On Soft Lebesgue Measure

S.Goldar[1] and S.Ray[2]

[1] Department of mathematics, Pingla Thana Mahavidyalaya, Maligram, West Midnapore, West Bengal - 721140, India,
Email : sujaygoldar@gmail.com
[2] Department of mathematics, Visva-Bharati university, Santiniketan - 731235, West Bengal, India,
Email : subhasis.ray@visva-bharati.ac.in

Abstract

In this article, we introduce the concept of soft intervals, soft ordering and sequences of soft real numbers, and some of their structural properties are studied. The notion of soft Lebesgue measure on the soft real numbers has been introduced. Also, a correspondence relationship has been established between the soft Lebesgue measure and the classical Lebesgue measure. Furthermore, we have studied some exciting results and relations between the soft Lebesgue measure and the Lebesgue measure of soft real sets.

Key words and phrases: Soft Set, Soft Element, Soft Real Number, Soft Interval, Soft Sequence, Soft Lebesgue Measure.

MSC 2010 No.: 54A40, 22A05, 54E15, 03E72, 08A72.

1 Introduction

Classical mathematical methods are not enough to solve the problems of daily life and also are not enough to meet the new requirements. Therefore the presence of uncertainty is one issue that arises in many scientific disciplines, including our daily problems. Several theories have been developed to reduce and retrieve information from the uncertainty, and few established theories such as vague sets, fuzzy sets and rough sets are made. These approaches were regarded as the most famous mathematical instruments in decision modelling. However, these approaches have their limitations due

 doi:10.20944/preprints202010.0374.v2

to the inadequacy of parameters. Fuzzy set theory has been generalized into soft set theory which is now one of the essential branches of modern mathematics. It provides a tool to administer various types of uncertainties arising in diverse problems in economics, environmental sciences, sociology etc.
Molodtsov [33] proposed the soft set theory considering ample enough parameters to direct uncertainties. Accordingly, problems with uncertainties become easier to tackle using the theory of soft sets. Later, Maji et al. [30] defined operations on soft sets in 2003 and studied the nature of soft sets. Molodtsov et al. [34] applied successfully in directions such as smoothness of functions, game theory, operations research, Riemann-Integration, Perron integration, probability and theory of measurement. The first practical application of soft set in decision-making problems is presented by Maji et al. [31]. Ali et al. [6] gave some new notions such as restricted intersection, restricted union, restricted difference, and the extended intersection of soft sets. Jun [24] applied Molodtsov's idea of soft sets to the theory of BCK/BCI-algebras and introduced the notion of soft BCK/BCI-algebras and soft subalgebras, and then investigated their basic properties. Also, the combination of soft sets and rough sets was first explored in [15]. Along this line of study, there are some more recent works such as [4], [5], [40].

In 2007, Aktas and Cagman [3] first introduced the notion of a soft group. It is worth mentioning that Aktas and Cagman introduced the definition of the soft group over the soft set defined by Molodtsov [33]. The study of Aktas and Cagman [3] includes soft subgroups, normal soft subgroups and soft homomorphisms. Wen [44], Yuan Xuehai's graduate student, presented the new definitions of soft subgroups and normal soft subgroups and obtained some preliminary results. And consequently, several other researchers have extended the idea of the soft group following the definition of the soft group by Aktas and Cagman. Most of the papers on the soft group are devoted to presenting the definition and properties of the soft groups analogous to that of ordinary groups. Also, Soft rings and soft ideals are defined by Acar et al. [1] in (2010) and discussed their basic properties. Since then, some researchers, Jun [24] and Celik et al. [12], have studied other soft algebraic structures and their properties.

In 2011, Cagman et al. introduced soft topology in [11], and Shabir, Naz defined soft topological spaces in [39]. They defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft closure, the soft neighbourhood of a point, soft T_i-spaces, for $i = 1; 2; 3; 4$, soft regular spaces, soft normal spaces and established their several properties. In 2011, Hussain and Ahmad [23] continued investigating the properties of soft open(closed), soft neighbourhood and soft closure. They also defined and discussed the properties of the soft interior, soft exterior and soft boundary. Also, in 2012, Ahmad and Hussain [2] explored the structures of soft topology using soft points. Aygunoglu and Aygun [7] introduced the soft continuity of soft mapping soft product topology, studied soft compactness, and generalized the Tychonoff theorem to the soft topological spaces. Min [32] gave some results on soft topological spaces. There are several literature available on the structure of soft topological spaces [8], [10], [17], [18], [22], [39], [41], [43] and extended the idea of soft topology according to the definition of soft topology by Shabir, Naz and Cagman et al.

In recent years the development of soft set theory and its application has been taking place at a rapid pace. Kharral and Ahmad [28] defined and discussed several properties of soft images and soft inverse images of soft sets. The notion of soft images and soft inverse images further applied to the problem of medical diagnosis in medical fields. Also, Kamaci et al. [25], Khan et al. [26], [27], Zhan et al. [47], Zhao et al., and [48] pointed out several other applications of soft set theory to solve various decision-making problems.

In this article, we have introduced soft real numbers as real functions of a real variable. So these classical functions behave like soft elements, and their collection behaves like the soft set. A soft topology is defined over the soft real sets. This topology is good enough to study the properties of soft Lebesgue measure, which is also introduced and studied in this manuscript. Furthermore, a correspondence relationship has been established between every soft Lebesgue measure and ordinary Lebesgue measure of a soft real set, using the notion of soft real numbers. Some exciting results and relations between soft Lebesgue measure and ordinary Lebesgue measure of soft real sets are studied. To the best of our knowledge, the Lebesgue measure on soft sets is defined here as the first of its kind.

