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Article

Geometric aspects of the isentropic liquid dynamics and vorticity invariants

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Abstract: We review a modern differential geometric description of the fluid isotropic motion and featuring it the diffeomorphism group structure, modelling the related dynamics, as well as its compatibility with the quasi-stationary thermodynamical constraints. There is analyzed the adiabatic liquid dynamics, within which, following the general approach, there is explained in detail, the nature of the related Poissonian structure on the fluid motion phase space, as a semidirect Banach groups product, and a natural reduction of the canonical symplectic structure on its cotangent space to the classical Lie-Poisson bracket on the adjoint space to the corresponding semidirect Lie algebras product. We also present a modification of the Hamiltonian analysis in case of the isotermal liquid dynamics. We study the differential-geometric structure of the adiabatic magneto-hydrodynamic superfluid phase space and its related motion within the Hamiltonian analysis and invariant theory. In particular, we construct an infinite hierarchies of different kinds of integral magneto-hydrodynamic invariants, generalizing those before constructed in the literature, and analyze their differential-geometric origins. A charged liquid dynamics on the phase space invariant with respect to an abelian gauge group transformation is also investigated, some generalization of the canonical Lie-Poisson type bracket is obtained.

Keywords: liquid flow; hydrodynamic Euler equations; diffeomorphism group; Lie-Poisson structure; isentropic hydrodynmaic invarinats; vortex invariants; charged liquid fluid dynamics; symmetry reduction

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0. Introduction

It is well known that often the same physical system is described using different sets of variables, related with their different physical interpretation. Simultaneously this same system is endowed with different mathematical structures deeply depending on the geometric scenario used for its description. In general these structures prove to be not equivalent but some special way connected to each other. In particular, such double descriptions commonly occur in systems with distributed parameters as hydrodrodynamics, magnetohydrodynamics and diverse gauge systems, which are effectively described by means of both symplectic and Poissonian structures on the suitable phase spaces. In particular, it was observed [3,19,28,32,35,35,36,53,54] that these structures are canonically

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related to each other. Mathematical properties, lying in a background of their analytical description, make it possible to study additional important parameters [5,12,22,25,26,33,34,39,40,42,49,50] of different hydrodynamic and magnetohydrodynamic systems, amongst which we will mention integral invariants, describing such internal fluid motion peculiarities as vortices, topological singularities and other different instability states, strongly depending [11,16] on imposed isentropic entropy constraints. Being interested in their general properties and mathematical structures, responsible for their existence and behavior, we present a detail enough differential geometrical approach to investigating thermodynamically quasi-stationary fluid motions, paying more attention to analytical argumentation of tricks and techniques used during the presentation.

In particular, we consider a compressible liquid filling a compact linearly-connected domain $M \subset \mathbb{R}^3$ with smooth boundary ∂M, and moving free of external forces. A configuration of this fluid is called the reference or Lagrangian configuration, its points are called material or Lagrangian points and denoted by $X \in M$ and are referred as material, or Lagrangian coordinates. We shall not for further be specific about the correct choices of the related functional spaces to be used and refer to works [10,24], where this is discussed in great detail. The manifold $M \subset \mathbb{R}^3$, thought of as the target space of a configuration $\eta \in Diff(M)$ of the fluid at a different time, is called the spatial or Eulerian configuration, whose points, called spatial or Eulerian points, will be denoted by small letters $x \in M$. Then a motion of the fluid is a time dependent family [1,3,4,21,24,29,32,34,50] of diffeomorphisms written as

$$M \ni x = \eta(X,t) := \eta_t(X) \in M \tag{1}$$

for any initial configuration $X \in M$ and some mapping $\eta_t \in Diff(M), t \in \mathbb{R}$. We also are given the mass density $\rho_0 \in \mathcal{R}(M) \subset C^\infty(M;\mathbb{R}_+)$ and the specific entropy $\sigma_0 \in \Sigma(M) \subset C^\infty(M;\mathbb{R}_+)$ of the fluid in the reference configuration, changing in time in such a way that

$$\rho_0(X) = \rho_t(x) J_{\eta_t}(x), \sigma_0(X) = \sigma_t(x), \tag{2}$$

where $J_{\eta_t}(x)$ denotes the standard Jacobian determinant of the motion $\eta_t \in Diff(M)$ at $x \in M$ and $\sigma_t(x)$ denotes the specific entropy for any $x = \eta_t(X) \in M$ and $t \in \mathbb{R}$. For a motion $x = \eta_t(X) \in M$ and arbitrary $X \in M, t \in \mathbb{R}$, one usually defines three velocities:

the material or Lagrangian velocity

$$V(X,t) = V_t(X) := \partial \eta_t(X)/\partial t, \tag{3}$$

the spatial or Eulerian velocity

$$v(x,t) = v_t(x) := v_t \circ \eta_t(X) \tag{4}$$

and convective or body velocity

$$\mathcal{V}(X,t) = \mathcal{V}_t(X) := -\partial X(x,t)/\partial t = -\partial \eta_t^{-1}(x)/\partial t, \tag{5}$$

being equivalent to the expression $\mathcal{V}_t = \eta_{t,*}^{-1} v_t$ for all $t \in \mathbb{R}$. Since the velocity $v_t : M \in T(M)$ is tangent to M for all $t \in \mathbb{R}$ at $x = \eta_t(X) \in M$, it determines a time dependent vector field on M. On the other hand, tangency of $V_t(X)$ and $\eta_t(X), X \in M$, means that the velocity V_t is a vector field over a configuration $\eta_t \in Diff(M)$ on M, that is $V_t : M \to T(M)$ is such a map that $V_t(X)$ is tangent to M not at $X \in M$, but at point $\eta_t(X) \in M$. Simultaneously the velocity $\mathcal{V}_t(X)$ is a tangent vector to M at $X \in M$, that is $\mathcal{V}_t$ is also a time dependent vector field on M. In what will follow we will think of the fluid as moving smoothly in the domain $M \subset \mathbb{R}^3$, at any time filling it and producing no shocks and cavitation.

In our review, based on this introductory Section, we present in Section 1 a modern differential geometric description of the isentropic fluid motion phase space and featuring it diffeomorphism

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group structure, modelling the related dynamics, as well as its compatibility with the quasi-stationary thermodynamical constraints. Section 2 is devoted to the Hamiltonian analysis of the adiabatic liquid dynamics, within which, following the general approach of [21,28,34], there is explained in detail, the nature of the related Poissonian structure on the fluid motion phase space, as a semidirect Banach groups product, and a natural reduction of the canonical symplectic structure on its cotangent space to the classical Lie-Poisson bracket on the adjoint space to the corresponding semidirect Lie algebras product. A modification of the Hamiltonian analysis in case of the isotermal liquid dynamics is presented in Sec 3. In Section 4 we proceed to studying the differential-geometric structure of the adiabatic magneto-hydrodynamic superfluid phase space and its related motion within the Hamiltonian analysis and invariant theory. We construct there an infinite hierarchies of different kinds of integral magneto-hydrodynamic invariants, generalizing those, before constructed in [18,21], and analyzed their differential-geometric origins. The last Section 5 presents a charged fluid dynamics on the phase space invariant with respect to an abelian gauge group transformation.

1. Ideal liquid dynamics and its geometry

It is well known that the motion of an ideal compressible and adiabatic fluid is governed by the Euler equations

$$\begin{gathered}\partial v/\partial t+\langle v|\nabla\rangle v+\rho^{-1}\nabla p^{(0)}=0,\\ \partial\rho/\partial t+<\nabla|\rho v>=0, \partial\sigma/\partial t+\langle v|\nabla\rangle\sigma=0,\end{gathered}\tag{6}$$

where $p_0 : M \to \mathbb{R}$ is the internal fluid pressure, $\sigma = \sigma(x_t, t) = \sigma_t(x_t)$ is the entropy at a spatial point $x_t = \eta_t(X) \in M$ for any $t \in \mathbb{R}$, which is fixed owing to the Euler equations (6), $\nabla := \partial/\partial x$ is the usual gradient on the space of smooth functions $C^\infty(M;\mathbb{R})$ and $< \cdot|\cdot >$ denotes the usual convolution on $T(M) \times T(M)$ subject to the usual metric in $\mathbb{R}^3$, reduced on the submanifold M. The evolution (6) is considered to be *a priori* thermodynamically quasi-stationary, that is the following *infinitesimal convective* relationship

$$\delta e_t(\rho_t(x_t),\sigma_t(x_t)) = T_t(x_t)\delta\sigma_t(x_t) + p_t^{(0)}(x_t)\rho_t^{-2}(x_t)\delta\rho_t(x_t)\tag{7}$$

holds for all $x_t \in M$ and $t \in \mathbb{R}$, where $e_t : \mathcal{R}(M) \times \Sigma(M) \to C^\infty(M \times \mathbb{R};\mathbb{R})$ denotes the internal specific fluid energy, $T_t : M \to \mathbb{R}_+$ denotes the fluid absolute temperature, $p_t^{(0)} : M \to \mathbb{R}$ is the internal liquid pressure and the variation sign "δ" means the change subject to both the temporal variable $t \in \mathbb{R}$ and the spatial variable $x_t \in M$.

Let us now analyze the internal mathematical structure of quantities $(\rho_t, \sigma_t) \in \mathcal{R}(M) \times \Sigma(M)$ as the *physical observables* subject to their evolution (6) with respect to the group diffeomorphisms $\eta_t \in Diff(M), t \in \mathbb{R}$, generated by the liquid motion vector field $dx_t/dt = v_t(x_t), x_t := \eta_t(X), t \in \mathbb{R}, X \in M :$

$$\begin{gathered}\mathcal{L}_{d/dt}(\rho_t d^3x_t\langle v_t|dx_t\rangle) = \rho_t d^3x_t(-\rho_t^{-1}dp_t^{(0)} + d|v_t|^2/2),\\ \mathcal{L}_{d/dt}(\rho_t d^3x_t) = 0, \ \ \mathcal{L}_{d/dt}\sigma_t = 0,\end{gathered}\tag{8}$$

where $\mathcal{L}_{d/dt} : \Lambda(M) \to \Lambda(M)$ denotes the corresponding Lie derivative with respect to the vector field $d/dt := \partial/\partial t + \langle v_t|\nabla\rangle \in \Gamma(M \times \mathbb{R}; T(M)), t \in \mathbb{R}$. The relationships (54) here simply mean that at every fixed $t \in \mathbb{R}$ the space of physical observables, being by definition, the adjoint space $\mathcal{G}^* := (\Lambda^1(M) \otimes \Lambda^3(M)) \oplus (\Lambda^3(M) \oplus \Lambda^0(M))$ of the extended configuration space is equal to $\mathcal{G} := \Gamma(M; T(M)) \times (\Lambda^0(M) \oplus \Lambda^3(M)) \simeq T_{Id}(G)$, the tangent space at the identity Id to the extended differential-functional group manifold $G := Diff(M) \times (\Lambda^0(M) \times \Lambda^3(M)) \simeq Diff(M) \times (\mathcal{R}(M) \times \Sigma(M))$, where we have naturally identified the abelian group product $\Lambda^0(M) \times \Lambda^3(M)$ with its direct tangent space sum $T(\Lambda^0(M)) \oplus T(\Lambda^3(M))$.

Consider now the natural action $Diff(M) \times G \to G$ of the $Diff(M)$-group on the constructed differential-functional manifold G :

$$(\eta \circ \varphi)(X) := \varphi(\eta(X)), (\eta \circ r)(X) := r(\eta(X)), \\ \eta \circ \ (s(X)d^3X\ ) := \eta^*(s(X)d^3X) \tag{9}$$

for $\eta \in Diff(M), X \in M$ and any $(\varphi; r, s) \in Diff(M) \times (\mathcal{R}(M) \times \Sigma(M))$. Then, taking into account the suitably extended action (55) on the differential-functional manifold G, one can formulate the following easily checkable and crucial for what will follow further proposition.

The functional manifold $G := Diff(M) \times (\mathcal{R}(M) \times \Sigma(M))$ in Eulerian coordinates is a smooth symmetry Banach group $G := Diff(M)(\mathcal{R}(M) \times \Sigma(M))$, equal to the semidirect product of the diffeomorphism group $Diff(M)$ and the direct product $\mathcal{R}(M) \times\ \Sigma(M)$ of abelian functional $\mathcal{R}(M) \simeq \Lambda^0(M)$, density $\Sigma(M) \simeq \Lambda^3(M)$ and one-form $\mathcal{B}(M) \simeq \Lambda^1(M)$ groups, endowed with the following right group multiplication law in Eulerian variables:

$$(\varphi_1; r_1, s_1 d^3x) \circ (\varphi_2; r_2, s_2 d^3x) = \\ = (\varphi_2 \cdot \varphi_1; r_1 + r_2 \cdot \varphi_1, s_1 d^3x + (s_2 d^3x) \cdot \varphi_1) \tag{10}$$

for arbitrary elements $\varphi_1, \varphi_2 \in Diff(M), r_1, r_2 \in \Lambda^0(M)$ and $s_1 d^3x, s_2 d^3x \in \Lambda^3(M)$.

This proposition allows a simple enough interpretation, namely, it means that the adiabatic mixing of the $G \ni (\varphi_2; r_2, s_2 d^3x)$-liquid configuration with the $G \ni (\varphi_1; r_1, s_2 d^3x)$-liquid configuration amounts to summation of their densities and entropies, simultaneously changing the common specific density owing to the fact, that some space of the domain M is already occupied by the first liquid configuration and the second one should be diffeomorphically shifted from this configuration to another free part of the spatial domain M, whose volume is assumed to be fixed and bounded.

