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Effective electrodynamics theory for the hyperbolic metamaterial consisting of metal-dielectric layers

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Abstract: In this work, we study the dynamical behaviors of the electromagnetic fields and material responses in the hyperbolic metamaterial consisting of periodically arranged metallic and dielectric layers. The thickness of each unit cell is assumed to be much smaller than the wavelength of the electromagnetic waves, so the effective medium concept can be applied. When electromagnetic (EM) fields are present, the responses of the medium in the directions parallel to and perpendicular to the layers are like that of Drude and Lorentz media, respectively. We derive the energy density of the EM fields and the power loss in the effective medium based on Poynting theorem and the dynamical equations of the polarization field. We also show that the Lagrangian density of the system can be constructed. The Euler-Lagrangian equations yield the correct dynamical equations of the electromagnetic fields and the polarization field in the medium. The canonical momentum conjugates to every dynamical field can be derived from the Lagrangian density via differentiation or variation with respect to that field. We apply Legendre transformation to this system, and find that the resultant Hamitonian density is identical to the energy density, up to an irrelevant divergence term.

Keywords: Metamaterial; Hyperbolic Metamaterial; Drude Model; Lorentz Model; Lagrangian; Hamiltonian

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1. Introduction

Metamaterials usually refer to artificially engineered structures for realizing various unusual optical/electromagnetic properties such as negative refraction [1,2], subwavelength imaging [3], indefinite permittivity [4], near-perfect absorption [5], or invisibility [6,7]. These unusual properties are mainly achieved through the resonance, conductivity, and directionality of the structural components such as split-ring resonators (SRRs), metallic rods array, or subwavelength dielectric-metal multilayers [8]. The resonance and directionality of the constituent components imply that the metamaterials are inherently dispersive, absorptive, and anisotropic. A fundamental problem concerning dispersive media is how to calculate the stored electromagnetic energy density [9-28]. For dispersive media with negligible absorption, the time-averaged energy density as a function of the (complex valued) electric and magnetic fields (in the frequency domain) can be derived by considering the adiabatically varying electromagnetic field [29]. However, such analysis does not work when finite absorption is present. Recently, two different approaches were proposed to resolve such non-trivial problem [16,18]. For the wire-SRR metamaterials, time-averaged energy density formula can be derived using equivalent circuit (EC) method [10,16,17]. On the other hand, instantaneous energy density formula (the time domain formula) can be obtained using the electrodynamics (ED) method with Poynting theorem [9,12-15,18-20,24,27,28]. However, quite some controversies exist in the literature, and they need to be resolved [16-20]. In our previous studies we found that the time domain formula for energy density can be uniquely determined by the ED approach provided we know how to identify the power loss [19,20]. Recently, we also developed the

Lagrangian field theory description for the wire-SRR and chiral metamaterials [30]. In this framework, the Hamiltonian density for the metamaterial can be obtained through the Legendre transformation. It is found that the Hamiltonian densities for the dynamical fields in these metamaterials are the same as the energy densities we already obtained, up to some irrelevant divergence terms.

Recently, hyperbolic metamaterials belonging to the category of anisotropic media have received more and more attention [31-35]. The hyperbolic dispersion of this kind of metamaterials are caused by the opposite signs of the two principal values of the permittivity tensor along and perpendicular to the optic axis. One of the most important applications of hyperbolic metamaterial is the hyperlens [35], which can image subwavelength objects in the far-field region, overcoming the shortcomings of the superlens [1,3] that can only image the same objects in the near-field region. The simplest example of a hyperbolic metamaterial is a dielectric-metal multilayer structure that operates under the long-wavelength limit [35]. Although it seems that such metamaterials are much simpler than the wire-SRR and chiral metamaterials, to the best of our knowledge, the energy density problem and the Lagrangian description for this kind of metamaterials have not yet been studied. This is the main motivation of our present study. In addition, a better understanding about the energy density, Lagrangian description, and Hamiltonain theory for the hyperbolic metamaterial can help researchers further explore the dynamical behaviors of EM waves in this kind of media without being restricted to the frequency domain phenomena. This may have practical importance in the future. The Lagrangian and Hamiltonian descriptions also provide a starting point for the development of quantum description of the electrodynamics in the metamaterials [30].

