(x)(Pu)(x)}{\sigma(x)}\bigg|_{x_k}^{x_{k+1}}$$

$$+ \int_{x_\infty}^{\infty}\left(-\frac{(Pu)''(x)}{\sigma(x)} + \frac{\sigma'(x)(Pu)'(x)}{\sigma(x)^2}\right)(Pu)(x)dx + \frac{(Pu)'(x)(Pu)(x)}{\sigma(x)}\bigg|_{x_\infty}^{\infty}$$

11

$$= \sum_{k=1}^{\infty} \left[(Pu)'(x_{k+1}^-) u(x_{k+1}) \frac{1}{\sigma(x_{k+1})} - (Pu)'(x_k^+) u(x_k) \frac{1}{\sigma(x_k)} \right] - \frac{(Pu)'(x_\infty) u(x_\infty)}{\sigma(x_\infty)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 - \frac{(Pu)'(x_\infty) u(x_\infty)}{\sigma(x_\infty)} \tag{3.17}$$

where $(Pu)'(x_k^+)$ and $(Pu)'(x_{k+1}^-)$ are the right derivative at x_k and the left derivative at x_{k+1} respectively.

Since

$$(Pu)'(x) = -\frac{\sigma(x)}{s([x_\infty, \infty))} u(x_\infty), \quad \forall x \in (x_\infty, \infty).$$

Therefore we obtain

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))}.$$

$\square$

3.3   Trace of the Dirichlet form related to one-dimensional diffusion process w.r.t mixed type measure

Let $(x_k)_k$ be a sequence of negative numbers which converges to 0 and so the set $V \subset (r_1, 0)$ has 0 as an accumulation point. We consider a new measure on (x_1, ∞) of mixed type, i.e. measure which has an absolutely continuous part and a discrete part as follows

$$\mu := \mu_{disc} + \mu_{abs},$$

where

$$\mu_{disc} = \sum_{k=1}^{\infty} a_k \delta_{x_k}, \ \forall a_k > 0, k \in \mathbb{N} \quad \text{and} \quad \mu_{abs} = \mathbf{1}_{(0,\infty)} \rho(x) dx.$$

Hence $F = \{x_k, \ k \in \mathbb{N}\} \cup [0, \infty)$ is the support of the measure μ. In order to compute the trace of $\mathcal{E}$ w.r.t measure μ we shall define the trace operator J by

$$J : \mathcal{D} \cap L^2(F, \mu) \to L^2(F, \mu), \ Ju = u_{|F}.$$

Obviously, we have

$$Ker(J) := \{u \in \mathcal{D} \ : \ u(x_k) = 0, \forall k \in \mathbb{N}, \ u_{|(0,\infty)} = 0\}.$$

Then the Sturm-Liouville problem has Pu a unique solution of

$$-\frac{1}{\sigma}(Pu)'' + \frac{\sigma'}{\sigma^2}(Pu)' = 0 \quad \text{in} \ \bigcup_{k=1}^{\infty}(x_k, x_{k+1}), \tag{3.18}$$

$$Pu = u \quad \text{on} \ V \cup (0, \infty).$$

We can express the general solution of Pu in the same way as the third section

$$Pu(x) = -\frac{\int_c^{x_k} \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])} \big(u(x_{k+1}) - u(x_k) \big) + u(x_k) + \frac{\int_c^x \sigma(\tau)\, d\tau}{s([x_k, x_{k+1}])} \big(u(x_{k+1}) - u(x_k) \big), \tag{3.19}$$

for all $x \in [x_k, x_{k+1}]$ and $k \in \mathbb{N}$, and c is an arbitrary fixed point in (r_1, ∞).

