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Solving Prandtl-Blasius boundary layer equation using Maple

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A solution for the Prandtl-Blasius equation is essential to all kinds of boundary layer problems. This paper revisits this classic problem and presents a general Maple code as its numerical solution. The solutions were obtained from the Maple code, using the Runge-Kutta method. The study also considers convergence radius expanding and an approximate analytic solution is proposed by curve fitting.

Keywords: Prandtl boundary layer, Prandtl-Blasius equation, numerical solution, Runge-Kutta method, Maple

I. INTRODUCTION

The boundary-layer theory can be traced back to Ludwig Prandtl (1904)[1]. In his famous study on the motion of liquids with very small friction, he presented the mathematical basis of flows for very large Reynolds numbers. In his 1904 study, he simplified the 2D Navier-Stokes equation into the following boundary layer equation

$$\rho(uu_{,x}+vu_{,y}) = -p_{,x}+\mu u_{,yy}, \quad (1)$$

$$p_{,y} = 0, \quad (2)$$

$$u_x+v_{,y} = 0. \quad (3)$$

Prandtl obtained the plate drag $drag = 1.1..b\sqrt{\mu\rho l U^2}$ for

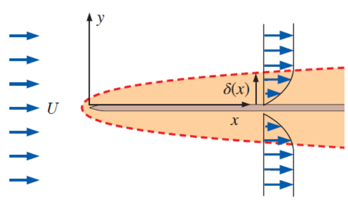

FIG. 1. The concept of Prandtl's boundary layer.

water flows at both sides of the plate, as shown in Figure 1 above. Later, in 1908, his first doctorate student, Paul Richard Heinrich Blasius[2] obtained a series solution for the boundary layer equation $f'''+\frac{1}{2}ff''=0$, and proposed an asymptotic solution, before modifying Prandtl's drag formula to its final version, namely $drag = 1.327b\sqrt{\mu\rho l U^2}$, where b referes to the plate width, refers to l plate length, refers to U free-steam velocity, refers to μ dynamical viscosity and refers to ρ flow density.

Blasius (1908) provides the power series solution as shown below

$$f(\eta) = \sum_{k}^{\infty}(-\frac{1}{2})^k \frac{C_k\alpha^{k+1}}{(3k+2)!}\eta^{3k+2},$$

where the coefficients C_k are calculated by a recurrence formula, $C_0 = C_1 = 1, C_2 = 11,\ C_3 = 375,\ C_4 = 27987,\ C_5 = 3817137,\ C_6 = 865874115,\ C_7 = 298013289795...$, and

$$C_k = \sum_{r=0}^{k-1}\frac{(3k-1)!}{(3r)!(3k-3r-1)!}C_{k-r-1}C_r,\ (k \geq 2).$$

This series is incomplete since the parameter $\alpha = f''(0)$ should be numerically computed. The next section presents $\alpha = 0.332057336270228$. Blasius's series converges in a region $|\eta| \leq 5.690$. Toepher (1912) and Howarth (1938) applied the Runge-Kutta method obtain their numerical results. With computers, Smith (1956)[7], Rosenhead (1963)[11] and Evans (1968)[12] obtained accurate numerical results for the Blasius equation.

Subsequent to Blasius's work[2], several scholars revisited the problem, for instance, Töpfer(1912)[3], Hartree(1912)[5], Goldstein (1930)[4], and Howarth (1937)[6]. Prandtl's student Schlichting (1950) who set out Blasius solution's application to almost all areas of fluid mechanics, most of them have been included into a well-known book, namely Boundary-Layer Theory[11,23], as well as reviews[13,14,16–18] and textbooks[20,25,26].

The momentum to solve the Blasius equation has not stopped. Different approaches have been tried, and approximate analytic solutions have been used, such as Perturbation methods[8–10,15] and the homotopy analysis method (HAM)[19,21,22,24], computer driven numerical solution using Matlab[29], as well as the B-spline method[27]. In particular, Liao[21,22] expanded the convergence radius by using his HAM method. Detailed reviews on various solutions of the Blasius equation can be found in Liao[21,22].