In this paper, we study the effective electrodynamics of dielectric-metal multilayer structures under the limitation of long wavelengths. Assuming that each metal layer is absorptive, we study energy density and power loss of the system. We first discuss the boundary conditions of the dynamic fields in each dielectric layer and metal layer, and derive the effective fields. We also derive the effective pemittivities in the frequency domain based on our theory of the effective fields. We then study the dynamical evolution equations of the effective electric, magnetic, displacement, and polarization fields and derive the effective energy density using the ED method. We discuss the loss effect and provide a suitable dissipation function [36] so that we can derive the correct equations of motion for the dynamic fields from the Euler-Lagrange equation with dissipation. The Hamiltonian density is derived by applying the Legendre transformation to this system. The resultant Hamiltonian density is found to be the same as the energy density obtained before, up to a divergence term.

2. The effective fields, energy density and power loss

2.1. Boundary conditions and the effective fields

We consider a periodic multilayer structure consisting of dielectric and metal layers. Each unit cell has one dielectric and one metal layer (see Fig. 1). The lattice constant (the thickness of the unit cell) is $a = a_m + a_d$, where $a_m = fa$ is the thickness of the metal layer, $a_d = (1-f)a$ is the thickness of the dielectric layer, and f is the filling fraction of the metal layer in one unit cell satisfying $0 < f < 1$. The dielectric is a nondispersive material of permittivity $\varepsilon_d = \varepsilon_0 \tilde{\varepsilon}_d$, whereas the metal has the Drude type permittivity $\varepsilon_m(\omega) = \varepsilon_0 \tilde{\varepsilon}_m = \varepsilon_0 \left(1 - \frac{\omega_{p0}^2}{\omega^2 + i\Gamma\omega}\right)$ at frequency ω. Here $\tilde{\varepsilon}_d = \varepsilon_d / \varepsilon_0$ and $\tilde{\varepsilon}_m = \varepsilon_m / \varepsilon_0$ are the relative (dimensionless) permittivities of the dielectric and metal layers. To make the effective medium theory reasonable, the operating wavelength should be much longer than the lattice constant a. Under this assumption any field quantity through one single layer does not change value along the direction normal to the layer. The boundary conditions for the $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$ fields at the dielectric-metal interfaces are the continuity of the tangential component of the $\mathbf{E}$ and $\mathbf{H}$ (E_t, H_t) fields and the continuity of the normal component of the $\mathbf{D}$ and $\mathbf{B}$ (D_n, B_n) fields. These boundary conditions are derivable from Maxwell's equations (without external sources) by applying the Stokes and divergence theorem, respectively. We also assume both the dielectric and metal are non-magnetic materials, so the permeability through the whole structure takes the same value μ_0 as in empty space, and the $\mathbf{B}$ field is related to the $\mathbf{H}$ field by the simple relation $\mathbf{B}=\mu_0\mathbf{H}$. The constitutive relation $\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$ between the displacement field $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization field $\mathbf{P}$ is assumed for a single layer as well as for the effective fields in the medium. However, since the boundary conditions for the $\mathbf{E}$ and $\mathbf{D}$ fields are of different types, the tangential and normal components of the effective fields will be evaluated separately.

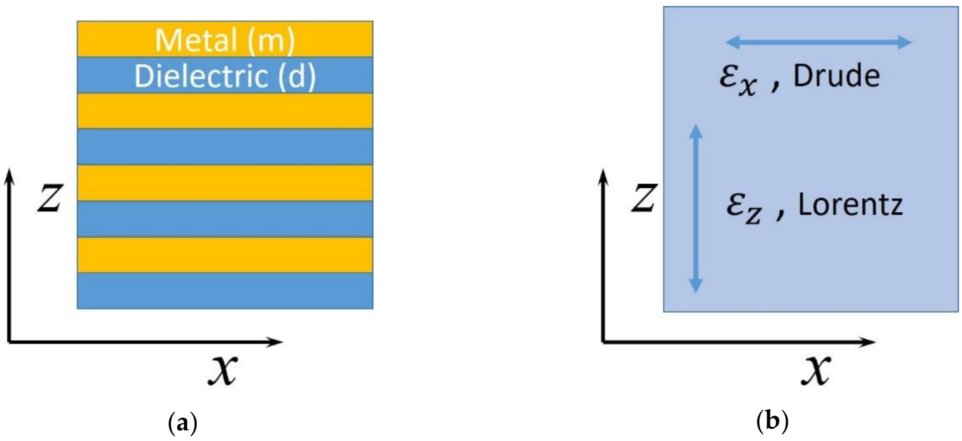

Figure 1. The effective medium consisting of periodically arranged dielectric and metal layers. (**a**) The original layer structure. Here the thickness of each layer as well as the lattice constant (the thickness of the unit cell) are assumed to be much smaller than the operating wavelength, so the effective medium theory can be constructed. (**b**) The effective medium has different dynamical properties in the direction parallel and perpendicular to the layers.

