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Weak Multiplier Hopf Algebras II

The source and target algebras

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Abstract

Let (A, Δ) be a *weak multiplier Hopf algebra* as introduced in [VD-W3] (see also [VD-W2]). It is a pair of a non-degenerate algebra A, with or without identity, and a coproduct $\Delta : A \longrightarrow M(A \otimes A)$, satisfying certain properties. In this paper , we continue the study of these objects and construct new examples.

A symmetric pair of the source and target maps ε_s and ε_t are studied, and their symmetric pair of images, the source algebra and the target algebra $\varepsilon_s(A)$ and $\varepsilon_t(A)$, are also investigated. We show that the canonical idempotent E (which is eventually $\Delta(1)$) belongs to the multiplier algebra $M(B \otimes C)$, where $(B = \varepsilon_s(A), C = \varepsilon_t(A))$ is the symmetric pair of source algebra and target algebra, and also that E is a *separability idempotent* (as studied in [VD4.v2]). If the weak multiplier Hopf algebra is regular, then also E is a *regular* separability idempotent.

We also see how for any weak multiplier Hopf algebra (A, Δ), it is possible to make $C \otimes B$ (with B and C as above) into a new weak multiplier Hopf algebra. In a sense, it forgets the 'Hopf algebra part' of the original weak multiplier Hopf algebra and only remembers symmetric pair of the source and target algebras. It is in turn generalized to the case of any symmetric pair of non-degenerate algebras B and C with a separability idempotent $E \in M(B \otimes C)$. We get another example using this theory associated to any discrete quantum group. Finally we also consider the well-known 'quantization' of the groupoid that comes from an action of a group on a set. All these constructions provide interesting new examples of weak multiplier Hopf algebras (that are not weak Hopf algebras introduced in [B-N-S]).

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0. Introduction

For an associative algebra A with a non-degenerate product, we say that $\lambda \in Hom(A, A)$ is a left multiplier of A if $\lambda(ab) = \lambda(a)b$ for all $a, b \in A$. We denote the space of left multipliers by $L(A)$. Symmetrically, we call $\rho \in Hom(A, A)$ is a right multiplier of A if $\rho(ab) = a\rho(b)$ for all $a, b \in A$. We denote the space of right multipliers by $R(A)$. We have two natural symmetric linear maps $L : A \longrightarrow L(A), L(a)(b) = \lambda_a(b) = ab$ and $R : A \longrightarrow R(A), R(a)(b) = \rho_a(b) = ba$.

The multiplier algebra $M(A)$ of A is the space of all *symmetric pairs* (λ, ρ) where $\lambda \in L(A)$ and $\rho \in R(A)$ such that $a\lambda(b) = \rho(a)b$ for all $a, b \in A$. The unit of $M(A)$ is denoted by 1, see [VD1, VD3, VD4.v1, VD4.v2].

Weak Hopf algebras were introduced in [B-N-S] by Bohm, Nill and Szlachanyi in 1999, and they have been of great interest in quantum algebra and mathematical physics. In previous work in [VD-W3] (see also [VD-W2]), we defined weak multiplier Hopf algebras, by extending the class of weak Hopf algebras. It is a pair of a non-degenerate algebra A, with or without identity, and a coproduct $\Delta : A \longrightarrow M(A \otimes A)$, satisfying certain properties. If the algebra has an identity and the coproduct is unital, then we have a Hopf algebra (see [A] or [S]). If the algebra has no identity, but if the coproduct is non-degenerate (which is the equivalent of being unital if the algebra has an identity), then (A, Δ) would be a multiplier Hopf algebra (see [VD1]– [VD3]). If the algebra has an identity, but the coproduct is not unital, we have a weak Hopf algebra (see [B-N-S] or [B-S]). In the general case, we neither assume A to have an identity nor do we assume Δ to be non-degenerate and so we work with a *genuine* weak multiplier Hopf algebra, see [VD-W2]–[VD-W5] (see also [B-G-L], [Z-W]). It is called *regular* if its antipode is a bijective map from A to itself.

The first fundamental example of a weak multiplier Hopf algebra is the algebra $A = K(G)$ of complex functions on G with finite support and pointwise product, where G is a groupoid. Here, the coproduct map Δ is not necessarily non-degenerate, while the existence of a certain canonical idempotent element $E \in M(A \otimes A)$ is assumed, which coincides with $\Delta(1)$ in the unital case.

Symmetrically, for the second example, we take the algebra B, defined as the groupoid algebra $\mathbb{C}G$ of G. If we use $p \mapsto \lambda_p$ for the canonical embedding of G in $\mathbb{C}G$, then if $p, q \in G$, we have $\lambda_p\lambda_q = \lambda_{pq}$ if pq is defined and 0 otherwise. Here the canonical idempotent E is given by $\sum \lambda_e \otimes \lambda_e$ where the sum is only taken over the units e of G. The antipode is given by $S(\lambda_p) = \lambda_{p^{-1}}$ for all $p \in G$.

These two examples are dual (symmetric) to each other. The duality is given by $\langle f, \lambda_p \rangle = f(p)$ whenever $f \in K(G)$ and $p \in G$. We give more details (about this duality) in [VD-W5] where we treat duality for regular weak multiplier Hopf algebras with integrals.

In the paper under review, we continue the study of these objects and construct new examples. If A is a weak multiplier Hopf algebra, a symmetric pair of the source and target maps are studied, and their symmetric pair of images, the source algebra and the target algebra, are also investigated. We show that the canonical idempotent E belongs

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to the multiplier algebra $M(B \otimes C)$, where (B, C) is the symmetric pair of source algebra and target algebra, and also that E is a separability idempotent. Several interesting new examples of weak multiplier Hopf algebras are constructed. Some of them are not weak Hopf algebras.

Content of the paper

In *Section* 1 we recall some of the basic notions and results on weak multiplier Hopf algebras as studied in our first papers on the subject ([VD-W2] and [VD-W3]). In particular, we will explain some of the covering properties as this will be important for the rest of the paper. We note that in the definition of a weak multiplier Hopf algebra, there are four symmetry concepts: (a) Multiplier algebra as explained in the introduction; (b) Full; (c) The properties of the counit and the separability idempotent; (d) the source and the target maps (algebras).

In the earlier papers on the subject, we briefly looked already at symmetric pair of the source and target maps ε_s and ε_t and their symmetric pair of images, the source and target algebras. In *Section* 2 we investigate these objects further. We recall the definitions and some of the basic properties that are found already in [VD-W3]. Notice that we make a *change in terminology*. We will now call the image $\varepsilon_s(A)$ of the source map the source algebra and the image $\varepsilon_t(A)$ of the target map the target algebra. In [VD-W4.v1] we used these terms for the multiplier algebras that can be characterized nicely in the regular case. Because now we are also studying the non-regular case, these multiplier algebras no longer seem to have the same characterization and this is what motivated us to change this terminology. We comment more on this in Section 2.

Indeed, in the regular case, we show that the multiplier algebras $M(\varepsilon_s(A))$ and $M(\varepsilon_t(A))$ of the images $\varepsilon_s(A)$ and $\varepsilon_t(A)$ of the source and target maps can be nicely characterized as certain subalgebras of the multiplier algebra $M(A)$.

In the general case, we show that the canonical idempotent E has all the properties of a separability idempotent (as studied in [VD4.v2]). It turns out to be a regular one if the weak Hopf algebra is regular. Finally we use the various results to show that the underlying algebra A of any weak multiplier Hopf algebra (A, Δ) has local units. Recall that in [VD-W3], we only could show this in the regular case.

In *Section* 3 we study special cases and examples. We start again with the two examples associated with a groupoid. We will be very short here as we include this mainly for completeness. These examples have been considered in earlier papers (see e.g. [VD-W3]). Then we consider any weak multiplier Hopf algebra (A, Δ) and we associate a new weak multiplier Hopf algebra (P, Δ_P) where the underlying algebra P is $\varepsilon_t(A) \otimes \varepsilon_s(A)$ and the coproduct is given by the formula

$$\Delta_P(c \otimes b) = c \otimes E \otimes b$$

for $b \in \varepsilon_s(A)$ and $c \in \varepsilon_t(A)$ and where E is the canonical multiplier in $M(A \otimes A)$. We also use this example further as a model for the construction of an abstract version of this case. Then we take any symmetric pair of non-degenerate algebras B and C and start with a so-called separability idempotent E in the multiplier algebra $M(B \otimes C)$. We take $P = C \otimes B$ and Δ_P as above. These two examples are 'quantizations' of the trivial groupoid

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G constructed from a set X by taking $G = X \times X$ with product $(z, y)(y, x) = (z, x)$ when $x, y, z \in X$.

This groupoid in turn is related with the case of a groupoid G constructed from a (left) action of a group H on a set X. Now G consists of triples (y, h, x) where $x, y \in X$ and $h \in H$ and $y = h \triangleright x$ and where $\triangleright$ is used to denote the action. The product is given by $(z, k, y)(y, h, x) = (z, kh, x)$. And finally, also this groupoid will be quantized (at least in a certain sense to be explained in this section).

The starting point is again a symmetric pair of non-degenerate algebras B and C with a separability idempotent E in the multiplier algebra $M(B \otimes C)$. Moreover there is a (regular) multiplier Hopf algebra Q that acts from the right on B and from the left on C in such a way that B is a right Q-module algebra and C a left Q-module algebra. These objects are related with the requirement that the right action of Q on C induces via E the left action of Q on B. See Section 3 for a more precise statement. The two-sided smash product P is defined as the algebra generated by B, C and Q with B and C commuting and the commutation rules between B and Q determined by the left action of Q on B and the ones between C and Q determined by the right action of Q on C. It carries a natural coproduct making P into a weak multiplier Hopf algebra.

Finally, in *Section* 4 we draw some conclusions and discuss possible further research on this subject.

In [VD-W5], the study of weak multiplier Hopf algebras is continued with the investigation of integrals and duality. The results obtained in the present paper are of *great importance* for the treatment of integrals and duality as it is done in [VD-W5].

The material studied in this paper is closely related with the theory of *(regular) multiplier Hopf algebroids*, as developed in [T-VD1], where the theory of weak multiplier Hopf algebras is treated within an algebroid framework. See also [T-VD2] for the relation between the two concepts.

We also like to refer to the paper on *weak multiplier bialgebras* by Böhm, Gómez-Torecillas and López-Centella (see [B-G-L]) where the notion of a weak multiplier bialgebra is developed. In this theory, the symmetric pair of source and target maps, as well as the symmetric pair of source and target algebras, play a crucial role. See also [K-VD] where a Larson-Sweedler type theorem is proven for these weak multiplier bialgebras.

Finally, we also notice that many other interesting works (see [B1]-[B2], [K-VD1]–[K-VD3], [T1]–[T2], [T-VD1]-[T-VD2] and [Z-W]) were motivated by the notion of weak multiplier Hopf algebras introduced in [VD-W2] (for an earlier background of this paper, see [W]).

Conventions and notations

We only work with algebras A over $\mathbb{C}$ (although we believe that this is not essential and that it is possible to obtain the same results for algebras over other, more general fields). We do not assume that they are unital but we need that the product is non-degenerate. We also assume our algebras to be idempotent (that is $A^2 = A$). In fact, it turns out that the algebras we encounter in this theory always have local units. We have seen this already in [VD-W3], in the regular case. Then of course, the product is automatically non-degenerate and also the algebra is idempotent.

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When A is such an algebra, we use $M(A)$ for the multiplier algebra of A. When m is in $M(A)$, then by definition we can define am and mb in A for all $a, b \in A$ and we have $(am)b = a(mb)$. The algebra A sits in $M(A)$ as an essential two-sided ideal and $M(A)$ is the largest algebra with identity having this property.

Recall that a homomorphism $\gamma : A \to M(B)$, where A and B are non-degenerate algebras, is called non-degenerate if $\gamma(A)B = B$ and $B\gamma(A) = B$. In that case, there is a unique extension of γ, still denoted by γ, to a unital homomorphism from $M(A)$ to $M(B)$. There is a similar result for non-degenerate anti-homomorphisms.

We consider $A \otimes A$, the tensor product of A with itself. It is again an idempotent, non-degenerate algebra and we can consider the multiplier algebra $M(A \otimes A)$. The same is true for a multiple tensor product. We use ζ for the flip map on $A \otimes A$, as well as for its natural extension to $M(A \otimes A)$.

We use 1 for the identity in any of these multiplier algebras. On the other hand, we mostly use ι for the identity map on A (or other spaces), although sometimes, we also write 1 for this map. The identity element in a group is denoted by e. If G is a groupoid, we will also use e for units. Units are considered as being elements of the groupoid and we use s and t for the source and target maps from G to the set of units.

When A is an algebra, we denote by A^{op} the algebra obtained from A by reversing the product. When Δ is a coproduct on A, we denote by Δ^{cop} the coproduct on A obtained by composing Δ with the flip map ζ.

For a coproduct Δ, as we define it in Definition 1.1 of [VD-W3], we assume that $\Delta(a)(1\otimes b)$ and $(a \otimes 1)\Delta(b)$ are in $A \otimes A$ for all $a, b \in A$. This allows us to make use of the *Sweedler notation* for the coproduct. The Sweedler notation is first explained in [Dr-VD], but only for the case of regular coproducts. In [VD3] an approach is developed in the case where the underlying algebras have local units. In the more recent paper [VD6], this condition is not assumed. However, it should be mentioned that the Sweedler notation is essentially just what is says, a notation. It is a way to denote formulas in a more transparent way. This point of view is explained in [VD6] and the reader is advised to look at that note for understanding the use of the Sweedler notation for weak multiplier Hopf algebras as in this paper.

