

Extended General Relativity for a Curved Universe

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Abstract: The Planck Legacy recent release revealed the presence of an enhanced lensing amplitude in the cosmic microwave background, which endorses the early universe positive curvature with a confidence level greater than 99%. Although General Relativity performs accurately in the present universe where spacetime is almost flat, its lost boundary term and the need of dark matter/energy might indicate its incompleteness. By utilising the Einstein–Hilbert action, this letter presents new extended field equations considering pre-existing universal curvatures and boundary contributions. The extended field equations are inclusive of Einstein field equations in addition to the boundary and the conformal curvature terms, which could remove the singularities from the theory.

Keywords: General Relativity, Curved Universe.

1. INTRODUCTION

There is strong evidence of early universe positive curvature with a confidence level greater than 99% based on Planck Legacy recent release [1]. In addition, the detected gravitational lensing within several galaxy clusters is an order of magnitude higher than that estimated by current flat standard model [2]. Enhancing General Relativity by considering the pre-existing universal curvature and the boundary contribution might address several unsolved problems of the universe.

This letter is organised as follows: In Section 2, the mathematical derivations are presented. Future works are projected in Section 3. A simulation case study is presented at the end of this letter.

2. Extended Field Equations

The pre-existing universal curvature signified by the scalar curvature \mathcal{R} is incorporated into the Einstein–Hilbert action. Nevertheless, to maintain the compatibility of the action, an intermediate metric-compatible invariant modulus $E_D = (T_u^v - T\delta_u^v/2)/R_u^v/\mathcal{R}$ is introduced based on the Theory of Elasticity [3]. The extended action is

$$S = \int \left[\frac{E_D R}{2} + \mathcal{L}_M \right] \sqrt{-g} d^4x \quad (1)$$

where \mathcal{L}_M is the matter Lagrangian density, g is the determinant of the metric tensor g_{uv} , T_u^v is the energy-momentum tensor of trace T , R_u^v is the Ricci curvature tensor, and δ_u^v is the Kronecker delta [4]. According to the principle of least action, the variation in the action should hold for any variation in the inverse metric as

$$\delta S = \int \left[\frac{E_D}{2} \left(\frac{\delta(R\mathcal{R}^{-1}\sqrt{-g})}{\delta g^{\mu\nu}} \right) + \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \quad (2)$$

$$\delta S = \int \left[\frac{E_D}{2} \left(\frac{\delta R\sqrt{-g}}{\mathcal{R}\delta g^{\mu\nu}} - \frac{R\delta\mathcal{R}\sqrt{-g}}{\mathcal{R}^2\delta g^{\mu\nu}} + \frac{R\delta\sqrt{-g}}{\mathcal{R}\delta g^{\mu\nu}} \right) + \frac{\delta\mathcal{L}_M\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_M\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \quad (3)$$

$$\delta S = \int \left[\frac{E_D}{2} \left(\frac{R_{uv}\delta g^{uv} + g^{uv}\delta R_{uv}}{\mathcal{R}\delta g^{\mu\nu}} - \frac{R(\mathcal{R}_{uv}\delta g^{\mu\nu} + g^{\mu\nu}\delta\mathcal{R}_{uv})}{\mathcal{R}^2\delta g^{\mu\nu}} \right) + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \right. \\ \left. + \frac{R\delta\sqrt{-g}}{\mathcal{R}\delta g^{\mu\nu}\sqrt{-g}} \right] + \frac{\mathcal{L}_M\delta\sqrt{-g}}{\delta g^{\mu\nu}\sqrt{-g}} \sqrt{-g}\delta g^{\mu\nu} d^4x \quad (4)$$

where $R = R_{uv}g^{uv}$ and $\mathcal{R} = \mathcal{R}_{uv}g^{uv}$ [4].

Regarding the boundary contribution to the action, the variation in the Ricci curvature tensor can be written in terms of the covariant derivative of the difference between two Levi-Civita connections [4], the Palatini identity:

$$\delta R_{\sigma\nu} = \nabla_\rho(\delta\Gamma_{\nu\sigma}^\rho) - \nabla_\nu(\delta\Gamma_{\rho\sigma}^\rho) \quad (5)$$

The variation in the Ricci curvature tensor with respect to the inverse metric tensor can be obtained utilising the metric compatibility of the covariant derivative, $\nabla_\sigma g^{\mu\nu} = 0$; with renaming the dummy indices as

$$g^{\sigma\nu}\delta R_{\sigma\nu} = \nabla_\rho(g^{\sigma\nu}\delta\Gamma_{\nu\sigma}^\rho - g^{\sigma\rho}\delta\Gamma_{\mu\sigma}^\mu) \quad (6)$$

