

Mean-field Theory of the Meissner effect in bulk revisited.

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Abstract

We show that the implementation of the $\frac{1}{c^2}$ transverse current-current interaction between electrons into the standard self-consistent electron BCS model in bulk under thermal equilibrium in the stable superconductive phase ensures the full compensation of a constant external magnetic field by the internal magnetic field created by the electrons.

keywords: BCS superconductivity, mean field, current-current interaction, internal/external fields, stability, Meissner effect

1 Introduction

Since its discovery by Kamerling-Onnes [1] at the beginning of the 20th century superconductivity became a very important field of physics with wide technical applications. The identification of superconductors as perfect diamagnets i.e. the Meissner effect [2] was discovered in the thirties and was soon followed by the beautiful phenomenological electromagnetic theory of London [3, 4]. The fundamental theoretical breakthrough however is due to Bardeen-Cooper-Schrieffer [5]. They have shown that the origin of the superconductive phase transition lies in the correlation between electrons of opposite momenta and spin resulted from phonon exchange. Although the most important and striking feature is the absence of resistance below a critical temperature, there is no deep understanding of it. Within the present unsatisfactory status of non-equilibrium statistical mechanics irreversibility, dissipation is introduced "by hand", although there are significant recent progresses in understanding the treatment of open systems [6]. Therefore the understanding at least of the equilibrium properties is a central point. The most important in this respect is the theory of the Meissner effect to which we devote our discussion.

The standard modeling of the BCS idea within self-consistent electron theories [7, 8, 9] offers no convincing results for the Meissner effect. Their failure is due to the incorrect implementation of electromagnetism and the insistence to relate the results to the London equation in the context of the bulk, where the magnetic field should be identically null. Starting from the non-relativistic quantum electrodynamics a current-current magnetic interaction may be derived [10, 11]. We show in this paper that its inclusion improves the standard bulk mean field theory explaining within this frame the perfect diamagnetism.

2 The standard theory of the Meissner effect in bulk.

The revelation of BCS that the source of superconductivity lies in the correlation of electrons of opposite momenta and spin due to interaction with phonons was decisive for the theory. However, to pursue this idea within the many-body theory including phonons seems difficult. Therefore one tried to construct pure electron many-body theories with a built-in "potential" giving rise to such correlations and implicitly to a superconducting phase transition.

Such a model Hamiltonian is due to Rickayzen [7, 8] we describe here. The specific version of Bogolyubov-de Gennes [9] is included in this frame.

Since one is concentrated on equilibrium properties in a μ -system one considers $\mathcal{H} \equiv H - \mu N$ instead of the Hamiltonian H . Rickayzen introduces besides the kinetic energy in the presence of classical time-independent given vector $\vec{\mathcal{A}}(\vec{x})$ and scalar $\phi(\vec{x})$ potentials, an "attractive" electron-electron interaction by a potential $W(\vec{x})$, that produces no bound states, but gives rise to correlations of BCS type:

$$\begin{aligned}
\mathcal{H} &\equiv H - \mu N & (1) \\
&= \sum_{\sigma=\pm\frac{1}{2}} \int d\vec{x} \psi_{\sigma}^{+}(\vec{x}) \left\{ \frac{1}{2m} \left(-i\hbar\nabla - \frac{e}{c} \vec{\mathcal{A}}(\vec{x}) \right)^2 + e\phi(\vec{x}) - \mu \right\} \psi_{\sigma}(\vec{x}) \\
&+ \frac{1}{2} \int d\vec{x} \int d\vec{x}' W(\vec{x} - \vec{x}') \left[\psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \psi_{\frac{1}{2}}(\vec{x}') \psi_{-\frac{1}{2}}(\vec{x}) + h.c. \right].
\end{aligned}$$

This Hamiltonian is invariant against time independent gauge transformations of the external vector potential $\vec{\mathcal{A}}(\vec{x})$. One keeps the discussion in a rather general frame without choosing a definite potential $W(\vec{x})$.