Lemma 3.10. *Let $u \in \mathcal{D}$. It holds*

$$\check{\mathcal{E}}[Ju] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \int_0^{\infty} (u'(x))^2 \frac{dx}{\sigma(x)}. \qquad (3.20)$$

Proof. A straightforward computation leads

$$
\begin{aligned}
\check{\mathcal{E}}[Ju] &= \int_{r_1}^{r_2} ((Pu)'(x))^2 \frac{dx}{\sigma(x)} = \int_{x_1}^{\infty} ((Pu)'(x))^2 \frac{dx}{\sigma(x)} \\
&= \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} ((Pu)'(x))^2 \frac{dx}{\sigma(x)} + \int_0^{\infty} ((Pu)'(x))^2 \frac{dx}{\sigma(x)} \\
&= \sum_{k=1}^{\infty} \left[(Pu)'(x_{k+1}^-) u(x_{k+1}) \frac{1}{\sigma(x_{k+1})} - (Pu)'(x_k^+) u(x_k) \frac{1}{\sigma(x_k)} \right] + \int_0^{\infty} (u'(x))^2 \frac{dx}{\sigma(x)} \\
&= \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \int_0^{\infty} (u'(x))^2 \frac{dx}{\sigma(x)}.
\end{aligned}
$$

$$(3.21)$$

$\square$

Remark 3.11. Let $u \in L^2(F, \mu)$. Set

$$Q[u] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \int_0^{\infty} (u'(x))^2 \frac{dx}{\sigma(x)}. \qquad (3.22)$$

Since Q is closed, then $\check{\mathcal{E}} = Q_{|ranJ}$.

We introduce $\check{\mathcal{D}}_{max}$ the space of the trace form $\check{\mathcal{E}}$ by

$$\check{\mathcal{D}}_{max} = \Big\{ u \in L^2(F, \mu) \ : \ u \in AC_s([0, \infty)), \ \sum_{k=1}^{\infty} \frac{\big(u(x_{k+1}) - u(x_k) \big)^2}{s(x_{k+1}) - s(x_k)} + \int_0^{\infty} \frac{(u'(x))^2 dx}{\sigma(x)} < \infty \Big\}.$$

We denote by $\check{\mathcal{E}}^{(J)}$ and $\check{\mathcal{E}}^{(c)}$ the quadratic forms such that

$$dom(\check{\mathcal{E}}^{(J)}) = dom(\check{\mathcal{E}}^{(c)}) = \check{\mathcal{D}}_{max}$$

and

$$\check{\mathcal{E}}^{(c)}[u] = \int_0^{\infty} (u'(x))^2 \frac{dx}{\sigma(x)},$$

$$\check{\mathcal{E}}^{(J)}[u] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2,$$

where $\check{\mathcal{E}}^{(c)}$ and $\check{\mathcal{E}}^{(J)}$ are the strongly local type and non-local type Dirichlet forms respectively.

We quote that the trace of the Dirichlet form $\check{\mathcal{E}}$ decomposed into $\check{\mathcal{E}}^{(c)}$ and $\check{\mathcal{E}}^{(J)}$. In fact, let us stress that this decomposition is mentioned by [BM20] for dimension $n = 3$ and $V = (0, 1) \cup \mathbb{N}$.

13

4 Conservativeness of traces of one-dimensional diffusions

In this section we further assume that $\mathcal{E}$ is conservative. Our aim is to establish necessary and sufficient conditions ensuring the trace form $\check{\mathcal{E}}$ to inherit conservativeness property. The form $\mathcal{E}$ is said to be conservative if

$$T_t 1 = 1 \text{ for some and for every } t > 0,$$

where T_t stands for L^∞-semi-group induced by the Dirichlet form $\mathcal{E}$.
Let us now start with the case where the set V has no accumulation point.

Theorem 4.1. *Assume that μ is infinite. Then the discrete Dirichlet form on the graph (V, b) is conservative if and only if*

$$\sum_{k=1}^{\infty} \big(s(x_{k+1}) - s(x_k) \big) \sum_{j=1}^{k} a_j = \infty. \tag{4.1}$$

Proof. The conservativeness of the Dirichlet form $\mathcal{E}$ is equivalent to the fact that the equation

$$\tilde{L}u + \alpha u = 0, \ \alpha > 0, \ u \in \ell^\infty, \tag{4.2}$$

has no nontrivial bounded solution (we refer the reader to [KL12]).