After a century of investigations of the Prandtl-Blasius equation $f'''+\frac{1}{2}ff''=0$, it is concluded that no closed solution has been founded, while numerical and approximate analytic solutions have been obtained. However, if one study literature on the Prandtl-Blasius equation carefully, it would not be difficult to find out that there is no simpler computer program, comprising 2 line code, be reported. It would be natural thinking that since the closed solution cannot be obtained and numerical solutions must be sought, it would be of academic value if the simplest calculation program can be provided.

To find a numerical solution for the equation $f'''+\frac{1}{2}ff''=0$, this study used a simple Maple code comprising 2 lines. We

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studied flow vorticity and found that the interaction of free-stream velocity, viscosity and the vorticity, was the source of drag. We expanded the convergence radius by changing the shooting boundary condition slightly. Based on our numerical solutions, we proposes a good approximate analytic solution by using a curve fitting. Besides the Prandtl-Blasius equation, we used several Maple codes to compute a few related problems. For ease of teaching and research, all Maple codes are provided. Some boundary layer related problems are solved and their relevant Maple codes are provided in appendix.

II. MAPLE CODE AND NUMERICAL SOLUTION FOR PRANDTL-BLASIUS EQUATION

Blasius[2] proposed a similar solution for the case in which the free stream velocity was constant, where $U(x) = constant$, corresponding to the boundary layer over a flat plate that is oriented parallel to the free flow. He introduced similar transformations, as shown below

$$\eta = y\sqrt{\frac{U}{\nu x}} \quad (4)$$

$$u(x,y) = Uf'(\eta) \quad (5)$$

$$v(x,y) = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\eta f'(\eta) - f(\eta)]. \quad (6)$$

and successfully transferred the Prandtl boundary equations in Eqs.1,2 and 3 into a single equation of $f(\eta)$, as follows

$$\begin{aligned} & f''' + \frac{1}{2}ff'' = 0, \\ & f(0) = 0, f'(0) = 0, \\ & f'(\eta \to \infty) = 1, \end{aligned} \quad (7)$$

where kinematical viscosity $\nu = \mu/\rho$ and $f' = df/d\eta$.

The solution of the Prandtl-Blasius equation Eq.7 has been studied intensively by a number of scholars[1–8,10–15,19,21–24,29]. To fully utilize the symbolic software, Maple, a simple Maple code is provided to solve the problem. The code is shown in Table I.

The boundary condition at $\eta \to \infty$ can not be materialized, the shooting method allows us to solve the problem by try and error. The shooting method can be convergent only at $\eta \leq 12.43$, while divergent beyond this point. The Blasius's series solution convergence radius is about $\eta = 6$, hence our numerical solution has a larger convergence radius than the series one.

With this simple Maple code, one can easily obtain a solution for $f(\eta)$ and its derivatives, as illustrated in Fig.2.

The numerical details are indicated in Table II below. An interesting outcome worth mentioning is that the convergence radius is expanded from $|\eta| \leq 5.690$ to $|\eta| \leq 12.43$ by try and error in shooting method.

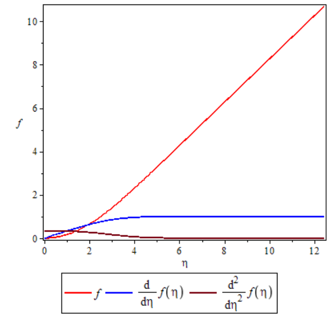

FIG. 2. Solution for Prandtl-Blasius equation

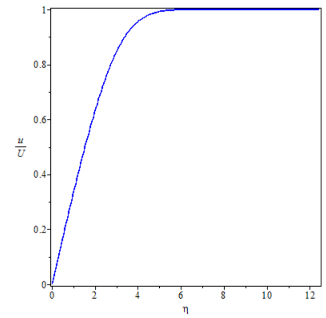

FIG. 3. $\frac{u}{U} = f'(\eta)$

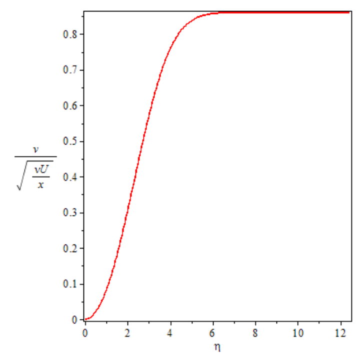

FIG. 4. $v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\eta f'(\eta) - f(\eta)]$