Coulomb interactions (including a positive background) we ignored here since they play no role in the next steps to follow. Further, one resorts to a self-consistent Hartree-Fock approximation including anomalous averages like $\langle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \rangle$:

$$\begin{aligned}
\mathcal{H}_{HF} &= \sum_{\sigma} \int d\vec{x} \psi_{\sigma}^{+}(\vec{x}) \left\{ \frac{1}{2m} \left(-i\hbar\nabla - \frac{e}{c} \vec{\mathcal{A}}(\vec{x}) \right)^2 + \phi(\vec{x}) - \mu \right\} \psi_{\sigma}(\vec{x}) & (2) \\
&+ \frac{1}{2} \int d\vec{x} \int d\vec{x}' W(\vec{x} - \vec{x}') \left[\langle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \rangle \psi_{-\frac{1}{2}}(\vec{x}') \psi_{\frac{1}{2}}(\vec{x}) \right. \\
&+ \left. \langle \psi_{-\frac{1}{2}}(\vec{x}') \psi_{\frac{1}{2}}(\vec{x}) \rangle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') - \langle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \rangle \langle \psi_{-\frac{1}{2}}(\vec{x}') \psi_{\frac{1}{2}}(\vec{x}) \rangle \right].
\end{aligned}$$

One may show within this frame, that in the absence of the fields $\vec{\mathcal{A}}(\vec{x})$, $\phi(\vec{x})$ a phase transition may occur under a critical temperature, provided the potential $W(\vec{x})$ ensures a non-vanishing solution for the symmetry breaking gap parameter ($\Delta(\vec{k}) \neq 0$) of the oft described "gap equation" we do not give here. This condition is equivalent to the vanishing of the first derivative of the free energy with respect to $\Delta(\vec{k})$.

The next step is to use equilibrium linear response theory to get the relationship between the average $\langle \vec{j}(\vec{x}) \rangle$ of the current density operator

$$\vec{j}(\vec{x}) \equiv \frac{e}{2m} \psi^{+}(\vec{x}) \left(-i\hbar\nabla + \frac{e}{c} \vec{\mathcal{A}}(\vec{x}) \right) \psi(\vec{x}) + h.c. \quad (3)$$

and the weak vector potential in the Coulomb gauge $\nabla \cdot \vec{\mathcal{A}}(\vec{x}) = 0$ and $\phi(\vec{x}) = 0$. A peculiarity of the linear response within self-consistent theories is that the deviation of the averages from their equilibrium value constitute so called induced perturbations. This means, that the true perturbation is

$$\mathcal{H}' = -\frac{1}{c} \int d\vec{x} \vec{j}(\vec{x}) \cdot \vec{\mathcal{A}}(\vec{x}) + \int d\vec{x} \int d\vec{x}' W(\vec{x} - \vec{x}') \left[\eta(\vec{x}, \vec{x}') \psi_{-\frac{1}{2}}(\vec{x}') \psi_{\frac{1}{2}}(\vec{x}) + h.c. \right] \quad (4)$$

with

$$\eta(\vec{x}, \vec{x}') \equiv \langle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \rangle - \langle \psi_{\frac{1}{2}}^{+}(\vec{x}) \psi_{-\frac{1}{2}}^{+}(\vec{x}') \rangle_0 \quad (5)$$

The resulting linear relationship between the Fourier Transforms of the two transverse vectors looks as

$$\langle \tilde{j}_{\mu}(\vec{k}) \rangle = \kappa(k) \tilde{\mathcal{A}}_{\mu}(\vec{k}). \quad (6)$$

with the scalar coefficient $\kappa(k)$ in an infinite homogeneous system being a function of only $k = \sqrt{\vec{k}^2}$. Its explicit expression however was calculated [7] only after neglecting the induced perturbations.

Actually, one is interested in this relationship only for small wave vectors (slowly varying behavior in the coordinate space!). If in the absence of the perturbation one had an anomalous superconducting phase with a non-vanishing gap $\Delta(\vec{k})$ at $\vec{k} = 0$ one gets [7] that $\kappa(0)$ is finite and strictly negative

$$\kappa(0) = -\frac{1}{c\Lambda} < 0. \quad (7)$$

It may be shown however [12], that under the condition of stability of the superconducting phase (which amounts to a non-negative second derivative of the free energy with respect to the gap parameter

$\Delta(\vec{k})$ at $k = 0$) Λ is indeed positive and the contribution produced by consideration of $\eta(\vec{x}, \vec{x}')$ does not change Rickayzen's result.

So far the calculations are OK. The problem resides in the interpretation of these results. According to the standard interpretation Eqs. 6, 7 have to be compared to the second London equation that reads as

$$\nabla \times \vec{i} = -\frac{1}{c\Lambda} \vec{B} \quad \text{or} \quad \vec{i} = -\frac{1}{c\Lambda} \vec{A} ,$$

with $\vec{i}(\vec{x})$ being the macroscopic super-current density and $\vec{A}(\vec{x})$ the total macroscopic field in the superconductor. Therefore, according to the usual interpretation, the Meissner effect would be explained within this model Hamiltonian. Even more, the parameter Λ might be interpreted as the London penetration length.

