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On Soft Lebesgue Measure

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Abstract

In this article, we introduce the concept of soft intervals, soft ordering and sequences of soft real numbers and study their properties and some interesting results. Also the notion of soft Lebesgue measure on the soft real numbers has been introduced. A correspondence relationship between the soft Lebesgue measure and the Lebesgue measure has been established.

Key words and phrases: Soft Set, Soft Element, Soft Real Number, Soft Interval, Soft Sequence and Soft Lebesgue Measure.

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1 Introduction

Classical mathematical methods are not enough to solve the problems of daily life and also are not enough to meet the new requirements. Therefore presence of uncertainty is one issue which arises in many scientific discipline including our day to day problems. In order to reduce and retrieve information from the uncertainty, there are several theories have been developing and few of established theories such as vague sets (Gau and Buehrer 1993), fuzzy sets (Zadeh 1965) and rough sets (Pawlak 1982) are made. These approaches were regarded as the most famous mathematical instruments in decision modelling. However, these approaches have their own limitations due to the inadequacy of parameters. Fuzzy set theory has been generalized

into soft set theory which is now one of the important branches of modern mathematics. It provides tool to administer various types of uncertainties arising in diverse problems in economics, environmental sciences, sociology etc.
Molodtsov [44](1999) proposed the soft set theory considering ample enough parameters to direct uncertainties. Accordingly, problems with uncertainties becoming easier to tackle using the theory of soft sets. Later, Maji et al. [39] defined operations on soft sets in 2003 and studied the nature of soft sets. Molodtsov et al. (2006) [45], he applied successfully in directions such as smoothness of functions, game theory, operations research, Riemann-Integration, Perron integration, probability and theory of measurement. The first practical application of soft set in decision making problems is presented by Maji et al. [40]. Ali et al. [7] gave some new notions such as restricted intersection, restricted union, restricted difference, and extended intersection of soft sets. Jun [31] applied Molodtsov's notion of soft sets to the theory of BCK/BCI-algebras and introduced the notion of soft BCK/BCI-algebras and soft subalgebras and then investigated their basic properties. Jun and Park [32] dealt with the algebraic structure of BCK/BCIalgebras by applying soft set theory. They introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras and gave several examples. Jun et al. [34] introduced the notion of soft p-ideals and p-idealistic soft BCI-algebras and investigated their basic properties.
In addition to the theory by Molodtsov [44] and Maji et al. [39] several other researchers studied the nature of soft sets and its applications in various real life problems. For example, [20], [17], [33], [35], [37], [40], [49].

In 2007, Aktas and Cagman [4] first introduced the notion of a soft group. It is worth mentioning that Aktas and Cagman introduced the definition of soft group over the soft set defined by Molodtsov [44]. The study of Aktas and Cagman [4] includes soft subgroups, normal soft subgroups and soft homomorphisms. Sezgin et al. [53] corrected some of the problematic cases in a previous paper by Aktas and Cagman [4], and introduced the concept of normalistic soft groups and the homomorphism of normalistic soft groups, studied their several related properties, and investigated some structures that are preserved under normalistic soft group homomorphisms. Sezgin and Atagün [54] proved that certain De Morgan's laws hold in soft set theory with respect to different operations and extended the theoretical aspect of operations on soft sets. The algebraic structure of set theories dealing with uncertainties has also been studied by various mathematicians. Wen [65], Yuan Xuehai's graduate student, presented the new definitions of soft subgroups and normal soft subgroups and obtained some primary results.

And consequently several other researchers ([5], [46], [48]) have extended the idea of soft group following the definition of soft group by Aktas and Cagman. Most of the papers on soft group, devoted to present the definition and properties of the soft groups analogous to that of ordinary groups. In 2008, Feng et al. [19] introduced the notation of soft semirings, soft ideals and idealistic soft semirings and investigated several related properties. Also Soft rings and soft ideals are defined by U.Acar, F.Koyuncu, B.Tanay [2] in (2010) and discussed their basic properties. Since then some researchers Y.B.Jun [31] and Celik ea al. [16] have studied other soft algebraic structures as well as their properties.