Basic preliminaries on soft sets and operations on soft sets are discussed in Section 2. Furthermore, soft real sets, soft real numbers and some definitions are discussed in Section 3. In Section 4, notion of sequence of soft real numbers are presented. Soft Lebesgue measure is defined in the Section 5. At the end, in section 6, a conclusion and future endeavours are drawn.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 December 2022 doi:10.20944/preprints202010.0374.v2

2 Preliminaries

Let X be an universe of discourse, A be parameter space and $P(X)$ be the power set of X.

Definition 2.1. [33] Let X be a universe of discourse and A be a set of parameters. Let $P(X)$ denote the power set of X and E be a subset of A. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$.

Definition 2.2. [30] For two soft sets (F, A) and (G, B) over a common universe of discourse X, we say that (F, A) is a soft subset of (G, B) if
(i) $A \subseteq B$, and
(ii)for all $t \in A$, $F(t)$ and $G(t)$ are identical approximations.
In this case we write $(F, A) \subset_s (G, B)$.
(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A) \supset_s (G, B)$.

Definition 2.3. [30] Two soft sets (F, A) and (G, B) over a common universe of discourse X are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4. [30] Let (F, A) and (H, B) be two soft sets over a universe of discourse X.
(i) The extended union $(F, A) \cup_s (H, B)$ is defined as the soft set (G, C), where $C = A \cup B$ and

$$\begin{aligned} G(t) &= F(t) \cup H(t) \quad \text{if} \quad t \in A \cup B \\ &= F(t) \quad \text{if} \quad t \in A - B \\ &= H(t) \quad \text{if} \quad t \in B - A \end{aligned}$$

(ii) The restricted intersection $(F, A) \cap_s (H, B)$ is defined as the soft set (G, C), where $C = A \cap B$ and
$G(t) = F(t) \cap H(t)$ for all $t \in C$.

Definition 2.5. [29] The soft sets are redefined as follows: Let A be the set of parameters and $E \subseteq A$. Then for each soft set (F, E) over X a soft set (H, E) is constructed over X, where $\forall \alpha \in A$,

$$H(\alpha) = \begin{cases} F(\alpha) & if\, \alpha \in E \\ \phi & if\, \alpha \in A \setminus E. \end{cases}$$

Thus the soft sets (F, E) and (H, A) are equivalent to each other and the usual set operations of the soft sets $(F_i, E_i), i \in \Delta$ is the same as those of the soft sets $(H_i, A), i \in \Delta$.

Definition 2.6. [35] Let $SS(X, A)$ be the collection of all soft sets over X with the parameter set A. Let $(F, A), (G, A) \in SS(X, A)$. Then

(i) (F, A) is said to be a soft subset of (G, A) if $F(t)$ is a subset of $G(t)$, $\forall t \in A$. It is denoted by $(F, A) \subseteq_s (G, A)$.

(ii) (F, A) is said to be soft equal to (G, A) if $F(t) = G(t)$, $\forall t \in A$. It is denoted by $(F, A) =_s (G, A)$.

(iii) the complement or relative complement of a soft set (F, A), denoted by $(F, A)^C$, is defined by $(F, A)^C =_s (F^C, A)$, where $F^C(t) = X \setminus F(t), \forall t \in A$.

(iv) the difference of two soft sets (F, A) and (G, A), denoted by $(F, A) -_s (G, A)$, is defined by $(F, A) -_s (G, A) = (F -_s G, A)$, where $[F -_s G](t) = F(t) \setminus G(t), \forall t \in A$.

(v) (F, A) is said to be the null soft set, denoted by (Φ, A), if $F(t) = \phi, \forall t \in A$.

(vi) (F, A) is said to be the absolute soft set, denoted by (X_s, A), if $F(t) = X, \forall t \in A$.

Definition 2.7. [35] Let (F_i, A) be a non-empty family of soft sets over a common universe X. Then their

(i) intersection, denoted by $\underset{i\in\triangle}{\cap_s}$, is defined by $\underset{i\in\triangle}{\cap_s}(F_i, A) = (\underset{i\in\triangle}{\cap_s} F_i, A)$, where $(\underset{i\in\triangle}{\cap_s} F_i)$
$(t) = \underset{i\in\triangle}{\cap}(F_i(t)), \forall t \in A$.

(ii) union, denoted by $\underset{i\in\triangle}{\cup_s}$, is defined by $\underset{i\in\triangle}{\cup_s}(F_i, A) = (\underset{i\in\triangle}{\cup_s} F_i, A)$, where $(\underset{i\in\triangle}{\cup_s} F_i)(t) = \underset{i\in\triangle}{\cup}(F_i(t)), \forall t \in A$.

Note 2.8. *[33] Every fuzzy set A may be interpreted as a soft set $F_A : [0, 1] \to X$ where $F_A(t) = \{x \in X : A(x) \geq t\}, t \in [0, 1]$. Thus from a fuzzy set A we get a soft set $(F_A, [0, 1])$.*
Further from the soft set $(F_A, [0, 1])$ the given fuzzy set A can be reconstructed

by :
$$A(x) = \sup_{\substack{t\in[0,1] \\ x\in F_A(t)}} t$$
Thus the idea of soft set is more general than the idea of fuzzy set.

Definition 2.9. [38] A function $a : A \to X$ is called soft element of a soft set F if $a(t) \in F(t)$ for all $t \in A$ and $F(t) \neq \phi$ for all $t \in A$.. In this case we write $a \in_s F$.

Definition 2.10. [38] Let F be a soft set then the collection of all soft elements of F is denoted by $SE(F)$.
That is $SE(F) = \{a : a : A \to X, a(t) \in F(t), \forall\, t \in A\ \}$. Hence $SE(F)$ is defined for those soft sets F such that $F(t) \neq \phi$ for all $t \in A$.