The second important observation concerns the variational one-form (7) which can be naturally interpreted as some constraint on the group manifold G for any fixed initial extended Lagrangian configuration $(\eta; \rho_0, \sigma_0 d^3X) \in G$, as it follows from the conditions (2)

$$J_{\eta_t}(X)\rho_t \circ \eta_t(X) := \rho_0(X),\ \sigma_t \circ \eta_t(X) := \sigma_0(X) \tag{11}$$

for all $X \in M$, $\eta_t \in Diff(M)$ and $t \in \mathbb{R}$. In addition, if to determine, owing to (7) and the constraint $\delta\sigma_t(x_t) = 0$ for all $t \in \mathbb{R}$, the specific energy density

$$e_t(\rho_t, \sigma_t) := w_t^{(0)}(\rho_t, \sigma_t) + c_t(\sigma_t) \tag{12}$$

for some still unknown mapping $c_t : \Sigma(M) \to C^\infty(M \times \mathbb{R}; \mathbb{R})$ and the internal potential energy function $w_t^{(0)} : \mathcal{R}(M) \times \Sigma(M) \to C^\infty(M; \mathbb{R})$ of the liquid under regard, the local energy conservation property

$$\frac{d}{dt} \int_{D_t} e_t(\rho_t, \sigma_t)\rho_t(x_t) d^3x_t = -\int_{D_t} < \nabla | p_t^{(0)}(x_t) v_t(x_t) > d^3x_t \tag{13}$$

holds for all $t \in \mathbb{R}$ and the domain $D_t := \eta_t(D) \subset M$, where a smooth submanifold $D \subset M$ is chosen arbitrary and $\eta_t : M \to M$ denotes the corresponding evolution subgroup of the diffeomorphism group $Diff_0(M)$, generated by the Euler evolution equations (6), becomes compatible with constraint (7) iff there holds the following equality:

$$p_t^{(0)}(x_t) = \rho_t(x_t)^2 \partial w_t^{(0)}(\rho_t, \sigma_t)/\partial \rho_t \tag{14}$$

for all $x_t \in M$ and $t \in \mathbb{R}$. In particular, from (13) and (14) the following global internal energy functional

$$H := \int_M [w_t^{(0)}(\rho_t, \sigma_t) + c_t(\sigma_t)]\ \rho_t(x_t) d^3x_t \tag{15}$$

is conserved, that is $dH/dt = 0$ for all $t \in \mathbb{R}$.

As the extended Lagrangian configuration $(\eta;\rho_0,\sigma_0 d^3X) \in G$ is fixed for all whiles of time $t \in \mathbb{R}$ and the dynamical variables $\rho_t \in \mathcal{R}(M)$ and $\sigma_t \in \Sigma(M)$ depend only on the evolution diffeomorphisms $\eta_t \in Diff(M)$, $t \in \mathbb{R}$, it is reasonable to consider the constraint (7) as an element of the cotangent space $T^*_{\eta_t}(Diff(M))$ to the diffeomorphism group $Diff(M)$ at the point $\eta_t \in Diff(M)$ for any $t \in \mathbb{R}$.

Determine first the tangent space $T_\eta(G)$ to the group manifold G at point $(\eta;\rho_0,\sigma_0 d^3X) \in G$, which will be the direct product of the tangent spaces $T_\eta(Diff(M)), T_{\rho_0}(\Lambda^0(M))$ and $T_{\sigma_0 d^3X}(\Lambda^3(M))$. The last two tangent spaces are isomorphic, respectively, to themselves, that is $T_{\rho_0}(\Lambda^0(M) \simeq \Lambda^0(M)$ and $T_{\sigma_0 d^3X}(\Lambda^3(M) \simeq \Lambda^3(M)$ at any $X \in M$. Their adjoint spaces are naturally determined as suitably constructed density and functional spaces on the manifold $M : T^*_{\rho_0}(\Lambda^0(M)) \simeq \Lambda^3(M)$ and $T_{\sigma_0 d^3X}(\Lambda^3(M)) \simeq \Lambda^1(M)$. Concerning the tangent space $T_\eta(Diff(M))$ at a configuration $\eta \in Diff(M)$ we will make use of the construction, devised before in [1,2,21]. Namely, let $\eta \in Diff(M)$ be a Lagrangian configuration and determine the tangent space $T_\eta(Diff(M))$ at $\eta \in Diff(M)$ as the collection of left invariant vectors $\xi_\eta := L_{\eta,*}\xi$ at $\eta \in Diff(M)$, where $L_\eta : Diff(M) \to Dif(M)$ is, by definition, the left shift on the diffeomorphism group $Diff(M)$ and $\xi \in T_{Id}(Diff(M))$ is a tangent vector at the unity $Id \in Diff(M)$. It is obvious that for all reference points $X \in M$ and any smooth curve $\mathbb{R} \ni \tau \to \eta_\tau \in Diff(M)$ of diffeomorphisms the set of right invariant vectors $\xi(X) = (\eta^{-1} \circ d\eta_t/d\tau)(X))|_{\tau=0} \in T_X(M)$ at point $X \in M$ defines a smooth vector field $\xi : M \to T(M)$ on the manifold M. Since, by definition, the tangent space $T_{Id}(Diff(M))$ coincides with the Lie algebra $diff(M)$ of the diffeomorphism group $Diff(M)$, strictly isomorphic to the Lie algebra $\Gamma(T(M))$ of right invariant vector fields on M, the dual space $T^*_{Id}(Diff(M))$ can be naturally determined from the geometric point of view as the space $diff^*(M)$, consisting of analytic functions on $diff(M)$ and coinciding with the set of one-form densities on M :

$$diff^*(M) \simeq \Lambda^1(M) \otimes |\Lambda^3(M)|. \tag{16}$$

Similarly, the cotangent space $T^*_\eta(Diff(M))$ consists of all one-form densities on M over $\eta \in Diff(M)$:

$$T^*_\eta(Diff(M)) = \{\alpha_\eta : M \to T^*(M) \otimes |\Lambda^3(M)| : \alpha_\eta(X) \in T^*_{\eta(X)}(M) \otimes |\Lambda^3(M)|\} \tag{17}$$

subject to the canonical nondegenerate convolution $(\cdot|\cdot)_c$ on $T^*_\eta(Diff(M)) \times T_\eta(Diff(M))$: if $\alpha_\eta \in T^*_\eta(Diff(M)), \xi_\eta \in T_\eta(Diff(M))$, where $\alpha_\eta|_X = \langle\alpha_\eta(X)|dx\rangle \otimes d^3X, \xi_\eta|_X = \langle\xi_\eta(X)|\partial/\partial x\rangle$, then

$$(\alpha_\eta|\xi_\eta)_c := \int_M \langle\alpha_\eta(X)|\xi_\eta(X)\rangle d^3X. \tag{18}$$

The construction above makes it possible to identify the cotangent bundle $T^*_\eta(Diff(M))$ at the fixed Lagrangian configuration $\eta \in Diff(M)$ to the tangent space $T_\eta(Diff(M))$, as much as the tangent space $T(M)$ is endowed with the natural internal tangent bundle metric $\langle\cdot|\cdot\rangle$ at any point $\eta(X) \in M$, identifying $T(M)$ with $T^*(M)$ via the isomorphism $\sharp : T^*(M) \to T(M)$. The latter can be also naturally lifted to $T^*_\eta(Diff(M))$ at $\eta \in Diff(M)$, namely: for any elements $\alpha_\eta, \beta_\eta \in T^*_\eta(Diff(M)), \alpha_\eta|_X = \langle\alpha_\eta(X)|dx\rangle \otimes d^3X$ and $\beta_\eta|_X = \langle\beta_\eta(X)|dx\rangle \otimes d^3X \in T^*_\eta(Diff(M))$ we can define the metric

$$(\alpha_\eta|\beta_\eta) := \int_M \rho_0(X) < \alpha^\sharp_\eta(X)|\beta^\sharp_\eta(X) > d^3X, \tag{19}$$

where, by definition, $\alpha^\sharp_\eta(X) := \sharp(\rho_0(X)^{-1}\langle\alpha_\eta(X)|dx\rangle), \beta^\sharp_\eta(X) := \sharp(\rho_0(X)^{-1}\langle\beta_\eta(X)|dx\rangle) \in T_{\eta(X)}(M)$ for any $X \in M$.

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The diffeomorphism group $Diff(M)$ can be naturally restricted to the factor - group $Diff_0(M) := Diff(M)/Diff_{\rho_0,\sigma_0}(M)$ subject to the stationary normal symmetry subgroup $Diff_{\rho_0,\sigma_0}(M) \subset Diff(M)$, where

$$Diff_{\rho_0,\sigma_0}(M) := \{\varphi \in Diff(M) : \rho_0(X) = J_{\varphi(X)}\rho_0(\varphi(X)), \sigma_0(X) = \sigma_0(\varphi(X))\} \tag{20}$$

for any $X \in M$. Based on the construction above one can proceed to constructing smooth flows and functionals on the specially extended group manifold $G_0 := Diff_0(M)\ (\Lambda^0(M)\times \Lambda^3(M))$ and consider their coadjoint action on the cotangent bundle $T^*_{g_\eta}(G_0), g_\eta := (\eta;\rho_0,\sigma_0) \in G_0$, and relate them some way to the evolution with respect to the Euler equations (6). Moreover, as the cotangent bundle $T^*_{g_\eta}(G_0), g_\eta \in G_0$, is *a priori* endowed with the canonical Poisson structure, one can study both the Hamiltonian flows on it, related with the Euler equations (6), and a hidden geometrical meaning of the differential constraints like (7).

2. Hamiltonian analysis: the adiabatic liquid dynamics

We have observed above that the liquid motion is adequately described by means of the symmetry diffeomorphism group $Diff_0(M)$, acting on the target manifold $M \subset \mathbb{R}^3$, and this way modeling liquid motion, generated by suitable vector fields on $Diff_0(M)$.This also means that the fluid motion strongly depends on the constraint (7) on the cotangent bundle $T^*_{g_\eta}(G_0)$, $g_\eta \in G_0$, and *a priori* possesses the canonical Poisson structure on it. Taking into account that the diffeomorphism group $Diff_0(M)$ acts on the extended group density manifold $G_0 := Diff_0(M)(\Lambda^0(M) \times \Lambda^3(M))$, fixed by the element $(\eta;\rho_0,\sigma_0 d^3X) \in G$, one can suitably construct the canonical Poisson bracket on the cotangent bundle $T^*_{g_\eta}(G_0), g_\eta \in G_0$, using the canonical coordinate variables on it. Namely, let $(\mu_\eta;\rho_0 d^3X,\sigma_0) \in T^*_{g_\eta}(G_0)$, $g_\eta \in G_0$, be coordinates on $T^*_{g_\eta}(G_0)$, where

$$\begin{aligned} \mu_\eta(X) &= \rho_0(X)[\flat V_\eta(X)]d^3X|_{x=\eta(X)} = \\ &= \rho_0(X)\flat v(\eta(X))]J_{\eta^{-1}}(x)d^3x := \rho(x)v(x)d^3x, \\ r_\eta(X) &= \rho_0(X)d^3X = \rho_0(X)d^3X|_{x=\eta(X)} := \rho(x)d^3x, \\ s_\eta(X) &= \sigma_0(X) = \sigma(\eta(X))|_{x=\eta(X)} := \sigma(x) \end{aligned} \tag{21}$$

and we denoted $\flat := \sharp^{-1}$, being suitably represented into the Eulerian spatial variables on $T^*_{g_\eta}(G_0)$ at point $(\eta;\rho,\sigma d^3x) \in G_0$. In particular, the quantities $\mu(x) := \rho(x)v(x)d^3x = (\eta^*\mu_\eta)(X)$, $r(x) := \rho(x)d^3x = (\eta^* r_\eta)(X)$ and $s(x) := \sigma(x) = (\eta^* s_\eta)(X)$ are called, respectively, the Eulerian momentum density, the Eulerian fluid density and entropy variables at point $x = \eta(X) \in M$. The corresponding metric on $T^*_{g_\eta}(G_0)$ is given by the expression

$$\begin{gathered} ((\alpha_{\eta,1};r_{\eta,1}s_{\eta,1})|(\alpha_{\eta,2};r_{\eta,2}s_{\eta,2})) := (\alpha_{\eta,1}|\alpha_{\eta,2})+ \\ +(r_{\eta,1}|r_{\eta,2}) + (s_{\eta,1}|s_{\eta,2}), \end{gathered} \tag{22}$$

where $(\alpha_{\eta,1}|\alpha_{\eta,2})$ for $\alpha_{\eta,1},\alpha_{\eta,2} \in T^*_\eta(Diff_0(M))$ is determined by (19) and for any $r_{\eta,1},r_{\eta,2} \in T^*_\eta(\Lambda^3(M))$ and $s_{\eta,1},s_{\eta,2} \in T^*_\eta(*^0(M))$ one determines, respectively, as

$$(r_{\eta,1}|r_{\eta,2}) := \int_M (\rho_1(x)\rho_2(x))\,d^3x,\ (s_{\eta,1}|s_{\eta,2}) := \int_M (\sigma_1(x)\sigma_2(x))\,d^3x. \tag{23}$$

Consider now the cotangent bundle $T^*_{g_\eta}(G_0)$ at point $g_\eta = (\eta;\rho,\sigma d^3x) \in G_0$ as a smooth manifold endowed with the canonical symplectic structure on it, equivalent to the corresponding canonical Poisson bracket on $T^*_{g_\eta}(G_0)$. Taking into account that the manifold $T^*_{g_\eta}(G_0)$, shifted by the right $R_{\eta^{-1}}$-action to the manifold $T^*_{Id}(G_0)$, $Id \in G_0$, becomes diffeomorphic to the adjoint space $\mathcal{G}^*$ to the Lie

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algebra $\mathcal{G}$ of the group G_0, as there was stated [34–36,53,54] still by S. Lie in 1887, this canonical Poisson bracket on $T^*_\eta(G_0)$ transforms [2,3,29,34,53,54] into the classical Lie-Poisson bracket on the adjoint space $\mathcal{G}^*$. Moreover, the orbits of the group G_0 on $T^*_{g_\eta}(G_0), g_\eta = (\eta; \rho, \sigma d^3x) \in G_0$, transform into the corresponding coadjoint orbits on the adjoint space $\mathcal{G}^*$, generated by elements of the Lie algebra $\mathcal{G}$. To construct this Lie-Poisson bracket, we formulate preliminary the following proposition.