Basic references

For the theory of Hopf algebras, we refer to the standard works of Abe [A] and Sweedler [S]. For multiplier Hopf algebras and integrals on multiplier Hopf algebras, we refer to [VD1] and [VD2]. Weak Hopf algebras have been studied in [B-N-S] and [B-S] and more results are found in [N] and [N-V1]. Various other references on the subject can be found in [Va]. In particular, we refer to [N-V2] because we will use notations and conventions from this paper when dealing with weak Hopf algebras.

For the theory of groupoids, we refer to [Br], [H], [P] and [R].

1. Preliminaries on weak multiplier Hopf algebras

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Let (A, Δ) be *a weak multiplier Hopf algebra* as in Definition 1.14 of [VD-W3]. In general, we do not assume that it is regular. On the other hand, we also recall some of the results that are only true in the regular case.

A is *an algebra* over $\mathbb{C}$, with or without identity but with a product that is non-degenerate (as a bilinear map). The algebra is also idempotent in the sense that $A = A^2$ (meaning that any element in A is a sum of products of elements of A). In Proposition 4.9 of [VD-W3], we showed that in the regular case, the underlying algebra automatically has local units. In fact, the result turns out to be true also in the non-regular case. We will obtain a proof in this paper (see Proposition 2.21 in Section 2). Remark that for an algebra with local units, the product is automatically non-degenerate and the algebra is idempotent.

There is a *coproduct* Δ on A. It is a homomorphism from A to the multiplier algebra $M(A \otimes A)$ of the tensor product $A \otimes A$ of A with itself. It is not assumed that it is non-degenerate (see further). The *canonical maps* T_1, T_2, T_3 and T_4 are linear maps defined on $A \otimes A$ by

$$\begin{aligned} T_1(a \otimes b) &= \Delta(a)(1 \otimes b) & \qquad T_2(c \otimes a) &= (c \otimes 1)\Delta(a) \\ T_3(a \otimes b) &= (1 \otimes b)\Delta(a) & \qquad T_4(c \otimes a) &= \Delta(a)(c \otimes 1). \end{aligned}$$

In general, it is assumed that T_1 and T_2 have range in $A \otimes A$. If also T_3 and T_4 map into $A \otimes A$, then the *coproduct* is called *regular*.

The coproduct is assumed to be *full*. This means that the smallest subspaces V and W of A satisfying symmetric properties:

$$\Delta(a)(1 \otimes b) \in V \otimes A \qquad \text{and} \qquad (c \otimes 1)\Delta(a) \in A \otimes W$$

for all $a, b \in A$ are A itself. If the coproduct is regular, then a similar property will also be true for the maps T_3 and T_4 and so both the flipped coproduct Δ^{cop} on A and the original coproduct on A^{op} will also be full coproducts.

Fullness of the coproduct implies that any element in A is a linear span of elements of the form $(\iota \otimes \omega)(\Delta(a)(1 \otimes b))$ where $a, b \in A$ and where ω is a linear functional on A. Similarly for the span of elements $(\omega \otimes \iota)((c \otimes 1)\Delta(a))$ with $a, c \in A$ and a linear functional ω on A. In fact, this property is equivalent with fullness of the coproduct. We have a result of the same type for fullness of a regular coproduct. See e.g. Proposition 1.6 in [VD-W1] and also Lemma 1.11 in [VD-W2]

Furthermore, it is assumed that there is *a counit*. This is a linear map $\varepsilon : A \to \mathbb{C}$ satisfying the following symmetric properties:

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab \qquad \text{and} \qquad (\iota \otimes \varepsilon)((c \otimes 1)\Delta(a)) = ca$$

for all a, b, c in A. Similar formulas will be true for the other canonical maps in the case of a regular coproduct.

Because the coproduct is assumed to be full, this counit is unique in the following sense. Assume that ε and ε' are linear maps such that

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab \qquad \text{and} \qquad (\iota \otimes \varepsilon')((c \otimes 1)\Delta(a)) = ca$$

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for all a, b, c in A. Then already $\varepsilon = \varepsilon'$. This is proven by applying $\iota \otimes \varepsilon \otimes \iota$ on the right hand side and $\iota \otimes \varepsilon' \otimes \iota$ on the left hand side of the equation that expresses coassociativity of the coproduct

$$(c \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(a)(1 \otimes b)) = (\iota \otimes \Delta)((c \otimes 1)\Delta(a))(1 \otimes 1 \otimes b).$$

In the two cases we get the same result, namely $(c \otimes 1)\Delta(a)(1 \otimes b)$. This is true for all $a, b, c \in A$ and from the fullness of the coproduct, it follows that $\varepsilon = \varepsilon'$.

It is not clear if there is a uniqueness result without the assumption that the coproduct is full. And it is also not clear if the existence of a counit, in the non-unital case, implies fullness of the coproduct. Remark that in general, the counit is not a homomorphism in the case of weak multiplier Hopf algebras.

It seems not possible to construct a counit, even given that the coproduct is full. Therefore, the existence of the counit is part of the axioms for weak multiplier Hopf algebras.

There is an idempotent element E in $M(A \otimes A)$, called the *canonical idempotent*, giving the ranges of the canonical maps T_1 and T_2 as the following symmetric properties:

$$\Delta(A)(1 \otimes A) = E(A \otimes A) \qquad \text{and} \qquad (A \otimes 1)\Delta(A) = (A \otimes A)E.$$

If the weak multiplier Hopf algebra is regular, we also have these properties for the ranges of the canonical maps T_3 and T_4. So in that case, we also have the following symmetric properties:

$$\Delta(A)(A \otimes 1) = E(A \otimes A) \qquad \text{and} \qquad (1 \otimes A)\Delta(A) = (A \otimes A)E$$

with the same idempotent. This element is uniquely determined and it satisfies

$$\Delta(a)E = \Delta(a) = E\Delta(a)$$

for all $a \in A$.

We see that the coproduct is degenerate if E is strictly smaller than 1. However, still the coproduct can be extended in a unique way to a homomorphism from $M(A)$ to $M(A \otimes A)$ (again denoted by Δ) provided we assume $\Delta(1) = E$. Similarly, the homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ have unique extension to $M(A \otimes A)$ such that, again using the same symbols for these extensions, we have the following symmetric properties:

$$(\Delta \otimes \iota)(1) = E \otimes 1 \qquad \text{and} \qquad (\iota \otimes \Delta)(1) = 1 \otimes E.$$

We use 1 for the identity, both in $M(A)$ and in $M(A \otimes A)$. We have $(\Delta \otimes \iota)(E) = (\iota \otimes \Delta)(E)$. It is further assumed that

$$(\Delta \otimes \iota)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1).$$

The last equality means, in a sense that can be made precise, that the left and the right legs of E commute.

The left and the right legs of E are also *big enough* in the following sense.

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Lemma 1.1. If $a \in A$ and if $E(1 \otimes a) = 0$, then $a = 0$. Similarly $a = 0$, if $(1 \otimes a)E = 0$, $(a \otimes 1)E = 0$ or if $E((a \otimes 1) = 0$.
Proof. Assume $a \in A$. If $E(1 \otimes a) = 0$ then $\Delta(b)(1 \otimes a) = 0$ for all $b \in A$. If we apply the counit ε on the first leg of this equality we find $ba = 0$ for all b and so $a = 0$. If $E(a \otimes 1) = 0$ we get $(c \otimes 1)\Delta(b)(a \otimes 1) = 0$ for all $b, c \in A$. Now we apply the counit on the second leg and we find $cba = 0$ for all $b, c \in A$. Again this implies $a = 0$. A similar argument works for the two other cases. □

There is a *unique antipode* S. It is a linear map from A to the multiplier algebra $M(A)$. It is an anti-algebra map in the sense that $S(ab) = S(b)S(a)$ for all $a, b \in A$ and it is an anti-coalgebra map meaning that $\Delta(S(a)) = \zeta(S \otimes S)\Delta(a)$ for all $a \in A$ (in an appropriate sense - see e.g. Proposition 3.7 and more comments in [VD-W3] for a correct formulation). Recall that we use ζ for the flip map. Moreover, the antipode satisfies the following symmetric formulas between S and ι:

$$\sum_{(a)} a_{(1)}S(a_{(2)})a_{(3)} = a \qquad \text{and} \qquad \sum_{(a)} S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a)$$

for all a in A. One has to multiply with an element of A, left or right, in order to be able to use the Sweedler notation, and so strictly speaking, the formulas hold in $M(A)$ (see also Remark 1.2 below).

We have the equalities

$$E(a \otimes 1) = \sum_{(a)} \Delta(a_{(1)})(1 \otimes S(a_{(2)})) \tag{1. 1}$$

$$(1 \otimes a)E = \sum_{(a)} (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}) \tag{1. 2}$$

for all a. These equations are equivalent with

$$\Delta(c)(a \otimes 1) = \sum_{(a)} \Delta(ca_{(1)})(1 \otimes S(a_{(2)})) \tag{1. 3}$$

$$(1 \otimes a)\Delta(b) = \sum_{(a)} (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}b) \tag{1. 4}$$

for all a, b, c. Observe that using the Sweedler notation in these formulas is just a matter of notation and nothing more. Indeed, the formula (1.3) above is a shorthand for the formula $\Delta(c)(a \otimes 1) = \sum_i \Delta(p_i)(1 \otimes S(q_i))$ where $\sum_i p_i \otimes q_i = (c \otimes 1)\Delta(a)$. This is true for all the formulas with the Sweedler notation we have here in this preliminary section. It illustrates a remark already made in the introduction.

In the regular case, we have that the antipode maps A to itself and is bijective. In fact, this property of the antipode characterizes the *regular* weak multiplier Hopf algebras.

In that case, we have the following counterparts of the formulas (1.1) and (1.2) above. We have

$$E(1 \otimes a) = \sum_{(a)} \Delta(a_{(2)})(S^{-1}(a_{(1)}) \otimes 1) \tag{1. 5}$$

$$(a \otimes 1)E = \sum_{(a)} (1 \otimes S^{-1}(a_{(2)}))\Delta(a_{(1)}) \tag{1. 6}$$

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for all a. Again these formulas can also be written as

$$\Delta(b)(1 \otimes a) = \sum_{(a)} \Delta(ba_{(2)})(S^{-1}(a_{(1)}) \otimes 1) \tag{1. 7}$$

$$(a \otimes 1)\Delta(c) = \sum_{(a)} (1 \otimes S^{-1}(a_{(2)}))\Delta(a_{(1)}c) \tag{1. 8}$$

for all a, b, c.

Observe the following peculiarity in these formulas. The formulas (1.3) and (1.4) are true in the non-regular case but the expressions need not be in $A \otimes A$. On the other hand, the formulas (1.7) and (1.8) only make sense in the regular case (as the inverse of S is involved), while now the expressions are true in $A \otimes A$.

We now make an important remark about the covering of the previous formulas.

Remark 1.2. i) First rewrite the (images of the) canonical maps T_1 and T_2, and of T_3 and T_4 in the regular case, using the Sweedler notation, as

$$\Delta(a)(1 \otimes b) = \sum_{(a)} a_{(1)} \otimes a_{(2)}b \qquad (c \otimes 1)\Delta(a) = \sum_{(a)} ca_{(1)} \otimes a_{(2)} \tag{1. 9}$$

$$(1 \otimes b)\Delta(a) = \sum_{(a)} a_{(1)} \otimes ba_{(2)} \qquad \Delta(a)(c \otimes 1) = \sum_{(a)} a_{(1)}c \otimes a_{(2)} \tag{1. 10}$$

where $a, b, c \in A$. In all these four expressions, either $a_{(1)}$ is covered by c and or $a_{(2)}$ by b. This is by the assumption put on the coproduct, requiring that the canonical maps have range in $A \otimes A$.

ii) Next consider the expressions

$$\sum_{(a)} a_{(1)} \otimes S(a_{(2)})b \qquad \text{and} \qquad \sum_{(a)} cS(a_{(1)}) \otimes a_{(2)} \tag{1. 11}$$

$$\sum_{(a)} a_{(1)} \otimes bS(a_{(2)}) \qquad \text{and} \qquad \sum_{(a)} S(a_{(1)})c \otimes a_{(2)} \tag{1. 12}$$

where $a, b, c \in A$. In the first two formulas (1.11), we have a covering by the assumption that the generalized inverses R_1 and R_2 of the canonical maps exist as maps on $A \otimes A$ with range in $A \otimes A$ (see [VD-W3]). In the second pair of formulas (1.12), we have a good covering only in the regular case. It follows by considering the expressions in (1.9) and using that S is a bijective anti-algebra map from A to itself. In the regular case, we can also consider the above expressions with S replaced by S^{-1}.

iii) If on the one hand, we first apply S in the first or the second factor of the expressions in (1.9) and multiply and if on the other hand we simply apply multiplication on the expressions in (1.11), we get the four elements

$$\sum_{(a)} S(a_{(1)})a_{(2)}b \qquad \text{and} \qquad \sum_{(a)} ca_{(1)}S(a_{(2)}) \tag{1. 13}$$

$$\sum_{(a)} a_{(1)}S(a_{(2)})b \qquad \text{and} \qquad \sum_{(a)} cS(a_{(1)})a_{(2)} \tag{1. 14}$$

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in A for all $a, b, c \in A$. This is used to define the source and target maps in the next section (see Definition 2.1 in the next section).

iv) Now, we combine the coverings obtained in i) and ii). Consider e.g. the two expressions

$$\sum_{(a)} \Delta(a_{(1)})(1 \otimes S(a_{(2)})b) \tag{1. 15}$$

$$\sum_{(a)} (cS(a_{(1)}) \otimes 1)\Delta(a_{(2)}) \tag{1. 16}$$

where $a, b, c \in A$. The first expression (1.13) is obtained by applying the canonical map T_1 to the first of the two expressions in (1.11). So this gives an element in $A \otimes A$ and we know that it is $E(a \otimes b)$ as we can see from the formula (1.1). Similarly, the second expression (1.14) is obtained by applying the canonical map T_2 to the second of the two expressions in (1.11). We know that this is $(b \otimes a)E$ as we see from the formula (1.2) above. Remark that $E(a \otimes b)$ and $(b \otimes a)E$ belong to $A \otimes A$ because by assumption $E \in M(A \otimes A)$, but that on the other hand, it is not obvious (as we see from the above arguments) that the expressions that we obtain for these elements belong to $A \otimes A$.

v) Finally, as a consequence of the above statements, also the four expressions

$$\sum_{(a)} S(a_{(1)})a_{(2)}S(a_{(3)})b \qquad \text{and} \qquad \sum_{(a)} ca_{(1)}S(a_{(2)})a_{(3)} \tag{1. 17}$$

$$\sum_{(a)} cS(a_{(1)})a_{(2)}S(a_{(3)}) \qquad \text{and} \qquad \sum_{(a)} a_{(1)}S(a_{(2)})a_{(3)}b \tag{1. 18}$$

are well-defined in A for all $a, b, c \in A$ (also in the non-regular case as $S : A \to M(A)$). This justifies a statement made earlier about the properties of the antipode. □
And once again, in all these cases, the Sweedler notation is just used as a more transparent way to denote expressions. We refer to the coverings just to indicate how the formulas with the Sweedler notation can be rewritten without the use of it.