In 2011, N.Cagman, S.Karatas, S.Enginogiu introduced soft topology in [15] and M.Shabir, M.Naz defined soft topological spaces in [55]. They defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft closure, soft neighborhood of a point, soft T_i-spaces, for $i = 1; 2; 3; 4$, soft regular spaces, soft normal spaces and established their several properties. In 2011, S. Hussain and B. Ahmad [29] continued investigating the properties of soft open(closed), soft neighborhood and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary. Also in 2012, B. Ahmad and S. Hussain [3] explored the structures of soft topology using soft points. A. Kharral and B. Ahmad [36], defined and discussed the several properties of soft images and soft inverse images of soft sets. They also applied these notions to the problem of medical diagnosis in medical systems. Aygunoglu and Aygun [8] introduced the soft continuity of soft mapping, soft product topology and studied soft compactness and generalized Tychono theorem to the soft topological spaces. Min [43] gave some results on soft topological spaces. Zorlutuna et al. [68] also investigated soft interior point and soft neighbourhood. There are several literature available on the structure of soft topological spaces [22], [23], [41], [48], [64], [6], [9], [10], [14], [22], [23], [27], [28], [30], [41], [55], [56], [58] [60], [63], [64], [**?**] and extended the idea of soft topology according to the definition of soft topology by M.Shabir, M.Naz and N.Cagman, S.Karatas, S.Enginogiu.

In this article, we have introduced soft real numbers as real functions of real variable. So these classical functions behave like soft elements and their collection behave like soft set. Now on this soft set a soft topology is defined. This topology is good enough to study the properties of soft Lebesgue measure, which is also introduced and studied. To the best of our knowledge, measure on soft sets is defined here is first of its kind. The

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introduction of classical measure theory can be referred [69], [70], [71].

2 Preliminaries

Let X be an universal set, A be parametric set and $P(X)$ be the power set of X.

Definition 2.1. [44] A soft set (F, A) is a mapping $F : A \to P(X)$. We write F for the soft set (F, A).

Definition 2.2. [51] A function $a : A \to X$ is called soft element of a soft set F if $a(t) \in F(t)$ for all $t \in A$ and $F(t) \neq \phi$ for all $t \in A$.. In this case we write $a \in_s F$.

Definition 2.3. [51] Let F be a soft set then the collection of all soft elements of F is denoted by $SE(F)$.
That is $SE(F) = \{a : a : A \to X, a(t) \in F(t), \forall t \in A \}$. Hence $SE(F)$ is defined for those soft sets F such that $F(t) \neq \phi$ for all $t \in A$.

Definition 2.4. [39] Let F and H be two soft sets over an universal set X then H is said to be a soft subset of F if $H(t) \subseteq F(t)$ for all $t \in A$. In this case we denote $H \subset_s F$.

Definition 2.5. [39] Let F and H be two soft sets then the soft union $F \cup_s H$ and the soft intersection $F \cap_s H$ are defined by $(F \cup_s H)(t) = F(t) \cup H(t)$ and $(F \cap_s H)(t) = F(t) \cap H(t)$ for all $t \in A$.

Definition 2.6. [39] Two soft sets F and H are said to be soft equal if $F(t) = H(t)$ for all $t \in A$ and it is denoted by $F =_s H$. Also $a, b \in_s F$ are said to be soft equal if $a(t) = b(t)$ for all $t \in A$.

3 Soft Real Sets, Soft Real Numbers and Some Definitions

From this section, we take $[0, 1]$ as a parametric set and $\mathbb{R}$ as a universal set. Also we have introduced the notation of collection of all soft real numbers, collection of all soft real set, soft interval and investigated its structural properties. Now, we define the soft real number, soft real set and soft intervals on the universal set $\mathbb{R}$ as follows.

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Definition 3.1. Let $N(\mathbb{R}) = \{\epsilon : \epsilon : [0,1] \to \mathbb{R}\}$ and $S(\mathbb{R}) = \{\delta : \delta : [0,1] \to \mathcal{P}(\mathbb{R})\}$. $N(\mathbb{R})$ is collection of all soft real numbers and $S(\mathbb{R})$ is a collection of all soft real set. Clearly $\mathbb{R} \subset N(\mathbb{R})$ and $P(\mathbb{R}) \subset S(\mathbb{R})$ [taking the constant as function].

Definition 3.2. Let $N(\mathbb{N}) = \{\epsilon : \epsilon : [0,1] \to \mathbb{N}\}$. Then $N(\mathbb{N})$ is a collection of all soft natural numbers. Here $\mathbb{N} \subset N(\mathbb{N})$ and $N(\mathbb{N}) \subset N(\mathbb{R})$.