The Lie algebra $\mathcal{G} \simeq \Gamma(M; T(M))(\Lambda^0(M)) \oplus \Lambda^3(M))$ is determined by the following Lie commutator relationships:

$$[(a_1; r_1, s_1), (a_2; r_2, s_2)] = ([a_1, a_2];$$
$$\langle a_1|\nabla r_2\rangle - \langle a_2|\nabla r_1\rangle, \langle\nabla|a_1 s_2\rangle - \langle\nabla|a_2 s_1\rangle) \tag{24}$$

for any vector fields $a_1, a_2 \in diff_0(M) \simeq \Gamma(M; T(M))$ and scalar quantities $r_1, r_2 \in \Lambda^0(M)$ and $s_1, s_2 \in \Lambda^3(M)$ on the manifold M.

Proof. Proof of the commutation relationships (24) easily follows from the group multiplication (10), if to take into account that tangent spaces $T(\Lambda^0(M)) \simeq \Lambda^0(M)$ and $T(\Lambda^3(M)) \simeq (\Lambda^3(M))$. □

As an example, we calculate, for brevity, the Poisson bracket on the cotangent space $T^*_\eta(Diff(\mathbb{T}^n))$ at any $\eta \in Diff(\mathbb{T}^n)$. Consider the cotangent space $T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n)$, the adjoint space to the tangent space $T_\eta(Diff(\mathbb{T}^n))$ of left invariant vector fields on $Diff(\mathbb{T}^n)$ at any $\eta \in Diff(\mathbb{T}^n)$, and take the canonical symplectic structure on $T^*_\eta(Diff(\mathbb{T}^n))$ in the form $\omega^{(2)}(\mu, \eta) := \delta\alpha(\mu, \eta)$, where the canonical Liouville form $\alpha(\mu, \eta) := (\mu|\delta\eta)_c \in \Lambda^1_{(\mu,\eta)}(T^*_\eta(Diff(\mathbb{T}^n)))$ at a point $(\mu, \eta) \in T^*_\eta(Diff(\mathbb{T}^n))$ is defined *a priori* on the tangent space $T_\eta(Diff(\mathbb{T}^n)) \simeq \Gamma(T(M))$ of right-invariant vector fields on the torus manifold $\mathbb{T}^n$. Having calculated the corresponding Poisson bracket of smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T^*_\eta(Diff(\mathbb{T}^n)); \mathbb{R})$ on $T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n), \eta \in Diff(\mathbb{T}^n)$, one can formulate the following proposition.

The Lie-Poisson bracket on the coadjoint space $T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n)$, is equal to the expression

$$\{f, g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c \tag{25}$$

for any smooth functionals $f, g \in C^\infty(\mathcal{G}^*; \mathbb{R})$.

Proof. By definition [1,3] of the Poisson bracket of smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T^*_\eta(Diff(\mathbb{T}^n)); \mathbb{R})$ on the symplectic space $T^*_\eta(Diff(\mathbb{T}^n))$, it is easy to calculate that

$$\{\mu(a), \mu(b)\} := \delta\alpha(X_a, X_b) =$$
$$= X_a(\alpha|X_b)_c - X_b(\alpha|X_a)_c - (\alpha|[X_a, X_b])_c, \tag{26}$$

where $X_a := \delta(\mu|a)_c/\delta\mu = a \in diff(\mathbb{T}^n), X_b := \delta(\mu|b)_c/\delta\mu = b \in diff(\mathbb{T}^n)$. Since the expressions $X_a(\alpha|X_b)_c = 0$ and $X_b(\alpha|X_a)_c = 0$ owing the right-invariance of the vector fields $X_a, X_b \in T_\eta(Diff(\mathbb{T}^n))$, the Poisson bracket (25) transforms into

$$\{(\mu|a)_c, (\mu|b)_c\} = -(\alpha|[X_a, X_b])_c =$$
$$= (\mu|[b, a])_c = (\mu|[\delta(\mu|b)_c/\delta\mu, \delta(\mu|a)_c/\delta\mu])_c \tag{27}$$

for all $(\mu, \eta) \in T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n), \eta \in Diff(\mathbb{T}^n)$ and any $a, b \in diff(\mathbb{T}^n)$. The Poisson bracket (25) is easily generalized to

$$\{f, g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c \tag{28}$$

for any smooth functionals $f, g \in C^\infty(\mathcal{G}^*; \mathbb{R})$, finishing the proof. □

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Proceed now to the Grassmann algebra $\Lambda(M)$ and the Hodge [52] star-isomorphism $*: \Lambda(M) \to \Lambda(M)$ subject to the usual metric on the tangent space $T(M)$ and determine the adjoint space to the abelian subalgebra $\mathcal{R}(M) \oplus \Sigma(M) \simeq \Lambda^0(M) \oplus \Lambda^3(M)$ as the space $*\Lambda^3(M) \oplus *\Lambda^0(M)$ with respect to the following scalar product on $\Lambda(M)$:

$$(\alpha^{(n)}|\beta^{(m)}) := \delta_{mn} \int_M (\alpha^{(n)} \wedge *\beta^{(m)}) \tag{29}$$

for any $\alpha^{(n)}, \beta^{(m)} \in \Lambda(M), m, n = \overline{0,3}$. Then the adjoint space $\mathcal{G}^*$, owing to the expressions (22) and (2), is described by means of the Eulerian variables $(\mu; \rho d^3x, \sigma) \in \mathcal{G}^* \simeq (\Lambda^1(M) \otimes |\Lambda^3(M)|)(\Lambda^3(M) \oplus \Lambda^0(M))$. The latter makes it possible to to calculate the corresponding Lie-Poisson bracket on the adjoint space $\mathcal{G}^*$ at a point $l := (\mu; \rho d^3x, \sigma) \in \mathcal{G}^*$, generalizing the Poisson bracket (27):

$$\begin{gathered}\{f,g\}(l) = (l|[\delta g/\delta l, \delta f/\delta l])_c = \\ = \int_M d^3x \left\langle m \left| \left[\left\langle \frac{\delta f}{\delta m} | \nabla \right\rangle \frac{\delta g}{\delta m} - \left\langle \frac{\delta g}{\delta m} | \nabla \right\rangle \frac{\delta f}{\delta m} \right] \right\rangle + \right. \\ + \int_M \rho d^3x \left[\left\langle \frac{\delta f}{\delta m} \left| \nabla \frac{\delta g}{\delta \rho} \right\rangle - \left\langle \frac{\delta g}{\delta m} \right| \nabla \frac{\delta f}{\delta \rho} \right\rangle \right] + \\ + \int_M \sigma \left[\left\langle \nabla \left| \frac{\delta f}{\delta m} \frac{\delta g}{\delta \sigma} \right\rangle - \left\langle \nabla \right| \frac{\delta g}{\delta m} \frac{\delta f}{\delta \sigma} \right\rangle \right] d^3x\end{gathered} \tag{30}$$

for any smooth functional $f, g \in C^\infty(\mathcal{G}^*; \mathbb{R})$, where we put, by definition, $\mu(x) := < m(x)|dx > d^3x$, $m(x) = \rho(x)v(x) \in T^*(M)$ for all $x \in M$ and any $t \in \mathbb{R}$.

Return now to the constraint (7) in the following variational form:

$$\delta e_t(\rho_t, \sigma_t)/\delta t = T_t(x_t)\delta\sigma_t(x_t)/\delta t + p_t^{(0)}(x_t)\rho_t^{-2}(x_t)\delta\rho_t(x_t)/\delta t, \tag{31}$$

which should hold at any $x_t \in M$ for all $t \in \mathbb{R}$. Insomuch as, owing to the Euler equations (6), the full (convective) derivative $\delta\sigma_t(x_t)/\delta t = 0$ at any $x_t \in M$ for all $t \in \mathbb{R}$, one checks once more that the expression (12) holds at any $x_t \in M$ for all $t \in \mathbb{R}$. To determine the energy density function (12), we consider the Euler equations (6) and check their Hamiltonian structure subject to the Poisson bracket (30), that is the existence of a Hamiltonian functional $H : \mathcal{G}^* \to \mathbb{R}$, for which

$$\frac{\partial}{\partial t}(m; \rho, \sigma) = \{H, (m, \rho, \sigma)\} \tag{32}$$

at any element $l = (m := \rho v; \rho, \sigma) \in \mathcal{G}^*$. By means of easy calculations one obtains from the system (32) the variational gradient vector

$$\delta H(l)/\delta l = (m\rho^{-1}; -|m|^2/(2\rho^2) + w^{(0)}(\rho, \sigma) + \rho\partial w^{(0)}(\rho, \sigma)/\partial\rho, \rho\partial w^{(0)}(\rho, \sigma)/\partial\sigma), \tag{33}$$

from which one derives [6,41,46] via the Volterra homotopy mapping

$$H = \int_0^1 (\delta H(\lambda l)/\delta l|l)_c d\lambda \tag{34}$$

the exact Hamiltonian expression

$$H = \int_M (|m|^2/(2\rho) + \rho w^{(0)}(\rho, \sigma)] d^3x, \tag{35}$$

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coinciding with the expression (15) at $c(\sigma) := |m|^2/(2\rho^2) = |v|^2/2$, as $m := \rho v$ for $v \in T(M)$. Thus, we obtain the internal energy density functional (12) as

$$e_t(\rho_t, \sigma_t) = |v_t|^2/2 + w_t^{(0)}(\rho_t, \sigma_t), \tag{36}$$

for all $\rho := \rho_t \in \mathcal{R}(M), \sigma := \sigma_t \in \Sigma(M)$ and $v_t \in T(M)$, satisfying simultaneously both the constraint (7) and the Euler evolution equations (6) for all $t \in \mathbb{R}$. Moreover, from the condition (13) one easily finds [21] the following important local differential relationship:

$$\begin{aligned} \partial[\rho_t(x_t)e_t(\rho_t,\sigma_t)]/\partial t + &< \nabla|\rho_t(x_t)v_t(x_t)(e_t(\rho_t,\sigma_t) + \\ +\rho_t(x_t)\partial w_t^{(0)}(\rho_t,\sigma_t)/\partial\rho_t\ ) &> = 0 \end{aligned} \tag{37}$$

satisfied for all $x_t \in M$ and $t \in \mathbb{R}$, also confirming the energy conservation (35)

3. Hamiltonian analysis: the isotermal liquid dynamics

Consider a liquid motion governed by the following Euler equations governed by the Euler equations

$$\begin{gathered} \partial v/\partial t + \langle v|\nabla\rangle v + \rho^{-1}\nabla p^{(0)} = 0, \\ \partial\rho/\partial t + \langle\nabla|\rho v\rangle = 0, \partial T/\partial t + \langle v|\nabla T\rangle = 0, \end{gathered} \tag{38}$$

and describing the ideal compressible and isotermal motion of an ideal compressible and adiabatic fluid in a spatial domain $M \subset \mathbb{R}^3$, as the temperature $T_t(x_t) = T_0(x_t)$ at any evolution point $x_t := \eta_t(X) \in M$ for all $X \in M$ and $t \in \mathbb{R}$. The evolution (38) is considered to be *a priori* thermodynamically quasi-stationary, that is the following, *infinitesimal convective* energy relationship

$$\delta\tilde{h}_t(\rho_t, T_t) \ = -\sigma_t(x_t)\ \delta T_t\ + p_t^{(0)}(x_t)\ \rho_t^{-2}\delta\rho_t \tag{39}$$

holds for all densities $\rho_t \in \mathcal{R}(M)$, temperature $T_t \in \mathcal{T}(M)$ and specific entropy $\sigma_t \in \Sigma(M)$, where $\tilde{h} : \mathcal{R}(M) \times \mathcal{T}(M) \to \mathbb{R}$ denotes the corresponding internal specific fluid "energy" and the variation sign "δ" means the change subject to both the temporal variable $t \in \mathbb{R}$ and the spatial variable $x_t \in M$. Under the imposed *isotermal condition* $\delta T_t = 0$ the expression (39) transforms into

$$\tilde{h}_t(\rho_t, T_t) = \ |v_t|^2/2 + \tilde{w}_t^{(0)}(\rho_t, T_t), \tag{40}$$

where $\tilde{w}_t^{(0)}(\rho_t, T_t) := w_t^{(0)}(\rho_t, \sigma_t)|_{\sigma_t := \tilde{\sigma}(\rho_t, T_t)} - T_t\sigma_t(\rho_t, T_t)$, is the specific potential liquid energy for *the isotermal flow*, determined at $\sigma_t := \sigma_t(\rho_t, T_t)$, solving the functional relation $T_t = \partial w_t^{(0)}(\rho_t, \sigma_t)/\partial\sigma_t \in \mathcal{T}(M)$ subject to the entropy argument $\sigma_t \in \Sigma(M)$, if the condition $\partial^2 w_t^{(0)}(\rho_t, \sigma_t)/\partial\sigma_t^2 \neq 0$ holds for all densities $\rho_t \in \mathcal{R}(M)$ and $t \in \mathbb{R}$.