In the regular case, we also have many other nice formulas (see Section 4 in [VD-W3]. One of them is $(S \otimes S)E = \zeta E$ (as expected because $E = \Delta(1)$). Other formulas that we will use, will be recalled later. In any case, they are all found in [VD-W3] and we refer to this paper for details.

2. The symmetric pair of source and target algebras

As in the previous section we consider a weak multiplier Hopf algebra (A, Δ). In general, we do not assume that it is regular. In the regular case, nicer results can be obtained, but we try to push the theory as far as possible in the general case.

We first recall the definition of the symmetric pair of *source* and *target maps* $\varepsilon_s : A \to M(A)$ and $\varepsilon_t : A \to M(A)$ and prove the first properties. We show among other things that the images are non-degenerate subalgebras of $M(A)$, sitting nicely in $M(A)$ so that also their multiplier algebras can be considered as subalgebras of $M(A)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 November 2020 doi:10.20944/preprints202011.0293.v1

The symmetric pair of source and target maps, together with their images, have been considered already in [VD-W3] and a few properties were proven, mainly for the purpose of studying the antipode. In this paper, we will continue this study.

Remark that in this paper, as we mentioned already in the introduction, *we will define the symmetric pair of source and target algebras as the symmetric pair of images of the source and target maps* (see Notation 2.9). We will explain later why we do this.

We will also study the behavior of the antipode S on the source and target algebras. Recall that S is an anti-homomorphism from A to $M(A)$. It is non-degenerate in the sense that $S(A)A = A$ and $AS(A) = A$ (see Proposition 3.6 in [VD-W3]). Therefore, as a consequence of a general property mentioned already in the introduction (see also the appendix of [VD1]), it has a unique extension to a unital anti-homomorphism from $M(A)$ to itself.

We consider the canonical idempotent E in $M(A \otimes A)$ as reviewed in the previous section and we use that the coproduct Δ can be extended to the multiplier algebra as we have mentioned earlier. We will show that E is a separability idempotent as studied in [VD4.v2].

The source and target algebras B and C

We first consider the symmetric pair of *source* and *target maps* $\varepsilon_s : A \to M(A)$ and $\varepsilon_t : A \to M(A)$. Recall Definition 3.1 from [VD-W3].

Definition 2.1. For $a \in A$ we define

$$\varepsilon_s(a) = \sum_{(a)} S(a_{(1)})a_{(2)} \qquad \text{and} \qquad \varepsilon_t(a) = \sum_{(a)} a_{(1)}S(a_{(2)})$$

where S is the antipode. The map ε_s is called the source map and the map ε_t is the target map. □

We have seen in Remark 1.2.iii that these maps have well-defined values in the multiplier algebra $M(A)$.

We will show that the images of the source and target maps are subalgebras of $M(A)$. Before we can do this, we need some elementary properties, also important for the further study of these subalgebras.

First we have that the range of ε_s coincides with the left leg of E and that the range of ε_t is the right leg of E. These statements are made precise in the following proposition.

Proposition 2.2. The range $\varepsilon_s(A)$ of the source map is spanned by elements of the form $(\iota \otimes \omega(a \cdot b))E$ where $a, b \in A$ and where ω is a linear functional on A. Symmetrically the range $\varepsilon_t(A)$ of the target map is spanned by elements of the form $(\omega(c \cdot a) \otimes \iota)E$ where $a, c \in A$ and with ω a linear functional on A.

Proof. By formula (1.2) in Section 1 we get for $a, b \in A$ that

$$(1 \otimes a)E(1 \otimes b) = \sum_{(a)} S(a_{(1)})a_{(2)} \otimes a_{(3)}b$$

and this belongs to $\varepsilon_s(A) \otimes A$. We can apply a linear functional ω on the second leg and we see that $(\iota \otimes \omega(a \cdot b))E$ is well-defined and belongs to $\varepsilon_s(A)$. The fullness of Δ guarantees

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that any element of A is a sum of elements of the form

$$(\iota \otimes \omega)(\Delta(a)(1 \otimes b))$$

where $a, b \in A$ and where ω is a linear functional (see Section 1). Hence it follows that $\varepsilon_s(A)$ is spanned by elements as in the formulation of the proposition. Similarly for the range $\varepsilon_t(A)$ of the target map. □
Because $E \otimes 1$ and $1 \otimes E$ commute, it follows that $\varepsilon_s(a)$ and $\varepsilon_t(b)$ will commute in $M(A)$ for all $a, b \in A$.

Also the following is an easy consequence of the previous result. The formulas in the proposition make sense in the multiplier algebra $M(A \otimes A)$.

Proposition 2.3. We have the following symmetric properties:

$$\Delta(x) = (x \otimes 1)E = E(x \otimes 1) \qquad \text{and} \qquad \Delta(y) = E(1 \otimes y) = (1 \otimes y)E$$

for $x \in \varepsilon_t(A)$ and $y \in \varepsilon_s(A)$.
Proof. Simply apply the appropriate linear functionals on the first, respectively the third factor of the equations

$$(\iota \otimes \Delta)E = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1) \tag{2. 1}$$
$$(\Delta \otimes \iota)E = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1). \tag{2. 2}$$

This completes the proof. □
The result above is the motivation for the following lemma.

Lemma 2.4. For an element $x \in M(A)$, the following are equivalent:
i) $\Delta(x) = (x \otimes 1)E$,
ii) $\Delta(x) = E(x \otimes 1)$.

Similarly, for an element $y \in M(A)$, the following are equivalent:
i) $\Delta(y) = E(1 \otimes y)$,
ii) $\Delta(y) = (1 \otimes y)E$.
Proof. First let $x \in M(A)$ and assume that $\Delta(x) = (x \otimes 1)E$. Take any $y \in \varepsilon_s(A)$. Then

$$\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes 1)E\Delta(y) = (x \otimes 1)\Delta(y) = (x \otimes y)E.$$

We have used that $\Delta(y) = (1 \otimes y)E$, proven in the previous proposition for elements y in $\varepsilon_s(A)$. On the other hand

$$\Delta(yx) = \Delta(y)\Delta(x) = (1 \otimes y)E\Delta(x) = (1 \otimes y)\Delta(x) = (x \otimes y)E$$

and we see that $\Delta(xy) = \Delta(yx)$. Multiply with $\Delta(a)$ for any $a \in A$ and apply the counit. This will give $xya = yxa$ and because this is true for all a, we have $xy = yx$.

Because this result is true for all elements y in the left leg of E, as a consequence we find that $(x \otimes 1)E = E(x \otimes 1)$ and hence also $\Delta(x) = E(x \otimes 1)$.

Similarly if $\Delta(x) = E(x \otimes 1)$ also $\Delta(x) = (x \otimes 1)E$ will be true. This proves the equivalence of i) and ii) in the first part of the lemma.

The second part is proven in a completely similar way. □

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We arrive at the following notation.

Notation 2.5. We will denote by A_s the set of elements $y \in M(A)$ satisfying $\Delta(y) = E(1 \otimes y)$ and by A_t the set of elements $x \in M(A)$ satisfying $\Delta(x) = (x \otimes 1)E$. □

The following is an immediate consequence of the lemma.

Proposition 2.6. The sets A_s and A_t are commuting subalgebras of $M(A)$.

Proof. It is immediately clear from the definitions that these sets are subalgebras of $M(A)$. Moreover, if $x \in A_t$ and $y \in A_s$, we have as in the first part of the proof of the lemma

$$\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes y)E \qquad \Delta(yx) = \Delta(y)\Delta(x) = (x \otimes y)E$$

where we have used the two equivalences of i) and ii) in the lemma. Hence $\Delta(xy) = \Delta(yx)$ and as before, $xy = yx$. □

From Proposition 2.3, we know that $\varepsilon_s(A) \subseteq A_s$ and $\varepsilon_t(A) \subseteq A_t$. However, we can now prove more.

Proposition 2.7. Assume that $x \in A_t$. Then for all $a \in A$ we have $\varepsilon_t(xa) = x\varepsilon_t(a)$ and $\varepsilon_s(ax) = S(x)\varepsilon_s(a)$. Symmetrically, if $y \in A_s$ we have $\varepsilon_s(ay) = \varepsilon_s(a)y$ and $\varepsilon_t(ya) = \varepsilon_t(a)S(y)$ for all $a \in A$.

Proof. Take $x \in M(A)$ and assume that $\Delta(x) = (x \otimes 1)E$. Let $a \in A$. Then $\Delta(xa) = (x \otimes 1)\Delta(a)$ and if we apply $m(\iota \otimes S)$ where m is multiplication, we find $\varepsilon_t(xa) = x\varepsilon_t(a)$. By Lemma 2.4, we know that also $\Delta(ax) = \Delta(a)(x \otimes 1)$ and now we apply $m(S \otimes \iota)$ to find $\varepsilon_s(ax) = S(x)\varepsilon_s(a)$. This proves the first part of the proposition.

The second part is proven in a completely similar way. □

Using techniques as above, we find other formulas of this type but we will not need these.

The result above has a few obvious, but important consequences.

Proposition 2.8. i) The sets $\varepsilon_s(A)$ and $\varepsilon_t(A)$ are subalgebras.
ii) The algebra $\varepsilon_s(A)$ is a right ideal of A_s and $\varepsilon_t(A)$ is a left ideal of A_t. □

Remark that the algebras A_s and A_t contain the identity of $M(A)$. This is not the case in general for the subalgebras $\varepsilon_s(A)$ and $\varepsilon_t(A)$. It is also not clear if, again in general, $\varepsilon_s(A)$ is also a left ideal of A_s and if $\varepsilon_t(A)$ is also a right ideal of A_t. All of this is related with the behavior of the antipode on these algebras (as we can see already from formulas in Proposition 2.7). In a subsequent item, we investigate this further.

First we look at the multiplier algebras of the images of the source and the target maps.

The multiplier algebras of the source and target algebras

We introduce the following notation and terminology. As mentioned already in the introduction, the terminology is different from the one originally used in [VD-W3], see further.

Notation 2.9. In what follows, we will denote the algebra $\varepsilon_s(A)$ by B and $\varepsilon_t(A)$ by C. We will call (B, C) the symmetric pair of *source algebra* and *target algebra*. □

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Recall that we do not expect these algebras to be unital. And we are interested in their multiplier algebras, if they exist.

We begin with some module properties giving more information about these algebras B and C and how they sit in $M(A)$.

Proposition 2.10. We have

$$A = AB \qquad \text{and} \qquad A = CA \tag{2. 3}$$
$$A = BA \qquad \text{and} \qquad A = AC. \tag{2. 4}$$

Proof. We know that

$$ba = \sum_{(a)} ba_{(1)}S(a_{(2)})a_{(3)} = \sum_{(a)} ba_{(1)}\varepsilon_s(a_{(2)})$$

for all a, b. The right hand side is in $A\varepsilon_s(A)$ and because $A^2 = A$ we find that $A = A\varepsilon_s(A)$. Similarly, from the formula

$$ab = \sum_{(a)} a_{(1)}S(a_{(2)})a_{(3)}b = \sum_{(a)} \varepsilon_t(a_{(1)})a_{(2)}b$$

for all a, b, we get $A = \varepsilon_t(A)A$.

If on the other hand, we start with the formula

$$bS(a) = \sum_{(a)} bS(a_{(1)})a_{(2)}S(a_{(3)}) = \sum_{(a)} bS(a_{(1)})\varepsilon_t(a_{(2)})$$

for all a, b, we find that $AS(A)$ is contained in $A\varepsilon_t(A)$ (recall Remark 1.2.ii in Section 1). Now, in Proposition 3.6 of [VD-W3], we have shown that $AS(A) = A$ and so we get also $A = A\varepsilon_t(A)$. Similarly $A = \varepsilon_s(A)A$. □

The results above say that A as a B-bimodule and as a C-bimodule is unital. If we combine the above result with the property in Proposition 2.7 we get the following.