3 Soft Real Sets, Soft Real Numbers and Some Definitions

From this section, we take $[0, 1]$ as a parameter space and $\mathbb{R}$ as a universe of discourse. Also, we have introduced the notation of all soft real numbers, collection of all soft real sets, and soft interval and investigated its structural properties. Now, we define the soft real number, soft real set and soft intervals on the universe of discourse $\mathbb{R}$ as follows.

Definition 3.1. Let $N(\mathbb{R}) = \{\epsilon : \epsilon : [0, 1] \to \mathbb{R}\}$ and $S(\mathbb{R}) = \{\delta : \delta : [0, 1] \to \mathcal{P}(\mathbb{R})\}$. $N(\mathbb{R})$ is a collection of all soft real numbers, and $S(\mathbb{R})$ is a collection of all soft real sets. Hence each soft real number is a real-valued function on $[0, 1]$ and soft real set is set function on $[0, 1]$. The range set of every member of $N(\mathbb{R})$ is a subset of $\mathbb{R}$ and the range set of every member of $S(\mathbb{R})$ is a subset of $\mathcal{P}(\mathbb{R})$.
Suppose $\epsilon \in N(\mathbb{R})$ then ϵ is a function from $[0, 1]$ to $\mathbb{R}$. If we define $\delta : [0, 1] \to \mathcal{P}(\mathbb{R})$ by $\delta(t) = \{\epsilon(t)\}$, the singleton set containing only $\epsilon(t)$, for all $t \in [0, 1]$ then $\delta \in S(\mathbb{R})$. So we can consider $N(\mathbb{R})$ as a subset of $S(\mathbb{R})$.
Similarly, for each real number α, considering $\alpha(t) = \alpha$ for all $t \in [0, 1]$, we can say $\mathbb{R} \subset N(\mathbb{R})$. Also, for any set $A \subset \mathbb{R}$, considering $A(t) = A$ for all $t \in [0, 1]$, we can conclude that $A \in S(\mathbb{R})$ so $\mathcal{P}(\mathbb{R}) \subset S(\mathbb{R})$.
According to the definition of the soft set, $S(\mathbb{R})$ is a collection of soft sets with parameter set $[0, 1]$; that is, for every $\delta \in S(\mathbb{R})$, $(\delta, [0, 1])$ is a soft set. If $F \subset N(\mathbb{R})$, define $F(t) = \{f(t) : f \in F\}$ for all $t \in [0, 1]$, then F is a soft set and in particular $(N(\mathbb{R}), [0, 1])$ is a soft set.

Definition 3.2. Let $N(\mathbb{N}) = \{\epsilon : \epsilon : [0,1] \to \mathbb{N}\}$. Then $N(\mathbb{N})$ is a collection of all soft natural numbers. Here $\mathbb{N} \subset N(\mathbb{N})$ and $N(\mathbb{N}) \subset N(\mathbb{R})$.

Note 3.3. *Since* $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, *so we define* $N(\mathbb{R}^*)$ *by* $N(\mathbb{R}^*) = \{\epsilon : \epsilon : [0,1] \to \mathbb{R}^*\}$.

Example 3.4. Any many-valued function defined on $[0,1]$ can be treated as a soft real set.

Example 3.5. $\sin^{-1} x$, $\cos^{-1} x$, $\log x$ with slide modification can be considered as a soft real sets.

Example 3.6. Suppose $\Im$ is the collection of all sub intervals $[0,1]$ with zero as lower end point and $\ell(I)$ is the length of the interval I then $\ell^{-1} : [0,1] \to \Im$ is soft real sets.

Theorem 3.7. $A \subset N(\mathbb{R})$ *if and only if* $A \in S(\mathbb{R})$.

Proof. Let $A \subset N(\mathbb{R})$. Define $\delta_A : [0,1] \to P(\mathbb{R})$ by $\delta_A(t) = \{\eta(t) : \eta \in A\}$. So $\delta_A \in S(\mathbb{R})$ and we write A for δ_A.
If $A \in S(\mathbb{R})$ then $\alpha \in_s A$ implies $\alpha(t) \in A(t)$ for all $t \in [0,1]$. Hence $\alpha : [0,1] \to \mathbb{R}$ and so $\alpha \in N(\mathbb{R})$. Therefore $A \subset N(\mathbb{R})$. □

Definition 3.8. Let $a, b \in N(\mathbb{R})$. We define $a \leq_s b$ if $a(t) \leq b(t)$ for all $t \in [0,1]$. If $a \leq_s b$ and $a(t) < b(t)$ for some t, we say $a <_s b$. Here $' \leq_s'$ is only partial order, not totally order.

Definition 3.9. Let $a, b \in N(\mathbb{R})$. We define

$$Max_s\{a,b\} = Max\{a(t), b(t)\} = a \vee b \text{ and}$$
$$Min_s\{a,b\} = Min\{a(t), b(t)\} = a \wedge b.$$

So $a \vee b, a \wedge b \in N(\mathbb{R})$. Clearly $a \vee b \leq_s a \wedge b$ also if $a \neq_s b$ (i.e. $a(t) \neq b(t)$ for some t) then $a \vee b <_s a \wedge b$.

Definition 3.10. Let $A \subset N(\mathbb{R})$, A is called bounded if A is uniformly bounded i.e. if there is $M \in \mathbb{R}$ such that $\mid \eta(t) \mid < M$ for all $\eta \in A$ and $t \in [0,1]$. Suppose A is bounded then define $\epsilon(t) = Sup\{\eta(t) : \eta \in A\}$ so $\varepsilon(t) \neq \infty$ for all t and $\epsilon \in \mathbb{R}$, ϵ is least upper bound of A. So $N(\mathbb{R})$ is complete.