Observe now that the third equation of (38) is exactly equivalent to the internal fluid kinetic energy conservation integral relationship

$$\frac{d}{dt}\int_{D_t} \rho_t(x_t)T_t(x_t)d^3x_t = 0 \tag{41}$$

over the domain $D_t := \eta_t(D) \subset M$, where a smooth submanifold $D = D_t|_{t=0} \subset M$ is chosen arbitrary and $\eta_t : M \to M, t \in \mathbb{R}$, denotes the corresponding evolution subgroup of the diffeomorphism group $Diff_0(M)$, generated by the Euler evolution equations (38). The relationship (41) simply means that if the density function $\rho_t \in \mathcal{R}(M)$ transforms under diffeomorphism group $Diff_0(M)$ action as the abelian functional group $\mathcal{R}(M) \simeq \Lambda^0(M)$, the corresponding transformation of the temperature $T_t \in \mathcal{T}(M)$ is induced by the diffeomorphism group $Diff_0(M)$ action on the related abelian group

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$\mathcal{T}(M) \simeq \Lambda^3(M)$. Concerning the energy density (40) one easily obtains the following differential relationship:

$$\partial[\rho_t(x_t)\tilde{h}_t(\rho_t, T_t)]/\partial t + \langle\nabla|\rho_t(x_t)v_t(x_t)\left[\tilde{h}_t(\rho_t, T_t)\rangle + \rho_t\partial\tilde{w}_t^{(0)}(\rho_t, T_t)/\partial\rho_t\right]\rangle = 0, \tag{42}$$

satisfied for all $t \in \mathbb{R}$. As a simple consequence of the relationship (42) one obtains that the following functional

$$\tilde{H} = \int_{D_t} \rho_t(x_t)\tilde{h}_t(\rho_t, T_t)d^3x_t \tag{43}$$

is conserved over the domain $D_t := \eta_t(D) \subset M$, where a smooth submanifold $D = D_t|_{t=0} \subset M$ is chosen arbitrary.

Similarly to reasonings of Section 2, one can construct now the differential-functional group space $Diff(M) \times (\mathcal{R}(M) \times \mathcal{T}(M))$ and formulate the following easily checkable proposition.

The differential-functional group functional manifold $Diff(M) \times (\mathcal{R}(M) \times \mathcal{T}(M))$ in Eulerian coordinates is a smooth Banach group $G := Diff(M)(\mathcal{R}(M) \times \mathcal{T}(M))$, equal to the semidirect product of the diffeomorphism group $Diff(M)$ and the direct product $\mathcal{R}(M) \times \mathcal{T}(M)$ of abelian functional $\mathcal{R}(M) \simeq \Lambda^0(M)$ and density $\mathcal{T}(M) \simeq \Lambda^3(M)$ groups, endowed with the following group multiplication law:

$$\begin{gathered}(\varphi_1; r_1, \tau_1 d^3x\ ) \circ (\varphi_2; r_2, \tau_2 d^3x) = \\ +(\varphi_2 \cdot \varphi_1; r_1 + r_2 \cdot \varphi_1, \tau_1 d^3x + (\tau_2 d^3x) \cdot \varphi_1)\end{gathered} \tag{44}$$

for arbitrary elements $\varphi_1, \varphi_2 \in Diff(M), r_1, r_2 \in \Lambda^0(M)$ and $\tau_1 d^3x, \tau_2 d^3x \in \Lambda^3(M)$.

This proposition allows a simple enough interpretation, namely, it means that the adiabatic mixing of the $G \ni (\varphi_2; r_2, \tau_2 d^3x)$ liquid configuration with the $G \ni (\varphi_1; r_1,\ \tau_1 d^3x)$ liquid configuration amounts to summation of their densities, simultaneously changing the common specific kinetic energy, proportional [8,23,37] to the liquid temperature, owing to the fact, that some space of the domain M is already occupied by the first liquid configuration and the second one should be diffeomorphically shifted from this configuration to another free part of the spatial domain M with fixed and bounded volume. The diffeomorphism group $Diff(M)$ can be naturally restricted to the factor - group $Diff_0(M) := Diff(M)/Diff_{\rho_0,T_0}(M)$ subject to the stationary normal symmetry subgroup $Diff_0(M) := Diff_{\rho_0,T_0}(M) \subset Diff(M)$, where

$$Diff_{\rho_0,T_0}(M) := \{\varphi \in Diff(M) : \rho_0(X) = J_{\varphi(X)}\rho_0(\varphi(X)), T_0(X) = T_0(\varphi(X))\} \tag{45}$$

for any $X \in M$. Based on the construction above one can proceed to studying the extended Banach group $G := Diff_0(M)(\Lambda^0(M) \times \Lambda^3(M))$ action on the cotangent bundle $T^*_{g_\eta}(G)$, $g_\eta := (\eta; \rho_0, T_0) \in G_0$, generated by the fluid evolution with respect to the Euler equations (38). The related fluid motion is naturally modelled by means of the coadjoint action of the corresponding Lie algebra $\mathcal{G} \simeq T_{g_\eta}(G_0) \simeq \Gamma(M; T(M))(\Lambda^0(M) \oplus \Lambda^3(M))$ of the group G_0 at $g_\eta = Id \in G_0$ on its adjoint space $\mathcal{G}^* \simeq (\Lambda^1(M) \otimes \Lambda^3(M))(*\Lambda^0(M) \oplus *\Lambda^3(M)) = (\Lambda^1(M) \otimes \Lambda^3(M))\ (\Lambda^3(M) \oplus \Lambda^0(M))$.

The related Lie structure on $\mathcal{G}$ easily ensues from the action (44):

$$\begin{gathered}[(a_1; r_1, \tau_1), (a_2; r_2, \tau_2)] = ([a_1, a_2]; \\ \langle a_1|\nabla r_2\rangle - \langle a_2|\nabla r_1\rangle, \langle\nabla|a_1\tau_2\rangle - \langle\nabla|a_2\tau_1\rangle)\end{gathered} \tag{46}$$

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for any representative elements $(a_1; r_1, \tau_1)$ and $(a_2; r_2, \tau_2) \in \mathcal{G}$. Moreover, as the cotangent bundle $T^*_{g_\eta}(G_0), g_\eta = Id \in G_0$, being diffeomorphic to the adjoint space $\mathcal{G}^*$ to the Lie algebra $\mathcal{G}$ of the Banach group G_0, is *a priori* endowed with the canonical Lie-Poisson structure

$$
\begin{gathered}
\{f, g\}(l) = (l|[\delta g/\delta l, \delta f/\delta l])_c = \\
= \int_M d^3x \left\langle m \left| \left[\left\langle \frac{\delta f}{\delta m} |\nabla \right\rangle \frac{\delta g}{\delta m} - \left\langle \frac{\delta g}{\delta m} |\nabla \right\rangle \frac{\delta f}{\delta m} \right] \right. \right\rangle + \\
+ \int_M \rho d^3x \left[\left\langle \frac{\delta f}{\delta m} \left| \nabla \frac{\delta g}{\delta \rho} \right. \right\rangle - \left\langle \frac{\delta g}{\delta m} \left| \nabla \frac{\delta f}{\delta \rho} \right. \right\rangle \right] + \\
+ \int_M T \left[\left\langle \nabla \left| \frac{\delta f}{\delta m} \frac{\delta g}{\delta T} \right. \right\rangle - \left\langle \nabla \left| \frac{\delta g}{\delta m} \frac{\delta f}{\delta T} \right. \right\rangle \right] d^3x
\end{gathered}
\tag{47}
$$

for any smooth functional $f, g \in C^\infty(\mathcal{G}^*; \mathbb{R})$, where we put, by definition, an element $l := (m; \rho, T) \simeq (\mu; \rho d^3x, T) \in \mathcal{G}^*$, $\mu(x) := < m(x)|dx > d^3x$, $m(x) = \rho(x)v(x) \in T^*(M)$ for all $x \in M$ and $t \in \mathbb{R}$, one can easily check that the flow (38) is Hamiltonian:

$$
dl/dt = \{\tilde{H}, l\} \tag{48}
$$

subject to the adjusted Hamiltonian functional (43):

$$
\tilde{H} := \int_M h_t(\rho_t, T_t) d^3x_t = \int_M \rho_t(|m_t|^2/2\rho_t^2 + \tilde{\varpi}_t^{(0)}(\rho_t, T_t)) d^3x. \tag{49}
$$

satisfying the conservative condition $d\tilde{H}/dt = 0$ for all $t \in \mathbb{R}$, following simultaneously both from (48) and from the differential relationship (42).

4. Hamiltonian analysis: the adiabatic magneto-hydrodynamic superfluid motion

4.1. Geometric description

We start with considering a quasi-neutral superfluid contained in a domain $M \subset \mathbb{R}^3$ and interacting with a "frozen" sourceless magnetic field $B \in \mathcal{B}(M) \subset C^\infty(M; \mathbb{E}^3)$, satisfying the superconductivity conditions

$$
\tilde{E} := E + v \times B = 0, \ \partial E/\partial t = \nabla \times B, \tag{50}
$$

where $\tilde{E} : M \to \mathbb{E}^3$ is the internal net superfluid electric field, $E = -\partial A/\partial t : M \to \mathbb{E}^3$ and $B = \nabla \times A : M \to \mathbb{E}^3$ are the internal electric and magnetic fields, respectively, generated by the corresponding magnetic vector field potential $A : M \to \mathbb{E}^3$, $v : M \longrightarrow T(M)$ is the superfluid velocity and "$\times$" denotes the usual vector product in the Euclidean space $\mathbb{E}^3$. The following natural boundary conditions $\langle n|v\rangle|_{\partial M} = 0$ and $\langle n|B\rangle|_{\partial M} = 0$ are imposed on the superfluid flow, where $n \in T^*(M)$ is the vector normal to the boundary ∂M, which is considered to be almost everywhere smooth.

Then in adiabatic magnetohydrodynamics (MHD) quasi-neutral superconductive superfluid motion is described by the following system of evolution equations:

$$
\begin{gathered}
\partial v/\partial t + \langle v|\nabla\rangle v + \rho^{-1}\nabla p - \rho^{-1}(\nabla \times B) \times B = 0, \\
\partial \rho/\partial t + \langle \nabla|\rho v\rangle = 0, \partial \sigma/\partial t + \langle u|\nabla \sigma\rangle = 0, \partial B/\partial t = \nabla \times (v \times B),
\end{gathered}
\tag{51}
$$

where, as before, $\rho := \rho_t \in \mathcal{R}(M)$ is the superfluid density, $B := B_t : M \longrightarrow \mathbb{E}^3$ is the "frozen" into the superfluid magnetic field, $p := p_t : M \longrightarrow \mathbb{R}$ is the internal liquid pressure and $\sigma := \sigma_t : M \longrightarrow \mathbb{R}$ is

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the specific superfluid entropy at time $t \in \mathbb{R}$. The latter is related with the internal MHD superfluid specific energy function $e = e_t(\rho_t, \sigma_t)$ owing to the first thermodynamic law:

$$T_t(\rho_t, \sigma_t)\ \delta\sigma_t = \delta e_t(\rho_t, \sigma_t) - p_t\rho_t^{-2}\delta\rho_t, \tag{52}$$

satisfied for any admissible variations of the phase space parameters $\rho_t \in \mathcal{R}(M), \sigma_t \in \Sigma(M)$, where $T_t = T_t(\rho_t, \sigma_t)$ is the internal absolute temperature in the superfluid for $t \in \mathbb{R}$. The adiabatic condition $\delta\sigma_t(x_t) = 0$, where $x_t := \eta_t(X) \in M$ for all $X \in M$ and the related to (51) evolution diffeomorphism $\eta_t \in Diff(M), t \in \mathbb{R}$, entails the following expression for the specific internal energy

$$e_t(\rho_t, \sigma_t) = \ w_t^{(0)}(\rho_t, \sigma_t) + c_t(\sigma_t, B_t), \tag{53}$$

where $w_t^{(0)}\ : \mathcal{R}(M) \times \Sigma(M)\ \rightarrow C^\infty(M;\mathbb{R})$ is the corresponding internal potential specific energy density and $c_t : \Sigma(M) \times \mathcal{B}(M) \rightarrow C^\infty(M;\mathbb{R})$ is some still unknown function, depending in general on the imposed magnetic field $B_t : M \longrightarrow \mathbb{E}^3, t \in \mathbb{R}$.

Let us now analyze, as before, the mathematical structure of quantities $(\rho_t, \sigma_t, B_t) \in \mathcal{R}(M) \times \Sigma(M) \times \mathcal{B}(M)$ as the *physical observables* subject to their evolution (51) with respect to the group diffeomorphisms $\eta_t \in Diff(M), t \in \mathbb{R}$, generated by the liquid motion vector field $dx_t/dt = v_t(x_t), x_t := \eta_t(X), t \in \mathbb{R}, X \in M:$

$$\begin{gathered}\mathcal{L}_{d/dt}(\langle\rho_t v_t|dx_t\rangle d^3x_t) = [-dp_t^{(0)} + \rho_t^{-1}d|v_t|^2/2 + \langle B_t|\nabla\rangle\langle B_t|dx_t\rangle)]\rho_t d^3x_t, \\ \mathcal{L}_{d/dt}(\rho_t d^3x_t) = 0, \quad \mathcal{L}_{d/dt}\sigma_t = 0, \quad \mathcal{L}_{d/dt}\,(*\langle B_t|dx_t\rangle) = 0,\end{gathered} \tag{54}$$

where $\mathcal{L}_{d/dt} : \Lambda(M) \rightarrow \Lambda(M)$ denotes the corresponding Lie derivative with respect to the vector field $d/dt := \partial/\partial t + \langle v_t|\nabla\rangle \in \Gamma(M \times \mathbb{R}; T(M)), t \in \mathbb{R}$. The relationships (54) mean that the the space of physical observables, being by definition, the adjoint space $\mathcal{G}_{em}^* :=\ \Lambda^1(M)d^3x \times (\Lambda^3(M) \oplus \Lambda^0(M) \oplus \Lambda^2(M))$ to the extended configuration space equal to $\mathcal{G}_{em} := diff(M) \times (\Lambda^0(M) \oplus \Lambda^3(M) \oplus \Lambda^1(M)) \simeq T_{Id}(G_{em})$, the tangent space at the identity Id to the extended differential-functional group manifold $G_{em} := Diff(M) \times \Lambda^0(M) \times\Lambda^3(M) \times \Lambda^1(M) \simeq Diff(M) \times \mathcal{R}(M) \times\Sigma(M) \times \mathcal{B}(M)$, where we have naturally identified the abelian group product $\Lambda^0(M) \times\Lambda^3(M) \times \Lambda^1(M)$ with its direct tangent space sum $T(\Lambda^0(M)) \oplus T(\Lambda^3(M)) \oplus T(\Lambda^1(M))$.