Following simple manipulation, all relevant quantise can easily be obtained by using Maple. The velocity in x direction $u = Uf'(\eta)$ profile as seen in Fig.3

The velocity in y direction $v = -\frac{1}{2}\sqrt{\frac{\nu U}{x}}[\eta f'(\eta) - f(\eta)]$ profile as shown in Fig.4

The shear stress on the flat plate

$$\tau = \mu(\frac{\partial u}{\partial y})|_{y=0} = \mu\sqrt{\frac{U^3}{\nu x}}f''(0) = 0.332048\mu\sqrt{\frac{U^3}{\nu x}}. \quad (8)$$

For a plate with a length L and width b, the plate drag *drag* for the water flows at both sides of plate, as shown in Figure

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TABLE I. Maple code for problem: $f''' + \frac{1}{2}ff'' = 0$, with $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 1$

Remarks	Maple code
Solution	solution := dsolve({$f''' + \frac{1}{2}ff'' = 0$, f(0) = 0, D(f)(0) = 0, D(f)(12.43) = 1}, numeric)
Plots	odeplot(solution, [eta, f(eta)], eta = 0 .. 12.43)

TABLE II. Numerical results for problem: $f''' + \frac{1}{2}ff'' = 0$, with $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 1$

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0	0	0	0.332057336270228
0.1	0.00166028097748329	0.0332054966058561	0.332048145748033
0.2	0.00664099782185387	0.0664077801596799	0.331983834255578
0.3	0.0149414623187717	0.0995985889897647	0.331809346697686
0.4	0.0265598832055768	0.132764155997273	0.331469843619160
0.5	0.0414928195150539	0.165885252325028	0.330910954899200
0.6	0.0597346375181186	0.198937252436665	0.330079127676020
0.7	0.0812769754437425	0.231890235983639	0.328922067860142
0.8	0.106108220767729	0.264709138163229	0.327389270302448
0.9	0.134213005526786	0.297353957812383	0.325432629177788
1	0.165571725783800	0.329780030672017	0.323007116916611
2	0.650024370215956	0.629765736670949	0.266751545690649
3	1.39680823153785	0.846044443888272	0.161360318755386
4	2.30574641937620	0.955518229353090	0.0642341216112545
5	3.28327366522910	**0.991541900689297**	0.0159067979373118
6	4.27962092307682	0.998972872289725	0.00240204010581148
7	5.27923881151476	0.999921604109742	0.000220169039772643
8	6.27921343179810	0.999996274564183	0.0000122408522222333
8.2	6.47921288713369	0.999998087480233	0.00000646792883303279
8.7	6.97921243151176	0.999999668030006	0.00000120272733477146
8.8	7.07921240368407	**0.999999769481724**	**8.46312294375786e-7**
9	7.27921237111197	0.999999890448371	4.12807423557125e-7
10	8.27921234339988	0.999999998015206	8.44248043699535e-9
11	9.27921234294946	0.999999999977930	1.04517612148937e-10
12	10.2792123429452	0.999999999998648	5.63589958345985e-12
12.43	10.7092123429449	**1**	0

1 is as follows

$$drag = 2b\int_0^L \tau dx = 2f''(0)b\sqrt{\mu\rho U^3}\int_0^L \frac{dx}{\sqrt{x}} = 1.328229345b\sqrt{\mu\rho LU^3}. \tag{9}$$