Corollary 2.11. The algebras B and C are idempotent. □

Indeed, for all a, b we have e.g. $\varepsilon_s(a\varepsilon_s(b)) = \varepsilon_s(a)\varepsilon_s(b)$. Similarly for $\varepsilon_t(A)$.

Later, we will see that the algebras B and C have local units. This implies that the bimodules are also non-degenerate. In fact, this already follows by a more general argument, which is part of the following, also more general result.

Lemma 2.12. Let R be a subalgebra of $M(A)$. Multiplication makes A into a R-bimodule. Assume that this module is unital. Then it is also a non-degenerate bimodule. The algebra R is a non-degenerate algebra and the embedding of R in $M(A)$ extends uniquely to an embedding of the multiplier algebra $M(R)$ of R in $M(A)$. Moreover we have, considering $M(R)$ as sitting inside $M(A)$,

$$M(R) = \{x \in M(A) \mid xr \in R \text{ and } rx \in R \text{ for all } r \in R\}. \tag{2. 5}$$

Proof. We first show that the module is non-degenerate. Take any $a \in A$ and assume that $ra = 0$ for all $r \in R$. Then $a'ra = 0$ for all $a' \in A$ and $r \in R$. Because we assume

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that $AR = A$, it follows that also $a'a = 0$ for all $a' \in A$. Then $a = 0$. Similarly on the other side. We get in that A is a non-degenerate R-bimodule.

We also claim that R is a non-degenerate subalgebra of $M(A)$. To show this assume that $r \in R$ and that $rs = 0$ for all $s \in R$. Multiply with an element $a \in A$ from the right and use that $RA = A$. This implies that $ra = 0$ for all $a \in A$. Then $r = 0$. Similarly on the other side. So the algebra R is non-degenerate and we can consider its multiplier algebra $M(R)$.

As A is assumed to be a unital R-bimodule, we have $RA = A$ and $AR = A$. So the embedding $j : R \to M(A)$ is a non-degenerate homomorphism and a standard result implies that it extends uniquely to a unital homomorphism $\overline{j} : M(R) \to M(A)$. It is not hard to show that in this case, this extension is still an embedding. Because obviously for any $x \in M(R)$ we have $xr \in R$ and $rx \in R$ for all $r \in R$, we find one inclusion of the statement (2.1). The other inclusion is proven by using again that the R-bimodule A is unital. □

We can apply this lemma and we obtain the following. Recall that we use B to denote the algebra $\varepsilon_s(A)$ and C for $\varepsilon_t(A)$ (cf. Notation 2.9).

Theorem 2.13. The algebras B and C are non-degenerate and idempotent. Their multiplier algebras $M(B)$ and $M(C)$ embed in $M(A)$. An element $x \in M(C)$ still satisfies

$$\Delta(x) = (x \otimes 1)E = E(x \otimes 1)$$

while

$$\Delta(y) = E(1 \otimes y) = (1 \otimes y)E$$

is still true for elements y of $M(B)$. So $M(B) \subseteq A_s$ and $M(C) \subseteq A_t$.

Proof. The conditions in Lemma 2.12 are fulfilled for the subalgebras B and C as shown in the Propositions 2.10. Therefore, B and C are non-degenerate algebras and they sit in $M(A)$ in such a way that the embeddings $B \subseteq M(A)$ and $C \subseteq M(A)$ extend to embedding of their multiplier algebras $M(B)$ and $M(C)$.

We have already explained that the algebras B and C are idempotent. There are various ways to prove that we still have the embeddings $M(B) \subseteq A_s$ and $M(C) \subseteq A_t$. Take e.g. $m \in M(C)$, $x \in C$ and $a \in A$. Then

$$\Delta(mxa) = (mx \otimes 1)\Delta(a) = (m \otimes 1)\Delta(xa).$$

As $CA = A$, it follows that $\Delta(ma) = (m \otimes 1)\Delta(a)$ for all $a \in A$ and hence $\Delta(m) = (m \otimes 1)E$. Similar arguments are used for the other equations. □

In the next item of this section, we study the behavior of the antipode on the algebras B and C.

The antipode on the source and target algebras

We begin with the following result about the symmetric pair of subalgebras A_s and A_t of $M(A)$. Recall that we can extend the antipode S to a unital anti-homomorphism from $M(A)$ to itself.

Proposition 2.14.
i) If $x, y \in M(A)$ and $(1 \otimes x)E = (y \otimes 1)E$, then $x \in A_t$ and $y \in A_s$.

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ii) If $x, y \in M(A)$ and $E(1 \otimes x) = E(y \otimes 1)$, then $x \in A_t$ and $y \in A_s$.
iii) If $x \in A_t$ then $S(x) \in A_s$ and $(1 \otimes x)E = (S(x) \otimes 1)E$.
iv) If $y \in A_s$ then $S(y) \in A_t$ and $E(y \otimes 1) = E(1 \otimes S(y))$.
Proof. i) Assume $x, y \in M(A)$ and that $(1 \otimes x)E = (y \otimes 1)E$. If we apply $\iota \otimes \Delta$ to this equation, we find

$$\begin{aligned}(1 \otimes \Delta(x))(E \otimes 1) &= (y \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)\\ &= (1 \otimes x \otimes 1)(E \otimes 1)(1 \otimes E)\\ &= (1 \otimes x \otimes 1)(1 \otimes E)(E \otimes 1).\end{aligned}$$

Now we use the property that $(1 \otimes a)E = 0$ implies that $a = 0$ (see Lemma 1.1 in Section 1). This will eventually give $\Delta(x) = (x \otimes 1)E$. This proves that $x \in A_t$. If we apply $\Delta \otimes \iota$ instead, we obtain that $y \in A_s$.

ii) The second property is proven in completely the same way.

iii) Let $x \in A_t$ so that $\Delta(x) = E(x \otimes 1)$. Then for all $a \in A$ we have $\Delta(ax) = \Delta(a)(x \otimes 1)$ and so

$$\begin{aligned}(1 \otimes ax)E &= \sum_{(ax)} (S((ax)_{(1)}) \otimes 1)\Delta((ax)_{(2)})\\ &= \sum_{(a)} (S(a_{(1)}x) \otimes 1)\Delta(a_{(2)})\\ &= \sum_{(a)} (S(x)S(a_{(1)}) \otimes 1)\Delta(a_{(2)})\\ &= (S(x) \otimes a)E\end{aligned}$$

This implies $(1 \otimes x)E = (S(x) \otimes 1)E$. It follows from i) that $S(x) \in A_s$.

iv) Similarly we get $S(y) \in A_t$ when $y \in A_s$ and $E(y \otimes 1) = E(1 \otimes S(y))$. □
Remark that it follows that S is injective on A_s and on A_t. However, it does not imply that these maps are surjective in the general case.

We now investigate the maps $S_B : B \to M(A)$ and $S_C : C \to M(A)$ that we obtain by restricting (the extension of) the antipode to the subalgebras B and C of $M(A)$. As a special case of the equations above, we have

$$(1 \otimes x)E = (S_C(x) \otimes 1)E \qquad \text{and} \qquad E(y \otimes 1) = E(1 \otimes S_B(y))$$

for $x \in B$ and $y \in C$. In particular, we know already that $S_B : B \to A_t$ and $S_C : C \to A_s$. In the next proposition, we get a stronger result.

Proposition 2.15. The map S_B is a non-degenerate anti-homomorphism from B to $M(C)$ and the map S_C is a non-degenerate anti-homomorphism from C to $M(B)$. Both maps are injective.
Proof. i) Take $x \in C$. Then $x \in A_t$ and from Proposition 2.7 we know that $\varepsilon_s(ax) = S(x)\varepsilon_s(a)$ for all a. Because now also $\varepsilon_s(aS(x)) = \varepsilon_s(a)S(x)$ for all a, we see that $S(x) \in M(B)$. Similarly $S(y) \in M(C)$ when $y \in B$. It follows that S_C is an anti-homomorphism from C to $M(B)$ and that S_B is an anti-homomorphism of B to $M(C)$.

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ii) As $BA = A$ and $\varepsilon_t(ya) = \varepsilon_t(a)S(y)$ for $y \in B$, we see that $CS(B) = C$. On the other hand we have

$$A = S(A)A = S(AB)A = S(B)S(A)A = S(B)A$$

and because $\varepsilon_t(S(y)a) = S(y)\varepsilon_t(a)$ for $y \in B$ we see that also $S(B)C = C$.

Hence $S_B : B \to M(C)$ and $S_C : C \to M(B)$ are non-degenerate anti-homomorphisms. □
From the general theory, we know that S_B and S_C have unique extensions to unital anti-homomorphism from $M(B)$ to $M(C)$ and from $M(C)$ to $M(B)$ respectively. These extensions are still the restrictions of the antipode S to the multiplier algebras $M(B)$ and $M(C)$ respectively.

In the *regular case*, we have the following stronger results.

Theorem 2.16. In the case of a regular weak multiplier Hopf algebra, we have that S_B is an anti-isomorphism from B tot C and S_C is an anti-isomorphism from C to B. The multiplier algebras $M(B)$ and $M(C)$ are respectively equal to the algebras A_s and A_t as defined in Notation 2.5.
Proof. We can use e.g. that $(S \otimes S)E = \zeta E$ in the case of a regular weak multiplier Hopf algebra (see Proposition 4.4 in [VD-W3]). As B is the left leg of E and C is the right leg of E, we find that S maps B to C and C to B. It also follows that these maps are surjective. As we know already that they are also injective, we find the first statement of the proposition.

The equation $(S \otimes S)E = \zeta E$ also implies that S maps A_s to A_t and vice versa. In Proposition 2.7 we have shown that

$$\varepsilon_s(ay) = \varepsilon_s(a)y \qquad \text{and} \qquad \varepsilon_s(ax) = S(x)\varepsilon_s(a)$$

for $y \in A_s$ and $x \in A_t$. It follows that the algebra B, the image $\varepsilon_s(A)$, is a two-sided ideal of A_s. And because we know already that the $M(B) \subseteq A_s$, it follows that $M(B) = A_s$.

Similarly we have $M(C) = A_t$. □
It is not completely clear what the situation is in the non-regular case. We have Proposition 2.15 saying that S_B embeds B in $M(C)$ and Theorem 2.13 saying that $M(C)$ is a subalgebra of A_t. Symmetrically S_C embeds C in $M(B)$ and $M(B)$ is a subalgebra of A_s.

For this reason, we have changed our terminology and are now calling the algebras B and C, the images of the source and target maps respectively, the source and target algebras. In an earlier version of this paper [VD-W4.v1] we have used these terms for A_s and A_t instead. This was motivated by the fact that, in the regular case, the can be identified with the multiplier algebras of B and C respectively. But this is not sure in the non-regular case that we are investigating in greater detail in this version of the paper.

The canonical idempotent E as a separability idempotent in $M(B \otimes C)$

We have the algebras B and C. They are non-degenerate and idempotent. The algebra B is the left leg of E and the algebra C is the right leg of E, in an appropriate sense, see Proposition 2.2. And because E is a multiplier of $A \otimes A$, we can expect that it is also a multiplier of $B \otimes C$. This turns out to be the case. Moreover it is a separability idempotent as defined and studied in [VD4.v2]. This is what we show next.

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The first step is the following result.

Lemma 2.17. We have the following symmetric properties:

$$E(1 \otimes a) \in B \otimes A \qquad \text{and} \qquad (a \otimes 1)E \in A \otimes C$$

for all $a \in A$.

Proof. For all a in A we can define a left multiplier $\varepsilon'_s(a)$ of A by the formula

$$\varepsilon'_s(a)b = (\iota \otimes \varepsilon)(E(b \otimes a))$$

where b is in A. We will see later why we use this notation.

Fix two elements a, a' in A. Write

$$\sum_{(a)} \varepsilon'_s(a_{(1)}) \otimes a_{(2)}a' = \sum_i t_i \otimes q_i$$

where t_i is a left multiplier of A and $q_i \in A$. Assume that the (q_i) are linearly independent.

For all b in A we find

$$\begin{aligned}
\sum_{(a)} \varepsilon'_s(a_{(1)})b \otimes a_{(2)}a' &= \sum_{(a)} (\iota \otimes \varepsilon \otimes \iota)((E \otimes 1)(b \otimes a_{(1)} \otimes a_{(2)}a')) \\
&= (\iota \otimes \varepsilon \otimes \iota)((E \otimes 1)(1 \otimes E)(b \otimes \Delta(a)(1 \otimes a'))) \\
&= (\iota \otimes \varepsilon \otimes \iota)((\iota \otimes \Delta)(E(b \otimes a))(1 \otimes 1 \otimes a')) \\
&= E(b \otimes a)(1 \otimes a') = E(b \otimes aa').
\end{aligned}$$

Therefore $E(b \otimes aa') = \sum_i t_i b \otimes q_i$ for all $b \in A$.

On the other hand, for all $c \in A$, we have also

$$\begin{aligned}
(1 \otimes c)E(1 \otimes aa') &= \sum_{(c)} S(c_{(1)})c_{(2)} \otimes c_{(3)}aa' \\
&= \sum_{(c)} \varepsilon_s(c_{(1)}) \otimes c_{(2)}aa'
\end{aligned}$$

and this belongs to $B \otimes A$.

If we combine this with the previous formulas, we find $\sum_i t_i \otimes cq_i \in B \otimes A$ for all $c \in A$. Now let ω be a linear functional on the space $L(B)$ of left multipliers of A that vanishes on elements in B. We find $\sum_i \omega(t_i)cq_i = 0$ for all c in A. By non-degeneracy of the product in A and because the elements (q_i) are linearly independent, it follows that $\omega(t_i) = 0$ for all i. Hence t_i is in B for all i and we find that $E(1 \otimes aa') \in B \otimes A$. Because A is idempotent, we get $E(1 \otimes A) \subseteq B \otimes A$.