Consider now the constructed differential-functional group manifold G_{em} in Eulerian variables, on which one naturally acts the $Diff(M)$-group $Diff(M) \times G_{em} \rightarrow G_{em}$ the standard way:

$$\begin{gathered}(\eta\circ\varphi)(X) := \varphi(\eta(X)), (\eta\circ r)(X) := r(\eta(X)), \\ \eta\circ\ (s(X)d^3X\ ) :=\ \eta^*(s(X)d^3X\ ), \\ \eta\circ\ \langle b(X)|dX\rangle\ :=\ \eta^*\langle b(X)|d^3X\rangle\end{gathered} \tag{55}$$

for $\eta \in Diff(M), X \in M$ and any $(\varphi; r, s, b) \in Diff(M) \times \mathcal{R}(M) \times \Sigma(M) \times \mathcal{B}(M)$. Then, taking into account the suitably extended action (55) on the differential-functional manifold G_{em}, one can formulate the following easily checkable and crucial for what will follow further proposition.

The differential-functional group manifold $G_{em} := Diff(M) \times \mathcal{R}(M)\times\ \Sigma(M)\ \times\mathcal{B}(M)$ in Eulerian coordinates is a smooth symmetry Banach group $G_{em} := Diff(M)(\mathcal{R}(M) \times \Sigma(M) \times \mathcal{B}(M))$, equal to the semidirect product of the diffeomorphism group $Diff(M)$ and the direct product $\mathcal{R}(M)\times \Sigma(M)\times\ \mathcal{B}(M)$ of abelian functional $\mathcal{R}(M) \simeq \Lambda^0(M)$, density $\Sigma(M) \simeq \Lambda^3(M)$ and one-form $\mathcal{B}(M)\ \simeq \Lambda^1(M)$ groups, endowed with the following group multiplication law in Eulerian variables:

$$\begin{gathered}(\varphi_1; r_1, s_1d^3x, \langle b_1|dx\rangle) \circ (\varphi_2; r_2, s_2d^3x, \langle b_2|dx\rangle) = \\ = (\varphi_2\cdot\varphi_1; r_1 + r_2\cdot\varphi_1, s_1d^3x + (s_2d^3x)\cdot\varphi_1, \langle b_1|dx\rangle + \langle b_2|dx\rangle\circ\varphi_1)\end{gathered} \tag{56}$$

for arbitrary elements $\varphi_1, \varphi_2 \in Diff(M), r_1, r_2 \in \Lambda^0(M),\ s_1d^3x, s_2d^3x \in \Lambda^3(M)$ and $\langle b_1|dx\rangle, \langle b_2|dx\rangle \in \Lambda^1(M)$.

Thus, one can proceed to studying the corresponding coadjoint action of the Lie algebra $\mathcal{G}_{em} \simeq T_{Id}(G_{em}), Id \in G_{em}$, on the adjoint space $\mathcal{G}^*_{em}$. As the Lagrangian configuration $\eta_0 \in Diff(M)$ and the entropy $\sigma_0 \in \Sigma(M)$ are assumed to be invariant under the Banach diffeomorphism group action $Diff(M)$, the resulting group action can be reduced to the factor-group $Diff_0(M) := Diff(M)/Diff_{\eta_0,\sigma_0}(M)$ action on the semidirect group product $G_{em,0} := Diff_0(M)\,(\mathcal{R}(M) \times \Sigma(M) \times \mathcal{B}(M))$. Based on the multiplication law (56) one easily calculates the following Lie algebra commutation relationships:

$$
\begin{gathered}
[(a_1;r_1,s_1,b_1),(a_2;r_2,s_2,b_2)] = ([a_1,a_2];\langle a_1|\nabla r_2\rangle - \\
-\langle a_2|\nabla r\rangle, \langle\nabla|a_1 b_2\rangle - \langle\nabla|a_2 s_1\rangle, \langle a_1|\nabla\rangle b_2 - \\
-\langle a_2|\nabla\rangle b_1 + \langle b_2|\circ\nabla a_1\rangle - \langle b_1|\circ\nabla a_2\rangle)
\end{gathered} \tag{57}
$$

for any elements $a_1,a_2 \in diff(M) \simeq T(M), r_1,r_2 \in \mathcal{R}(M) \simeq \Lambda^0(M)$, $s_1,s_2 \in \Sigma(M) \simeq \Lambda^3(M)$ and $b_1,b_2 \in \mathcal{B}(M) \simeq \Lambda^1(M)$.

The adjoint space to the semidirect product Lie algebra $\mathcal{G}_{em,0} = diff(M)(\mathcal{R}(M) \oplus \Sigma(M) \oplus \mathcal{B}(M))$ can be, naturally, written symbolically as the space $\mathcal{G}^*_{em,0} = (\Lambda^1(M) \otimes \Lambda^3(M)) \times (*\Lambda^0(M) \oplus *\Lambda^3(M) \oplus *\Lambda^1(M)) = diff^*(M) \times (\Lambda^3(M) \oplus \Lambda^0(M) \oplus \Lambda^2(M))$, where as before, the mapping $* : \Lambda(M) \to \Lambda(M)$ denotes the Hodge isomorphism. Then, taking into account the the adjoint space $\mathcal{G}^*_{em,0}$ to the Lie algebra $\mathcal{G}_{em,0}$ is endowed with the following [19–21,28,34,51] canonical Lie-Poisson bracket

$$
\begin{aligned}
&\{f,g\} := \int_M \langle m|\langle \tfrac{\delta f}{\delta m}|\nabla\rangle \tfrac{\delta g}{\delta m} - \langle \tfrac{\delta g}{\delta m}|\nabla\rangle \tfrac{\delta f}{\delta m}\rangle d^3x + \\
&+\int_M \rho\left(\langle \tfrac{\delta f}{\delta m}|\nabla \tfrac{\delta g}{\delta \rho}\rangle - \langle \tfrac{\delta g}{\delta m}|\nabla \tfrac{\delta f}{\delta \rho}\rangle\right) d^3x + \int_M \sigma\langle\nabla|(\tfrac{\delta f}{\delta m}\tfrac{\delta g}{\delta \sigma} - \tfrac{\delta g}{\delta m}\tfrac{\delta f}{\delta \sigma})\rangle d^3x + \\
&+\int_M \left(\langle B|\langle \tfrac{\delta f}{\delta m}|\nabla\rangle \tfrac{\delta g}{\delta B} - \left\langle \tfrac{\delta g}{\delta m}|\nabla\right\rangle \tfrac{\delta f}{\delta B}\rangle + \langle \tfrac{\delta f}{\delta B}, \langle B|\nabla\rangle \tfrac{\delta g}{\delta m}\rangle - \langle \tfrac{\delta g}{\delta B}, \langle B|\nabla\rangle \tfrac{\delta f}{\delta m}\rangle\right) d^3x
\end{aligned} \tag{58}
$$

for any smooth functionals $f,g \in \mathcal{D}(\mathcal{G}^*_{em,0})$ on the adjoint space $\mathcal{G}^*$, where, as before, we denoted by $m := \rho v \in T^*(M)$ the specific momentum of the superfluid. The bracket (58) naturally ensues from the canonical symplectic structure on the cotangent phase space $T^*(G_{em,0})$, as it was before demonstrated in Section 2.

Write down now the first two equations of the Euler MHD system (51) as the local fluid mass and momentum conservation laws in the integral Ampere–Newton form

$$
\begin{gathered}
\tfrac{d}{dt}\int_{D_t} \rho_t d^3x_t = 0, \quad \tfrac{d}{dt}\int_{D_t} \rho_t v_t\, d^3x_t + \\
+\int_{\partial D_t} p_t^{(0)}(x_t) d^2S_t - \int_{D_t} \langle B_t(x_t)|\nabla\rangle B_t(x_t) d^3x_t = 0,
\end{gathered} \tag{59}
$$

which is completely equivalent to the relationships (54) and where $p_t : M \to \mathbb{R}_+$ is the net internal superfluid pressure, $(\nabla \times B_t(x_t)) \times B_t(x_t) : M \to C^\infty(M;\mathbb{E}^3)$ is the spatially distributed Lorentz force on the superfluid, d^2S_t is the respectively oriented surface measure on the boundary ∂D_t for the domain $D_t := \eta_t(D) \subset M, t \in \mathbb{R}$, and a smooth submanifold $D \subset M$ is chosen arbitrary. Taking into account that $(\nabla \times B_t(x_t)) \times B_t(x_t) = \langle B_t|\nabla\rangle B_t - \nabla\langle B_t|B_t\rangle/2$ for any $B_t \in \mathcal{B}(M)$, the second integral relationship (59) becomes equivalent to the following:

$$
\partial v_t/\partial t + \langle v_t|\nabla\rangle v_t + \rho_t^{-1}\nabla p_t^{(0)}(\rho_t,\sigma_t) - \rho_t^{-1}\langle B_t|\nabla\rangle B_t = 0, \tag{60}
$$

where we have represented the internal superfluid pressure quantity as

$$
p_t(x_t) := p_t^{(0)}(\rho_t,\sigma_t) - \langle B_t|B_t\rangle/2 \tag{61}
$$

for some mapping $p_t^{(0)}:\mathcal{R}(M)\times\Sigma(M)\to C^\infty(M;\mathbb{R})$, strictly depending only on the internal liquid configuration $\eta_t\in Diff(M)$ for all $t\in\mathbb{R}$.

Based on the Poisson bracket expression (58), one can now easily determine the Hamiltonian function $H:M\to\mathbb{R}$, corresponding to the Euler evolution equations (51) on the adjoint space $\mathcal{G}^*$:

$$\begin{gathered} H=\int_M\rho_t(|m_t|^2/(2\rho_t^2)+\ w_t^{(0)}(\rho_t,\sigma_t)+ \\ +|B_t|^2/(2\rho_t))dx_t^3:=\int_M\rho(x_t)e_t(\rho_t,\sigma_t)d^3x_t, \end{gathered} \tag{62}$$

where the quantity

$$\begin{aligned} e_t(\rho_t,\sigma_t) &= |m_t|^2/(2\rho_t^2)+\ w_t^{(0)}(\rho_t,\sigma_t)+ \\ +|B_t|^2/(2\rho_t) &: =|m_t|^2/(2\rho_t^2)+\ w_t(\rho_t,\sigma_t;t) \end{aligned} \tag{63}$$

denotes the specific internal superfluid energy, modified by means of the "frozen" magnetic field $B_t\in\mathcal{B}(M),t\in\mathbb{R}$, replacing the before defined in 2 internal specified potential energy $w_t^{(0)}(\rho_t,\sigma_t)$ by the shifted specified potential energy quantity $w_t(\rho_t,\sigma_t,t):=w_t^{(0)}(\rho_t,\sigma_t)+|B_t|^2/(2\rho_t)$. In particular, the equation (60) reduces to the equivalent Hamilton expression

$$\partial m_t/\partial t=\{H,m_t\} \tag{64}$$

for $m_t\in T^*(M)\simeq diff^*(M)$ and all $t\in\mathbb{R}$. It is also seen that if $B_t\to 0$ uniformly with respect to time $t\in\mathbb{R}$, the internal energy expression (63) brings about that (36). Recall now that the following quasi-stationary first thermodynamic energy conservation law

$$\delta e_t(\rho_t,\sigma_t)=\rho_t^{-2}p_t(x_t)\delta\rho_t+T_t(x_t)\delta\sigma_t \tag{65}$$

holds for all admitted superfluid variations $\delta\rho_t\in\mathcal{R}(M)$ and $\delta\sigma_t\in\Sigma(M),t\in\mathbb{R}$. As, by adiabatic assumption, $\delta\sigma_t=0$ for all $t\in\mathbb{R}$, for the internal pressure one easily obtains the expression $p_t(x_t)=\rho_t^2\partial w_t^{(0)}(\rho_t,\sigma_t)/\partial\rho_t-\langle B_t|B_t\rangle/2$, exactly coinciding with that of (61).

The Hamiltonian function (62) satisfies evidently the conservation condition $dH/dt=0$ for all $t\in\mathbb{R}$. To check this directly, it is enough to observe [21] that the following differential relationship

$$\partial e_t(\rho_t,\sigma_t)/\partial t+\langle\nabla|\rho_t v_t\left[e_t(\rho_t,\sigma_t)+\rho_t\partial w_0(\rho_t,\sigma_t)/\partial\rho_t-|B_t|^2/2\right]\rangle=0 \tag{66}$$

holds for all $t\in\mathbb{R}$ and whose integration over the domain $M\subset\mathbb{R}^3$ easily gives rise to the conservation of the Hamiltonian function (62).

4.2. *Magneto-hydrodynamic invariants and their geometry*

It is long ago stated [2,4,21,38] the importance of spatial invariants describing the stability [21] of MHD superfluid motion. Based on the modern symplectic theory of differential–geometric structures on manifolds, we devise a unified approach to study MHD invariants of compressible superfluid flow, related with specially constructed symmetry structures and commuting to each other vector fields on the phase space.