Hereafter we denote the direction parallel to and normal to the layers as x and z, respectively. The effective polarization field is the averaged dipole density, given by

$$\mathbf{P}=(1-f)\mathbf{P}^d+f\mathbf{P}^m, \tag{1}$$

here the superscript "d" and "m" denote the corresponding medium. Similarly the effective $\mathbf{E}$ field is defined by

$$\mathbf{E}=(1-f)\mathbf{E}^d+f\mathbf{E}^m. \tag{2}$$

The boundary conditions for E_t and D_n lead to the relations

$$E_x=E_x^d=E_x^m=D_x^d/\varepsilon_d, \quad D_z=D_z^d=D_z^m=\varepsilon_d E_z^d. \tag{3}$$

In addition, the relation $\mathbf{D}^d=\varepsilon_0\mathbf{E}^d+\mathbf{P}^d=\varepsilon_d\mathbf{E}^d$ implies

$$P_x^d=(\tilde{\varepsilon}_d-1)\varepsilon_0 E_x, \quad P_z^d=\left(1-\frac{1}{\tilde{\varepsilon}_d}\right)D_z. \tag{4}$$

From Eq.(1) to Eq.(4) and the relation $\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$, we find

$$\begin{aligned} D_x &= \varepsilon_0 E_x+(1-f)P_x^d+fP_x^m \\ &= \varepsilon_0 E_x+(1-f)(\tilde{\varepsilon}_d-1)\varepsilon_0 E_x+fP_x^m \\ &= \varepsilon_0\left[f+(1-f)\tilde{\varepsilon}_d\right]E_x+fP_x^m \end{aligned} \tag{5}$$

Similarly, the z component of the $\mathbf{D}$ field can be derived as

$$\begin{aligned} D_z &= \varepsilon_0 E_z + (1-f) P_z^d + f P_z^m \\ &= \varepsilon_0 E_z + (1-f)(D_z - \varepsilon_0 E_z) + f P_x^m \\ &= \varepsilon_0 E_z + (1-f)\left(1 - \frac{1}{\tilde{\varepsilon}_d}\right) D_z + f P_z^m \,. \\ &= \frac{\tilde{\varepsilon}_d \left(\varepsilon_0 E_z + f P_z^m\right)}{f\tilde{\varepsilon}_d + (1-f)} \end{aligned} \tag{6}$$

We now define two coefficients α_x, α_z and a new field $\mathbf{Q} = [Q_x, Q_z]$ as

$$\alpha_x = f + (1-f)\tilde{\varepsilon}_d, \quad \alpha_z = \frac{\tilde{\varepsilon}_d}{f\tilde{\varepsilon}_d + (1-f)} \,. \tag{7}$$

and

$$Q_x = f P_x^m, \quad Q_z = \alpha_z f P_z^m, \tag{8}$$

Using these notations, the effective $\mathbf{D}$ field in Eq.(5) and Eq.(6) can be expressed as

$$D_x = \alpha_x \varepsilon_0 E_x + Q_x, \quad D_z = \alpha_z\left(\varepsilon_0 E_z + f P_z^m\right) = \alpha_z \varepsilon_0 E_z + Q_z \,. \tag{9}$$

We have not yet analyzed the dynamical behavior of the $\mathbf{P}^m$ field. The dynamical behavior of $\mathbf{P}^m$ field will determine the dynamical behavior of the effective medium. The Drude type permittivity $\varepsilon_m(\omega) = \varepsilon_0\left(1 - \frac{\omega_{p0}^2}{\omega^2 + i\Gamma\omega}\right)$ of the metal implies the equation of motion for $\mathbf{P}^m$:

$$\ddot{\mathbf{P}}^m + \Gamma\dot{\mathbf{P}}^m = \varepsilon_0 \omega_{p0}^2 \mathbf{E}^m \,. \tag{10}$$

Taking the x component of Eq.(3), Eq.(8), and Eq.(10), we get the dynamical equation for Q_x

$$\ddot{Q}_x + \Gamma\dot{Q}_x = \varepsilon_0 f \omega_{p0}^2 E_x = \alpha_x \varepsilon_0 \omega_p^2 E_x \,. \tag{11}$$

Here the effective plasma frequency ω_p of the effective medium is defined by the relation

$$\omega_p^2 = \frac{f\omega_{p0}^2}{\alpha_x} \,. \tag{12}$$

Similarly, taking the z component of Eq.(10) we get the following dynamical equation

$$\ddot{Q}_z + \Gamma\dot{Q}_z = \alpha_z \varepsilon_0 \omega_{p0}^2 f E_z^m \,. \tag{13}$$