In a completely similar way we can prove that also $(A \otimes 1)E \in A \otimes C$. This proves the lemma. □

From the proof we see that $\sum_{(a)} \varepsilon'_s(a_{(1)}) \otimes a_{(2)}a' \in B \otimes A$ and from the fullness of the coproduct, it follows that $\varepsilon'(a) \in B$ for all $a \in A$. This in turn follows of course also from the property that $E(1 \otimes A) \subseteq B \otimes A$.

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We will give more comments on this result later. First we use the lemma to prove the following main result.

Theorem 2.18. The canonical idempotent of a weak multiplier Hopf algebra is a separability idempotent in $M(B\otimes C)$ where (B,C) is the symmetric pair of source and target algebras.

Proof. i) By the lemma, we find that $E(1\otimes a)$ belongs to $B\otimes A$. We therefore can apply ε_t on the second leg of this expression. We know that the second leg of E belongs to $\varepsilon_t(A)$ and this is a subalgebra of A_t. In Proposition 2.7 we have shown that $\varepsilon_t(xa) = x\varepsilon_t(a)$ for all $x \in A_t$. Therefore $(\iota\otimes\varepsilon_t)(E(1\otimes a)) = E(1\otimes\varepsilon_t(a))$. We conclude that $E(1\otimes\varepsilon_t(a)) \in B\otimes C$ for all a and so $E(1\otimes C) \subseteq B\otimes C$.

In a completely similar way, we find that $(B\otimes 1)E \subseteq B\otimes C$. It follows not only that $E \in M(B\otimes C)$, but also that it satisfies the first requirements for a separability idempotent (see Section 1 of [VD4.v2]).

ii) We will now show that E is full in the sense of Definition 1.1 of [VD4.v2]. For this, assume that V is a subspace of B so that $E(1\otimes x) \subseteq V\otimes C$ for all $x \in C$. Then $(1\otimes b)E(1\otimes xa) \in V\otimes A$ for all $a,b \in A$ and $x \in C$. In Proposition 2.10 we showed that $CA = A$ and in Proposition 2.2 that B is spanned by elements of the form $(\iota\otimes\omega(a\cdot b))E$ where $a,b \in A$ and where ω is a linear functional on A. Then we must have $V = B$ proving that the left leg of E (as an idempotent in $M(B\otimes C)$) is still all of B. Similarly for the right leg. Hence E is full.

iii) Finally, we know already from Proposition 2.15 that the antipode is a non-degenerate anti-homomorphism from B to $M(C)$ as well as a non-degenerate anti-homomorphism from C to $M(B)$. As in Proposition 2.14 they satisfy

$$(1\otimes x)E = (S(x)\otimes 1)E \qquad \text{and} \qquad E(y\otimes 1) = E(1\otimes S(y))$$

when $x \in C$ and $y \in B$. This is the final requirement in Definition 1.4 of [VD4.v2] and shows that E is a separability idempotent in $M(B\otimes C)$. This completes the proof. □

Remark that in item iii) of the proof above, we find $E(y\otimes 1) = E(1\otimes S(y))$ for all $y \in B$. Then $E(1\otimes S(y)x) = E(y\otimes x)$ for all $x \in C$ and $y \in B$. From the fact that $E \in M(B\otimes C)$ and that S is a non-degenerate anti-homomorphism from B to $M(C)$, it would also follow that $E(1\otimes C) \subseteq B\otimes C$.

In the *regular case*, we have the following expected result.

Proposition 2.19. If the weak multiplier Hopf algebra is regular, then E is a regular separability idempotent.

Proof. There are different ways to prove this. If we start with the definition of regularity for a weak multiplier Hopf algebra (as e.g. in Definition 4.1 of [VD-W3]), then we assume that $(A,\Delta^{\rm cop})$ also satisfies the axioms of a weak multiplier Hopf algebra. The canonical idempotent now is ζE where E is the canonical idempotent of the original weak multiplier Hopf algebra. Remember that ζ is the flip map on $A\otimes A$ and extended to $M(A\otimes A)$.

Because B and C are the left and the right legs of E, we get that C and B are the left and the right legs of ζE. Applying Theorem 2.18 to the new weak multiplier Hopf algebra $(A,\Delta^{\rm cop})$, we obtain that ζE is a separability idempotent in $M(C\otimes B)$. Then E is indeed

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a regular separability idempotent by the very definition of regularity for a separability idempotent (see Definition 2.4 of [VD4.v2]). □
In an earlier version of this paper (reference [VD-W4.v1]), we only considered regular weak multiplier Hopf algebras and this result was obtained already, see Section 2 in [VD-W4.v1]).

Let us now consider some of the results we have proven for general and regular separability idempotents in [VD4.v2] and see what they give in the case of the canonical idempotent of a weak multiplier Hopf algebra. Recall the distinguished linear functionals φ_B and φ_C on B and C respectively, defined and characterized by the formulas

$$(\varphi_B \otimes \iota)E = 1 \qquad \text{and} \qquad (\iota \otimes \varphi_C)E = 1;$$

see Proposition 1.9 in [VD4.v2].

Proposition 2.20. The distinguished linear functionals φ_B and φ_C, obtained for the separability idempotent E, satisfy the following symmetric properties:

$$\varphi_B(\varepsilon_s(a)) = \varepsilon(a) \qquad \text{and} \qquad \varphi_C(\varepsilon_t(a)) = \varepsilon(a)$$

for all $a \in A$.
Proof. We have the formula

$$(1 \otimes a)E(1 \otimes b) = \sum_{(a)} \varepsilon_s(a_{(1)}) \otimes a_{(2)}b$$

for all $a, b \in A$ (see e.g. in the proof of Lemma 2.17). If we apply φ_B on the first factor, we obtain

$$\varphi_B(\varepsilon_s(a_{(1)}))a_{(2)}b = ab.$$

If we apply a linear functional ω we find $\varphi_B(\varepsilon_s(a')) = \omega(ab)$ with

$$a' = (\iota \otimes \omega)(\Delta(a)(1 \otimes b).$$

Because $\varepsilon(a') = \omega(ab)$ we see that $\varphi_B(\varepsilon_s(a')) = \varepsilon(a')$. By the fullness of the coproduct, any element of A is of the form $(\iota \otimes \omega)(\Delta(a)(1 \otimes b)$. This proves the first formula of this proposition. The other one is proven in a similar way. □

Existence of local units

From the general theory of (possibly non-regular) separability idempotents, we know that there exist local units (cf. Proposition 1.10 in [VD4.v2]). As a consequence we get the following result.

Proposition 2.21 The algebra A has local units.
Proof. Let $a \in A$ and assume that ω is a linear functional on A so that $\omega(ba) = 0$ for all $b \in A$. Then

$$(\iota \otimes \omega)((1 \otimes b)(\iota \otimes S)((c \otimes 1)\Delta(p))(1 \otimes a)) = 0$$

for all $b, c, p \in A$. We use that $(c \otimes 1)\Delta(p) \in A \otimes A$. We know that $((\iota \otimes S)\Delta(p))(1 \otimes a)$ belongs to $A \otimes A$. Therefore, we can cancel c in the above equation and get

$$(\iota \otimes \omega)((1 \otimes b)((\iota \otimes S)\Delta(p))(1 \otimes a)) = 0.$$

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Write $((\iota \otimes S)\Delta(p))(1 \otimes a)$ as $\sum_i p_i \otimes q_i$ and assume that the elements (p_i) are linearly independent. We find $\omega(bq_i) = 0$ for all i and all $b \in A$. Replace b by p_i and take the sum over i. Because $\sum p_i q_i = \varepsilon_t(p)a$ we get $\omega(\varepsilon_t(p)a) = 0$ for all $p \in A$. This means that $\omega(xa) = 0$ for all $x \in C$.

We know that $A = CA$ and because we have left local units in C, there exists an element $x \in C$ so that $xa = a$. Then we see that $\omega(a) = 0$. This means that $a \in Aa$ and we know that this implies that A has left local units. In a similar way, we find that A also has right local units. This completes the proof. □
We see in the proof that we only need that B has right local units and that C has left local units. These results have a more easy proof in [VD4.v2].

Recall also that in earlier work on weak multiplier Hopf algebras, the existence of local units was only obtained in the case of a regular weak multiplier Hopf algebra, see Proposition 4.9 in [VD-W3].

We finish this section with a couple of remarks.

Remarks 2.22. i) As we see from the proof of Lemma 2.17 and from earlier arguments, we find that $(\iota \otimes \varepsilon)((1 \otimes a)E) = \varepsilon_s(a)$ when $a \in A$. The formula makes sense as an equality of left multipliers of A. Remark that we do not expect $(1 \otimes a)E$ to belong to $B \otimes A$. Similarly, we find $(\varepsilon \otimes \iota)(E(a \otimes 1)) = \varepsilon_t(a)$ for a in A, now as right multipliers of A. Again we do not expect $E(A \otimes 1) \subseteq A \otimes C$.

ii) On the other hand, we do have $E(1 \otimes A) \subseteq B \otimes A$ and $(A \otimes 1)E \subseteq A \otimes C$ as we have shown in the lemma. As we have seen before, if we apply ε on the second leg in the first case and on the first leg in the second case, we get

$$(\iota \otimes \varepsilon)(E(1 \otimes a)) = \varepsilon'_s(a) \qquad \text{and} \qquad (\varepsilon \otimes \iota)((a \otimes 1)E) = \varepsilon'_t(a)$$

where $\varepsilon'_s : A \to B$ and $\varepsilon'_t : A \to C$.

iii) From the proof of the lemma, we see that the range of ε'_s is the same as the range of ε_s, namely B. Indeed, we have

$$\sum_{(c)} \varepsilon_s(c_{(1)}) \otimes c_{(2)}aa' = \sum_{(a)} \varepsilon'_s(a_{(1)})b \otimes a_{(2)}a'$$

and using the fullness of the coproduct, we see that the range of ε_s is contained in the range of $\varepsilon' s$. Similarly, we can define ε'_t by $\varepsilon'_t(a) = (\varepsilon \otimes \iota)((a \otimes 1)E)$ and also ε'_t and ε_t have the same range, namely C.

iv) In the regular case, we get

$$\varepsilon'_s(a) = \sum_{(a)} a_{(2)}S^{-1}(a_{(1)}) \qquad \text{and} \qquad \varepsilon'_t(a) = \sum_{(a)} S^{-1}(a_{(2)})a_{(1)}$$

for $a \in A$. We see that then

$$S(\varepsilon'_t(a)) = \varepsilon_t(a) \qquad \text{and} \qquad S(\varepsilon'_t(a))\varepsilon_s(a)$$

for all a. □

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It is somewhat remarkable that in general, the maps ε'_s and ε'_t exist and have the same range as the maps ε_s and ε_t respectively, while it is not expected that the inverse of S exists. We make more comments on this peculiarity in Section 3, where we discuss further possible research.

These four counital maps are also considered in e.g. [B-G-L] but the notations are different. For the convenience of the reader, in [K-VD] is included an appendix with a dictionary. It includes the following formulas relating our notation with the ones used in [B-G-L]:

$$\varepsilon_s(a) = \sqcap^R(a), \qquad \varepsilon'_s(a) = \overline{\sqcap}^R(a), \tag{2. 6}$$

$$\varepsilon_t(a) = \sqcap^L(a), \qquad \varepsilon'_t(a) = \overline{\sqcap}^L(a). \tag{2. 7}$$

3. Examples and special cases

In this section we will treat some examples and special cases. The main purpose is to illustrate results in Section 2 about the source and target algebras. However we will also use some of the examples for the illustration of the general theory of weak multiplier Hopf algebras because this has not yet been done in the earlier papers we wrote on the subject.

The groupoid examples

For completeness we begin with a very brief review of the two basic motivating examples associated with a groupoid. We will not give details as they can be found in our earlier papers on the subject (see [VD-W2] and [VD-W3]). On the other hand, we use these examples to illustrate some of the statements we made earlier in this paper, as well as for some other examples further in this section.

Example 3.1. i) Consider a groupoid G. *First* there is the algebra A, defined as the space $K(G)$ of complex functions on G with finite support and pointwise product. Recall that the coproduct Δ on $K(G)$ is defined by

$$\Delta(f)(p,q) = \begin{cases} f(pq) & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

The pair (A, Δ) is a regular weak multiplier Hopf algebra (in the sense of Definitions 1.14 and 4.1 in [VD-W3]). The canonical idempotent E in $M(A \otimes A)$ is given by the function on pairs (p, q) in $G \times G$ that is 1 if pq is defined and 0 if this is not the case. The antipode S is defined by $(S(f))(p) = f(p^{-1})$ whenever $f \in K(G)$ and $p \in G$.