We start from a useful differential-geometric observation that the magneto-hydrodynamic Euler equations $\Gamma(M;T(M))$ action on the adjoint space to the Lie algebra $\mathcal{G}$ of the Banach group $G=Diff(M)(\Lambda^0(M)\oplus\Lambda^3(M)\oplus *^1(M))$, generated by the following vector field differential relationship:

$$dx_t/dt=v_t(x_t), \tag{67}$$

where $x_t=\eta_t(X)\in M$, $X\in M$, and $v_t:M\to T(M),t\in\mathbb{R}$, is an acceptable time-dependent vector field on the domain M, describing the adiabatic superfluid and superconductive motion via

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the diffeomorphism subgroup mappings $\eta_t \in Diff(M), \eta_t|_{t=0} = \eta_0 \in Diff_0(M)$. Taking into account that the initial superfluid configuration $\eta_0 \in Diff(M)$ is fixed, one can define, following reasonings from [43], a new differential relationship

$$dx_\tau/dt = u_t(x_\tau) \tag{68}$$

on the domain M with respect to the evolution variable $\tau \in \mathbb{R}$, parameterized by the time parameter $t \in \mathbb{R}$, where $u_t : M \to T(M)$, is a τ-independent vector field on M, generating the diffeomorphism subgroup $\psi_t \in Diff(M)$, $x_\tau := \psi_\iota(\eta_0(X)), X \in M$, commuting to that generated by the vector field (67), that is $\eta_t \circ \psi_\iota = \psi_t \circ \eta_\iota$ for all $t, \tau \in \mathbb{R}$. The action of the diffeomorphism subgroup $\psi_t \in Diff(M)$ at any fixed time $t \in \mathbb{R}$ can be naturally interpreted as rearranging the particle configurations in the superfluid not changing its other dynamic characteristics. If to denote the corresponding Lie derivatives with respect to the vector fields (67) and (68) by differential expressions $\mathcal{L}_{d/dt} := \partial/\partial t + \langle v_t|\nabla\rangle\circ : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ and $\mathcal{L}_{u_t} := \langle u_t|\nabla\rangle\circ : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$, the commutation condition $\eta_t \circ \psi_\iota = \psi_t \circ \eta_\iota$ for all $t, \tau \in \mathbb{R}$ is equivalently rewritten as the operator commutator

$$[\mathcal{L}_{d/dt}, \mathcal{L}_{u_t}] = 0. \tag{69}$$

Consider now an arbitrary integral invariant of the MHD superfluid, governed by the Euler system (51):

$$I = \int_{D_t} \rho_t(x_t)\gamma_t(m_t;\rho_t,\sigma_t,B_t)d^3x_t, \tag{70}$$

generated by some specific density functional $\gamma_t : \mathcal{G}^* \to C^\infty(M\times\mathbb{R};\mathbb{R})$ and held over the domain $D_t = \eta_t(D)$ for any domain $D \subset M$, corresponding to the diffeomorphism subgroup $\eta_t \in Diff(M), t \in \mathbb{R}$, generated by flow (67). Taking into account that there holds the following density relationship

$$\mathcal{L}_{d/dt}(\rho_t(x_t)d^3x_t) = 0 \tag{71}$$

for any $t \in \mathbb{R}$, one easily derives from (70) and (71) that also

$$\mathcal{L}_{d/dt}\gamma_t(m_t;\rho_t,\sigma_t,B_t) = 0 \tag{72}$$

for any $t \in \mathbb{R}$. Thus, based on the commutation relationship (69) one can formulate the following important lemma.

Let vector fields (67) and (68) commute to each other and a density functional $\gamma_0 : \mathcal{G}^* \times \mathbb{R} \to C^\infty(M\times\mathbb{R};\mathbb{R})$ satisfies for all $t \in \mathbb{R}$ the condition

$$\mathcal{L}_{d/dt}\gamma_0(m_t;\rho_t,\sigma_t,B_t) = 0, \tag{73}$$

then the following expressions

$$I_{n,k} = \int_{D_t} \rho_t(\mathcal{L}^n_{u_t}\gamma_0(m_t;\rho_t,\sigma_t,B_t)^k d^3x_t \tag{74}$$

over the domain $D_t = \eta_t(D)$, generated by the corresponding to the flow (67) diffeomorphism subgroup $\eta_t \in Diff(M), t \in \mathbb{R}$, and arbitrary domain $D \subset M$, are for all integers $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, the MHD invariants of the superfluid flow (51).

Proof. A proof easily follows from the commutation condition (69) and the superfluid density relationship (71). □

As examples, let us take, following [21,43], the vector field $u_t := \rho_t^{-1}B_t \in \Gamma(T(M))$, commuting to the vector field $v_t \in \Gamma(T(M))$, and $\gamma_0 = i_{u_t}\langle A_t\,|dx_t\rangle = \langle A_t\,|\rho_t^{-1}B_t\rangle \in C^\infty(M\times\mathbb{R};\mathbb{R})$, where the magnetic vector potential $A_t \in C^\infty(M;\mathbb{R}), t \in \mathbb{R}$, satisfies the classical Maxwell relationships: the magnetic field $B_t = \nabla \times A_t$ and the electric field $E_t = -\partial A_t/\partial t = -v_t \times B_t$, owing to the net electric field superconductivity (50) condition $\tilde{E}_t = E_t + v_t \times B_t = 0$. Really, the commutativity condition condition (69) means that

$$\mathcal{L}_{d/dt}(\rho_t^{-1}B_t) - \langle \rho_t^{-1}B_t|\nabla > v_t = 0, \tag{75}$$

which is satisfied, owing to the second and forth equations of the Euler MHD system (51), as well as to the invariance

$$\begin{gathered}\mathcal{L}_{d/dt}\gamma_0 = \mathcal{L}_{d/dt}i_{u_t}\langle A_t\, |dx_t\rangle = [\mathcal{L}_{d/dt}, i_{u_t}]\langle A_t\, |dx_t\rangle + \\ +i_{u_t}\mathcal{L}_{d/dt}\langle A_t\, |dx_t\rangle = i_{[d/dt,u_t]}\langle A_t\, |dx_t\rangle + i_{u_t}\mathcal{L}_{d/dt}\langle A_t\, |dx_t\rangle = 0,\end{gathered} \tag{76}$$

which holds owing to the algebraic relationship

$$[\mathcal{L}_{d/dt}, i_{u_t}] = i_{[\partial/\partial t + v_t v_t, u_t]}, \tag{77}$$

commutativity of vector fields u_t and $v_t \in \Gamma(M)$ and the integral relationship

$$\begin{gathered}\frac{d}{dt}\int_{\partial S_t}\langle A_t\, |dx_t\rangle = \int_{\partial S_t}\mathcal{L}_{d/dt}\langle A_t\, |dx_t\rangle = \\ = \int_{\partial S_t}[\langle \mathcal{L}_{d/dt}A_t\, |dx_t\rangle + \langle A_t\, |\mathcal{L}_{d/dt}dx_t\rangle] = \\ = \int_{\partial S_t}[\langle \mathcal{L}_{d/dt}A_t\, |dx_t\rangle + \langle A_t\, |dv_t\rangle] = \\ = \int_{\partial S_t}[\langle v_t \times B + \langle v_t|\nabla\rangle A_t\, |dx_t\rangle + \langle A_t\, |dv_t\rangle] = \\ = \int_{\partial S_t}[\langle v_t \times (\nabla \times A) + \langle v_t|\nabla\rangle A_t\, |dx_t\rangle + \langle A_t\, |dv_t\rangle] = \\ = \int_{\partial S_t}[\langle dA_t|v_t\rangle + \langle A_t\, |dv_t\rangle] = \int_{\partial S_t}[d\langle A_t\, |v_t\rangle] = 0,\end{gathered} \tag{78}$$

equivalent to the condition $\mathcal{L}_{d/dt}\langle A_t\, |dx_t\rangle = 0$ for all $t \in \mathbb{R}$. The same statement we obtain from the slightly simpler reasoning:

$$\begin{gathered}\frac{d}{dt}\int_{\partial S_t}\langle A_t\, |dx_t\rangle = \frac{d}{dt}\int_{S_t}\langle \nabla \times A_t|dS_t^2\rangle = \\ = \frac{d}{dt}\int_{S_t}\langle B_t|dS_t^2\rangle := -\int_{\partial S_t}\langle \tilde{E}_t|dx_t\rangle = 0,\end{gathered} \tag{79}$$

following from the net electric field $\tilde{E}_t = 0$ superconductivity condition (50) along the boundary ∂S_t, where $S_t := \eta_t(S_0) \subset M$ is the surface, generated by the diffeomorphism subgroup $\eta_t \in Diff(M), t \in \mathbb{R}$, and an arbitrarily chosen surface $S_0 = S_t|_{t=0} \subset M$. The latter is, evidently, equivalent to the equality $\mathcal{L}_{d/dt}\langle A_t\, |dx_t\rangle = 0$ modulo the gauge transformation $A_t \to A_t + \nabla\xi_t$, where $\mathcal{L}_{d/dt}\xi_t + \langle A_t|v_t\rangle = 0$ for some function $\xi_t \in C^\infty(M;\mathbb{R})$ and all $t \in \mathbb{R}$. Thus, one can formulate [21,43] the following proposition.

The functionals

$$I_{n,k}^{(B)} = \int_{D_t}\rho_t\left(\mathcal{L}^n_{\rho_t^{-1}B_t}\langle A|\rho_t^{-1}B_t\rangle\right)^k d^3x_t \tag{80}$$

over the domain $D_t = \eta_t(D)$, generated by the corresponding to the flow (67) diffeomorphism subgroup $\eta_t \in Diff(M), t \in \mathbb{R}$, and arbitrary domain $D \subset M$, are for all integers $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, the MHD invariants of the superfluid flow (51). In particular, the following relationships $\{H, I_{n,k}^{(B)}\} = 0$ hold for all $n \in \mathbb{Z}_+$.

It is natural here to mention [18,21] that the specific entropy functional $\gamma_0 = \sigma_t : M \to C^\infty(M \times \mathbb{R};\mathbb{R})$ satisfies the sufficient condition $\mathcal{L}_{d/dt}\sigma_t = 0, t \in \mathbb{R}$, a priori generates for the superfluid flow (51) the infinite hierarchy

$$I_{n,k}^{(\sigma)} = \int_{D_t}\rho_t\left(\mathcal{L}^n_{\rho_t^{-1}B_t}\sigma_t(x_t)\right)^k d^3x_t, \tag{81}$$

$n \in \mathbb{Z}_+, k \in \mathbb{Z}$, of the MHD invariants over the domain $D_t = \eta_t(D)$, generated by the corresponding to the flow (67) diffeomorphism subgroup $\eta_t \in Diff(M), t \in \mathbb{R}$, and arbitrary domain $D \subset M$.

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To construct other MHD invariants, depending on the superfluid velocity $v_t \in \Gamma(T(M)), t \in \mathbb{R}$, let us consider, following [43], two differential one-forms $\langle \alpha_t | dx_t \rangle, \langle \beta_t | dx_t \rangle \in \Lambda^1(M)$, $x_t := \eta_t(X)$, $X \in M$, satisfying for all $t \in \mathbb{R}$ the following identity:

$$\mathcal{L}_{d/dt} \langle \alpha_t | dx_t \rangle = dh_t \ + \mathcal{L}_{u_t} \langle \beta_t | dx_t \rangle, \tag{82}$$

for some function $h_t \in \Lambda^0(M)$, where the vector field

$$dx_t / d\tau = u_t(x_t) \tag{83}$$

is uniform with respect to the evolution parameter $\tau \in \mathbb{R}$ and satisfies the following constraints:

$$[\mathcal{L}_{d/dt}, \mathcal{L}_{u_t}] = 0, \quad \langle \nabla | \rho_t u_t \rangle = 0 \tag{84}$$

and $u_t \parallel \partial M$ at almost all points $x_t \in \partial M$ for all evolution parameters $t, \tau \in \mathbb{R}$. Then one can formulate the following general proposition.

The following integral expressions

$$\begin{aligned} I_0^{(\alpha,\beta)} &= \int_M \rho_t \langle \alpha_t | u_t \rangle d^3 x_t, I_1^{(\alpha,\beta)} = \int_M \rho_t [\langle \alpha_t | v_t \rangle + h_t] d^3 x_t, \\ I_2^{(\alpha,\beta)} &= \int_M \rho_t \langle \mathcal{L}_{d/dt} \alpha_t | u_t \rangle d^3 x_t \end{aligned} \tag{85}$$

over the whole domain $M \subset \mathbb{R}^3$ are for all integers $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, the global MHD invariants.

Proof. Consider, for example, a proof that $I_0^{(\alpha,\beta)} : \mathcal{G} \to \mathbb{R}$ is an invariant: taking into account that $\mathcal{L}_{d/dt}(\rho_t d^3 x_t) = 0$, one obtains the expression:

$$\begin{gathered} dI_0^{(\alpha,\beta)}/dt = \int_M \rho_t \mathcal{L}_{d/dt} \langle \alpha_t | u_t \rangle d^3 x_t \ = \\ = \int_M \rho_t i_{u_t} \left(dh_t + \mathcal{L}_{u_t} \langle \beta_t | dx_t \rangle \right) d^3 x_t = \\ = \int_M \rho_t \left(i_{u_t} dh_t^{(n)} + i_{u_t} (i_{u_t} d + d i_{u_t}) \langle \beta_t | dx_t \rangle \right) d^3 x_t = \\ = \int_M \rho_t i_{u_t} d \left(h_t^{(n)} + \langle \beta_t | u_t \rangle \right) d^3 x_t = \\ = \int_M \langle \nabla | \tilde{h}_t \rho_t u_t \rangle d^3 x_t = \int_{\partial M} \langle \tilde{h}_t^{(n)} \rho_t u_t | dS_t^2 \rangle = 0 \end{gathered} \tag{86}$$

for all $t \in \mathbb{R}$, where we put, by definition, $\tilde{h}_t := (h_t + \langle \beta_t | u_t \rangle)$, denoted dS_t^2 the surface measure on the boundary ∂M, used the Cartan representation $\mathcal{L}_{u_t} = (i_{u_t} d + d i_{u_t})$ and the natural boundary tangency condition $\rho_t u_t \partial M$, thus proving the proposition. Exactly similar calculations ensue for the next two invariant $I_k^{(\alpha,\beta)} : \mathcal{G} \to \mathbb{R}, k = \overline{1,2}$, on which we will not stop here. . $\square$

As a simple example, one can put $\alpha_t^{(0)} := \flat(v_t) \simeq v_t, \beta_t := B_t$, the vector field $u_t = \rho_t^{-1} B_t : M \to T(M), t \in \mathbb{R}$, and show by easy calculations, using the variational equality (52), that

$$\mathcal{L}_{d/dt} \langle v_t | dx_t \rangle = d(|v_t|^2/2 \ - h_t - |B_t|^2/\rho_t) + \mathcal{L}_{u_t} \langle B_t | dx_t \rangle + T_t d\sigma_t, \tag{87}$$

where, we have denoted the specific enthalpy [8,23,37] function $h_t := e_t + p_t \rho_t^{-1}$. As a consequence of equality (87), under the spatial temperature constancy $\nabla T_t = 0$ condition for all $t \in \mathbb{R}$, one obtains the following MHD superfluid invariant:

$$I_0^{(v,B)} := \int_M \langle v_t | B_t \rangle d^3 x_t = \int_M \langle m_t | \rho_t^{-1} B_t \rangle, \tag{88}$$

where $m_t \simeq \langle m_t(x_t)|dx_t\rangle \otimes d^3x_t \in diff^*(M)$ and $\rho_t^{-1}B_t \simeq \langle \rho_t^{-1}(x)B_t|\partial/\partial x\rangle \in T(M)$, coinciding with the MHD invariant, presented before in [21,43]. If the above temperature condition does not hold, the equality (87) reduces to the differential relationship

$$\partial\langle v_t|B_t\rangle/\partial t + \langle\nabla|[v_t\langle v_t|B_t\rangle + B_t(h_t - |v_t|^2/2]\rangle + \rho_t T_t\langle\rho_t^{-1}B_t|\nabla\sigma_t\rangle, \tag{89}$$

satisfied for all $x_t \in M$ and $t \in \mathbb{R}$.