Definition 3.11. For any $a, b \in N(\mathbb{R})$ where $a \neq_s b$. We define the soft open interval $(a,b)_s =_s \{\epsilon \in N(\mathbb{R}) : a \wedge b <_s \epsilon <_s a \vee b\}$. By Theorem 3.7, the soft open interval is a soft set. Representation of the soft open interval is not unique i.e. it may happens that $(a,b)_s =_s (c,d)_s$ but $a \neq_s c, b \neq_s d$.

Definition 3.12. The soft identity real number I_s defined by $I_s = 1$ for all $t \in [0,1]$.

Definition 3.13. Let $a, b \in N(\mathbb{R})$ then
(i) the soft addition of a and b is $a \oplus b$ and it is defined by $(a \oplus b)(t) = a(t) + b(t)$ for all $t \in [0,1]$.
(ii) the soft substraction of a and b is $a \ominus b$ and it is defined by $(a \ominus b)(t) = a(t) - b(t)$ for all $t \in [0,1]$.
(iii) the soft multiplication of a and b is $a \odot b$ and it is defined by $(a \odot b)(t) = a(t) \cdot b(t)$ for all $t \in [0,1]$.
(iv) the soft division of a and b is $a \oslash b$ and it is defined by $(a \oslash b)(t) = \frac{a(t)}{b(t)}$ for all $t \in [0,1]$.

Definition 3.14. Let $\beta = \{(a,b)_s : a, b \in N(\mathbb{R})\}$ then β is a basis of some topology τ of $N(\mathbb{R})$. For if $x \in N(\mathbb{R})$ then $x \in (x \ominus I_s, x \oplus I_s)_s$ and I_s is a constant function so $(x \ominus I_s, x \oplus I_s)_s$ is a soft open interval. So $\bigcup_{A \in \beta} A = N(\mathbb{R})$.
Again if $(a,b)_s, (c,d)_s$ are two soft open interval and $P \in (a,b)_s \cap_s (c,d)_s$ then $a \wedge b <_s P <_s a \vee b$ and $c \wedge d <_s P <_s c \vee d$.
Clearly, $\alpha = (a \wedge b) \vee (c \wedge d) <_s P <_s (a \vee b) \wedge (c \vee d) = \gamma$. Hence $P \in (\alpha, \gamma)_s \subset_s (a,b)_s \cap_s (c,d)_s$.

Definition 3.15. Let $N(\mathbb{R}) = \{\epsilon : \epsilon : [0,1] \to \mathbb{R}\}$. Therefore $\mathbb{R} \subset N(\mathbb{R})$. So $N(\mathbb{R})$ contains $\mathbb{R}$ as a subspace, and the subspace topology on $\mathbb{R}$ induced by the topology of $N(\mathbb{R})$ is a usual topology of $\mathbb{R}$.

Proposition 3.16. *$(N(\mathbb{R}), \tau)$ is Hausdorff topological space.*

Proof. Let ϵ_1 and ϵ_2 be any two distinct elements in $N(\mathbb{R})$. So $\exists\ t_1 \in [0,1]$ such that $\epsilon_1(t_1) \neq \epsilon_2(t_1)$. Suppose $\epsilon_1(t_1) < \epsilon_2(t_1)$. Let $k = \frac{\epsilon_2(t_1) - \epsilon_1(t_1)}{2}$. So the open intervals $(\epsilon_1(t_1) - k, \epsilon_1(t_1) + k)$ and $(\epsilon_2(t_1) - k, \epsilon_2(t_1) + k)$ are disjoint. Hence the soft open intervals $(\epsilon_1 \ominus k_s, \epsilon_1 \oplus k_s)_s$ and $(\epsilon_2 \ominus k_s, \epsilon_2 \oplus k_s)_s$, where $k_s : [0,1] \to \mathbb{R}$ such that $k_s(t) = k$ for all $t \in [0,1]$, are disjoint and contain ϵ_1 and ϵ_2 respectively. Hence $(N(\mathbb{R}), \tau)$ is Hausdorff topological space. □

Proposition 3.17. *$(N(\mathbb{R}), \tau)$ is 1^{st} countable topological space.*

Proof. Let $a \in N(\mathbb{R})$. For each $n \in \mathbb{N}$ define soft real number $1 \oslash n : [0,1] \to \mathbb{R}$ by $(1 \oslash n)(t) = \frac{1}{n}$ for all $t \in [0,1]$. Let U_n be soft open intervals defined by $U_n =_s (a \ominus (1 \oslash n), a \oplus (1 \oslash n))_s$. Then $\{U_n : n \in \mathbb{N}\}$ forms a countable local base at a. Thus $(N(\mathbb{R}), \tau)$ is 1^{st} countable topological space. □

Definition 3.18. Let $a, b \in N(\mathbb{R})$. We define length of the interval $(a, b)_s$ as a soft real no $a \vee b \ominus a \wedge b$.

Definition 3.19. A soft real number $a \in N(\mathbb{R})$ is said to be positive soft real number if $a(t) \geq 0$ for all $t \in [0, 1]$ and $a(t) > 0$ for at least one t.
Similarly, a soft real number $a \in N(\mathbb{R})$ is said to be negative soft real number if $a(t) \leq 0$ for all $t \in [0, 1]$ and $a(t) < 0$ for at least one t.

Definition 3.20. The elements of $N(\mathbb{R}) - N(\mathbb{R}^*)$ will be called soft zero elements, where $\mathbb{R}^* = \mathbb{R} - \{0\}$.
Let $\epsilon : [0, 1] \to \mathbb{R}$ be a soft real number. If $\epsilon(t) = 0$ for all $t \in [0, 1]$ then ϵ is said to be absolute soft zero element and it is denoted by 0_s.