In this example, the algebra A_s is the algebra of all complex functions on G so that $f(p) = f(q)$ whenever $p, q \in G$ satisfy $s(p) = s(q)$. It is naturally identified with the algebra of all complex functions on the set G_0 of units in G. The *source map* ε_s from A to A_s is defined by $(\varepsilon_s(f))(p) = f(p^{-1}p)$ whenever $p \in G$ and $f \in K(G)$. The image of the source map $\varepsilon_s(A)$, what we called in this paper the *source algebra*, is identified with the algebra of complex functions with finite support on the units. Symmetrically, the algebra A_t consists of functions f on G so that $f(p) = f(q)$ if $t(p) = t(q)$ for $p, q \in G$. It is also identified with the space of all complex functions on the units. The *target map* ε_t from

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A to A_t is defined by $(\varepsilon_t(f))(p) = f(pp^{-1})$ for all p and $f \in K(G)$. The *target algebra*, i.e. the image $\varepsilon_t(A)$ of the target map, is again identified with the space of functions with finite support on the units. Recall that these two algebras are subalgebras of the multiplier algebra $M(A)$ (here the algebra of all complex functions on G). Observe also that the source and target algebras $\varepsilon_s(A)$ and $\varepsilon_t(A)$, can be strictly smaller than the algebras A_s and A_t respectively. This happens when the set of units is infinite. In that case, we see that A_s is indeed the multiplier algebra $M(\varepsilon_s(A))$ of $\varepsilon_s(A)$ and similarly for the target map.

ii) For the *second case*, we take the groupoid algebra $\mathbb{C}G$ of G. If we use $p \mapsto \lambda_p$ for the canonical embedding of G in $\mathbb{C}G$, then if $p, q \in G$, we have $\lambda_p\lambda_q = \lambda_{pq}$ if pq is defined and 0 otherwise. The coproduct on $\mathbb{C}G$ is given by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all $p \in G$. The idempotent E is $\sum \lambda_e \otimes \lambda_e$ where the sum is only taken over the units e of G. The antipode is given by $S(\lambda_p) = \lambda_{p^{-1}}$ for all $p \in G$.

The symmetric pair of *source* and *target maps* is given by $\varepsilon_s(\lambda_p) = \lambda_e$ where $e = s(p)$ and $\varepsilon_t(\lambda_p) = \lambda_e$ where now $e = t(p)$ for $p \in G$. Here the *source* and *target algebras* coincide and it is the algebra of the span of elements of the form λ_e where e is a unit of G. Also here the source and target algebras need not be unital and so can be strictly smaller then their multiplier algebras. □

Recall that these two cases are dual (symmetric) to each other. The duality is given by $\langle f, \lambda_p \rangle = f(p)$ whenever $f \in K(G)$ and $p \in G$. We will give more details (about this duality) in [VD-W5] where we treat duality for (regular) weak multiplier Hopf algebras with integrals.

Examples associated with separability idempotents

For the next example, we start with any separability idempotent. Later, we will consider two special cases of this. The most important one will be constructed from the separability idempotent that is the canonical idempotent of a given weak multiplier Hopf algebra. In some sense, we isolate the source and target algebras with what remains of the original coproduct.

These examples illustrate very well the use of different properties of the source and target algebras, obtained in the previous section.

Recall from [VD4.v2] that a *separability idempotent* is an idempotent in the multiplier algebra $M(B \otimes C)$ of the tensor product of two non-degenerate algebras B and C with certain properties. In particular there exist non-degenerate anti-homomorphisms $S_B : B \to M(C)$ and $S_C : C \to M(B)$ characterized by the formulas

$$E(b \otimes 1) = E(1 \otimes S_B(b)) \qquad \text{and} \qquad (1 \otimes c)E = (S_C(c) \otimes 1)E$$

whenever $b \in B$ and $c \in C$. There are also the unique linear functionals φ_B and φ_C on B and C respectively, characterized by

$$(\varphi_B \otimes \iota)(E(1 \otimes c)) = c \qquad \text{and} \qquad (\iota \otimes \varphi_C)((b \otimes 1)E) = b$$

for all $b \in B$ and $c \in C$. We refer to [VD4.v2] for details.

We now construct a weak multiplier Hopf algebra from a separability idempotent in the next proposition.

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Theorem 3.2. Let (B, C) be a symmetric pair of non-degenerate algebras and assume that E is a separability idempotent in $M(B \otimes C)$. Let $P = C \otimes B$. There is a coproduct Δ_P on P defined by

$$\Delta_P(c \otimes b) = c \otimes E \otimes b$$

for $c \in C$ and $b \in B$. The pair (P, Δ_P) is a weak multiplier Hopf algebra. The counit ε_P is given by $\varepsilon_P(c \otimes b) = \varphi_B(S_C(c)b)$. We also have $\varepsilon_P(c \otimes b) = \varphi_C(cS_B(b))$. The canonical idempotent E_P of (P, Δ_P) is $1 \otimes E \otimes 1$. The antipode S_P is given by $S_P(c \otimes b) = S_B(b) \otimes S_C(c)$ when $b \in B$ and $c \in C$. The source and target algebras are $1 \otimes B$ and $C \otimes 1$ respectively and the source and target maps are

$$\varepsilon_s^P(c \otimes b) = 1 \otimes S_C(c)b \qquad \text{and} \qquad \varepsilon_t^P(c \otimes b) = cS_B(b) \otimes 1$$

for all $b \in B$ and $c \in C$. In these formulas, 1 is the identity in $M(C)$ and $M(B)$ respectively.

Proof. We will systematically use ι_P, 1_P, etc. for objects related with P. For the objects related with the original algebras, we will use no index.

i) The algebra P is non-degenerate and idempotent because this is true for its components B and C.

ii) Because $E \in M(B \otimes C)$ we have that $\Delta_P(c \otimes b)$, defined as $c \otimes E \otimes b$, belongs to $M(P \otimes P)$. Because $E^2 = E$, it is clear that Δ_P is a homomorphism. By assumption, we have that $E(1 \otimes C)$ and $(B \otimes 1)E$ are subsets of $B \otimes C$. Therefore

$$\Delta_P(P)(1_P \otimes P) \subseteq P \otimes P \qquad \text{and} \qquad (P \otimes 1_P)\Delta_P(P) \subseteq P \otimes P.$$

The coproduct Δ_P is coassociative and $(\Delta_P \otimes \iota_P)\Delta_P(c \otimes b) = c \otimes E \otimes E \otimes b$ for all $b \in B$ and $c \in C$. This coproduct is full because E is assumed to be full (as in Definition 1.1 of [VD4.v2]).

iii) Now we prove that there is a counit ε_P on (P, Δ_P). First define $\varepsilon_P(c \otimes b) = \varphi_C(cS_B(b))$. For all $b \in B$ and $c \in C$ we have that

$$\begin{aligned}(\iota_P \otimes \varepsilon_P)\Delta_P(c \otimes b) &= (\iota_P \otimes \varepsilon_P)(c \otimes E \otimes b)\\ &= (\iota_P \otimes \varphi_C)(c \otimes E(1 \otimes S_B(b)))\\ &= (\iota_P \otimes \varphi_C)(c \otimes E(b \otimes 1)) = c \otimes b.\end{aligned}$$

On the other hand, if we define $\varepsilon'_P(c \otimes b) = \varphi_B(S_C(c)b)$ we will find similarly

$$(\varepsilon'_P \otimes \iota)\Delta_P(c \otimes b) = c \otimes b$$

for all $b \in B$ and $c \in C$. Then, from the general theory, we know that ε_P and ε'_P must be the same (see e.g. the argument we gave in the preliminary section of this paper). In the regular case we treat later, we will give another argument for this fact (see a remark after the proof of Proposition 3.3). This proves the existence of the counit.

iv) Take any elements $b, b' \in B$ and $c, c' \in C$. Then

$$\Delta_P(c \otimes b)(1 \otimes 1 \otimes c' \otimes b') = (1 \otimes E \otimes 1)(c \otimes 1 \otimes c' \otimes bb').$$

If we replace c' by elements of the form $S_B(b'')c''$, the right hand side will be

$$(1 \otimes E \otimes 1)(c \otimes b'' \otimes c'' \otimes bb').$$

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Next we use that B is idempotent and that the map S_B is non-degenerate. Then we can conclude from this that $\Delta_P(P)(1_P \otimes P) = E_P(P \otimes P)$ with $E_P = 1 \otimes E \otimes 1$. Similarly we find $(P \otimes 1_P)\Delta_P(P) = (P \otimes P)E_P$ and it follows that E_P is the canonical idempotent for (P, Δ_P).

It is straightforward to verify that the legs of E_P commute. Moreover

$$(\iota_P \otimes \Delta_P)(E_P) = 1 \otimes E \otimes E \otimes 1$$

and this is clearly $(1_P \otimes E_P)(E_P \otimes 1_P)$.

v) We now define $S_P(c \otimes b) = S_B(b) \otimes S_C(c)$ for all b and c and we show that all the conditions of Theorem 2.9 of [VD-W3] are fulfilled. This will complete the proof.

We consider the candidate for the generalized inverse R_1 of the canonical map T_1 using this expression for S_P. We get, using formally $E_{(1)} \otimes E_{(2)}$ for E, that

$$\begin{aligned} R_1(c \otimes b \otimes c' \otimes b') &= ((\iota_P \otimes S_P)(c \otimes E \otimes b))(1 \otimes 1 \otimes c' \otimes b') \\ &= c \otimes E_{(1)} \otimes S_B(b)c' \otimes S_C(E_{(2)})b'. \end{aligned}$$

That this maps $P \otimes P$ to itself is a consequence of the property, obtained in Proposition 1.9 of [VD4.v2], saying that $E_{(1)} \otimes S_C(E_{(2)})b'$ is in $B \otimes B$.

Using this candidate for the antipode, we can calculate the candidates for the source and target maps ε_s^P and ε_t^P. We find

$$\begin{aligned} \varepsilon_t^P(c \otimes b) &= (c \otimes E_{(1)})(S_B(b) \otimes S_C(E_{(2)})) = cS_B(b) \otimes 1 \\ \varepsilon_s^P(c \otimes b) &= (S_B(E_{(1)}) \otimes S_C(c))(E_{(2)} \otimes b) = 1 \otimes S_C(c)b \end{aligned}$$

for all $b \in B$ and $c \in C$. We have again used the Sweedler type notation for E and that $E_{(1)}S_C(E_{(2)}) = 1$ and $S_B(E_{(1)})E_{(2)} = 1$ (see Proposition 1.6 in [VD4.v2]).

Finally we have to show that

$$\sum_{(a)} \varepsilon_t^P(a_{(1)})a_{(2)} = a \qquad \text{and} \qquad \sum_{(a)} \varepsilon_s^P(a_{(1)})S_P(a_{(2)}) = S_P(a)$$

for all $a = c \otimes b$. We find

$$\varepsilon_t^P(c \otimes E_{(1)})(E_{(2)} \otimes b) = cS_B(E_{(1)})E_{(2)} \otimes b = c \otimes b$$

proving the first equation. And

$$\varepsilon_s^P(c \otimes E_{(1)})S_P(E_{(2)} \otimes b) = (1 \otimes S_C(c)E_{(1)})(S_B(b) \otimes S_C(E_2)) = S_B(b) \otimes S_C(c).$$

Finally, we have to show that $T_1R_1(p \otimes p') = E_P(p \otimes p'$ for all $p, p' \in P$ where T_1 is the canonical map $p \otimes p' \mapsto \Delta_P(p)(1 \otimes p')$ and where R_1 it its generalized inverse constructed with the antipode S_P as above. With $p = c \otimes b$ and $p' = c' \otimes b'$ we find

$$\begin{aligned} T_1R_1(p \otimes p') &= (c \otimes E_{(1)} \otimes E_{(2)}S_B(b) \otimes 1)(1 \otimes 1 \otimes c' \otimes b') \\ &= (c \otimes E_{(1)}b \otimes E_{(2)} \otimes 1)(1 \otimes 1 \otimes c' \otimes b') \\ &= (1 \otimes E \otimes 1)(c \otimes b \otimes c' \otimes b') \end{aligned}$$

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and this is what we need because $E_P = (1 \otimes E \otimes 1)$. In a similar way, we find $T_2 R_2(p \otimes p') = (p \otimes p')E_P$ where T_2 is the other canonical map and R_2 its generalized inverse construct with the antipode S_P

This proves that the candidate for the antipode satisfies all the requirements needed for Theorem 2.9 of [VD-W3] and it completes the proof. □

We now consider the regular case. The result is as expected.

Theorem 3.3. If E is a regular separability idempotent in $M(B \otimes C)$, then the weak multiplier Hopf algebra (P, Δ_P), constructed in the previous proposition, is a regular weak multiplier Hopf algebra.

Proof. There are different ways to prove this. We will use the original definitions of regularity in both cases.

Recall that E is called regular if ζE is a separability idempotent in $M(C \otimes B)$ where as before ζ is the flip map. Assume that this is the case. We then have to show that the pair (P, Δ^{cop}) is also a weak multiplier Hopf algebra. Here the algebra P is $C \otimes B$ as before while

$$\Delta^{\text{cop}}(c \otimes b) = E_{(2)} \otimes b \otimes c \otimes E_{(1)}$$

for $b \in B$ and $c \in C$. Define the isomorphism $\gamma : B \otimes C \to P$ by $\gamma(b \otimes c) = c \otimes b$. Then the coproduct Δ^{cop} yields a coproduct Δ' on $B \otimes C$ given by

$$\Delta'(b \otimes c) = b \otimes E_{(2)} \otimes E_{(1)} \otimes c$$

for $b \in B$ and $c \in C$. Because ζE is a separability idempotent in $M(C \otimes B)$, it follows from the previous proposition that $(B \otimes C, \Delta')$ is a weak multiplier Hopf algebra. Then this is also true for the pair (P, Δ^{cop}). This completes the proof. □

Observe the following. Given $b \in B$ and $c \in C$ we have

$$(S_C(c) \otimes 1)E(b \otimes 1) = (1 \otimes c)E(1 \otimes S_B(b)) \tag{3. 1}$$

and if we apply $\varphi_B \otimes \varphi_C$ we find that $\varphi(S_C(c)b) = \varphi_C(cS_B(b))$. This illustrates the equality of the two forms of the counit in the formulation of Theorem 3.2. This argument however only seems to work for a (semi-)regular separability idempotent because only in that case we know that the elements in the equation (3.1) belong to $B \otimes C$.