Note 3.3. *Since* $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, *so we define* $N(\mathbb{R}^*)$ *by* $N(\mathbb{R}^*) = \{\epsilon : \epsilon : [0,1] \to \mathbb{R}^*\}$.

Example 3.4. Any many valued function defined on $[0,1]$ can be treated as a soft real set.

Example 3.5. $\sin^{-1} x$, $\cos^{-1} x$, $\log x$ with slide modification can be considered as a soft real sets.

Example 3.6. Suppose $\Im$ is the collection of all sub intervals $[0,1]$ with zero as lower end point and $\ell(I)$ is the length of the interval I then $\ell^{-1} : [0,1] \to \Im$ is soft real sets.

Theorem 3.7. $A \subset N(\mathbb{R})$ *if and only if* $A \in S(\mathbb{R})$.

Proof. Let $A \subset N(\mathbb{R})$. Define $\delta_A : [0,1] \to P(\mathbb{R})$ by $\delta_A(t) = \{\eta(t) : \eta \in A\}$. So $\delta_A \in S(\mathbb{R})$ and we write A for δ_A.
If $A \in S(\mathbb{R})$ then $\alpha \in_s A$ implies $\alpha(t) \in A(t)$ for all $t \in [0,1]$. Hence $\alpha : [0,1] \to \mathbb{R}$ and so $\alpha \in N(\mathbb{R})$. Therefore $A \subset N(\mathbb{R})$. □

Definition 3.8. Let $a, b \in N(\mathbb{R})$. We define $a \leq_s b$ if $a(t) \leq b(t)$ for all $t \in [0,1]$. If $a \leq_s b$ and $a(t) < b(t)$ for some t, we say $a <_s b$. Here $' \leq_s'$ is only partial order not totally order.

Definition 3.9. Let $a, b \in N(\mathbb{R})$. We define

$$Max_s\{a,b\} = Max\{a(t), b(t)\} = a \vee b$$
$$and\ Min_s\{a,b\} = Min\{a(t), b(t)\} = a \wedge b.$$

So $a \vee b, a \wedge b \in N(\mathbb{R})$. Clearly $a \vee b \leq_s a \wedge b$ also if $a \neq_s b$ (i.e. $a(t) \neq b(t)$ for some t) then $a \vee b <_s a \wedge b$.

Definition 3.10. Let $A \subset N(\mathbb{R})$, A is called bounded if A is uniformly bounded i.e. if there is $M \in \mathbb{R}$ such that $\mid \eta(t) \mid < M$ for all $\eta \in A$ and $t \in [0,1]$. Suppose A is bounded then define $\epsilon(t) = Sup\{\eta(t) : \eta \in A\}$ so $\varepsilon(t) \neq \infty$ for all t and $\epsilon \in \mathbb{R}$, ϵ is least upper bound of A. So $N(\mathbb{R})$ is complete.

Definition 3.11. For any $a, b \in N(\mathbb{R})$ where $a \neq_s b$. We define the soft open interval $(a,b)_s =_s \{\epsilon \in N(\mathbb{R}) : a \wedge b <_s \epsilon <_s a \vee b\}$. By Theorem 3.7, soft open interval is a soft set. Representation of the soft open interval is not unique i.e. it may happens that $(a,b)_s =_s (c,d)_s$ but $a \neq_s c, b \neq_s d$.

Definition 3.12. The soft identity real number I_s defined by $I_s = 1$ for all $t \in [0,1]$.

Definition 3.13. Let $a, b \in N(\mathbb{R})$ then
(i) the soft addition of a and b is $a \oplus b$ and it is defined by $(a \oplus b)(t) = a(t) + b(t)$ for all $t \in [0,1]$.
(ii) the soft substraction of a and b is $a \ominus b$ and it is defined by $(a \ominus b)(t) = a(t) - b(t)$ for all $t \in [0,1]$.
(iii) the soft multiplication of a and b is $a \odot b$ and it is defined by $(a \odot b)(t) = a(t) \cdot b(t)$ for all $t \in [0,1]$.
(iv) the soft division of a and b is $a \oslash b$ and it is defined by $(a \oslash b)(t) = \frac{a(t)}{b(t)}$ for all $t \in [0,1]$.