However, the right hand side of Eq.(12) must be replaced by the effective fields. This can be done by noting that Eq.(2), Eq.(3), and Eq.(10) tell us

$$f E_z^m = E_z - (1-f)\frac{D_z}{\varepsilon_d} = \alpha_z f E_z - \frac{(1-f)}{\varepsilon_d} Q_z \,. \tag{14}$$

Substituting Eq.(14) into Eq.(13), we get

$$\ddot{Q}_z + \Gamma\dot{Q}_z + \omega_0^2 Q_z = F \alpha_z \varepsilon_0 \omega_0^2 E_z, \tag{15}$$

here the resonance frequency ω_0 and the factor F are defined as

$$\omega_0^2 = \frac{(1-f)\alpha_z}{\tilde{\varepsilon}_d}\omega_{p0}^2, \qquad F = \frac{\omega_{p0}^2}{\omega_0^2} - 1 = \frac{f\tilde{\varepsilon}_d}{1-f}. \tag{16}$$

We are now ready to derive the energy density of the effective medium system. Before doing so, let's check what do these equations tell us about the principal permittivity $\varepsilon_x(\omega)$ and $\varepsilon_z(\omega)$. Consider harmonic fields of frequency ω, and replace all fields with their complex vector representations with time factor $e^{-i\omega t}$. Under this consideration, Eq.(11) and Eq.(15) yield

$$\tilde{Q}_x = -\frac{\alpha_x\varepsilon_0\omega_p^2}{\omega^2 + i\Gamma\omega}\tilde{E}_x, \quad \tilde{Q}_z = -\frac{F\alpha_z\varepsilon_0\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}\tilde{E}_z \ \text{(complex rep.)}. \tag{17}$$

Here $\tilde{Q}_x$, $\tilde{Q}_z$, $\tilde{E}_x$, $\tilde{E}_z$ are the complex representations of Q_x, Q_z, E_x, E_z.

Substituting Eq.(17) into Eq.(9), and applying the relations $D_x = \varepsilon_x E_x$, $D_z = \varepsilon_z E_z$, we find

$$\varepsilon_x = \alpha_x\varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 + i\Gamma\omega}\right), \quad \varepsilon_z = \alpha_z\varepsilon_0\left(1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}\right). \tag{18}$$

These results are exactly the same as those obtained directly by using the effective permittivity formula $\varepsilon_x = f\varepsilon_m + (1-f)\varepsilon_d$ and $1/\varepsilon_z = f/\varepsilon_m + (1-f)/\varepsilon_d$ at the direction parallel and normal to the layers, as can be easily checked. According to Eq.(17) and Eq.(18), the $\mathbf{Q}$ field tends to zero as the frequency becomes higher and higher. In fact, the $\mathbf{Q}$ field is the dynamical part of the $\mathbf{P}$ field that does not react immediately to the change of the $\mathbf{E}$ field.

2.2. Poynting theorem, energy density, and power loss

To derive the energy density, we first derive from Maxwell's equations the following equations

$$-\nabla\cdot\mathbf{S} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}, \tag{19}$$

Now, if the right-hand side of Eq.(19) can be written as $\frac{\partial W}{\partial t} + P_{loss}$, and P_{loss} can be identified as the power loss density, then $W = W_e + W_b$ is the desired energy density of the system, and Eq.(19) represents the Poynting theorem (the energy conservation law). Using the simple relation $\mathbf{B} = \mu_0\mathbf{H}$, we find $\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu_0}{2}H^2\right)$, so the magnetic energy density is $W_b = \frac{\mu_0}{2}H^2$. Furthermore, using Eq.(9), Eq.(11), and Eq.(15), we get

$$\begin{aligned}
\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} &= E_x\frac{\partial D_x}{\partial t} + E_z\frac{\partial D_z}{\partial t} = E_x\frac{\partial}{\partial t}(\alpha_x\varepsilon_0E_x + Q_x) + E_z\frac{\partial}{\partial t}(\alpha_z\varepsilon_0E_z + Q_z) \\
&= \frac{\partial}{\partial t}\left[\frac{\varepsilon_0}{2}\left(\alpha_xE_x^2 + \alpha_zE_z^2\right)\right] + \frac{(\ddot{Q}_x + \Gamma\dot{Q}_x)\dot{Q}_x}{\alpha_x\varepsilon_0\omega_p^2} + \frac{(\ddot{Q}_z + \Gamma\dot{Q}_z + \omega_0^2Q_z)\dot{Q}_z}{F\alpha_z\varepsilon_0\omega_0^2} \\
&= \frac{\partial}{\partial t}\left[\frac{\varepsilon_0}{2}\left(\alpha_xE_x^2 + \alpha_zE_z^2\right) + \frac{\dot{Q}_x^2}{2\alpha_x\varepsilon_0\omega_p^2} + \frac{\dot{Q}_z^2 + \omega_0^2Q_z^2}{2F\alpha_z\varepsilon_0\omega_0^2}\right] + \Gamma\left(\frac{\dot{Q}_x^2}{\alpha_x\varepsilon_0\omega_p^2} + \frac{\dot{Q}_z^2}{F\alpha_z\varepsilon_0\omega_0^2}\right)
\end{aligned}, \tag{20}$$