In Theorem 3.2 we have associated a weak multiplier Hopf algebra to any separability idempotent. On the other hand, we know that conversely, the canonical idempotent E of a weak multiplier Hopf algebra is a separability idempotent in $M(B \otimes C)$ where now B and C are the source and target algebras. This is proven in Section 2 (Theorem 2.18). What happens when we then apply the construction of Theorem 3.2 again is explained the following proposition.

Proposition 3.4. Let (A, Δ) be a weak multiplier Hopf algebra. Consider the canonical idempotent E as sitting in $M(B \otimes C)$ where (B, C) is the symmetric pair of source and target algebras. Associate a new weak multiplier Hopf algebra (P, Δ_P) as in Theorem 3.2. Define $\gamma : P \to M(A)$ by $\gamma(x \otimes y) = xy$ for $x \in C$ and $y \in B$. Then γ is a non-degenerate homomorphism. It satisfies $\Delta \circ \gamma = (\gamma \otimes \gamma) \circ \Delta_P$ and $S \circ \gamma = \gamma \circ S_P$.

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Proof. Because the source and target algebras B and C are commuting subalgebras of $M(A)$, it follows that γ is an algebra homomorphism from P to $M(A)$. The image is CB. Because of Proposition 2.10, we have $CBA = A = ACB$ and so γ is non-degenerate. It extends to a unital homomorphism on the multiplier algebra of P.

For all $y \in B$ and $x \in C$ we have

$$\Delta(\gamma(x \otimes y)) = \Delta(xy) = (x \otimes y)E$$

while on the other hand

$$(\gamma \otimes \gamma)\Delta_P(x \otimes y) = (\gamma \otimes \gamma)(x \otimes E \otimes y) = (x \otimes 1)E(1 \otimes y).$$

These expressions are the same as the element y commutes with the second leg of E.

For the antipode we find

$$\gamma(S_P(x \otimes y)) = \gamma(S_B(y) \otimes S_C(x)) = S(y)S(x) = S(xy) = S(\gamma(x \otimes y))$$

where we have again used that the element x of C and the element y of B commute. □

Remark that in general, the map γ is not injective. Take e.g. the weak multiplier Hopf algebra constructed from a set X. The algebra A is the algebra $K(X)$ of complex functions with finite support and $\Delta(f)(p,q) = 0$ when p and q are different while $\Delta(f)(p,p) = f(p)$. This is a weak multiplier Hopf algebra. The canonical idempotent is the function $X \otimes X$ that is 1 on the diagonal and 0 everywhere else. Clearly the left and the right legs are all of $K(X)$. In particular $B = C$. The map γ is the multiplication map from $K(X \times X)$ to $K(X)$ and this is not injective.

If the algebra A is unital, we can also show that $\gamma \circ \varepsilon_t^P = \varepsilon_t \circ \gamma$ and $\gamma \circ \varepsilon_s^P = \varepsilon_s \circ \gamma$. Indeed, for all a in A and x, y in C and B respectively, we have by Proposition 2.7

$$\varepsilon_t(xya) = x\varepsilon_t(a)S(y).$$

If $a = 1$ we get $\varepsilon_t(a) = 1$ and so $\varepsilon_t(xy) = xS(y)$. This means $\gamma(\varepsilon_t^P(x \otimes y)) = xS_B(y) = \varepsilon_t(\gamma(x \otimes y))$. Similarly for ε_s. If the algebra is not unital, we can not argue like this because the counital maps ε_s and ε_t have no obvious extensions from A to the multiplier algebra $M(A)$.

In [VD-W5], where we treat integrals and duality, we will consider this example again and show that integrals on (P, Δ_P) automatically exist and therefore that we can obtain a dual version of this example.

Discrete quantum groups

In what follows, we will use the term *discrete quantum group* for a regular multiplier Hopf algebra (A, Δ) of discrete type with a (left) cointegral h satisfying the extra condition that $\varepsilon(h) = 1$ (where ε is the counit). This is the case when h is an idempotent. Then $S(h) = h$ (where S is the antipode). Symmetrically, h is also a right cointegral.

It is shown in Proposition 3.11 of [VD4.v2] that $\Delta(h)$ is a separability idempotent in $M(A \otimes A)$. So here B and C coincide with the original algebra A. The antipodal maps S_B and S_C are both nothing else but the antipode S on A. The distinguished linear

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functionals φ_B and φ_C are the right and the left integrals ψ and φ on (A, Δ), normalized so that $\varphi(h) = \psi(h) = 1$.

Then as a consequence of Theorem 3.2, we get the following.

Proposition 3.5. Let (A, Δ) be a discrete quantum group and h the normalized cointegral. The algebra P defined as $A \otimes A$ is a regular weak multiplier Hopf algebra for the coproduct Δ_P defined by $\Delta_P(a \otimes b) = a \otimes \Delta(h) \otimes b$ with $a, b \in A$. The counit ε_P is given by the linear map $a \otimes b \mapsto \varphi(aS(b))$. We also have $\varepsilon_P(a \otimes b) = \psi(S(a)b)$. The canonical multiplier E_P is $1 \otimes \Delta(h) \otimes 1$. The antipode S_P is given by $S_P(a \otimes b) = S(b) \otimes S(a)$ when $a, b \in A$. The source and target algebras are

$$\varepsilon_s^P(P) = 1 \otimes A \qquad \text{and} \qquad \varepsilon_t^P(P) = A \otimes 1$$

and the source and target maps are given by

$$\varepsilon_s^P(a \otimes b) = 1 \otimes S(a)b \qquad \text{and} \qquad \varepsilon_t^P(a \otimes b) = aS(b) \otimes 1$$

for all $a, b \in A$. Here 1 is the identity in $M(A)$. □

Again we have integrals and we can construct the dual. This will be done in [VD-W5].

A quantization of the groupoid associated with a group action

Let us start by considering the weak multiplier Hopf algebra associated with a groupoid in Example 3.1.i. We can apply the result of Proposition 3.3.

Denote the space of units by X. The source and target algebras B and C are identified with the algebra $K(X)$ of complex functions with finite support on X. Then the canonical idempotent is a separability idempotent in $C(X \times X)$, the algebra of all complex functions on $X \times X$. It is the function with value 1 on the diagonal and 0 on other elements.

We get for P the algebra $K(X \times X)$ of all complex functions with finite support on $X \times X$. The element E_P is the function of four variables x, u, v, y in X that is 1 if $u = v$ and 0 if $u \neq v$. The antipodal maps S_B and S_C on B and C are given by the identity map on the algebra $K(X)$. Therefore, the antipode S_P on $K(X \otimes X)$ is given by the flip map. In fact, the weak multiplier Hopf algebra we get in this way, is nothing else but the algebra of functions on the trivial groupoid $X \times X$ where the product of two elements (x, u) and (v, y) is only defined when $u = v$ and then is (x, y).

It is also interesting to see what happens when we apply Proposition 3.4 in this case. We leave it as an excercise to the reader.

We see that this has very little to do anymore with the original groupoid. And of course, we end up with a special case of Proposition 3.2. For this, we just take any set X and look at the above construction.

Let us now consider the groupoid that results from a group action on a set. So let X be any set and assume that a group H acts on X, say from the left. Denote the action as $h \triangleright x$ for $x \in X$ and $h \in H$. Then there is a groupoid G associated as follows. One has

$$G = \{(y, h, x) \mid x, y \in X \text{ and } h \in H \text{ so that } y = h \triangleright x\}.$$

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The product of two elements (z,k,y') and (y,h,x) is defined if $y=y'$ and then

$$(z,k,y)(y,h,x)=(z,kh,x).$$

The set of units is X and the source and target maps are given by

$$s(y,h,x)=x \qquad \text{and} \qquad t(y,h,x)=y.$$

The set of units is considered as a subset of G by the embedding $x\to(x,e,x)$ where e is the identity in H.

We can construct the weak multiplier Hopf algebras, associated with this groupoid, as in Example 3.1. In the case where the group is trivial, we then get the example we just mentioned above. If on the other hand, the space X is trivial (i.e. it consists only of one point), then we get the multiplier Hopf algebras associated with the group H.

There is however another way to associate a weak multiplier Hopf algebra. It is a special case of a construction that we will consider next.

The *starting point* is as in Theorem 3.2. We have a separability idempotent E in the multiplier algebra $M(B\otimes C)$ of the tensor product of two non-degenerate idempotent algebras B and C. It need not be regular. Furthermore, we have a multiplier Hopf algebra (Q,Δ). Here we assume that it is regular. We will explain why we need this condition for the multiplier Hopf algebra.

We assume that Q acts from the left on C and from the right on B. The actions are denoted by $q\triangleright c$ and $b\triangleleft q$ when $b\in B$, $c\in C$ and $q\in Q$. It is assumed that B is a right Q-module algebra and that C is a left Q-module algebra. In particular, the two actions are unital. Moreover, these data are required to satisfy

$$(E_{(1)}\triangleleft q)\otimes E_{(2)}=E_{(1)}\otimes(q\triangleright E_{(2)}) \tag{3. 2}$$

where we use the Sweedler type notation $E=E_{(1)}\otimes E_{(2)}$ and where the equation is given a meaning by multiplying with an element b of B in the first factor from the left and with an element c of C in the second factor from the right.

The underlying algebra P that we use in this example is a *two-sided smash product* of Q with B and C. The construction has been studied for Hopf algebras (see e.g. [B-P-VO]) but not yet for multiplier Hopf algebras. However, the results and the arguments are very similar to the theory of smash products as developed in [Dr-VD-Z]. Therefore, in the following proposition, we do not give all the details. We concentrate on the correct statements and briefly indicate how things are proven. Remark that the construction only works fine in the case of a regular multiplier Hopf algebra. This is the reason why we need regularity for (Q,Δ).

Proposition 3.6. As above, assume that Q is a regular multiplier Hopf algebra, that B is a right Q-module algebra and that C is a left Q-module algebra. Then the tensor product $C\otimes Q\otimes B$ is an associative algebra P with the product defined as

$$(c\otimes q\otimes b)(c'\otimes q'\otimes b')=\sum_{(q)(q')} c(q_{(1)}\triangleright c')\otimes q_{(2)}q'_{(1)}\otimes(b\triangleleft q'_{(2)})b' \tag{3. 3}$$

where $b,b'\in B$, $c,c'\in C$ and $q,q'\in Q$. □

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Remark that the actions are assumed to be unital and therefore they provide the necessary coverings in (3.3).

The proof of this result is straightforward. Also remark that the algebra P is idempotent if this is the case for B and C.

The two-sided smash product can be considered in two ways as a twisted product in the sense of [VD-VK]. First one considers the twisting of the algebras C and QB (where QB is the ordinary smash product of Q and B). In this case, the twist map is given by the formula

$$qb \otimes c \mapsto \sum_{(q)} (q_{(1)} \triangleright c) \otimes q_{(2)} b$$

where $b \in B$, $c \in C$ and $q \in Q$. For the second possibility, one takes the twisting of the algebras CQ and B (where CQ is the smash product of C and Q). Now the twist map is given by the formula

$$b \otimes cq \mapsto \sum_{(q)} cq_{(1)} \otimes (b \triangleleft q_{(2)})$$

where again $b \in B$, $c \in C$ and $q \in Q$. In the two cases, one now has to verify that the twist map is compatible with the product in the two algebras (ensuring that the result is an associative algebra). One easily verifies that the two constructions give the same algebra and that the result is also the same as in the proposition above.

Just as in the case of smash products, one has obvious embeddings of B, C and Q in the multiplier algebra of P and if we identify these three algebras with their images in $M(P)$, we see that P is the linear span of elements cqb with $b \in B$, $c \in C$ and $q \in Q$ and that we have the commutation rules:
i) B and C commute,
ii) $bq = \sum_{(q)} q_{(1)}(b \triangleleft q_{(2)})$ for all $b \in b$ and $q \in Q$,
iii) $qc = \sum_{(q)} (q_{(1)} \triangleright c) q_{(2)}$ for all $c \in C$ and $q \in Q$.
Therefore we can view P as the algebra generated by B, C and Q subject to these commutation rules.

By definition we have that the map $c \otimes q \otimes b \mapsto cqb$ is a linear bijection from $C \otimes Q \otimes B$ to P. However one also has various other maps that are also bijective. One can consider e.g. the maps

$$b \otimes q \otimes c \mapsto bqc$$
$$b \otimes c \otimes q \mapsto bcq$$
$$q \otimes b \otimes c \mapsto qbc$$

where always $b \in B$, $c \in C$ and $q \in Q$. This property will be used in the proof of Proposition 3.7 below.

Also remark that this construction reduces to well-known constructions in the following three special situations. If the multiplier Hopf algebra Q is trivial, then we obtain for P simply the tensor product algebra $C \otimes B$. If the algebra B is trivial we obtain the smash product $C\#Q$, constructed with the right action of Q on C while if C is trivial, we get the smash product $Q\#B$, for the left action of B on C. Recall that in the original paper [Dr-V-Z], we developed the theory for left actions. The reader can also have a look at

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Section 1 of the expanded version of [De-VD-W] found on the arXiv where the two types of smash products are reviewed.

Then we are ready for the following example.

Proposition 3.7. Assume that B and C are non-degenerate idempotent algebras and that E is a separability idempotent in $M(B \otimes C)$. Let Q be a regular multiplier Hopf algebra and assume that B is a right Q-module algebra and C a left Q-module algebra. Moreover assume the compatibility relation (3.2) as above.