Definition 3.14. Let $\beta = \{(a,b)_s : a, b \in N(\mathbb{R})\}$ then β is a basis of some topology τ of $N(\mathbb{R})$. For if $x \in N(\mathbb{R})$ then $x \in (x \ominus I_s, x \oplus I_s)_s$ and I_s is a constant function so $(x \ominus I_s, x \oplus I_s)_s$ is a soft open interval. So $\bigcup\limits_{A \in \beta} A = N(\mathbb{R})$.
Again if $(a,b)_s, (c,d)_s$ are two soft open interval and $P \in (a,b)_s \cap_s (c,d)_s$ then $a \wedge b <_s P <_s a \vee b$ and $c \wedge d <_s P <_s c \vee d$.
Clearly, $\alpha = (a \wedge b) \vee (c \wedge d) <_s P <_s (a \vee b) \wedge (c \vee d) = \gamma$. Hence $P \in (\alpha, \gamma)_s \subset_s (a,b)_s \cap_s (c,d)_s$.

Definition 3.15. Let $N(\mathbb{R}) = \{\epsilon : \epsilon : [0,1] \to \mathbb{R}\}$. Therefore $\mathbb{R} \subset N(\mathbb{R})$. So $N(\mathbb{R})$ contains $\mathbb{R}$ as subspace and the subspace topology on $\mathbb{R}$ induced by the topology of $N(\mathbb{R})$ is a usual topology of $\mathbb{R}$.

Proposition 3.16. *$(N(\mathbb{R}), \tau)$ is Hausdorff topological space.*

Proof. Let ϵ_1 and ϵ_2 be any two distinct elements in $N(\mathbb{R})$. So $\exists\, t_1 \in [0,1]$ such that $\epsilon_1(t_1) \neq \epsilon_2(t_1)$. Suppose $\epsilon_1(t_1) < \epsilon_2(t_1)$. Let $k = \frac{\epsilon_2(t_1) - \epsilon_1(t_1)}{2}$. So the open intervals $(\epsilon_1(t_1) - k, \epsilon_1(t_1) + k)$ and $(\epsilon_2(t_1) - k, \epsilon_2(t_1) + k)$ are disjoint. Hence the soft open intervals $(\epsilon_1 \ominus k_s, \epsilon_1 \oplus k_s)_s$ and $(\epsilon_2 \ominus k_s, \epsilon_2 \oplus k_s)_s$, where $k_s : [0,1] \to \mathbb{R}$ such that $k_s(t) = k$ for all $t \in [0,1]$, are disjoint and contain ϵ_1 and ϵ_2 respectively. Hence $(N(\mathbb{R}), \tau)$ is Hausdorff topological space. $\square$

Proposition 3.17. *$(N(\mathbb{R}), \tau)$ is 1^{st} countable topological space.*

Proof. Let $a \in N(\mathbb{R})$. For each $n \in \mathbb{N}$ define soft real number $1 \oslash n : [0,1] \to \mathbb{R}$ by $(1 \oslash n)(t) = \frac{1}{n}$ for all $t \in [0,1]$. Let U_n be soft open intervals defined by $U_n =_s (a \ominus (1 \oslash n), a \oplus (1 \oslash n))_s$. Then $\{U_n : n \in \mathbb{N}\}$ forms a countable local base at a. Thus $(N(\mathbb{R}), \tau)$ is 1^{st} countable topological space. □

Definition 3.18. Let $a, b \in N(\mathbb{R})$. We define length of the interval $(a,b)_s$ as a soft real no $a \vee b \ominus a \wedge b$.

Definition 3.19. A soft real number $a \in N(\mathbb{R})$ is said to be positive soft real number if $a(t) \geq 0$ for all $t \in [0,1]$ and $a(t) > 0$ for at least one t.
Similarly, a soft real number $a \in N(\mathbb{R})$ is said to be negative soft real number if $a(t) \leq 0$ for all $t \in [0,1]$ and $a(t) < 0$ for at least one t.

Definition 3.20. The elements of $N(\mathbb{R}) - N(\mathbb{R}^*)$ will be called soft zero elements, where $\mathbb{R}^* = \mathbb{R} - \{0\}$.
Let $\epsilon : [0,1] \to \mathbb{R}$ be a soft real number. If $\epsilon(t) = 0$ for all $t \in [0,1]$ then ϵ is said to be absolute soft zero element and it is denoted by 0_s.

Definition 3.21. Let $\epsilon \in \mathbb{R}$. Suppose $\epsilon_c : [0,1] \to \mathbb{R}$ such that $\epsilon_c(t) = \epsilon$ for all $t \in [0,1]$. Such soft real numbers are called constant soft real numbers.