Definition 3.21. Let $\epsilon \in \mathbb{R}$. Suppose $\epsilon_c : [0, 1] \to \mathbb{R}$ such that $\epsilon_c(t) = \epsilon$ for all $t \in [0, 1]$. Such soft real numbers are called constant soft real numbers.

Definition 3.22. ∞_s is a set of functions defined by $\infty_s = \{a : A \to \mathbb{R}^+ \cup \{\infty\}$ such that $a(t) = \infty$ for at least one $t \in A\}$.

Definition 3.23. Let $x, y \in N(\mathbb{R})$. Then the addition, difference, product and division are defined as usual considering x, y are functions.

Note 3.24. *Let $x, y \in N(\mathbb{R})$ and $x >_s y >_s 0_s$ so $x(t) > y(t) > 0$ for some $t \in [0, 1]$. Let $n_t \in \mathbb{N}$ such that $y(t).n_t > x(t)$. Define $n : [0, 1] \to \mathbb{N}$ such that*

$$n(t) = \begin{cases} n_t, & \text{for some } t \in [0,1] \text{ such that } x(t) > y(t); \\ I_s, & \text{otherwise.} \end{cases}$$

Then clearly $y \odot n >_s x$, which is Archimedean property on soft real numbers.

Definition 3.25. Let $x \in N(\mathbb{R})$. For any $n \in [0, 1]$ we define the n^{th} integral power x^n defined by $x^n(t) = [x(t)]^n$ for all $t \in [0, 1]$. x^n is a soft real number.

Theorem 3.26. *[38] For any $x \in N(\mathbb{R}^*)$, $x^{-1}(t) = [x(t)]^{-1}$ for all $t \in [0, 1]$.*

4 Sequences of Soft Real Numbers

In this section, we have studied the sequence of soft real numbers and investigated some interesting results.

Definition 4.1. A sequence of soft real numbers is a function from $\mathbb{N}$ to $N(\mathbb{R})$.

Definition 4.2. Let $\epsilon > 0$ be a real number. Suppose $\epsilon_c : [0,1] \to \mathbb{R}$ such that $\epsilon_c(t) = \epsilon$ for all $t \in [0,1]$. Let $\{x_n\}$ be a sequence of soft real numbers. $\{x_n\}$ is said to be convergent to a soft real number l if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s$ for all $n \geq N$.
We say that l is a soft limit of the sequence of soft real numbers $\{x_n\}$ and defined by $x_n \to_s l$.

Definition 4.3. A sequence of soft real numbers $\{x_n\}$ is bounded if there exists two soft real numbers m and M such that $m \leq_s x_n \leq_s M$.

Theorem 4.4. *Let $\{x_n\}$ be a sequence in $N(\mathbb{R})$. Then $\{x_n\}$ converges to a soft real number l if and only if $x_n(t) \to l(t)$ for all $t \in [0,1]$.*

Proof. Let $\{x_n\}$ converges to a soft real number l so for the $\epsilon_c(t) = \epsilon$ for all $t \in [0,1]$, $(\epsilon > 0)$ there exists $N \in \mathbb{N}$ such that $x_n \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s$ for all $n \geq N$. So $x_n(t) \in (l \ominus \epsilon_c, l \oplus \epsilon_c)_s(t)$ for all $t \in [0,1]$ and for all $n \geq N$ which implies that $x_n(t) \in (l(t) - \epsilon, l(t) + \epsilon)$ for all $t \in [0,1]$ and for all $n \geq N$. Hence $x_n(t) \to l(t)$ as $n \to \infty$.
For the converse part, the proof is easy and follows from Definition 4.2. □

Note 4.5. *The soft real numbers l is a unique, since $l(t)$ is unique for each $t \in [0,1]$. Also as $\{x_n(t)\}$ is bounded, we get for each two real numbers m_t and M_t such that $m_t \leq x_n(t) \leq M_t$. Now let two soft real numbers m and M which is defined by $m(t) = m_t$ and $M(t) = M_t$ for all $t \in [0,1]$. Then m and M are bounds of $\{x_n\}$. So we can conclude: $\{x_n\}$ is bounded if and only if $\{x_n(t)\}$ is bounded.*

Theorem 4.6. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $N(\mathbb{R})$ converges to the soft real numbers x and y respectively, then*
(1) $x_n \oplus y_n \to_s x \oplus y$
(2) $x_n \ominus y_n \to_s x \ominus y$
(3) $x_n \odot y_n \to_s x \odot y$
(4) $x_n \oslash y_n \to_s x \oslash y$

Proof. The proof of the theorem follows from Theorem 4.4. □

5 Soft Lebesgue Measure

In this section, we have introduced the notation of soft Lebesgue measure and studied its analogous properties.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 December 2022 doi:10.20944/preprints202010.0374.v2

Definition 5.1. Let $A \subset N(\mathbb{R})$, a collection $\{(a_n, b_n)_s : n \in I\}$ is soft open cover of A if $A \subset_s \bigcup_{n \in I} (a_n, b_n)_s$.
Each soft real number is a collection of real numbers. Each soft set is collection of real sets and each soft open interval $(a, b)_s$ is collection of open intervals $\{((a \wedge b)(t), (a \vee b)(t)) : t \in [0, 1]\}$.