We rewrite

$$\tilde{L}u(x_k) + \alpha u(x_k) = \frac{1}{a_k} \sum_j \big(s(x_k) - s(x_j) \big) \big(u(x_k) - u(x_j) \big) + \alpha u(x_k) = 0. \tag{4.3}$$

This leads to,

$$u(x_2) = \big(1 + \alpha a_1(s(x_2) - s(x_1)) \big) u(x_1), \tag{4.4}$$

and

$$\frac{\big(s(x_{k+1}) - s(x_k) \big) \big(u(x_k) - u(x_{k+1}) \big)}{a_k} + \frac{\big(s(x_k) - s(x_{k-1}) \big) (u(x_k) - u(x_{k-1}))}{a_k} + \alpha u(x_k) = 0, \tag{4.5}$$

for all $k \geq 2$.

Thus by induction and the recursive formula we get

$$
\begin{aligned}
u(x_{k+1}) - u(x_k) &= \frac{s(x_{k+1}) - s(x_k)}{s(x_k) - s(x_{k-1})}\big(u(x_k) - u(x_{k-1})\big) + a_k \alpha\big(s(x_{k+1}) - s(x_k)\big)u(x_k) \\
&\vdots \\
&= \frac{s(x_{k+1}) - s(x_k)}{s(x_2) - s(x_1)}\big(u(x_2) - u(x_1)\big) + \alpha\big(s(x_{k+1}) - s(x_k)\big)\sum_{j=2}^{k} a_j \, u(x_j) \\
&= a_1\alpha\big(s(x_{k+1}) - s(x_k)\big)u(x_1) + \alpha\big(s(x_{k+1}) - s(x_k)\big)\sum_{j=2}^{k} a_j \, u(x_j) \\
&= \alpha\big(s(x_{k+1}) - s(x_k)\big)\sum_{j=1}^{k} a_j \, u(x_j), \ \forall\, k \geq 1.
\end{aligned} \tag{4.6}
$$

The latter formula gives rise to two observations (which can be proved by induction):

1. $u(x_k)$ has the sign of $u(x_1)$ for all $k \in \mathbb{N}$. This is if $u(x_1) > 0$, then $u(x_k) > 0$, for all $k \in \mathbb{N}$ and if $u(x_1) < 0$, hence $u(x_k) < 0$, for all $k \in \mathbb{N}$.

2. $u(x_k)$ is monotone, depending on the sign of $u(x_1)$.

Hence without loss of generality we may and shall assume that $u(x_1) > 0$. In this case $(u(x_k))_{k\in\mathbb{N}}$ is positive and strictly monotone increasing sequence.
Accordingly, making use of formula (4.6) we derive

$$
u(x_{k+1}) - u(x_k) \leq \left(\alpha\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j\right)u(x_k), \ \forall\, k \geq 1, \tag{4.7}
$$

and

$$
\frac{u(x_{k+1})}{u(x_k)} \leq 1 + \alpha\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j, \ \forall\, k \geq 1. \tag{4.8}
$$

Finally we achieve

$$
u(x_{N+1}) \leq u(x_1)\prod_{k=1}^{N+1}\left(1 + \alpha\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j\right). \tag{4.9}
$$

Obviously the latter product is finite provided $\sum_{k=1}^{\infty}\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j < \infty$ and then we get a bounded non-zero solution.
In the other sense, we suppose that $\sum_{k=1}^{\infty}\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j = \infty$. Then summing over k in formula (4.6) and keeping in mind that the sequence $(u(x_k))_{k\in\mathbb{N}}$ is increasing. We obtain

$$
u(x_{N+1}) - u(x_1) = \alpha\sum_{k=1}^{N}\big[s(x_{k+1}) - s(x_k)\big]\sum_{j=1}^{k} a_j u(x_j). \tag{4.10}
$$