Unfortunately, such a reasoning is **misleading**. The total magnetic field \vec{B} as well as the super-current are identically null in the bulk if Meissner effect occurs! The reference to the London equation is totally misplaced here. On the other hand, the classical field \vec{A} in this model has undefined sources and no magnetic field produced by the electrons is present at all in this theory. Therefore it can not be identified with the total macroscopic magnetic vector potential \vec{A} . Besides, this Hamiltonian is invariant with respect to time independent gauge transformations and therefore \vec{A} has to be identified obviously with the external field. In the frame of the non-relativistic QED the Hamiltonian is defined in a fixed gauge, namely in the Coulomb gauge for the quantized electromagnetic field.

The task of the microscopical theory in the bulk is just to show, that in thermal equilibrium a homogeneous constant internal magnetic field compensates completely the external one. Obviously some ingredients are still missing. One is at the range of validity of the ordinary quantum mechanics, that takes no magnetic interactions between the electrons into account. Steps towards the non-relativistic quantum electrodynamics of charged particles are compulsory. This criticism of the electromagnetic aspects is pertinent also to the Ginzburg-Landau non-linear theory of superconductivity [13], although that is not a Hamiltonian theory.

3 The current-current interaction and the theory of superconductivity

Already 100 years ago Darwin [14] has argued in the frame of the classical electrodynamics of point-like electrons, that up to order $1/c^2$ one might separate the motion of the particles from that of the electromagnetic fields. From this separation emerges a magnetic electronic current-current interaction. Since the classical electrodynamics of point-like charged particles is a vicious theory having neither Lagrangian, nor Hamiltonian formulation, his derivation lacked any rigor, nevertheless it contains a grain of truth and was considered later also by Landau and Lifshitz [15]. They have shown that Darwin implicitly has chosen a certain very strange non-linear choice of gauge. Such a gauge however imposes constraints on the velocities of the particles and therefore no ordinary canonical formalism is allowed. See in this context Dirac's theory of canonical formalism with constraints [16, 17].

Actually one needs a new analysis of the problem in the frame of the non-relativistic QED (see [18, 19] for a modern presentation). A natural choice of the gauge is here the Coulomb one. Indeed, in this gauge one has to do only with the two physical degrees of freedom of the photons and man is free from artificial constraints. One has to restrict the attention to the subspace of states without photons in order to have a pure electron theory. The next necessary step is to neglect retardation effects and this restricts the validity of the results to order $1/c^2$. A good object to perform such a discussion is either the S-matrix, or the theory of Green functions with the help of the Feynman diagrams. In an early paper Holstein, Norton and Pincus [20] have shown, that without retardation the photon propagator reduces to the Coulomb potential and the most important Feynman diagram is a transverse current-current interaction between the electrons. Recently one has shown [10, 11] that the two ingredients : restriction to the electronic subspace and ignoring the retardation in the photon propagator leads to an electronic Hamiltonian, that besides the usual Coulomb interaction between the electrons contains also a magnetic transverse current-current interaction

$$-\frac{1}{2} \int d\vec{x} \int d\vec{x}' \frac{N \left[\vec{j}_\perp(\vec{x}) \vec{j}_\perp(\vec{x}') \right]}{c^2 |\vec{x} - \vec{x}'|} . \quad (8)$$

This is nothing else but the well-known Biot-Savart law. One may argue that due to the smallness of the velocities in the condensed matter such an $1/c^2$ term may be neglected. This is obviously false. Our everyday experience teaches us, that a macroscopic number of very slow electrons may create enormous magnetic fields. We shall show, that exactly this term is necessary to round up the theory of the preceding Section leading to the exact cancellation of the total magnetic field in the bulk.