Definition 3.22. ∞_s is a set of functions defined by $\infty_s = \{a : A \to \mathbb{R}^+ \cup \{\infty\}$ such that $a(t) = \infty$ for at least one $t \in A\}$.

Definition 3.23. Let $x, y \in N(\mathbb{R})$. Then the addition, difference, product and division are defined as usual considering x, y are functions.

Note 3.24. *Let $x, y \in N(\mathbb{R})$ and $x >_s y >_s 0_s$ so $x(t) > y(t) > 0$ for some $t \in [0,1]$. Let $n_t \in \mathbb{N}$ such that $y(t).n_t > x(t)$. Define $n : [0,1] \to \mathbb{N}$ such that*

$$\begin{aligned} n(t) &= n_t, \quad \textit{for} \quad \textit{some} \quad t \in [0,1] \quad \textit{such} \quad \textit{that} \quad x(t) > y(t) \\ &= I_s, \quad \textit{otherwise} \end{aligned}$$

Then clearly $y \odot n >_s x$, which is Archimedean property on soft real numbers.

Definition 3.25. Let $x \in N(\mathbb{R})$. For any $n \in [0,1]$ we define the n^{th} integral power x^n defined by $x^n(t) = [x(t)]^n$ for all $t \in [0,1]$. Obviously, x^n is a soft real number.

Theorem 3.26. *[51] For any $x \in N(\mathbb{R}^*)$, $x^{-1}(t) = [x(t)]^{-1}$ for all $t \in [0,1]$.*

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4 Sequences of Soft Real Numbers

In this section, we have studied sequence of soft real numbers and investigated some interesting results on it.

Definition 4.1. A sequence of soft real numbers is a function from $\mathbb{N}$ to $N(\mathbb{R})$.

Definition 4.2. Let $\epsilon > 0$ be a real number. Suppose $\epsilon_c : [0,1] \to \mathbb{R}$ such that $\epsilon_c(t) = \epsilon$ for all $t \in [0,1]$. Let $\{x_n\}$ be a sequence of soft real numbers. $\{x_n\}$ is said to be convergent to a soft real number l if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s$ for all $n \geq N$.
We say that l is a soft limit of the sequence of soft real numbers $\{x_n\}$ and defined by $x_n \to_s l$.

Definition 4.3. A sequence of soft real numbers $\{x_n\}$ is bounded if there exists two soft real numbers m and M such that $m \leq_s x_n \leq_s M$.

Theorem 4.4. *Let $\{x_n\}$ be a sequence in $N(\mathbb{R})$. Then $\{x_n\}$ converges to a soft real number l if and only if $x_n(t) \to l(t)$ for all $t \in [0,1]$.*

Proof. Let $\{x_n\}$ converges to a soft real number l so for the $\epsilon_c(t) = \epsilon$ for all $t \in [0,1]$, $(\epsilon > 0)$ there exists $N \in \mathbb{N}$ such that $x_n \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s$ for all $n \geq N$. So $x_n(t) \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s(t)$ for all $t \in [0,1]$ and for all $n \geq N$ which implies that $x_n(t) \in (l(t) - \epsilon, l(t) + \epsilon)$ for all $t \in [0,1]$ and for all $n \geq N$. Hence $x_n(t) \to l(t)$ as $n \to \infty$.
For the converse part, the proof is easy and follows from Definition 4.2. □

Note 4.5. *The soft real numbers l is a unique, since $l(t)$ is unique for each $t \in [0,1]$. Also as $\{x_n(t)\}$ is bounded, we get for each two real numbers m_t and M_t such that $m_t \leq x_n(t) \leq M_t$. Now let two soft real numbers m and M which is defined by $m(t) = m_t$ and $M(t) = M_t$ for all $t \in [0,1]$. Then m and M are bounds of $\{x_n\}$. So we can conclude: $\{x_n\}$ is bounded if and only if $\{x_n(t)\}$ is bounded.*

Theorem 4.6. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $N(\mathbb{R})$ converges to the soft real numbers x and y respectively, then*
(1) $x_n \oplus y_n \to_s x \oplus y$
(2) $x_n \ominus y_n \to_s x \ominus y$
(3) $x_n \odot y_n \to_s x \odot y$
(4) $x_n \oslash y_n \to_s x \oslash y$

Proof. The proof of the theorem follows from Theorem 4.4. □

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5 Soft Lebesgue Measure

In this section, we have introduced the notation of soft Lebesgue measure and studied its analogous properties.