Denoting the Reynolds number, $Re_x = \frac{Ux}{\nu}$, the drag coefficient is defined as

$$C_f = \frac{drag}{\frac{1}{2}\rho U^2 2bL} = \frac{1.328229345}{\sqrt{Re_x}}. \tag{10}$$

If we define the boundary layer thickness as $\delta = \delta|_{u=0.99U}$, we can see that, $\eta \approx 5$, as indicated in Table II. Hence,

$$\delta \approx 5\sqrt{\frac{\nu x}{U}}. \tag{11}$$

If we consider he thickness definition to be, $\delta = \delta|_{u=U}$, then $\eta = 8.8$ or $\eta = 12.43$, while the boundary layer thickness can be extended to $\delta \approx 8.8\sqrt{\frac{\nu x}{U}}$ or $\delta \approx 12.43\sqrt{\frac{\nu x}{U}}$.

The boundary layer flow generates vorticity $\boldsymbol{\omega} = \omega\boldsymbol{k}$, amounting to the following

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -(\frac{1}{4Re_x}\eta + 1)f''(\eta)\sqrt{\frac{U^3}{\nu x}}. \tag{12}$$

The vorticity for a given Reynolds number, is illustrated in Fig.5

Since validation of the boundary layer theory requires the Reynolds number $Re_x > 100$, the team $\frac{1}{Re_x}\eta + 1 \approx 1$ implies that the Reynolds number has little influence on the vorticity, hence the vorticity obeys the law of "$-f''(\eta)$".

Since $\eta|_{y=0} = 0$ leads to $\omega(0) = -f''(0)\sqrt{\frac{U}{\nu x}} = -0.332057\sqrt{\frac{U}{\nu x}}$, the vorticity is mainly on the boundary and decays rapidly away from the boundary.

The local friction is as follows

$$\tau = \mu(\frac{\partial u}{\partial y})|_{y=0} = \mu\sqrt{\frac{U^3}{\nu x}}f''(0) = -\mu\omega(0), \tag{13}$$

which indicates the interaction between vorticity and viscosity, μ is the source of drag of the fluid acting on the plate.

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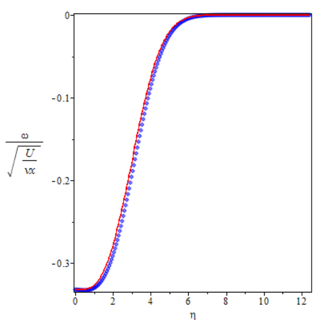

FIG. 5. Vorticity, Blue line for $Re_x = 500$, red point for $Re_x = 2000$

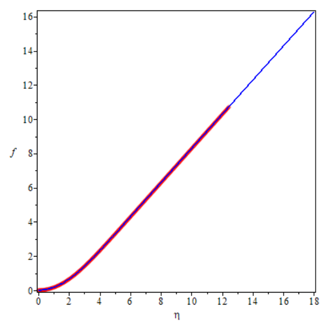

FIG. 6. Blue line is for the expanded convergence radius

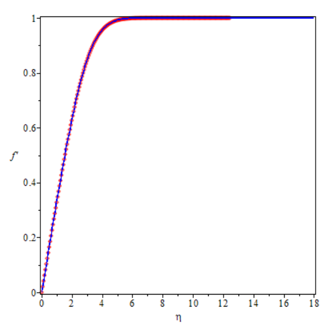

FIG. 7. Blue line is for the expanded convergence radius

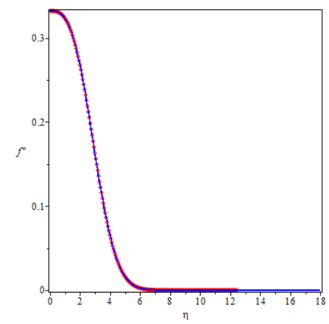

FIG. 8. Blue line is for the expanded convergence radius

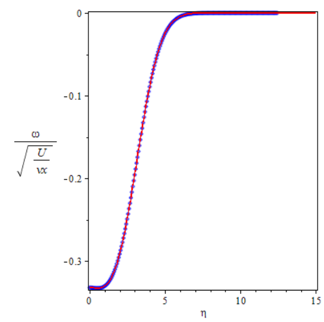

FIG. 9. Vorticity at $Re_x = 500$, Blue line is for the expanded convergence radius