Lemma 5.2. *Let $A \subset N(\mathbb{R})$ and $\{(a_n, b_n)_s : n \in I\}$ be a soft open cover of A then $\{(a_n, b_n)_s(t)\} = \{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ be a cover of $A(t)$ for all $t \in [0, 1]$.*

Proof. Let $\alpha \in A(t)$, so there exists $y \in_s A$ such that $y(t) = \alpha$. Since $\{(a_n, b_n)_s : n \in I\}$ is a soft cover of A then $y \in_s (a_n, b_n)_s$ for some n. Therefore $\alpha = y(t) \in (a_n, b_n)_s(t) = ((a_n \wedge b_n)(t), (a_n \vee b_n)(t))$. Now since $\alpha \in A(t)$ is arbitrary then $\{(a_n, b_n)_s(t)\} = \{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ is a cover of $A(t)$ for all $t \in [0, 1]$. □

Definition 5.3. Let $A \subset N(\mathbb{R})$, then soft Lebesgue outer measure of A is defined by $\mu_s^*(A) = inf\{\sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n)_s : \{(a_n, b_n)_s\}$ is a collection of soft open intervals which cover $A\} \cup_s \{\infty_s\}$.

Note 5.4. *As $(a_n \vee b_n \ominus a_n \wedge b_n)$ is a function from $[0, 1]$ to $\mathbb{R}$, the summation is well defined and is a soft real number. So the infimum is taken over a set of soft real numbers. Since this set is uniformly bounded by zero, the infimum exists. Hence $\mu_s^* : S(\mathbb{R}) \to N(\mathbb{R}^+) \cup_s \{\infty_s\}$ where*
$N(\mathbb{R}^+) = \{\epsilon : \epsilon : [0, 1] \to \mathbb{R} \text{ and } \epsilon(t) \geq 0 \; \forall \; t\}$.

Theorem 5.5. *Let $A \subset N(\mathbb{R})$, $\mu_s^*(A)$ be soft Lebesgue outer measure of A and $\mu^*(A(t))$ be Lebesgue outer measure of $A(t)$. Then $\mu_s^*(A)(t) = \mu^*(A(t))$ for all $t \in [0, 1]$.*

Proof. Let $t_0 \in [0, 1]$ be fixed. Let $\{(\alpha_n, \beta_n)\}$ be a cover of $A(t_0)$ and $\{(a_n, b_n)_s\}$ be a soft open cover of A. For each $n \in \mathbb{N}$ define soft real numbers c_n and d_n by

$$
\begin{aligned}
c_n(t) &= ((a_n \wedge b_n)(t) \quad \text{if} \quad t \neq t_0 \\
&= \alpha_n \quad \text{if} \quad t = t_0 \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
d_n(t) &= ((a_n \vee b_n)(t) \quad \text{if} \quad t \neq t_0 \\
&= \beta_n \quad \text{if} \quad t = t_0.
\end{aligned}
$$

Then clearly $\{(c_n, d_n)_s\}$ be a collection of soft intervals and is a soft cover of A. So $\mu_s^*(A)(t_0) \leq (\sum_{n=1}^{\infty}(d_n \ominus c_n))(t_0) = \sum_{n=1}^{\infty}(d_n(t_0) - c_n(t_0)) = \sum_{n=1}^{\infty}(\beta_n - \alpha_n)$. Taking infimum of all such collection $\{(\alpha_n, \beta_n)\}$ that cover $A(t_0)$, we get $\mu_s^*(A)(t_0) \leq \mu^*(A(t_0))$. Since $t_0 \in [0,1]$ is arbitrary, we have $\mu_s^*(A)(t) \leq \mu^*(A(t))$ for all $t \in [0,1]$.

Now suppose $\{(a_n, b_n)_s\}$ be a soft open cover of A then $\{(a_n \wedge b_n)(t), (a_n \vee b_n)(t)\}$ be a cover of $A(t)$ for all $t \in [0,1]$. So $\mu^*(A(t)) \leq (\sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n))(t)$ for all $t \in [0,1]$. Consider a soft real number s such that $s(t) = \mu^*(A(t))$. Hence $s \leq_s \sum_{n=1}^{\infty}(a_n \vee b_n \ominus a_n \wedge b_n)$. Taking infimum of all collection of soft open intervals $\{(a_n, b_n)_s\}$ that cover the soft real set A, we conclude that $s \leq_s \mu_s^*(A)$ i.e., $\mu^*(A(t)) \leq \mu_s^*(A)(t)$ for all $t \in [0,1]$. This completes the proof. □

Corollary 5.6. *Let $A \subset N(\mathbb{R})$. $\mu_s^*(A)$ is soft Lebesgue outer measure of A if and only if $\mu^*(A(t))$ is Lebesgue outer measure of $A(t)$ for all $t \in [0,1]$.*

Proof. The proof follows from Theorem 5.5. □

Theorem 5.7. *If μ_s^* is soft Lebesgue outer measure, then*
(i) $\mu_s^*(\Phi) =_s 0_s$.
(ii) $A \subset_s B \subset N(\mathbb{R})$ then $\mu_s^*(A) \leq_s \mu_s^*(B)$.
(iii) If $\{A_n\}$ is a sequence of subsets of $N(\mathbb{R})$ then $\mu_s^*(\bigcup_{n=1}^{\infty} A_n) \leq_s \sum_{n=1}^{\infty} \mu_s^*(A_n)$.

Proof. (i) From Theorem 5.5, we have $\mu_s^*(\Phi)(t) = \mu^*(\Phi(t)) = \mu^*(\phi) = 0 = 0_s(t)$ for all $t \in [0,1]$. Therefore $\mu_s^*(\Phi) =_s 0_s$.
(ii) $A \subset_s B$ implies that $A(t) \subset B(t)$ for all $t \in [0,1]$. Then by Theorem 5.5, we have $\mu_s^*(A)(t) = \mu^*(A(t)) \leq \mu^*(B(t)) = \mu_s^*(B)(t)$ for all $t \in [0,1]$. Therefore $\mu_s^*(A) \leq_s \mu_s^*(B)$.
(iii) The results follows from Theorem 2.4 and Theorem 5.5. □

Definition 5.8. Let $A \subset N(\mathbb{R})$. Then A is said to be soft Lebegue measurable if for each $A_1 \subseteq N(\mathbb{R})$ such that $\mu_s^*(A_1) =_s \mu_s^*(A \cap A_1) + \mu_s^*(A^c \cap A_1)$.