Definition 5.1. Let $A \subset N(\mathbb{R})$, a collection $\{(a_n, b_n)_s : n \in I\}$ is soft open cover of A if $A \subset_s \bigcup_{n\in I}(a_n, b_n)_s$.

Now we have, each soft real number is an uncountable collection of real numbers. Each soft set is collection of real sets and each soft open interval $(a, b)_s$ is collection of open intervals $\{((a \wedge b)(t), (a \vee b)(t)) : t \in \mathbb{R}\}$.

Lemma 5.2. *Let $A \subset N(\mathbb{R})$ and $\{(a_n, b_n)_s : n \in I\}$ be a soft open cover of A then $\{(a_n, b_n)_s(t)\} = \{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ be a cover of $A(t)$ for all $t \in [0, 1]$.*

Proof. Let $\alpha \in A(t)$, so there exists $y \in_s A$ such that $y(t) = \alpha$. Since $\{(a_n, b_n)_s : n \in I\}$ is a soft cover of A then $y \in_s (a_n, b_n)_s$ for some n. Therefore $\alpha = y(t) \in (a_n, b_n)_s(t) = ((a_n \wedge b_n)(t), (a_n \vee b_n)(t))$. Now since $\alpha \in A(t)$ is arbitrary then $\{(a_n, b_n)_s(t)\} = \{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ is a cover of $A(t)$ for all $t \in [0, 1]$. □

Definition 5.3. Let $A \subset N(\mathbb{R})$, soft Lebesgue outer measure of A is defined by $\mu_s^*(A) = inf\{\sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n)_s : \{(a_n, b_n)_s\}$ is a collection of soft open intervals which cover $A\}$.

Note 5.4. *As $(a_n \vee b_n \ominus a_n \wedge b_n)$ is a function from $[0, 1]$ to $\mathbb{R}$, the summation is well defined and is a soft real number. So the infimum is taken over a set of soft real numbers. Since this set is uniformly bounded by zero, the infimum exists. Hence $\mu_s^* : S(\mathbb{R}) \to [0_s, \infty_s)_s$.*

Theorem 5.5. *Let $A \subset N(\mathbb{R})$, $\mu_s^*(A)$ be soft Lebesgue outer measure of A and $\mu^*(A(t))$ be Lebesgue outer measure of $A(t)$. Then $\mu_s^*(A)(t) = \mu^*(A(t))$ for all $t \in [0, 1]$.*

Proof. Let $t_0 \in [0, 1]$ be fixed. Let $\{(\alpha_n, \beta_n)\}$ be a cover of $A(t_0)$ and $\{(a_n, b_n)_s\}$ be a soft open cover of A. For each $n \in \mathbb{N}$ define soft real numbers c_n and d_n by

$$\begin{aligned} c_n(t) &= ((a_n \wedge b_n)(t) \quad \text{if} \quad t \neq t_0 \\ &= \alpha_n \quad \text{if} \quad t = t_0 \qquad \text{and} \end{aligned}$$

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$$d_n(t) = ((a_n \vee b_n)(t) \quad \text{if} \quad t \neq t_0$$
$$= \beta_n \quad \text{if} \quad t = t_0.$$

Then clearly $\{(c_n, d_n)_s\}$ be a collection of soft intervals and is a soft cover of A. So $\mu_s^*(A)(t_0) \leq (\sum_{n=1}^{\infty}(d_n \ominus c_n))(t_0) = \sum_{n=1}^{\infty}(d_n(t_0) - c_n(t_0)) = \sum_{n=1}^{\infty}(\beta_n - \alpha_n)$. Taking infimum of all such collection $\{(\alpha_n, \beta_n)\}$ that cover $A(t_0)$, we get $\mu_s^*(A)(t_0) \leq \mu^*(A(t_0))$. Since $t_0 \in [0,1]$ is arbitrary, we have $\mu_s^*(A)(t) \leq \mu^*(A(t))$ for all $t \in [0,1]$.
Now suppose $\{(a_n, b_n)_s\}$ be a soft open cover of A then $\{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ be a cover of $A(t)$ for all $t \in [0,1]$. So $\mu^*(A(t)) \leq (\sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n))(t)$ for all $t \in [0,1]$. Consider a soft real number s such that $s(t) = \mu^*(A(t))$. Hence $s \leq_s \sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n)$. Taking infimum of all collection of soft open intervals $\{(a_n, b_n)_s\}$ that cover the soft real set A, we conclude that $s \leq_s \mu_s^*(A)$ i.e., $\mu^*(A(t)) \leq \mu_s^*(A)(t)$ for all $t \in [0,1]$. This complete the proofs. □