Using Eq.(8), Eq.(12), and Eq.(16), we find that the final term of Eq. (20) can be re-expressed as a quantity proportional to $\left(\dot{\mathbf{P}}^m\right)^2$:

$$P_{loss} = \Gamma\left(\frac{\dot{Q}_x^2}{\alpha_x \varepsilon_0 \omega_p^2} + \frac{\dot{Q}_z^2}{F\alpha_z \varepsilon_0 \omega_0^2}\right) = \frac{f\,\Gamma}{\varepsilon_0 \omega_{p0}^2}\left(\dot{\mathbf{P}}^m\right)^2 . \tag{21}$$

Since $\dot{\mathbf{P}}^m$ is the polarization current density, thus $\frac{\Gamma}{\varepsilon_0 \omega_{p0}^2}\left(\dot{\mathbf{P}}^m\right)^2$ is the Joule heat rate density in the metal layer. It is therefore reasonable to identify the quantity in Eq.(21) as the power loss density of the effective medium. With this identification, we can identify the electric energy density W_e as

$$W_e = \frac{\varepsilon_0}{2}\left(\alpha_x E_x^2 + \alpha_z E_z^2\right) + \frac{\dot{Q}_x^2}{2\alpha_x \varepsilon_0 \omega_p^2} + \frac{\dot{Q}_z^2 + \omega_0^2 Q_z^2}{2F\alpha_z \varepsilon_0 \omega_0^2} . \tag{22}$$

The total energy density is thus given by

$$W = W_e + W_b = \frac{\varepsilon_0}{2}\left(\alpha_x E_x^2 + \alpha_z E_z^2\right) + \frac{\mu_0}{2}H^2 + \frac{\dot{Q}_x^2}{2\alpha_x \varepsilon_0 \omega_p^2} + \frac{\dot{Q}_z^2 + \omega_0^2 Q_z^2}{2F\alpha_z \varepsilon_0 \omega_0^2} . \tag{23}$$

This result indicates that the total energy density of the system is definitely positive and is consisting of two parts: the non-dispersive part

$$W_{EH} = \frac{\varepsilon_0}{2}\left(\alpha_x E_x^2 + \alpha_z E_z^2\right) + \frac{\mu_0}{2}H^2 , \tag{24}$$

and the dispersive part

$$W_Q = \frac{\dot{Q}_x^2}{2\alpha_x \varepsilon_0 \omega_p^2} + \frac{\dot{Q}_z^2 + \omega_0^2 Q_z^2}{2F\alpha_z \varepsilon_0 \omega_0^2} . \tag{25}$$

The different dispersion characters are caused by the facts: W_{EH} is the contribution from the electricmagnetic fields themselves and the part of the polarization field that follws the change of the fields immediately, whereas W_Q is originated from the material response that does not follow the fields immediately because a conduction electron in the metal has nonzero innertial mass.

For harmonic fields, the time average of a product $a(t)b(t)$ (energy density or Poynting vector) can be evaluated by using the formula [37]

$$\langle a(t)b(t)\rangle = \frac{1}{T}\int_0^T a(t)b(t)dt = \frac{1}{2}\mathrm{Re}\left(\tilde{a}\tilde{b}^*\right), \tag{26}$$

where $T = 2\pi/\omega$ is the oscillation periodic, while $\tilde{a}$ and $\tilde{b}$ are the complex representations of the fields $a(t) = \mathrm{Re}\left(\tilde{a}e^{-i\omega t}\right)$ and $b(t) = \mathrm{Re}\left(\tilde{b}e^{-i\omega t}\right)$.