15

Hence

$$u(x_{N+1}) \geq \alpha u(x_1) \sum_{k=1}^{N} \left[s(x_{k+1}) - s(x_k) \right] \sum_{j=1}^{k} a_j \to \infty \text{ as } N \to \infty,$$

which finishes the proof. □

Theorem 4.2. *If μ is a finite measure. Then $\check{\mathcal{E}}[u]$ is not conservative.*

Proof. Since μ is finite, i.e., $\mu(\{x_k, \ k \in \mathbb{N}\}) = \sum_{k=1}^{\infty} a_k < \infty$, then conservativeness and recurrence are equivalent. We have already compute the trace of a transient Dirichlet form $\mathcal{E}$. Therefore, according to [FOT11, Lemma 6.2.2., p.317] transience property is inherited by the trace form which yields that $\check{\mathcal{E}}$ is transient and hence it is not conservative. □

The case where V is a finite set can be resolved easily. For the case where V has x_{∞} as an accumulation point, we remark that the trace form $\check{\mathcal{E}}$ has a killing part. Hence owing to theoretical results (see [BM20]) it can't be conservative.

Theorem 4.3. *Assume that V is a finite set. Then there exists $N \in \mathbb{N}$ such that $x_{\infty} = x_N$, $\check{\mathcal{D}} = \mathbb{R}^N$ and for each $u \in \mathbb{R}^N$*

$$\check{\mathcal{E}}[u] = \sum_{k=1}^{N-1} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \frac{u(x_N)^2}{s([x_N, \infty))}$$

is not conservative.

Proof. Non-conservativeness of the trace form $\check{\mathcal{E}}$ follows from the fact that $1 \in \check{\mathcal{D}}$ and $\check{\mathcal{E}}[1] = \frac{1}{s([x_N, \infty))} \neq 0$. □

Theorem 4.4. *Assume that set V is infinite and accumulates at $x = x_{\infty}$. Then*

$$\check{\mathcal{E}}[u] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \frac{u(x_{\infty})^2}{s([x_{\infty}, \infty))} \tag{4.11}$$

for each $u \in \check{\mathcal{D}}$. Moreover $\check{\mathcal{E}}$ is not conservative.

Proof. For every $u \in \ell^2(V, \mu)$. We set

$$Q[u] = \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \frac{u(x_{\infty})^2}{s([x_{\infty}, \infty))}$$

which is acting on

$$\text{dom}\, Q = \left\{ u \in \ell^2(V, \mu) \ : \ \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)} \big(u(x_{k+1}) - u(x_k) \big)^2 + \frac{u(x_{\infty})^2}{s([x_{\infty}, \infty))} < \infty \right\}.$$

It is easy to check that Q is closed, then the trace form $\check{\mathcal{E}}$ is the restriction of Q to $ran J$. Furthermore if $\sum_{k=1}^{\infty} a_k = \infty$, we get from [KL12, Theorem 6] that $\check{\mathcal{D}} := D(\check{\mathcal{E}}) = \text{dom}\, Q$ and $\check{\mathcal{E}}[u] = Q[u]$ for all $u \in \check{\mathcal{D}}$.

16

For any sequence $(u_n)_{n \in \mathbb{N}} \subset \check{\mathcal{D}}$ with $0 \leq u_n \leq 1$ and $u_n \uparrow 1 \ \mu - a.e.$, we have for every $v \in \check{\mathcal{D}}$

$$\lim_{n \to \infty} \check{\mathcal{E}}(u_n, v) = \lim_{n \to \infty} \left[\sum_{k=1}^{\infty} \frac{\big(u_n(x_{k+1}) - u_n(x_k)\big)\big(v(x_{k+1}) - v(x_k)\big)}{s(x_{k+1}) - s(x_k)} + \frac{u_n(x_\infty)v(x_\infty)}{s([x_\infty, \infty))} \right].$$

By dominated convergence theorem we get

$$\lim_{n \to \infty} \check{\mathcal{E}}(u_n, v) = 0 + \frac{v(x_\infty)}{s([x_\infty, \infty))} \neq 0.$$