Supplementing the Hamiltonian of Eq. 1 with the above term and taking into account the presence of time independent external fields \vec{A}_{ext} and V_{ext} the complete BCS-Hamiltonian looks now as

$$\begin{aligned} \mathcal{H}^{new} &= - \int d\vec{x} \psi^+(\vec{x}) \left\{ \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A}_{ext}(\vec{x}) \right)^2 + V_{ext}(\vec{x}) - \mu \right\} \psi(\vec{x}) \\ &+ \frac{1}{2} \int d\vec{x} \int d\vec{x}' \frac{N[\rho(\vec{x})\rho(\vec{x}')] }{|\vec{x} - \vec{x}'|} - \frac{1}{2} \int d\vec{x} \int d\vec{x}' \frac{N[\vec{j}_\perp(\vec{x})\vec{j}_\perp(\vec{x}')] }{c^2|\vec{x} - \vec{x}'|} \\ &+ \frac{1}{2} \int d\vec{x} \int d\vec{x}' W(\vec{x} - \vec{x}') \left[\psi_{\frac{1}{2}}^+(\vec{x})\psi_{-\frac{1}{2}}^+(\vec{x}')\psi_{\frac{1}{2}}(\vec{x}')\psi_{-\frac{1}{2}}(\vec{x}) + h.c. \right] . \end{aligned} \quad (9)$$

For sake of completeness and symmetry we included here also the Coulomb interaction between the electrons and used the notation $N[\dots]$ for the normal product of operators. Here

$$\vec{j}(\vec{x}) = \frac{e}{2m} \left(\psi^+(\vec{x}) \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A}_{ext}(\vec{x}) \right) \psi(\vec{x}) + h.c. \right) \quad (10)$$

and \vec{j}_\perp designs the transverse part of the current.

Then, adding the s.c. Hartree approximation of the new term Eq. 8 into Eq. 2 we get

$$\begin{aligned} \mathcal{H}_{HF}^{new} &= \int d\vec{x} \psi^+(\vec{x}) \left\{ \left(\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A}_{ext}(\vec{x}, t) \right)^2 + eV_{ext}(\vec{x}, t) - \mu \right\} \psi(\vec{x}) \\ &+ \int d\vec{x} \int d\vec{x}' \psi^+(\vec{x}) \frac{e\langle \rho(\vec{x}', t) \rangle}{|\vec{x} - \vec{x}'|} \psi(\vec{x}) - \int d\vec{x} \int d\vec{x}' \frac{\langle \vec{j}_\perp(\vec{x}, t) \rangle \langle \vec{j}_\perp(\vec{x}', t) \rangle}{c^2|\vec{x} - \vec{x}'|} \\ &+ \frac{1}{2} \int d\vec{x} \int d\vec{x}' W(\vec{x} - \vec{x}') \left[\langle \psi_{\frac{1}{2}}^+(\vec{x})\psi_{-\frac{1}{2}}^+(\vec{x}') \rangle \psi_{-\frac{1}{2}}(\vec{x}')\psi_{\frac{1}{2}}(\vec{x}) \right. \\ &+ \left. \langle \psi_{-\frac{1}{2}}(\vec{x}')\psi_{\frac{1}{2}}(\vec{x}) \rangle \psi_{\frac{1}{2}}^+(\vec{x})\psi_{-\frac{1}{2}}^+(\vec{x}') - \langle \psi_{\frac{1}{2}}^+(\vec{x})\psi_{-\frac{1}{2}}^+(\vec{x}') \rangle \langle \psi_{-\frac{1}{2}}(\vec{x}')\psi_{\frac{1}{2}}(\vec{x}) \rangle \right] . \end{aligned} \quad (11)$$

In this expression one may already identify the internal fields

$$V_{int}(\vec{x}, t) = \int d\vec{x}' \frac{\langle \rho(\vec{x}', t) \rangle}{|\vec{x} - \vec{x}'|}; \quad \vec{A}_{int}(\vec{x}, t) = \int d\vec{x}' \frac{\langle \vec{j}_\perp(\vec{x}', t) \rangle}{|\vec{x} - \vec{x}'|}, \quad (12)$$

or in Fourier transforms:

$$\tilde{V}^{int}(\vec{k}) = \frac{4\pi}{k^2} \langle \tilde{\rho}(\vec{k}) \rangle \quad \tilde{A}_\mu^{int}(\vec{k}) = \frac{4\pi}{k^2} \langle \tilde{j}_\mu(\vec{k}) \rangle . \quad (13)$$