Applying Eq.(26) to the energy density and power loss density, we get

$$\begin{aligned}\langle W\rangle &= \frac{\varepsilon_0}{4}\left(\alpha_x |\tilde{E}_x|^2 + \alpha_z |\tilde{E}_z|^2\right) + \frac{\mu_0}{4}|\tilde{H}|^2 + \frac{\omega^2 |\tilde{Q}_x|^2}{4\alpha_x\varepsilon_0\omega_p^2} + \frac{\left(\omega^2+\omega_0^2\right)|\tilde{Q}_z|^2}{4F\alpha_z\varepsilon_0\omega_0^2} \\ &= \frac{\alpha_x\varepsilon_0}{4}\left(1+\frac{\omega_p^2}{\omega^2+\Gamma^2}\right)|\tilde{E}_x|^2 + \frac{\alpha_z\varepsilon_0}{4}\left[1+\frac{F\omega_0^2\left(\omega^2+\omega_0^2\right)}{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}\right]|\tilde{E}_z|^2 + \frac{\mu_0}{4}|\tilde{H}|^2\end{aligned} \tag{27}$$

and

$$\begin{aligned}\langle P_{loss}\rangle &= \frac{\Gamma\varepsilon_0}{2}\left[\frac{\alpha_x\omega_p^2}{\omega^2+\Gamma^2}|\tilde{E}_x|^2 + \frac{\alpha_z F\omega_0^2\omega^2}{\left(\omega^2-\omega_0^2\right)^2+\Gamma^2\omega^2}|\tilde{E}_z|^2\right]. \\ &= \frac{\omega}{2}\mathrm{Im}\left(\varepsilon_x|\tilde{E}_x|^2 + \varepsilon_z|\tilde{E}_z|^2\right)\end{aligned} \tag{28}$$

The second line of Eq.(28) is consistent with the generally accepted concept that it is the imaginary part of the permittivity who corresponds to the energy loss in the absorptive medium. Another interesting observation is that when we "turn off" the absorption (i.e., $\Gamma \to 0$), the time averaged energy density becomes

$$\begin{aligned}\langle W\rangle_{\Gamma\to 0} &= \frac{\alpha_x\varepsilon_0}{4}\left(1+\frac{\omega_p^2}{\omega^2}\right)|\tilde{E}_x|^2 + \frac{\alpha_z\varepsilon_0}{4}\left[1+\frac{F\omega_0^2\left(\omega^2+\omega_0^2\right)}{\left(\omega^2-\omega_0^2\right)^2}\right]|\tilde{E}_z|^2 + \frac{\mu_0}{4}|\tilde{H}|^2 \\ &= \frac{1}{4}\left[\frac{\partial(\omega\varepsilon_x)_{\Gamma\to 0}}{\partial\omega}|\tilde{E}_x|^2 + \frac{\partial(\omega\varepsilon_z)_{\Gamma\to 0}}{\partial\omega}|\tilde{E}_z|^2\right]\end{aligned}. \tag{29}$$

The second line of Eq.(29) is the prediction to a dispersive medium with negligible absorption, which can be derived by considering the adiabatic variation of the field amplitudes [29].

3. The effective Lagrangian density and Hamiltonian density

3.1. Lagrangian density and Euler-Lagrangian equations

In this section we will construct the Lagrangian density for the effective medium system as function of the scalar potential φ, the vector potential $\mathbf{A}$, the $\mathbf{Q}$ field, and their time and space derivatives. It is clear that in Maxwell's equations the Gauss law $\nabla\cdot\mathbf{B}=0$ for the $\mathbf{B}$ field and the Faraday's induction law $\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t$ are automatically satisfied because $\mathbf{B}=\nabla\times\mathbf{A}$ implies $\nabla\cdot\mathbf{B}=0$ and $\mathbf{E}=-\nabla\varphi-\dot{\mathbf{A}}$ implies $\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t$. We will show that the other two Maxwell's equations $\nabla\cdot\mathbf{D}=0$ and $\nabla\times\mathbf{H}=\partial\mathbf{D}/\partial t$ as well as the equations of motion for the $\mathbf{Q}$ fields (Eq.(11) and Eq.(15)) can be derived from the Euler-Lagrangian equations (with dissipation)

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi_\alpha)}\right)+\partial_j\left(\frac{\partial\mathcal{L}}{\partial(\partial_j\psi_\alpha)}\right)-\frac{\partial\mathcal{L}}{\partial\psi_\alpha}=-\frac{\partial\mathcal{F}}{\partial(\partial_t\psi_\alpha)}. \tag{30}$$

Here $\psi_\alpha=[\varphi,A_x,A_z,Q_x,Q_z]$ are the dynamical fields involved in the Lagrangian density, and the relations $D_x=\alpha_x\varepsilon_0E_x+Q_x$ and $D_z=\alpha_z\varepsilon_0E_z+Q_z$ were used. The dissipation function density $\mathcal{F}$ in Eq.(30) is defined as

$$\mathcal{F}=\frac{\Gamma}{2}\left(\frac{\dot{Q}_x^2}{\alpha_x\varepsilon_0\omega_p^2}+\frac{\dot{Q}_z^2}{F\alpha_z\varepsilon_0\omega_0^2}\right)=\frac{1}{2}P_{loss} \tag{31}$$

which has the value equal to one half of the power loss.