Since $v \neq 0$ on I, we get the non-conservativeness of $\check{\mathcal{E}}$. $\qquad \square$

5 Application : traces of the one-dimensional diffusion related to Bessel's process

For each $n \in \mathbb{N}$, $n \geq 2$. We consider the speed measure m defined on $I = (0, \infty)$ by

$$dm(x) = 2x^{2\nu+1}dx, \quad \text{where} \quad \nu = \frac{n}{2} - 1.$$

We define the scaling function s as follows

$$ds(x) = \frac{1}{x^{2\nu+1}}dx.$$

We shall be concerned with the Dirichlet form $\mathcal{E}$ with domain $\mathcal{D} \subset L^2(I, 2x^{2\nu+1}dx)$ defined by

$$\mathcal{D} := \mathcal{D}_0 \cap L^2(I, 2x^{2\nu+1}dx), \ \ \mathcal{E}[u] := \int_0^\infty (u'(x))^2 x^{2\nu+1}dx \quad \text{for all } u \in \mathcal{D}$$

where

$$\mathcal{D}_0 := \left\{ u \colon (0, \infty) \to \mathbb{R} : \ u \text{ is abs. cont. w.r.t } ds, \ \ \int_0^\infty (u'(x))^2 x^{2\nu+1}dx < \infty \right\}.$$

Since $n \geq 2$ the Bessel process is transient [CF12, p.126] which yields that associated Dirichlet form is transient too.

We can easily check that for $n \geq 3$, (i.e. $\nu \geq \frac{1}{2}$) we obtain $r_1 = 0$ is a non-approachable boundary, i.e., $s(0) = \infty$. Whereas the boundary point $r_2 = \infty$ is an approachable boundary, i.e., $s(\infty) < \infty$.

According to the Feller's boundary classification, 0 is an entrance boundary. Indeed

$$\Gamma_1(0) = \infty \text{ and } \Sigma_1(0) = \frac{c^2}{2\nu + 2} < \infty \text{ for all constant } c > 0.$$

17

In these circumstances, the selfadjoint adjoint operator related to $\check{\mathcal{E}}$, which we denote by L is the generator of the Bessel process of index ν on the half-line. Moreover we have the following description of L. Set

$$\mathcal{L} := -\frac{1}{2}\frac{d^2}{dx^2} - \frac{2\nu+1}{2x}\frac{d}{dx}, \quad \text{for all} \quad \nu > -\frac{1}{2}.$$

Then

$$D(L) = \left\{ u \in \mathcal{D} : \ u' \in AC_{loc}(I), \ \lim_{x\downarrow 0^+} x^{2\nu+1}u'(x) = 0, \ \mathcal{L}u = -\frac{1}{2}u'' - \frac{2\nu+1}{2x}u' \in L^2(I,m) \right\}$$

$$Lu = \mathcal{L}u \quad \text{for all } u \in D(L).$$

We start with the case that the sequence $(x_k)_{k\in\mathbb{N}}$ diverges and so it has no accumulation point. Accordingly we consider the discrete measure defined as the first section by

$$\mu = \sum_{k\in\mathbb{N}} a_k \delta_{x_k},$$

which is supported by an infinite countable set $V = \{x_k, \ k \in \mathbb{N} \} \subset (0,\infty)$.

Remark 5.1. In our case for $n \geq 2$, $r_1 = \{0\}$ is an entrance boundary and it is well known that $Cap(\{0\}) = 0$ (we refer to [JYC09] for more details) and for each element $x_k \in V$ we have $Cap(\{x_k\}) > 0$.