Keeping only terms of first order in \vec{A}^{ext} the only difference to the previous approach is the replacement in the perturbation Eq. 4

$$\vec{j}\vec{A} \Rightarrow \vec{j}(\vec{A}^{ext} + \vec{A}^{int}) .$$

(However, it is important to stress, that this holds only in the first order terms!). Therefore, due to the new induced term now we get with the previously defined $\kappa(k)$

$$\langle \tilde{j}_\mu(\vec{k}) \rangle = \kappa(k) (\tilde{A}_\mu^{ext}(\vec{k}) + \frac{4\pi}{k^2} \langle \tilde{j}_\mu(\vec{k}) \rangle) \quad (14)$$

or

$$\langle \tilde{j}_\mu(\vec{k}) \rangle = \frac{\kappa(k)}{1 - \frac{4\pi}{k^2} \kappa(k)} \tilde{A}_\mu^{ext}(\vec{k}) \quad (15)$$

and

$$\tilde{B}_\mu^{int}(k) = \frac{\frac{4\pi}{k^2}\kappa(k)}{1 - \frac{4\pi}{k^2}\kappa(k)}\tilde{B}_\mu^{ext}(k) . \quad (16)$$

This last equation could have been obtained also by Zubarev's reasoning [21] based just on the macroscopic Maxwell equations, without any reference to a Hamiltonian. He obtains the linear response to the field \tilde{B} from the known linear response to the field \tilde{H} .

Since it was proven already [7, 12] that under the condition of a stable phase superconductive phase $\kappa(0)$ is finite, it follows

$$\tilde{B}_\mu^{int}(0) = -\tilde{B}_\mu^{ext}(0) . \quad (17)$$

This proves that in the frame of this s.c. BCS model in thermal equilibrium in the stable superconductive phase no constant magnetic field in the bulk may survive! However, the coefficient $\kappa(0)$ cannot be related to the London penetration length and its sign is irrelevant for the proof.

4 Conclusions

By improving the standard bulk model of BCS superconductivity in its electromagnetic aspects, we have proven that in its mean-field approximation, if the superconductive phase transition is stable, the Meissner effect results in the sense of the perfect compensation of a constant external magnetic field by the internal magnetic field. This was achieved by taking into account the magnetic transverse current-current interaction deduced from the non-relativistic QED. Over the whole derivation no definite potential $W(\vec{x})$ was considered, but just the general requirement of the minimum of the free energy with non-vanishing gap parameter.

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References

- [1] H.K. Onnes, Comm. Phys. Lb. Univ. Leiden, Nos. 119,120,122 (1911)
- [2] W. Meissner and R. Ochsenfeld, Naturwiss. **21**, 787 (1933)
- [3] F. London and H. London, Proc. Roy. Soc., 149 (1935)
- [4] Fritz London, Superfluids (Volume 1 Macroscopic Theory of Superconductivity), Dover Publications, Inc. New York 1960
- [5] J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys. Rev. **108**,1175 (1957)
- [6] H. Spohn, Rev. Mod. Phys. **53**, 569 (1980)
- [7] G.Rickayzen, Phys.Rev. **115**,795 (1959)
- [8] G.Rickayzen, Theory of Superconductivity (Interscience 1965)
- [9] P. de Gennes, Rev. Mod. Phys. **36**,(1) 225 (1964)
- [10] L.A. Bányai, arXiv:1905.02390v5 (2019)
- [11] Ladislaus Alexander Bányai, A Compendium of solid-state theory, Second Edition (Springer International 2020)
- [12] L. Bányai and P. Gartner, Phys. Rev. **B 29**, 4992 (1983)

- [13] V.L. Ginzburg, L. D. Landau , Zh. Exp. Teor. Fiz. **20**,1064 (1964)
- [14] C.G. Darwin, Phil. Mag. ser. 6. **36**, 537 (1920)
- [15] L.D. Landau and E. M. Lifshitz, The Classical Theory of Fields. Pergamon Press, 1971
- [16] P. A. M. Dirac, Can. J. Math. **2**,147 (1950)
- [17] P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science (Yeshiva University, New York (1964)
- [18] Bányai, L.A.; Bundaru, M. Quantum Mechanics and Electromagnetism. Non-Relativistic Theory.. Preprints 2020, 2020050083 (doi: 10.20944/preprints202005.0083.v1).
- [19] Ladislaus Alexander Bányai, Lectures on Non-Equilibrium Theory of Condensed Matter. Second Edition. (World Scientific 2020)
- [20] T. Holstein, R.E. Norton and P. Pincus, Phys. Rev. **B** 2649 (1973)
- [21] D. N. Zubarev, Nonequilibrium Statistical Thermodynamics, Consultants Bureau, 1974