Observing Eq.(24) and Eq.(25) it is not difficult to guess the Lagrangian density. It should be a sum of three kinds of terms

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_Q + \mathcal{L}_C , \tag{32}$$

where $\mathcal{L}_{EH}$ describes the electromagnetic fields

$$\mathcal{L}_{EH} = \frac{\varepsilon_0}{2}\left[\alpha_x\left(\partial_x\varphi + \dot{A}_x\right)^2 + \alpha_z\left(\partial_z\varphi + \dot{A}_z\right)^2\right] - \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2 , \tag{33a}$$

$\mathcal{L}_Q$ describes the $\mathbf{Q}$ field

$$\mathcal{L}_Q = \frac{\dot{Q}_x^2}{2\alpha_x\varepsilon_0\omega_p^2} + \frac{\dot{Q}_z^2 - \omega_0^2 Q_z^2}{2F\alpha_z\varepsilon_0\omega_0^2} , \tag{33b}$$

and $\mathcal{L}_C$ describes the matter-field coupling

$$\mathcal{L}_C = E_x Q_x + E_z Q_z = -\left(\partial_x\varphi + \dot{A}_x\right)Q_x - \left(\partial_z\varphi + \dot{A}_z\right)Q_z . \tag{33c}$$

As mentioned before, using this Lagrangian density $\mathcal{L}$ and the dissipation function density $\mathcal{F}$ in Eq.(31), one can derive all the dynamical equations of the effective fields from Eq.(30). It can be checked that for $\psi_\alpha = \varphi$ we get the Gauss law $\nabla\cdot\mathbf{D} = 0$; for $\psi_\alpha = \mathbf{A}$ we get the Ampere's law $\nabla\times\mathbf{H} = \partial\mathbf{D}/\partial t$; and for $\psi_\alpha = \mathbf{Q}$ we get the equations of motion for the $\mathbf{Q}$ field (i.e., Eq.(11) and Eq.(15)).

3.2. Canonical momenta, Legendre transformation and Hamiltonian density

In order to derive the Hamiltonian density, we have to derive all the Canonical momenta first. The Canonical momentum π_α conjugate to the dynamical field ψ_α is defined by $\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha}$. For $\psi_\alpha = [\varphi, A_x, A_z, Q_x, Q_z]$ we find $\pi_\alpha = [0, -D_x, -D_z, \dot{Q}_x/(\alpha_x\varepsilon_0\omega_p^2), \dot{Q}_z/(\alpha_z F\varepsilon_0\omega_0^2)]$. It is interesting to note that the canonical momentum π_φ conjugate to φ is equal to zero. This is a common feature of electrodynamics systems which reflects the fact that φ is a redundant dynamical variable.

The Hamiltonian density $\mathcal{H}$ is given by the Legendre transformation

$$\mathcal{H} = \pi_\alpha\dot{\psi}_\alpha - \mathcal{L} = \frac{\varepsilon_0}{2}\left(\alpha_x E_x^2 + \alpha_z E_z^2\right) + \frac{\mu_0}{2}H^2 + \frac{\dot{Q}_x^2}{2\alpha_x\varepsilon_0\omega_p^2} + \frac{\dot{Q}_z^2 + \omega_0^2 Q_z^2}{2F\alpha_z\varepsilon_0\omega_0^2} + \mathbf{D}\cdot\nabla\varphi . \tag{34}$$

According to this result the Hamiltonian density is almost the same as the energy density we obtained in Eq.(23). The only difference is the new term $\mathbf{D}\cdot\nabla\varphi = \nabla\cdot(\varphi\mathbf{D}) - \varphi\nabla\cdot\mathbf{D} = \nabla\cdot(\varphi\mathbf{D})$. Here we have used the Gauss law $\nabla\cdot\mathbf{D} = 0$. This additional divergence term can be dropped from Eq.(34) because it is only a "surface term" and will not influence the dynamical equations for the effective fields. After dropping this surface term the final term in Eq.(34) can be replaced by a term of zero value: $-\varphi\nabla\cdot\mathbf{D}$. In the framework of Dirac's treatment to the constraint systems [38] we can treat φ as a Lagrangian multiplier, and $\nabla\cdot\mathbf{D} = 0$ is the constraint to the canonical momentum $\boldsymbol{\pi}_\mathbf{A} = -\mathbf{D}$.