To compute the trace of the general Bessel's Dirichlet form $\check{\mathcal{E}}$ with domain $\check{\mathcal{D}} \subset \ell^2(V,\mu)$ we have to apply Theorem 3.7. with scaling function $ds(x) = \frac{1}{x^{2\nu+1}}\,dx$ to obtain the following expression

$$\begin{aligned}
\check{\mathcal{E}}[Ju] &= \sum_{k=1}^{\infty} \frac{1}{s([x_k,x_{k+1}])}\big(u(x_{k+1}) - u(x_k)\big)^2 \\
&= \sum_{k=1}^{\infty} \frac{1}{\left(\int_{x_k}^{x_{k+1}} \frac{1}{x^{2\nu+1}}\,dx\right)}\big(u(x_{k+1}) - u(x_k)\big)^2 \\
&= \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu} . x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2.
\end{aligned}$$

Let $u \in \ell^2(V,\mu)$. Set

$$q[u] = \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu} . x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2.$$

We have $\check{\mathcal{E}} = q_{|_{ranJ}}$. Indeed, q is a closed quadratic form together with the fact that $\check{\mathcal{E}}$ is the closure of q restricted to $ranJ$. Hence, $\mu(V) = \infty$ leads to

$$\check{\mathcal{D}} := D(\check{\mathcal{E}}) = \left\{ u \in \ell^2(V,\mu) \ : \ \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu} . x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2 < \infty \right\}$$

$$\check{\mathcal{E}}[u] = q[u] \quad \text{for all } u \in D(\check{\mathcal{E}}). \tag{5.1}$$

18

In this case, we can determine the discrete Bessel operator associated with trace form $\check{\mathcal{E}}$ as follows
for each $k \in \mathbb{N}$

$$\check{L}u = -\frac{\nu \; x_k^{2\nu} x_{k-1}^{2\nu} \; u(x_{k-1})}{a_k(x_k^{2\nu} - x_{k-1}^{2\nu})} + \frac{\nu}{a_k}\Big(\frac{x_k^{2\nu} x_{k+1}^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})} + \frac{x_k^{2\nu} x_{k-1}^{2\nu}}{(x_k^{2\nu} - x_{k-1}^{2\nu})}\Big)u(x_k) - \frac{\nu \; x_k^{2\nu} x_{k+1}^{2\nu} \; u(x_{k+1})}{a_k(x_{k+1}^{2\nu} - x_k^{2\nu})}.$$
(5.2)

For the conservativeness property of the general Bessel's Dirichlet forms we have following result as an application of the Theorem 3.1.
If μ is infinite, then $\check{\mathcal{E}}$ is conservative if and only if

$$\sum_{k=1}^{\infty} \frac{(x_{k+1}^{2\nu} - x_k^{2\nu})}{x_k^{2\nu} x_{k+1}^{2\nu}} \sum_{j=1}^{k} a_j = \infty.$$
(5.3)

Finally we consider the case where the sequence $(x_k)_{k\in\mathbb{N}}$ converges to x_∞. According to Lemma 3.9. we obtain

$$
\begin{aligned}
\check{\mathcal{E}}[Ju] &= \sum_{k=1}^{\infty} \frac{1}{s(x_{k+1}) - s(x_k)}\big(u(x_{k+1}) - u(x_k)\big)^2 + \frac{u(x_\infty)^2}{s([x_\infty, \infty))} \\
&= \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu}.x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2 + 2\nu.x_\infty^{2\nu}u(x_\infty)^2.
\end{aligned}
$$
(5.4)

For every $u \in \ell^2(V, \mu)$. Put

$$Q[u] = \sum_{k=1}^{\infty} 2\nu \frac{x_{k+1}^{2\nu}.x_k^{2\nu}}{(x_{k+1}^{2\nu} - x_k^{2\nu})}\big(u(x_{k+1}) - u(x_k)\big)^2 + 2\nu.x_\infty^{2\nu}u(x_\infty)^2.$$

We can easily check that trace form $\check{\mathcal{E}}$ is the closure of $Q_{|ranJ}$. We assume that $\mu(V) = \infty$. Then

$$\check{\mathcal{D}} = D(\check{\mathcal{E}}) = \{u \in \ell^2(V, \mu) \; : \; \check{\mathcal{E}}[u] < \infty\}, \quad \check{\mathcal{E}}[u] = Q[u].$$

Regarding conservativeness property, according to Theorem 4.4., the trace form $\check{\mathcal{E}}$ is not conservative.

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