Corollary 5.6. *Let $A \subset N(\mathbb{R})$. $\mu_s^*(A)$ is soft Lebesgue outer measure of A if and only if $\mu^*(A(t))$ is Lebesgue outer measure of $A(t)$ for all $t \in [0,1]$.*

Proof. The proof follows from Theorem 5.5. □

Theorem 5.7. *If μ_s^* is soft Lebesgue outer measure then*
(i) $\mu_s^*(\Phi) =_s 0_s$.
(ii) $A \subset_s B \subset N(\mathbb{R})$ then $\mu_s^*(A) \leq_s \mu_s^*(B)$.
(iii) If $\{A_n\}$ is a sequence of subsets of $N(\mathbb{R})$ then $\mu_s^*(\bigcup_{n=1}^{\infty} A_n) \leq_s \sum_{n=1}^{\infty} \mu_s^*(A_n)$.

Proof. (i) From Theorem 5.5, we have $\mu_s^*(\Phi)(t) = \mu^*(\Phi(t)) = \mu^*(\phi) = 0 = 0_s(t)$ for all $t \in [0,1]$. Therefore $\mu_s^*(\Phi) =_s 0_s$.
(ii) $A \subset_s B$ implies that $A(t) \subset B(t)$ for all $t \in [0,1]$. Then by Theorem 5.5, we have $\mu_s^*(A)(t) = \mu^*(A(t)) \leq \mu^*(B(t)) = \mu_s^*(B)(t)$ for all $t \in [0,1]$. Therefore $\mu_s^*(A) \leq_s \mu_s^*(B)$.
(iii) The results follows from Theorem 2.5 and Theorem 5.5. □

Definition 5.8. Let $A \subset N(\mathbb{R})$. Then A is said to be soft Lebegue measurable if for each $A_1 \subseteq N(\mathbb{R})$ such that $\mu_s^*(A_1) =_s \mu_s^*(A \cap A_1) + \mu_s^*(A^c \cap A_1)$. This definition shows that if a soft set A is soft Lebesgue measurable then

$A(t)$ is lebesgue measurable for all $t \in [0,1]$ and the converse is also true. So the next theorems are trivial.

Theorem 5.9. *(i) Φ and $N(\mathbb{R})$ are both soft measurable and A is soft measurable implies A^c is also soft measurable.*
(ii) Let $A \subset N(\mathbb{R})$. If $\mu_s^*(A) =_s 0_s$ then A is soft measurable.
(iii) Let $x_0 \in N(\mathbb{R})$ and $A \subset N(\mathbb{R})$. If A is soft measurable then $A \oplus x_0$ is also soft measurable.
(iv) Every soft interval is soft measurable.
(v) If $\{A_i\}$ is a sequence of soft measurable sets in $N(\mathbb{R})$ then $\bigcap\limits_{i=1}^{\infty} A_i$ and $\bigcup\limits_{i=1}^{\infty} A_i$ are soft measurable and moreover if $\{A_i\}$ are arbitrary sequence of disjoint soft measurable sets in $N(\mathbb{R})$ then $\mu_s^*(\bigcup\limits_{i=1}^{\infty} A_i) =_s \sum\limits_{i=1}^{\infty} \mu_s^*(A_i)$.

Theorem 5.10. *Let $\{A_n\}$ be an arbitrary sequence of soft measurable sets in $N(\mathbb{R})$.*
(a) If $A_n \subseteq_s A_{n+1}$ $\forall$ $n \in \mathbb{N}$ and $A =_s \bigcup\limits_{i=1}^{\infty} A_i$ then $\mu_s(A) =_s \lim\limits_{n\to\infty} \mu_s(A_n)$.
(b) Suppose that $\mu_s(A_1)$ is finite. If $A_{n+1} \subseteq_s A_n$ $\forall$ $n \in \mathbb{N}$ and $A =_s \bigcap\limits_{i=1}^{\infty} A_i$ then $\mu_s(A) =_s \lim\limits_{n\to\infty} \mu_s(A_n)$.

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