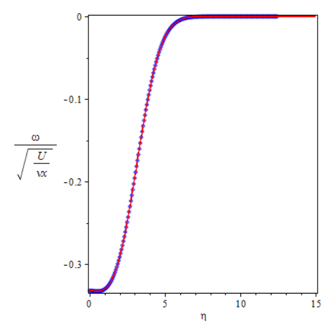

FIG. 10. Vorticity at $Re_x = 2000$, Blue line is for the expanded convergence radius

III. EXPANDING OF CONVERGENCE RADIUS UP TO $\eta = 251.67409$

In order to expand the convergence radius further, we replace one of boundary condition's, $f'(\infty) = 1$ with $f''(0)0.332057336270228$. The corresponding Maple code is provided in Table III.

The numerical results from this boundary are shown in Table IV.

The results comparisons are illustrated in Figures 6, 7,8,9 and 10.

If one compares the data in Table IV with that in Table II, one will be surprised to see that these correspond well with each other. The convergence radius is also extended to $|\eta| \leq 251.67409$. Beyond this point, the Maple code solution is divergent.

IV. DATA FITTING OF FUNCTION $f(\eta)$

For those who have not installed Maple, he polynomial fitting of function f and its primes are shown below. The fittings are valid for the entire convergence radius domain instead of the reported segmental fitting.

Using our numerical results from Table II, and fitting function in Maple, we can obtain three polynomial fittings as shown in Table V. It must be pointed out here that each fitting is done independently, and they are not connected.

TABLE III. Maple code for problem: $f''' + \frac{1}{2}ff'' = 0$, with $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0.332057336270228$

Remarks	Maple code
Solution	sol := dsolve({$f''' + \frac{1}{2}ff'' = 0$, f(0) = 0, D(f)(0) = 0, (D@@2)(f)(0) = 0.332057336270228}, numeric)
Plots	odeplot(sol, [eta, f(eta)], eta = 0 .. 251.67409)

TABLE IV. Numerical results for problem: $f''' + \frac{1}{2}ff'' = 0$, with $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0.332057336270228$

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0	0	0	0.332057336270228
0.1	0.00166027458322798	0.0332055040025001	0.332048148689987
0.2	0.00664099931612186	0.0664077922858582	0.331983835730853
0.3	0.0149414561579939	0.0995985986154332	0.331809350002633
0.4	0.0265598831096885	0.132764160936823	0.331469841408596
0.5	0.0414928112672231	0.165885253704228	0.330910957499993
0.6	0.0597346350066992	0.198937252124230	0.330079123728363
0.7	0.0812769666214984	0.231890236407141	0.328922070845611
0.8	0.106108213788000	0.264709136865312	0.327389267403343
0.9	0.134213005526786	0.297353957812383	0.325432629177788
1	0.165571709136145	0.329780027181432	0.323007121193428
2	0.650024303919321	0.629765748039948	0.266751498635688
3	1.39680803662046	0.846044371467904	0.161360256584747
4	2.30574609856618	0.955518154129491	0.0642340491293285
5	3.28327323651022	**0.991541790191322**	0.0159067535454309
6	4.27962036694372	0.998972750398667	0.00240200945704558
7	5.27923812288718	0.999921476426273	0.000220146810341452
8	6.27921260844824	0.999996143625627	0.0000122237028980494
8.7	6.97921151316917	0.999999535579582	0.00000118836098191137
8.8	7.07921147186480	**0.999999636412204**	**8.33454279337949e-7**
9	7.27921141219011	0.999999756564889	4.02602312724170e-7
10	8.27921124822598	0.999999862326453	3.81474083316453e-9
11	9.27921111130705	0.999999862641100	2.41403422546277e-9
12	10.2792109741436	0.999999864411131	-6.64589718802765e-9
13	11.2792108373132	0.999999864278211	-6.12017975548945e-9
...	...	...	...
20	18.2792098795010	0.999999863330679	-1.51917323973897e-9
...	...	...	...
251.67409	249.953268174194	0.999999863204860	-7.24965271393400e-9