4. Discussion

In this section we discuss some important results we encountered in the previous sections. First we want to study the meaning of the factors α_x and α_z appearing in the Eq.(9). A real dielectric material usually has $\tilde{\varepsilon}_d > 1$, so we get $\alpha_x = f + (1-f)\tilde{\varepsilon}_d > 1$ and $\alpha_z = f + (1-f)/\tilde{\varepsilon}_d < 1$. Consider a harmonic $\mathbf{E}$ field operating at a frequency ω much higher than ω_0 and ω_p. Then according to

Eq.(17) the $\mathbf{Q}$ field vanishes and the $\mathbf{D}$ field can be approximated as $[D_x, D_z] = \varepsilon_0[\alpha_x E_x, \alpha_z E_z]$. In one unit cell, the dielectric layer contributes $(1-f)\varepsilon_d E_x$ to the effective displacement field D_x, while the metal layer contributes $f\varepsilon_0 E_x$, so the sum of them gives $D_x = [f\varepsilon_0 + (1-f)\varepsilon_d]E_x = \alpha_x \varepsilon_0 E_x$. This consideration also explains the form $Q_x = fP_x^m$, which is the dynamical part of the polarization field contributed by the metal layer.

On the other hand, the α_z factor is smaller than 1 . This is a consequence of the fact that the applied $\mathbf{E}$ field induces surface charges at the top and bottom surfaces of the dielectric layer, and the surface charges build an internal depolarization field inside the layer pointing to the opposite direction of the applied field. This depolarization field cancels a part of the applied field so we get a reduction factor $\alpha_z < 1$. Using the constitutive relation $D_z = \varepsilon_0 E_z + P_z = \varepsilon_0 E_z + (1-f)P_z^d + fP_z^m$ we now can derive the expression of D_z in Eq.(9) more straightforwardly

$$D_z = \frac{D_z\left[D_z - (1-f)P_z^d\right]}{D_z - (1-f)P_z^d} = \frac{\varepsilon_0 E_z + fP_z^m}{1-(1-f)\left(1-\dfrac{1}{\tilde{\varepsilon}_d}\right)} = \frac{\varepsilon_0 E_z + fP_z^m}{f + \dfrac{(1-f)}{\tilde{\varepsilon}_d}} = \alpha_z\left(\varepsilon_0 E_z + fP_z^m\right). \tag{35}$$

This derivation also gives us the form of $Q_z = \alpha_z fP_z^m$ automatically. This indicates that the depolarization effect also happens in the metal layer. Besides, the depolarization field plays the role of the restoring force acting to the " Q_z oscillator" and gives a nonzero ω_0 (see Eq.(15), Eq.(23), Eq.(25), and Eq.(33b)).

The energy density can also be obtained by calculating the contributions from the dielectric layer and the metal layer separately, and sum them up. For example, the energy density corresponding to E_x is given by

$$W_X = \frac{1}{2}(1-f)\varepsilon_d E_x^2 + \frac{1}{2}f\varepsilon_0 E_x^2 + fN\frac{m}{2}\dot{x}^2 = \frac{\varepsilon_0}{2}\alpha_x E_x^2 + \frac{1}{2\alpha_x\varepsilon_0\omega_p^2}\dot{Q}_x^2. \tag{36}$$

Here N, m, and x stands for the concentration, mass and displacement of the conduction electrons in the metal, and the relations $P_x^m = Nqx$, $Q_x = fP_x^m$ and $\omega_{p0}^2 = Nq^2/m\varepsilon_0$ have been used. The more involved result corresponding E_z can also be derived in a similar way. However, the derivation is tedious but not very inspiring, so we stop here and do not discuss this problem further.

5. Conclusions

In this paper we have completed the derivations of the effective fields, energy density, Lagrangian density, and Hamiltonian density for the electrodynamics in the effective medium that consists of dielectric-metal layers. We have also discussed how to obtain the frequency domain quantities such the permittivities and time-averaged energy density from our time domain formulas. It is found that the Hamiltonian density is the same as the energy density, up to an irrelevant divergence term. The Lagrangian field theory is a systematic method which yields the correct equations of motion for the polarization field and the vector potential through the Euler-Lagrangian equations. Since the system is dissipative, a dissipation function was introduced which takes care of the effective related to energy loss. The Lagrangian/Hamiltonian field theory framework can provide the essential knowledge for further developing the quantum theory of this kind of media.

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