It is worth to remark here that the following baroclinic relationship

$$\nabla\rho_t^{-1} \times \nabla p_t = -\nabla T_t \times \nabla\sigma_t \tag{90}$$

holds for all $x_t \in M$ and $t \in \mathbb{R}$.

Similarly we also easily obtain the following invariant

$$I_1^{(v,B)} = \int_M \rho_t[|m_t|^2/\left(2\rho_t^2\right) + w_t^{(0)}(\rho_t,\sigma_t) + |B_t|^2/\left(2\rho_t\right)]d^3x_t = H, \tag{91}$$

coinciding exactly with the Hamiltonian function for the flow (51) on the phase space $\mathcal{G}^*$. The third invariant is, eventually, closely related with the vorticity vector $\xi_t := \nabla \times v_t : M \to \mathbb{E}^3, t \in \mathbb{R}$, and needs a more detail analysis.

It is instructive now to analyze the existence of integral invariants for the pure hydrodynamic case when the magnetic field $B_t = 0$, $t \in \mathbb{R}$, following the approach, devised before in [43]. In particular, owing to the relationship (90), there holds the following integral expression for the vorticity $\xi_t := \nabla \times v_t, t \in \mathbb{R}$:

$$\mathcal{L}_{d/dt}\xi_t - \langle\xi_t|\nabla\rangle v_t = \nabla T_t \times \nabla\sigma_t \tag{92}$$

and define the vector field

$$u_t := \rho_t^{-1}\xi_t \exp f_t(x_t) \tag{93}$$

for some scalar smooth mapping $f_t : M \to \mathbb{R}$, which we will choose from the assumed commutation condition

$$[\mathcal{L}_{d/dt}, \mathcal{L}_{u_t}] = 0. \tag{94}$$

The latter gives rise to the equality $\xi_t\mathcal{L}_{d/dt}f_t(x_t) = -\nabla T_t \times \nabla\sigma_t$ at any $x_t := \eta_t(X) \in M$, $X \in M$, or

$$\dot{f}_t\,(\nabla \times v_t) + \nabla T_t \times \nabla\sigma_t = 0, \tag{95}$$

where we took into account that $\mathcal{L}_{d/dt}f_t(x_t) = df_t(x_t)/dt := \dot{f}_t(x_t)$, $x_t \in M$, with respect the temporal parameter $t \in \mathbb{R}$. From (95) one obtains that the mapping $f_t : M \to \mathbb{R}$ should satisfy the following constraints:

$$\nabla\dot{f}_t = k_t v_t, \quad \dot{f}_t v_t = \rho_t^{-1}\nabla p(t) + \nabla\omega_t \tag{96}$$

for some scalar smooth functions k_t and $\omega_t : M \to \mathbb{R}, t \in \mathbb{R}$. It is easy to check that the system (96) is compatible, that is the quasi-stationary thermodynamic relationship $p_t^{(0)} = \rho_t^2\partial w_0(\rho_t,\sigma_t)/\partial\rho_t$ jointly with Euler equations (6) makes it possible to determine these unknown scalar smooth functions k_t and $\omega_t : M \to \mathbb{R}$ for all $t \in \mathbb{R}$.

Consider now, following [43], a slightly modified expression (87) at the magnetic field $B_t = 0$:

$$\mathcal{L}_{d/dt}\langle v_t \exp f_t|dx_t\rangle = \exp f_t d(\omega_t + |v_t|^2/2) \tag{97}$$

and calculate the related integral expression:

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_M \rho_t\,(i_{u_t}\langle v_t|dx_t\rangle)\,d^3x_t &= \textstyle\int_M \rho_t\mathcal{L}_{d/dt}\,(i_{u_t}\langle v_t|dx_t\rangle)\,d^3x_t = \\ = \textstyle\int_M \rho_t\,(i_{u_t}\mathcal{L}_{d/dt}\langle v_t|dx_t\rangle)\,d^3x_t &= \textstyle\int_M \rho_t\,(i_{u_t}d\tilde{h})\,d^3x_t = \\ = \textstyle\int_M (i_{\rho_t u_t}d\tilde{h})\,d^3x_t = \int_M\langle\nabla\tilde{h}_t|\rho_t u_t\rangle d^3x_t &= \textstyle\int_M\langle\nabla\tilde{h}_t|\xi_t\exp f_t(x_t)\rangle d^3x_t, \end{aligned} \tag{98}$$

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where we put, by definition, the function $\tilde{h}_t := \omega_t + |v_t|^2/2$.

If now to put that the mapping $f_t : M \to \mathbb{R}$ satisfies for all $t \in \mathbb{R}$ the constraint $\langle \nabla f_t | \xi_t \rangle = 0$, the integral expression (98) reduces to

$$\begin{gathered} \frac{d}{dt} \int_M \rho_t \ (i_{u_t} \langle v_t | dx_t \rangle)\, d^3x_t = \int_M \langle \nabla | \left(\exp f_t(x_t) \tilde{h}_t \xi_t \right) \rangle d^3x_t = \\ = \int_{\partial M} \langle \exp f_t(x_t) \tilde{h}_t \xi_t | d^2 S_t \rangle = 0, \end{gathered} \tag{99}$$

where there is assumed the vorticity vector tangency $\xi_t || \partial M$ constraint. Thus, under conditions assumed above, the following vortex functional

$$I = \int_M \langle v_t | \nabla \times v_t \rangle d^3x_t \tag{100}$$

persists to be conserved for all $t \in \mathbb{R}$.

If the function $f_t : M \to \mathbb{R}$, being defined by relationships (96), satisfies for all $t \in \mathbb{R}$ the scalar constraint $\langle \nabla f_t | \xi_t \rangle = 0$, one easily derives the following differential relationship:

$$\begin{aligned} \mathcal{L}_{d/dt} \langle \nabla f_t | \xi_t \rangle &= k_t \langle v_t | \xi_t \rangle + \langle \nabla | f_t \nabla T_t \times \nabla \sigma_t \rangle = \\ &= < \nabla \dot{f}_t | \xi_t \rangle + \langle \nabla | f_t \nabla T_t \times \nabla \sigma_t \rangle = 0, \end{aligned} \tag{101}$$

or, equivalently, in the integral form

$$\begin{gathered} \frac{d}{dt} \int_{D_t} \langle \nabla f_t | \xi_t \rangle \rho_t d^3x_t = \int_{D_t} \mathcal{L}_{d/dt} \langle \nabla f_t | \xi_t \rangle \rho_t d^3x_t = \\ = \int_{D_t} \left[\langle \nabla \dot{f}_t | \xi_t \rangle + \langle \nabla | f_t \nabla T_t \times \nabla \sigma_t \rangle \right] \rho_t d^3x_t = \\ = \int_{D_t} \left[\langle \nabla \dot{f}_t | \xi_t \rangle - \langle \nabla f_t | \nabla \times \rho_t^{-1} \nabla p_t^{(0)} \rangle \right] \rho_t d^3x_t \\ = \int_{D_t} \left[\langle \nabla \dot{f}_t | \xi_t \rangle \rho_t - \rho_t \langle \nabla \rho_t^{-1} | \nabla \times p_t^{(0)} \nabla f_t \rangle \right] d^3x_t = \\ = \int_{D_t} \left[\langle \nabla \dot{f}_t | \xi_t \rangle \rho_t + \langle \nabla \ln \rho_t | \nabla \times p_t^{(0)} \nabla f_t \rangle \right] d^3x_t = \\ = \int_{D_t} \langle \nabla \dot{f}_t | \xi_t \rangle \rho_t d^3x_t, \end{gathered} \tag{102}$$

where we took into account that for the adiabatic fluid flow under regard there holds the tangency $\nabla \rho_t || \partial D_t$ condition for all $t \in \mathbb{R}$. If the right hand side (102) proves to be zero, that is $\langle \nabla \dot{f}_t | \xi_t \rangle = 0, t \in \mathbb{R}$, this will mean that the constraint $\langle \nabla f_t | \xi_t \rangle = 0$ for all $t \in \mathbb{R}$, if $\langle \nabla f_t | \xi_t \rangle|_{t=0} = 0$ at $t = 0$, thus producing the vortex conservation functional (100).

5. The isentropic flows on phase spaces with gauge symmetry

In this Section we are interested in description of isentropic charged liquid flows on phase spaces with gauge symmetry, imposed by an external electromagnetic field. Unlike the Section , when the external magnetic field was completely frozen into the charged superfluid and completely governed by its dynamics, the case under present regard strongly differs from the latter and should take into account two independent yet interacting to each other dynamical systems. As the phase space under regard is endowed with gauge type electromagnetic field symmetry, the common dynamics of the coupled fluid and electromagnetic field should be properly considered on the related principal fiber bundle $P(Q_{em}, F)$ over the reduced fluid base space $Q_{em} \simeq (\Lambda^2(M) \times d\Lambda^1(M)) \times diff^*(M) \times (\Lambda^3(M) \times \Lambda^0(M))$ with the abelian structure group $F \simeq d\Lambda^0(M)$, acting on the fiber bundle P from the right via the gauge type transformation. We assume that locally the principal fiber bundle $P(Q_{em}, F)|_{loc} \simeq T^*(G) \times (*\Lambda^1(M) \times \Lambda^1(M))$, where the group $G = Diff(M)(\Lambda^0(M) \times \Lambda^3(M))$, the space $\Lambda^1(M)$ models the magnetic vector potential on M, its factor space $\Lambda^1(M)/F \simeq d\Lambda^1(M) \subset \Lambda^2(M)$ models the ambient

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magnetic field on M, the product $T^*(G) \times (*\Lambda^1(M) \times \Lambda^1(M))$ of cotangent spaces models the moving charged liquid under the ambient electromagnetic field and determines the Hamiltonian function

$$H_{em} = \int_M d^3x(|\mu - \theta\rho A|^2/(2\rho) + \rho w^{(0)}(\rho,\sigma) + (|E|^2 + |\nabla \times A|^2)/2), \tag{103}$$

where $E := -Y, (Y, A) \in *\Lambda^1(M) \times \Lambda^1(M)), \mu = \rho v, (\mu; \rho, \sigma) \in T^*(G), w^{(0)} : T^*(G) \to C^\infty(M; \mathbb{R})$ denotes the internal potential energy function and $\theta \in \mathbb{R}_+$ denotes the corresponding charge/mass ratio of the fluid under regard. The resulting evolution equations of the liquid motion with respect to the temporal parameter $t \in \mathbb{R}$ look in noncanonical variables as follows:

$$\begin{aligned} \partial v/\partial t + \langle v|\nabla\rangle v &= \theta(E + v \times B) - \rho^{-1}\nabla p^{(0)}, \\ \partial\rho/\partial t + \langle\nabla|\rho v\rangle &= 0, \quad \partial\sigma/\partial t + \langle\nabla|v\rangle\sigma = 0, \langle\nabla|E\rangle = \theta\rho, \\ \partial E/\partial t &= \nabla \times B - \theta\rho v, \;\; B = \nabla \times A, \end{aligned} \tag{104}$$

where, in general, the pressure $p^{(0)} := \rho^2(\partial w^{(0)}(\rho,\sigma)/\partial\rho + \sigma/\rho\partial w^{(0)}(\rho,\sigma)/\partial\sigma)$ and phase space points $(E, B; \rho v; \rho, \sigma) \in Q_{em}$ belong here to the base manifold of the fiber bundle $P(Q_{em}, F)$.

To proceed further in more detail, we begin by reviewing some backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. Some of the material is partly available in [24,31,32,34], so here it will be only sketched in notations suitable for us.