Remark 5.9. This definition shows that if a soft set A is soft Lebesgue measurable, then $A(t)$ is Lebesgue measurable for all $t \in [0,1]$, and the converse is also true. So the next theorems are trivial.

 doi:10.20944/preprints202010.0374.v2

Theorem 5.10. *(i) Φ and $N(\mathbb{R})$ are both soft measurable and A is soft measurable implies A^c is also soft measurable.*
(ii) Let $A \subset N(\mathbb{R})$. If $\mu_s^*(A) =_s 0_s$ then A is soft measurable.
(iii) Let $x_0 \in N(\mathbb{R})$ and $A \subset N(\mathbb{R})$. If A is soft measurable, then $A \oplus x_0$ is also soft measurable.
(iv) Every soft interval is soft measurable.
(v) If $\{A_i\}$ is a sequence of soft measurable sets in $N(\mathbb{R})$ then $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$ are soft measurable and moreover if $\{A_i\}$ are arbitrary sequence of disjoint soft measurable sets in $N(\mathbb{R})$ then $\mu_s^*(\bigcup_{i=1}^{\infty} A_i) =_s \sum_{i=1}^{\infty} \mu_s^*(A_i)$.

Theorem 5.11. *Let $\{A_n\}$ be an arbitrary sequence of soft measurable sets in $N(\mathbb{R})$.*
(a) If $A_n \subseteq_s A_{n+1}\ \forall\ n \in \mathbb{N}$ and $A =_s \bigcup_{i=1}^{\infty} A_i$ then $\mu_s(A) =_s \lim_{n\to\infty} \mu_s(A_n)$.
(b) Suppose that $\mu_s(A_1)$ is finite. If $A_{n+1} \subseteq_s A_n\ \forall\ n \in \mathbb{N}$ and $A =_s \bigcap_{i=1}^{\infty} A_i$ then $\mu_s(A) =_s \lim_{n\to\infty} \mu_s(A_n)$.

6 Conclusion

In this study, the soft real numbers, soft real sets, soft ordering, sequence of soft real numbers and soft Lebesgue measure have been coined along with some excellent results establishing an interconnection between the classical Lebesgue measure theory and the soft Lebesgue measure theory. This work opens an avenue to study the classical measure-theoretic concepts such as soft Lebesgue measurable functions and soft Lebesgue integrals, which could be potential interest in future endeavours.

Ethical approval:

This article does not contain any studies with human participants or animals performed by any of the authors.

Funding:

The authors do not receive any funding towards the present study.

Conflict of Interest:

We declare that there is no conflict of interest.

References

[1] Acar, U., Koyuncu, F., Tanay, B., (2010) *Soft sets and soft rings,* Computers and mathematics with Applications, **59**, 3458-3463.

[2] Ahmad, B., Hussain, S., (2012) *On Some Structure of Soft Topology,* Mathematical Sciences, **6**, 7 pages.

[3] Aktas, H., Cagman, N., (2007), *Soft sets and soft groups,* Information Sciences **177**, 2726-2735.

[4] Alcantud, J. C. R., Feng, F., and Yager, R. R., (2020), *An N-Soft Set Approach to Rough Sets,* IEEE Transactions on Fuzzy Systems, **28**(11), 2996-3007.

[5] Ali, A., Ali, M.I., Rehman, N., (2019), *Soft dominance based rough sets with applications in information systems,* International Journal of Approximate Reasoning, **113**, 171-195.

[6] Ali, M.I., Feng, F., Liu, X., Min, W.K. and Shabir,M. (2009), *On Some New Operation in Soft Set Theory,* Computers and mathematics with Applications, **9**, 1547-1553.

[7] Aygunoglu, A., Aygun, H., (2012), *Some Notes on Soft Topological Spaces,* Neural Computing and Applications, **21**, 113-119.

[8] Babitha, K.V., John, S.J., (2010) *Studies on Soft Topological Spaces,* Journal of Intelligent and Fuzzy Systems, **28**, 1713-1722.

[9] Babitha K. V. and Sunil J. J., (2010) *Soft set relations and functions*, Comput. Math. Appl. *60* 1840-1849.

[10] Cagman, N., Enginoglu, S., Karatas, S., Aydin, T., (2015) *On Soft Topology,* El-Cezeri Journal of science and Engineering, **2**, 23-38.

[11] Cagman, N., Karatas, S., Enginoglu, S., (2011) *Soft Topology,* Computers and mathematics with Applications, **61**, 351-358.

[12] Celik, Y., Ekiz, C., Yamak, S., (2011) *A new view on soft rings,* Hacettepe Journal of Mathematics and Statistics, **40**, 273-286.

[13] Cheng-Fu Yang, (2003) *A note on Soft Set Theory,* Computers and Mathematics with Applications,**45**, 555-562.

[14] Das S. and Samanta S. K., (2012) *Soft real sets, soft real numbers and their properties*, The Journal of Fuzzy mathematics *20*(3) 551-576.

[15] Feng, F., Liu, X., Leoreanu-Fotea, V.,, Jun,Y.B. , (2011), *Soft sets and soft rough sets,* Information Sciences **181**, 1125-1137.

[16] Feng, F., Li, Y., (2013), *Soft subsets and soft product operations,* Information Sciences **232**, 44-57.