By utilising Stokes' theorem, the total derivative integration for any tensor density is transformed as

$$\begin{aligned} & \iiint_V \nabla_\mu (\sqrt{-g} (g^{\sigma\nu}\delta\Gamma_{\nu\sigma}^\mu - g^{\sigma\mu}\delta\Gamma_{\mu\sigma}^\mu)) dV \\ &= \oint_S \partial_\mu \sqrt{|q|} (g^{\sigma\nu}\delta\Gamma_{\nu\sigma}^\mu - g^{\sigma\mu}\delta\Gamma_{\mu\sigma}^\mu \hat{n}_u) dS = \int K \hat{n}_u \sqrt{|q|} d^3x \end{aligned} \quad (7)$$

where q is the determinant of the induced metric tensor, \hat{n}_u is the normal on the manifold boundary and $K = K_{uv}q^{\mu\nu}$ is the trace of the extrinsic curvature in terms of the second fundamental form. Accordingly, the action in Eq. (1) can be rewritten in parts as

$$S_c + S_b + S_m = \frac{E_D}{2} \left(\int \frac{1}{\mathcal{R}} ([R_{no-boundary}] \sqrt{-g} d^4x + [K] \hat{n}_u \sqrt{|q|} d^3x) \right) + \int [\mathcal{L}_M] \sqrt{-g} d^4x \quad (8)$$

where S_c , S_b , and S_m denote the contributions to the action from the curvature only without the boundary, the boundary alone and the matter fields respectively. Upon the variation in the boundary part in Eq. (8), the variation in the entire action in Eq. (4) is then written as

$$\delta S = \int \left[\frac{E_D}{2} \left(\frac{\frac{R_{uv}\delta g^{\mu\nu}}{\mathcal{R}\delta g^{\mu\nu}} + \frac{(K_{uv}\delta q^{\mu\nu} + q^{\mu\nu}\delta K_{uv} + K\delta q/2q)\hat{n}_u d^{-1}x}{\mathcal{R}\delta q^{\mu\nu}}}{\frac{-\mathcal{R}_{uv}\delta g^{\mu\nu}R}{\mathcal{R}^2\delta g^{\mu\nu}} - \frac{(K_{uv}\delta q^{\mu\nu} + q^{\mu\nu}\delta K_{uv} + K\delta q/2q)R\hat{n}_u d^{-1}x}{\mathcal{R}^2\delta q^{\mu\nu}}} + \frac{R\delta g}{2\mathcal{R}g\delta g^{\mu\nu}} \right) + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \right] \sqrt{-g}\delta g^{\mu\nu} d^4x \quad (9)$$

where $\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g$ according to the Jacobi's formula [5]. The stress energy-momentum tensor is proportional to the Lagrangian term in the action by the definition [4] as

$$\frac{T_{uv}}{2} := \frac{g_{uv}\mathcal{L}_M}{2} - \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = -\frac{\delta g\mathcal{L}_M}{2g\delta g^{\mu\nu}} - \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \quad (10)$$

where the determinant differentiation according to the Jacobi's formula is $\delta g = g g^{\mu\nu}\delta g_{\mu\nu}$, as well as $\delta g^{\mu\nu}g_{\mu\nu} = -\delta g_{\mu\nu}g^{\mu\nu}$ [5]. According to the definition of the modulus E_D , which gives

$$\frac{E_D}{2} = \frac{T_u^v - T\delta_u^v/2}{2R_u^v/\mathcal{R}} = \frac{\mathcal{R}c^4}{16\pi G} \quad (11)$$

By substituting Eqs. (11,10) in Eq. (9), using the principle of least action and choosing the normal \hat{n}_u as a timelike, the extended field equations are

$$R_{uv} - \frac{1}{2}Rg_{uv} - \frac{R}{\mathcal{R}}\mathcal{R}_{uv} + \frac{R-\mathcal{R}}{\mathcal{R}}(K_{uv} - \frac{1}{2}Kq_{uv}) = \frac{8\pi G}{c^4}T_{uv} \quad (12)$$

The conformal curvature term $\frac{R}{\mathcal{R}}\mathcal{R}_{uv}$ is a crucial element in the understanding of the galaxy rotation curves because it accounts for the universal curvature evolution over the conformal time; a simulation

study is presented in Appendix 1. The boundary term behaviour is compatible with black hole entropy calculations [6] and could remove the singularities from the theory. The extended field equations can be categorised into:

1. In a flat/local universe where $\mathcal{R}_{uv} \cong 0$ and $R \approx \mathcal{R}$, they are transformed into the well-known Einstein field equations

$$R_{uv} - \frac{1}{2}Rg_{uv} = \frac{8\pi G}{c^4}T_{uv} \quad (13)$$

2. In a curved universe such in the early universe of a positive pre-existing curvature or at galaxy scales, they could be reduced to

$$R_{uv} - \frac{R}{\mathcal{R}}\mathcal{R}_{uv} - \frac{1}{2}Rg_{uv} = \frac{8\pi G}{c^4}T_{uv} \quad (14)$$

3. For black holes where $R \gg \mathcal{R}$, they can be reduced to

$$\frac{R - \mathcal{R}}{\mathcal{R}}(K_{uv} - \frac{1}{2}Kq_{uv}) = \frac{8\pi G}{c^4}T_{uv} \quad (15)$$

The boundary term in Eqs. (15, 12) is the second approximation of the field equations. The boundary part in Eq. (8) should be divided in parts into the possible induced submanifolds on the spacetime boundary as $S_b = \int S_{b1}d^3x + \int S_{b2}d^2x + \int S_{b3}dx$. However, this only influences the boundary term contribution.

3. Future Work

In the future, the evolution of the conformal curvature tensor \mathcal{R}_{uv} and its scalar curvature \mathcal{R} over the conformal time will be investigated utilising Hamilton's Ricci flow to derive extended Friedmann equations.

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References

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Appendix 1

The conformal curvature term in the new extended field equations represents the background/pre-existing universal curvature evolution over the conformal time. Accordingly, a fluid simulation study has been performed [7] where the background/pre-existing universal curvature evolution over the conformal time is modelled as shown in Figure 1. A perfect fluid was assumed to represent the spacetime continuum, where the divergence in the fluid flux was traced to estimate the generated external momentums on galaxies.

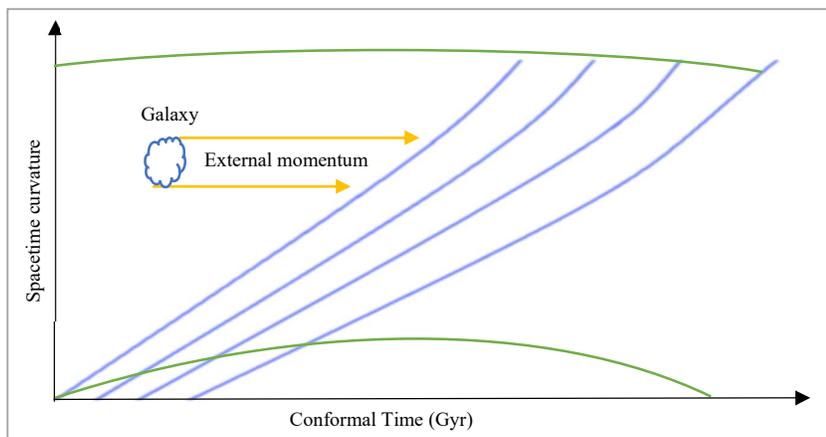


Figure 1. External momentum exerted on a galaxy due to the divergence of spatial curvature through the conformal time over the age of the universe. Green curves represent the curvature of spacetime worldlines. Blue curves represent the simulated continuum flux.

Using the resultant momentums, a simulation of a spiral galaxy as a forced vortex has been created as shown in Figure 2.

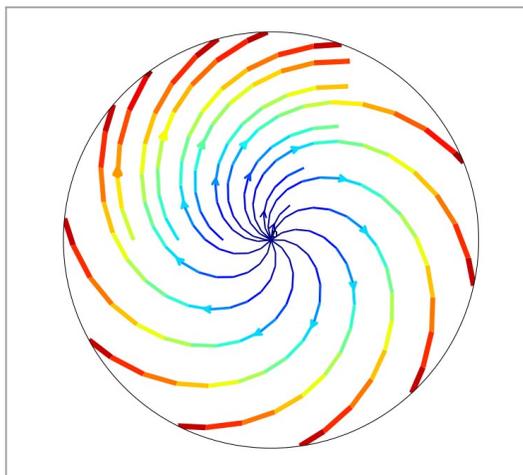


Figure 2. Spiral galaxy rotation. Blue represents the slowest tangential speeds, and the red represents the fastest speeds.

The simulation showed that the tangential speeds of outer parts of the spiral galaxy are faster in comparison with the speeds of inner parts. Based on these results, it could be concluded that the spacetime background/pre-existing curvature evolution over the age of the universe can be responsible for the high speed of stars in galaxies, which can explain the effects attributed to dark matter.