Consider a principal fiber bundle $P(Q_{em}, F)$ over the base space Q_{em} with the projection $\pi : P \to Q_{em}$ and the abelian structure group $F \simeq \Lambda^0(M)$, acting from the right on P by means of a smooth mapping $\chi : F \times P \to P$. Taking into account that $\Lambda^1(M) \simeq \Lambda^2(M) \simeq d\Lambda^1(M) \oplus \Lambda^2(M)/d\Lambda^1(M)$, $\Lambda^2(M)/d\Lambda^1(M) \simeq d\Lambda^0(M) \simeq$ calF owing to the classical [1,3,48] Helmgoltz representation, for each $h \in F$ a group diffeomorphism $\chi_h : P \to P$ generates for any fixed $u \in P$ the induced mapping $\hat{u} : F \to P$, where

$$\hat{u}(h) := \chi_h(u) \tag{105}$$

for all $h \in F$, being equal to the usual gauge transformed expression

$$\hat{u}(h) = (\mu + \theta\rho(x)d\ln h]; \rho(x)d^3x, \sigma; (*\langle Y|dx\rangle, \langle A|dx\rangle + d\ln h)), \tag{106}$$

where we made used of the local coordinate representation of P in coordinates $u := (\mu; \rho(x)d^3x, \sigma; (*\langle Y|dx\rangle, \langle A|dx\rangle)) \in P$, $\pi(R_h u) = \pi(u) := (\mu; \rho(x)d^3x, \sigma; (*\langle(\nabla\times)^{-1}Y|dx\rangle, \langle B|dx\rangle)), x \in M$, for any $h \in F$, $\pi(A) := B = \nabla \times A$ and suitable vector $Y \in C^\infty(M; \mathbb{E}^3)$. We here also assume that the gauge type transformation $\chi_h : P \to P$, $h \in F$, is equivariant [1,19,20,32] subject to the canonical product Poisson bracket $\{\cdot,\cdot\} := \{\cdot,\cdot\}_{T^*(G)} \otimes \{\cdot,\cdot\}_{T^*(\Lambda^1(M))}$ on P, that is for any smooth functionals $g_1, g_2 : P \to \mathbb{R}$ the invariance relationship

$$\{g_1 \circ \chi_h, g_2 \circ \chi_h\} = \{g_1, g_2\} \circ \chi_h \tag{107}$$

holds for all $h \in F$.

Let $e \in F$ be the unit element of the structure group F and denote by $d\Lambda^0(M) \simeq \mathcal{F}$ the corresponding Lie algebra (abelian) of the structure group F as the tangent space $T_e(F) \simeq$ calF at $e \in F$. The tangent mapping to (105) acts as $\hat{u}_*(e)$: calF $\to T_u(P)$ for $u \in P$ and is equal in local coordinates to the tangent vector expression

$$\hat{u}_*(e)(df) = (\theta df \otimes \rho(x)d^3x; 0, 0; (0, \langle\nabla f|dx\rangle)) \tag{108}$$

at $u \in P$, where $df \in$ calF $\simeq d\Lambda^0(M)$ and $x \in M$, where we took into account the corresponding action of the abelian group F on the group The mapping (106) makes it possible to define on the principal fiber bundle $P(T^*(G_{em}), F)$ a connection $\Gamma(\mathcal{A})$ by means of constructing [27,53,54] a morphism $\mathcal{A}$:

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$(T(P), \chi_{h,*}) \rightarrow$ (calF, $Ad_{h^{-1}}), h \in F$, such that for each $u \in P$ the corresponding mapping $\mathcal{A}(u)$: $T_u(P) \rightarrow$ calF is a left inverse one to the mapping

$$\hat{u}_*(e) : \mathcal{F} \rightarrow T_u(P), \tag{109}$$

that is

$$\mathcal{A}(u)\hat{u}_*(e) = Id, \tag{110}$$

where, by definition, for some vector-functions Φ and $Z \in C^\infty(P; \mathbb{E}^3)$

$$\mathcal{A}(u) = \langle d\Phi|\delta\mu\rangle + \langle *(dx - \nabla \times dZ - \theta\rho d\Phi)|\langle\delta A|dx\rangle\rangle \tag{111}$$

at $u \in P, x \in M$. Really, by definition, for any fundamental vertical vector field

$$\widetilde{df} = \hat{u}_*(e)(df) = (\theta df \otimes \rho(x)d^3x; 0, 0; (0, \langle\nabla f|dx\rangle)), \tag{112}$$

generated by an element $\nabla \ln h \simeq df \in$ calF, there should be $\langle\mathcal{A}(u)|\widetilde{df}\rangle = df$:

$$\begin{aligned} \langle\mathcal{A}(u)|\widetilde{df}\rangle = \langle *(\theta df) \otimes \rho(x)d^3x|d\Phi\rangle + \\ + \langle(dx - \theta\rho d\Phi)|\nabla f\rangle d^3x \simeq \langle dx|\nabla f\rangle = df, \end{aligned} \tag{113}$$

being completely satisfied. The needed invariance $R_{h*}\mathcal{A}(u) = Ad_{h^{-1}}\mathcal{A}(u)$ is also satisfied automatically for any $h \in F$.

The induced by mapping (109) Lie algebra action $\hat{u}_*(e) : \mathcal{F} \times P \rightarrow T_u(P)$ naturally generates [1,2,6,9,36] the *momentum mapping* $l : T_u^*(P) \rightarrow$ calF* at $u \in P$ for any $f \in F$ that for the vertical vector field $\widetilde{df} = d\langle A|dx\rangle/d\tau = \langle\nabla f|dx\rangle \in$cal F

$$\begin{aligned} \langle l(\alpha^{(1)}(u))|\widetilde{df}\rangle = \langle\hat{u}^*(e)\alpha^{(1)}(u)|\langle\nabla f|dx\rangle\rangle = \\ = \langle\alpha^{(1)}(u)|\hat{u}^*(e)\langle\nabla f|dx\rangle\rangle = \langle J(\mu;\rho) + Y|\nabla f\rangle d^3x, \end{aligned} \tag{114}$$

where $\alpha^{(1)}(u) := \langle *\langle Y|dx\rangle|\langle\delta A|dx\rangle\rangle + \langle\alpha(\mu;\rho,\sigma)|(\delta\mu;\delta\rho,\delta\sigma)\rangle \in T_u^*(P)$ for some element $\alpha(\mu;\rho,\sigma) \in T^*(T^*(G_{em}))$ is the corresponding Liouville form on the cotangent bundle $T^*(G_{em})$, the following determining equalities hold:

$$\begin{aligned} H_f(\mu;\rho)d^3x \quad &: \quad = \langle *\langle J(\mu;\rho)|dx\rangle|\langle\nabla f|dx\rangle\rangle, \\ (\theta\rho(x)\nabla f; 0, 0) \quad &: \quad = \{H_f(\mu;\rho), (\mu;\rho,\sigma)\}_{T^*(G)} \end{aligned} \tag{115}$$

with respect to the canonical Lie-Poisson bracket (25) on the cotangent bundle $T^*(G)$ to the group manifold $G = Diff(M)(\Lambda^0(M) \times \Lambda^3(M))$.

Fix now the momentum mapping value

$$l(\alpha^{(1)}(u)) = *\langle J(\mu;\rho) + Y|dx\rangle := *\langle\xi|dx\rangle \in \mathcal{F}^*,$$

equivalent to the condition $-\langle\nabla|\xi\rangle = \bar{\xi} \in C^\infty(M;\mathbb{R})$, construct the submanifold $P_\xi := \{u \in P : l(\alpha^{(1)}(u)) = \xi \in$calF$^*\}$ and consider the reduced phase space $\bar{P}_\xi := P_\xi/F_\xi$, where $F_\xi := \{h \in F : Ad_h^* *\langle\xi|dx\rangle = *\langle\xi|dx\rangle\}$ is the stationary subgroup of the element $\xi \in$ calF*. Taking into account that in our case $F_\xi = F$ for any element $\xi \in$ calF*, one can formulate the following theorem, characterizing [6,27] the related gauge symmetry symplectic structure reduction on the reduced manifold $\bar{P}_\xi, \xi \in$ calF*.

Given a principal fiber F-bundle with a connection $\Gamma(\mathcal{A})$ on the principal fiber bundle $P(Q_{em}; F)$ and an F-invariant element $\xi \in$ calF*, then every such connection $\Gamma(\mathcal{A})$ defines a symplectomorphism

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$\nu_\xi : \bar{P}_\xi \rightarrow Q_{em}$ between the reduced phase space $\bar{P}_\xi$ and the base manifold Q_{em}. Moreover, the following equality

$$(d\,\beta^{(1)}_{Q_{em}} + \Omega^{(2)}_\xi|_{Q_{em}}) = \nu^*_\xi\,\alpha^{(1)}\Big|_{\bar{P}_\xi} \tag{116}$$

holds for the canonical Liouville forms $\beta^{(1)} \in \Lambda^1(Q_{em})$ and $\alpha^{(1)} \in \Lambda^1(P)$, where $\Omega^{(2)}_\xi := \langle \xi|\Omega^{(2)}\rangle$ is the ξ-component of the corresponding curvature form $\Omega^{(2)} \in \Lambda^{(2)}(Q_{em}))\otimes$calF.

The statements above make it possible to construct a true symplectic structure on the cotangent bundle $T^*(G_{em})$. Namely, making use of the connection form (111) and symplectic structure expression (116), one derives the resulting reduced symplectic structure on the base manifold Q_{em} :

$$\begin{gathered}\omega^{(2)}_\xi(\pi(u_\xi)) = \langle \delta\alpha| \wedge (\delta\mu;\delta\rho,\delta\sigma)\rangle|_{\mu\rightarrow\mu-\theta\rho A} + \\ +\langle\nabla|\xi\rangle\langle\delta Z\wedge\delta B\rangle + \langle(\nabla\times)^{-1}\delta Y|\wedge\delta B\rangle + \\ +\langle(\nabla\times)^{-1}\left[\theta\langle\nabla|\xi\rho\rangle\delta\Phi + \theta\langle\nabla|\xi\delta\rho\rangle\Phi\right]|\wedge\delta B\rangle = \omega^{(2)}(\mu;\rho,\sigma)|_{\mu\rightarrow\mu-\theta\rho A} + \\ +\langle\langle\nabla|\xi\rangle\delta Z + (\nabla\times)^{-1}\left[\delta Y + \theta\langle\nabla|\xi\rho\rangle\delta\Phi + \theta\langle\nabla|\xi\delta\rho\rangle\Phi\right]|\wedge\delta B\rangle\end{gathered} \tag{117}$$

at $\pi(u_\xi) = (Y,B)\times(\mu;\rho,\sigma) \in Q_{em}$ for the fixed element $*\langle\xi|dx\rangle \in \mathcal{F}^*$, where the expression $\omega^{(2)}(\mu;\rho,\sigma) := \langle\delta\alpha|\wedge(\delta\mu;\delta\rho,\delta\sigma)\rangle$ is generated by the canonical Lie-Poisson bracket (25). The reduced Poisson structure on the base manifold Q_{em}, corresponding to the symplectic structure (117) and calculated at the vanishing vectors $Z = 0, \Phi = 0$, can be easily written down as

$$\begin{gathered}\{g_1,g_2\}(\mu;\rho,\sigma;E,B) = \int_M d^3x\left[\left\langle\mu|\langle\tfrac{\delta g_2}{\delta\mu}|\nabla\rangle\tfrac{\delta g_1}{\delta\mu} - \langle\tfrac{\delta g_1}{\delta\mu}|\nabla\rangle\tfrac{\delta g_2}{\delta\mu}\right\rangle\right] + \\ +\int_M d^3x\rho\left[\langle\tfrac{\delta g_2}{\delta\mu}|\nabla\tfrac{\delta g_1}{\delta\rho}\rangle - \langle\tfrac{\delta g_1}{\delta\mu}|\nabla\tfrac{\delta g_2}{\delta\rho}\rangle\right] + \\ +\int_M d^3x\sigma\left[\langle\tfrac{\delta g_2}{\delta\mu}|\nabla\tfrac{\delta g_1}{\delta\sigma}\rangle - \langle\tfrac{\delta g_1}{\delta\mu}|\nabla\tfrac{\delta g_2}{\delta\sigma}\rangle\right] + \\ +\int_M d^3x\left[\langle\tfrac{\delta g_1}{\delta E}|\nabla\times\tfrac{\delta g_2}{\delta B}\rangle - \langle\tfrac{\delta g_2}{\delta E}|\nabla\times\tfrac{\delta g_1}{\delta B}\rangle\right] + \\ +\int_M d^3x\left[\langle\tfrac{\delta g_1}{\delta\mu}|\tfrac{\delta g_2}{\delta E}\rangle - \langle\tfrac{\delta g_2}{\delta\mu}|\tfrac{\delta g_1}{\delta E}\rangle + \theta\rho\langle B|\tfrac{\delta g_1}{\delta\mu}\times\tfrac{\delta g_2}{\delta\mu}\rangle\right]\end{gathered} \tag{118}$$

for any smooth functionals $g_1, g_2 : P \rightarrow \mathbb{R}$, where $(\mu;\rho,\sigma;E,B) \in Q_{em}$. The related with the Hamiltonian function (103) subject to the Poisson bracket (118) evolution equations

$$\frac{\partial}{\partial t}(\mu;\rho,\sigma;E,B) = \{H_{em},(\mu;\rho,\sigma;E,B)\} \tag{119}$$

coincide exactly with those (104), constructed directly from the classical mechanics and electromagnetic laws. We need to remark here that the Poisson bracket structure, related with the obtained above reduced symplectic structure (117) on the base manifold Q_{em}, generalizes the Poisson bracket [32,34], and can be eventually used for analyzing nonregular charged fluid dynamics with singularities, including vortices and boundary topological peculiarities. It is easily also generalized to describing a multicomponent charged liquid dynamics.

6. Conclusion

In our review we presented a detail enough differential geometric description of the isentropic fluid motion phase space and featuring it the diffeomorphism group structure, modelling the related dynamics, as well as its compatibility with the quasi-stationary thermodynamical constraints. There

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was analyzed the adiabatic liquid dynamics, within which, following the general approach, there is explained in detail, the nature of the related Poissonian structure on the fluid motion phase space, as a semidirect Banach groups product, and a natural reduction of the canonical symplectic structure on its cotangent space to the classical Lie-Poisson bracket on the adjoint space to the corresponding semidirect Lie algebras product. We also presented a modification of the Hamiltonian analysis in case of the isotermal liquid dynamics. Some material was devoted to studying the differential-geometric structure of the adiabatic magneto-hydrodynamic superfluid phase space and its related motion within the Hamiltonian analysis and invariant theory. In particular, we constructed an infinite hierarchies of different kinds of integral magneto-hydrodynamic invariants, generalizing those, before constructed in [18,21], and analyzed their differential-geometric origins. We also investigated a charged liquid dynamics on the phase space invariant with respect to an abelian gauge group transformation.

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