[17] Georgiou, D.N., Megaritis, A.C., Petropoulos, V.I., (2013) *On Soft Topological Spaces,* Appl. Math. Inf. Sci. **7**, 1889-1901.

[18] Georgiou, D.N., Megaritis, A.C., (2014) *Soft Set Theory and Topology,* Appl.Gen. Topo. **15**, 93-109.

[19] Goldar, S., Ray, S., (2017) *A Study of Soft Topology from Classical View Point* , Proceedings of IMBIC, **6** , 108-116.

[20] Goldar, S., Ray, S., (2019) *On Soft Ring And Soft Ideal,* Journal of Applied Science and Computations, **6**, 1457-1467.

[21] Goldar, S., Ray, S., (2019) *A Study of Soft Topological Axioms and Soft Compactness by Using Soft Elements,* Journal of New Results in Scince, **8**, 53-66.

[22] Hussain, S., (2015) *A note on Soft connectedness,* Journal of the Egyptian Mathematical Society **23**, 6-11.

[23] Hussain S. and Ahmad B., (2011) *Some properties of soft topological spaces*, Comput. Math. Appl. *62* 4058-4067.

[24] Jun Y. B., (2008) *Soft BCK/BCI- algebras*, Comput. Math. Appl. *56*, 1408-1413.

[25] Kamaci,H., Atagun,A.O., and Sonmezoglu, A., (2018) *Row-products of soft matrices with applications in multiple-disjoint decision making*, Applied Soft Computing *62* , 892-914

[26] Khan,A., Zhu,Y., (2019) *An improved algorithm for normal parameter reduction of soft set*, Journal of Intelligent and Fuzzy Systems , *37*, 2953-2968.

[27] Khan,A., Zhu,Y., (2020) *New algorithms for parameter reduction of intuitionistic fuzzy soft sets*, Computational and Applied Mathematics, *39*, 131.

[28] Kharal A. and Ahmad B., (2011) *Mappings on soft classes*, New Mathematics and Natural Computation *7* (3) 471-481.

[29] Ma Z., Yang W. and Hu B. Q., (2010) *Soft set theory based on its extension*, Fuzzy Information and Engineering *4* 423-432.

[30] Maji, P.K., Biswas, R., Roy, A.R., (2003) *Soft Set Theory,* Computers and mathematics with Applications, **45**, 555-562.

[31] Maji, P. K., Roy, A. R., and Biswas, R., *An application of soft sets in a decision making problem,* Computers and Mathematics with Applications **44** (2002), 1077-1083.

[32] Min, W.K., (2011) *A Note on Soft Topological Spaces* Computers and mathematics with Applications, **62**, 3524-3528.

[33] Molodtsov, D., (1999) *Soft set theory - First results,* Computers and mathematics with Applications, **37(4/5)**, 19-31.

[34] Molodtsov, D., Leonov, V.Y., kovkov, D.V., (2006) *Soft sets technique and its application,* Nechetkic Sistemy i Myagkie Vychisteniya, **1**, 8-39.

[35] Nazmul Sk. and Samanta S. K., (2014) *Group soft topology*, The Journal of Fuzzy Mathematics *22* (2), 435-450.

[36] Pawlak, Z., *Hard set and Soft set,* ICS Research Report, Institute of Computer Sciences, Poland,

[37] Pawlak, Z., (1982) *Rough set,*, International Journal of Computers and Information Sciences, **11** , 341-356.

[38] Ray, S., Goldar, S., (2017) *Soft Set and Soft Group from Classical View Point,* The Journal of the Indian Mathematical Society, **84** , 273-286.

[39] Shabir, M., Naz, M., (2011) *On Soft Topological Spaces,*Computers and mathematics with Applications, **61**, 1786-1799.

[40] Shaheen, T., Mian, B., Shabir, M., Feng, F. , (2019) *A Novel Approach to Decision Analysis Using Dominance-Based Soft Rough Sets,* International Journal of Fuzzy Systems, **21**(3), 954-962.

[41] Thakur, S.S., Rajput, A.S., (2018) *Connectedness Between Soft Sets,* New Mathematics and Natural Computation, **14**, 53-71.

[42] Uluçay, V., Öztekin Ö., Sahin M., Olgun N., Kargın A., (2016) *Soft representation of soft groups* New Trends in Mathematical Science, 4(2), 23-29.

[43] Varol, B.P., Aygun, H., (2013) *On Soft Hausdorff Spaces,* Annals of Fuzzy Math. and Info., **5**, 15-24.

[44] Wen, Y.C., (2008) *Study on Soft Sets,* Liaoning: Liaoning Normal University (in chinese).

[45] Yin, X., Liao, Z., (2013) *Study on Soft Groups,* Journal of Computers, **8**, 960-967.

[46] Zadeh L. A., (1965) *Fuzzy sets*, Information and Control *8* 338-353.

[47] Zhan,J., and Zhu,K., (2015) *Reviews on decision making methods based on (fuzzy) soft sets and rough soft sets*, Journal of Intelligent and Fuzzy Systems *29*(3), 1169-1176.

[48] Zhao,H., Ma,W., and Sun.B., (2017) *A novel decision making approach based on intuitionistic fuzzy soft sets*, International Journal of Machine Learning and Cybernetics, *8*(4):1107-1117.

[49] A.M. Bruckner, J.B. Bruckner, B.S. Thomson, *Real Analysis,* Prentice-Hall International(UK), London, 2008.

[50] E. Emelyanov, *Introduction to measure theory and Lebesgue integration,* Middle East Technical University Press, Ankara, 2007.

[51] P. Halmos, *Measure theory,* D. Van Nostrand Company Inc., New York p314, 1950.