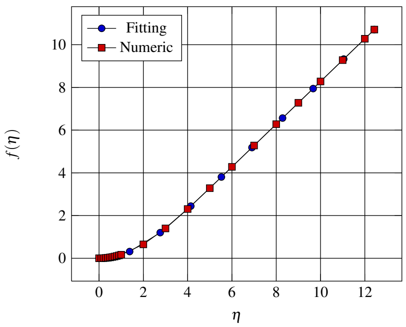

FIG. 11. $f(\eta)$

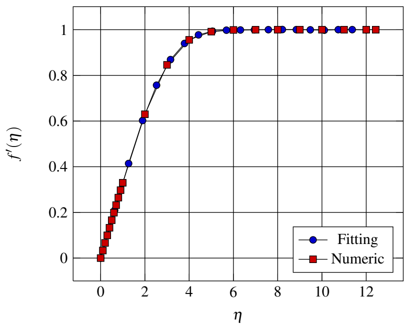

FIG. 12. $f'(\eta)$

Comparisons with the data in Table II are illustrated in Figures 11,12 and 13, respectively.

V. CONCLUSIONS

This study generated a simple Maple code, comprising 2 lines, and computed the Prandtl-Blasius equation

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TABLE V. Data fitting of $f(\eta)$, the 1st prime of $f(\eta)$ and the 2nd prime of $f(\eta)$

$f(\eta)$
$0.169590122015878\eta^2 - 0.00975243043702233\eta^3$ $+0.00949920484878554\eta^4 - 0.00490827418566491\eta^5$ $+0.00107974724248178\eta^6 - 0.000128524622004544\eta^7$ $+0.873236904505410 \times 10^{-5}\eta^8 - 3.20151483330390 \times 10^{-7}\eta^9$ $+4.93213211515512 \times 10^{-9}\eta^{10}$;
$f'(\eta)$
$0.337538516034295\eta - 0.0224722593243771\eta^2$ $+0.0309005339811106\eta^3 - 0.0211885940031025\eta^4$ $+0.00562296515736044\eta^5 - 0.000773827415658246e\eta^6$ $+0.0000591838415872827\eta^7 - 0.239635130719565 \times 10^{-5}\eta^8$ $+4.01841870085965 \times 10^{-8}\eta^9$;
$f''(\eta)$
$0.332057336270228 - 0.0264531212585649\eta$ $+0.0663932833156139\eta^2 - 0.0689310639889838\eta^3$ $+0.0232453293176898\eta^4 - 0.00381202903000096\eta^5$ $+0.000334793989126382\eta^6 - 0.0000151821911347140\eta^7$ $+2.79989280616388 \times 10^{-7}\eta^8$.

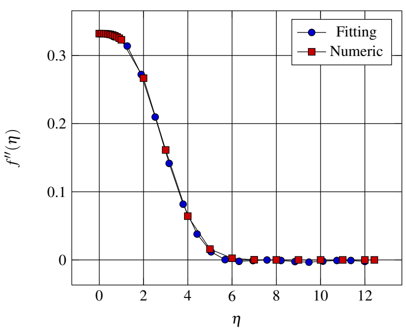

FIG. 13. $f''(\eta)$

$f'' + \frac{1}{2}ff'' = 0$ in detail. We have modified the boundary condition to extend the convergence radius. The modified results produce high-order accuracy and included a much larger convergence radius. We proposed a good fitting for the function, $f(\eta)$, which can be used by those who have not installed Maple.

Availability of data: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interests: The author declares that he/she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

B.-H. Sun: Conceptualization, Methodology, Formulations, Formal analysis, Funding acquisition, Investigation, Writing- Original draft preparation, Writing- Reviewing and Editing and all relevant works.

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