Consider the two-sided smash product P as given in the previous proposition. Then $\Delta(q)$ and E commute in the multiplier algebra of $P \otimes P$ for all $q \in Q$ and the two-sided smash product P can be equipped with a coproduct Δ_P, defined by

$$\Delta_P(cqb) = (c \otimes 1)\Delta(q)E(1 \otimes b) \tag{3. 4}$$

whenever $b \in B$, $c \in C$ and $q \in Q$.

It makes of the pair (P, Δ_P) a weak multiplier Hopf algebra. The canonical idempotent E_P is E, considered as sitting in $M(P \otimes P)$. The counit ε_P is given by the linear map

$$qcb \mapsto \varepsilon(q)\varphi_C(cS_B(b)))$$

where φ_C is the distinguished linear functional on C satisfying $(\iota \otimes \varphi_C)E = 1$ and where S_B is used for the antipodal from B to $M(C)$ associated with the separability idempotent E. The counit ε_P is also given by

$$cbq \mapsto \varphi_B(S_B(c)b)\varepsilon(q)$$

where now φ_B is the distinguished linear functional on B and S_B the antipodal map from C to $M(B)$. Here ε is the counit on Q. The antipode S_P is given by $S_P(cqb) = S_B(b)S(q)S_C(c)$ when $b \in B$, $c \in C$ and $q \in Q$. Here S is the antipode of the multiplier Hopf algebra Q.

The source and target algebras for P are again the algebras B and C, as sitting in $M(P)$ and the source and target maps are given by

$$\varepsilon_s^P(cqb) = (S_C(c) \triangleleft q)b \qquad \text{and} \qquad \varepsilon_t^P(cqb) = c(q \triangleright S_B(b))$$

for all $b \in B$, $c \in C$ and $q \in Q$. Observe that we use the extensions of the actions to the multiplier algebras.

Proof. First we remark that in the proof below, the coproduct, the counit, the antipode for the regular multiplier Hopf algebra Q will be denoted as Δ, ε and S, without the subscript Q. For the coproduct, the counit and the antipode for the new weak multiplier Hopf algebra P, we will use subscripts and write Δ_P, ε_P and S_P. We will use superscripts for the counital maps and write ε_s^P and ε_t^P. For the antipodal maps associated with E we write S_B and S_C. We also use φ_B and φ_C for the distinguished linear functionals on B and C respectively.

i) First, it is not hard to show that E and $\Delta(q)$ for all $q \in Q$ are elements of $M(P \otimes P)$. This is a consequence of the fact that the multiplier algebras of B, C and Q all sit in $M(P)$ and similarly for tensor products.

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ii) We now show that E and $\Delta(q)$ commute in $M(P \otimes P)$. Using the Sweedler notation, both for E as before and for $\Delta(q)$ we get

$$\begin{aligned} E\Delta(q) &= \sum_{(q)} E_{(1)}q_{(1)} \otimes E_{(2)}q_{(2)} \\ &= \sum_{(q)} q_{(1)}(E_{(1)} \triangleleft q_{(2)}) \otimes E_{(2)}q_{(3)} \\ &= \sum_{(q)} q_{(1)}E_{(1)} \otimes (q_{(2)} \triangleright E_{(2)})q_{(3)} \\ &= \sum_{(q)} q_{(1)}E_{(1)} \otimes q_{(2)}E_{(2)} = \Delta(q)E. \end{aligned}$$

In the above calculation, we first have used the commutation rule between B and Q (as the first leg of E is in B), then the relation of the actions of Q on E as in formula (3.2) and finally the commutation rule between C and Q (as the second leg of E is in C). Of course, to make things precise, we need to cover at the right places with the right elements. This can be done if we multiply from the left in the first factor with bp and from the right in the second factor with rc, where $b \in B$, $c \in C$ and $p, r \in Q$.

Then we can define Δ_P on P by the formula (3.4) in the formulation of the proposition. Using the commutation rules, the fact that E is an idempotent, that it commutes with elements $\Delta(q)$ and that Δ is a coproduct on Q, it can be shown that Δ_P is a coproduct on P. It is full.

It is also clear that E, as sitting in $M(P \otimes P)$ has to be the canonical idempotent for Δ_P.

iii) We now prove that there is a counit and that it is given by the formulas in the formulation of the proposition.

First define ε_P on P by $\varepsilon_P(qcb) = \varepsilon(q)\varphi_C(cS_B(b))$ for b, c, q in B, C, Q respectively. Observe that we use a different order of the elements b, c, q in this definition. Then we get for all b, c, q that

$$\begin{aligned} (\iota_P \otimes \varepsilon_P)\Delta_P(cqb) &= (\iota_P \otimes \varepsilon_P)((c \otimes 1)\Delta(q)E(1 \otimes b)) \\ &= \sum_{(q)} cq_{(1)}E_{(1)}\varepsilon_P(q_{(2)}E_{(2)}b) \\ &= \sum_{(q)} cq_{(1)}E_{(1)}\varepsilon(q_{(2)})\varphi_C(E_{(2)}S_B(b)) \\ &= cqE_{(1)}b\varphi_C(E_{(2)}) = cqb. \end{aligned}$$

If on the other hand, we define ε'_P on P by the formula $\varepsilon'_P(cbq) = \varphi_B(S_C(c)b))\varepsilon(q)$, a similar calculation will give then that

$$(\varepsilon'_P \otimes \iota_P)\Delta_P(cqb) = cqb$$

for all b, c, q.

It then follows from the general theory that $\varepsilon'_P = \varepsilon_P$ and that this is the counit.

In the regular case, we consider after the proof of this proposition, we can give a direct argument for the equality of these two expressions for the counit, just as we did in the simpler case in Theorem 3.2 (see a remark after the proof of Theorem 3.3).

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This takes care of the counit.

iv) Let us now look at the *antipode* and the *source* and *target maps*. It is expected that the antipode S_P must coincide with S_B, S_C, S_Q on B, C, Q respectively.

It can be verified that S_P defined in this way, is an anti-homomorphism from P to $M(P)$. For this one has to argue that the definition is compatible with the commutation rules between the component B, C, Q. We will need to use this further in our calculations.

In order to use Theorem 2.9 of [VD-W3] again to prove that (P, Δ_P) is a weak multiplier Hopf algebra, we first must show that the candidates for the maps R_1 and R_2, constructed with the candidate for the antipode map $P \otimes P$ to itself. We do this for R_1.

We have

$$\begin{aligned} R_1(cqb \otimes c'q'b') &= \sum_{(q)} cE_{(1)}q_{(1)} \otimes S_P(E_{(2)}q_{(2)}b)c'q'b' \\ &= \sum_{(q)} cE_{(1)}q_{(1)} \otimes S_B(b)S(q_{(2)})S_C(E_{(2)})c'q'b' \end{aligned}$$

for $c, c' \in C$, $b, b' \in B$ and $q, q' \in Q$. Then we first use that $E_{(1)} \otimes S_C(E_{(2)})b''$ is in $B \otimes B$ for all $b'' \in B$ as we proved in Proposition 1.9 of [VD4.v2]. We use that also $\sum_{(q)} q_{(1)} \otimes S(q_{(2)})q'$ is in $Q \otimes Q$ for all $q, q' \in Q$. All the time, we have to shuffle elements of B, C and Q but this will not present problems. We finally get that $R_1(cqb \otimes c'q'b') \in P \otimes P$. The argument for R_2 is similar.

In order to prove the next conditions, we first calculate the candidates for the counital maps ε_s^P and ε_S^P. For all b, c, q we find

$$\begin{aligned} \varepsilon_s^P(cqb) &= \sum_{(q)} S_P(cE_{(1)}q_{(1)})E_{(2)}q_{(2)}b \\ &= \sum_{(q)} S_P(E_{(1)}cq_{(1)})E_{(2)}q_{(2)}b \\ &= \sum_{(q)} S(q_{(1)})S_C(c)S_B(E_{(1)})E_{(2)}q_{(2)}b \\ &= \sum_{(q)} S(q_{(1)})S_C(c)q_{(2)}b \\ &= \sum_{(q)} S(q_{(1)})q_{(2)}(S_C(c) \triangleleft q_{(3)})b \\ &= (S_C(c) \triangleleft q)b. \end{aligned}$$

In a similar way find

$$\varepsilon_t^P(cqb) = c(q \triangleright S_B(b))$$

for all b, c, q in B, C, Q respectively. We use here the extension of an action to the multiplier algebra. If e.g. $q \in Q$ and $m \in M(C)$ we can define $q \triangleleft m$ by the requirement $q \triangleleft (mc) = \sum_{(q)}(q_{(1)} \triangleleft m)q_{(2)} \triangleleft c$ (see Proposition 4.7 in [Dr-VD-Z]).

Next we verify that T_1R_1 is given by left multiplication by E. For this, it is enough to

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verify that $E(cqb \otimes 1) = (\iota \otimes \varepsilon_t^P)\Delta_P(cqb)$ for all b, c, q. For the left hand side we have

$$\begin{aligned}
E(cqb \otimes 1) &= cE_{(1)}qb \otimes E_{(2)} \\
&= \sum_{(q)} cq_{(1)}(E_{(1)} \triangleleft q_{(2)})b \otimes E_{(2)} \\
&= \sum_{(q)} cq_{(1)}E_{(1)}b \otimes q_{(2)} \triangleright E_{(2)} \\
&= \sum_{(q)} cq_{(1)}E_{(1)} \otimes q_{(2)} \triangleright (E_{(2)}S_B(b)) \\
&= \sum_{(q)} cq_{(1)}E_{(1)} \otimes (q_{(2)} \triangleright E_{(2)})(q_{(3)} \triangleright S_B(b)) \\
&= \sum_{(q)} cq_{(1)}(E_{(1)} \triangleleft q_{(2)}) \otimes E_{(2)}(q_{(3)} \triangleright S_B(b)) \\
&= \sum_{(q)} cE_{(1)}q_{(1)} \otimes E_{(2)}(q_{(2)} \triangleright S_B(b)) \\
&= \sum_{(q)} E_{(1)}cq_{(1)} \otimes E_{(2)}(q_{(2)} \triangleright S_B(b)).
\end{aligned}$$

We find precisely $(\iota \otimes \varepsilon_t^P)\Delta_P(cqb)$. In a similar way we find that T_2R_2 is given by right multiplication with E.

v) Finally, the only thing left is to show that

$$\sum_{(p)} p_{(1)}S_P(p_{(2)})p_{(3)} = p \qquad \text{and} \qquad \sum_{(p)} S_P(p_{(1)})p_{(2)}S_P(p_{(3)}) = S_P(p)$$

for all $p \in P$. We do this e.g. for the first one. We use that

$$\sum_{(p)} p_{(1)}S_P(p_{(2)})p_{(3)} = \sum_{(p)} \varepsilon_t(p_{(1)})S_P(p_{(2)}).$$

Now, if $p = cqb$ we get using the Sweedler notation for E that

$$\begin{aligned}
\sum_{(p)} \varepsilon_t^P(p_{(1)})S_P(p_{(2)}) &= \sum_{(q)} \varepsilon_t^P(cq_{(1)}E_{(1)})q_{(2)}E_{(2)}b \\
&= \sum_{(q)} cq_{(1)} \triangleright (S_B(E_{(1)}))q_{(2)}E_{(2)}b \\
&= \sum_{(q)} cqS_B(E_{(1)})E_{(2)}b = cqb.
\end{aligned}$$

The other formula is proven in a similarly way. This completes the proof. □

Of course, the result in Theorem 3.2 is a special case of the above. Just remark that we have to reformulate the formulas in Theorem 3.3 by considering the algebra P, defined as $C \otimes B$ as the algebra generated by B and C, subject to the commutation of elements of B and elements of C as in i) above. Elements in P are then linear combinations of products cb with $b \in B$ and c in C. The coproduct Δ_P is now given as $\Delta_P(cb) = (c \otimes 1)E(1 \otimes b)$

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in $M(P \otimes P)$. Also P_s and P_t are identified with $M(B)$ and $M(C)$, as sitting in $M(P)$ whereas the source and target maps are

$$\varepsilon_s^P(cb) = S_C(c)b \qquad \text{and} \qquad \varepsilon_t^P(cb) = cS_B(b)$$

when $b \in B$ and $c \in C$.

Consider now the regular case. The following is again expected.

Proposition 3.8. If E is a regular separability idempotent, then (P, Δ_P) is a regular weak multiplier Hopf algebra.

Proof. We could give a direct argument as for the proof of Theorem 3.3. However, here we choose another, simpler way.

If E is regular, we know that the antipodal maps S_B and S_C are anti-isomorphisms from B to C and from C to B respectively. Because Q is also assumed to be a regular multiplier Hopf algebra, also its antipode S is bijective from Q to itself. This all implies that S_P will map P into itself and that it will be bijective. This is equivalent with saying that (P, Δ_P) is a regular weak multiplier Hopf algebra. □

We finish by giving another argument for the equality of the two expressions for the counit in the regular case.

For all b, c, q we have, using again the Sweedler type notation for E,

$$E_{(1)}b \otimes c(q \triangleright E_{(2)}) = (E_{(1)} \triangleleft q)b \otimes cE_{(2)}.$$

This implies that

$$E_{(1)} \otimes c(q \triangleright (E_{(2)}S_B(b))) = ((S_C(c)E_{(1)}) \triangleleft q)b \otimes E_{(2)}.$$

If we apply $\varphi_B \otimes \varphi_C$, we find

$$\varphi_C(c(q \triangleright S_B(b))) = \varphi_B((S_C(c) \triangleleft q)b). \tag{3. 5}$$

for all b, c, q. This is one equation we will use.

If again we start with equation (3.2), apply φ_B on the first factor and use fullness of E we find that $\varphi_B(b \triangleleft q) = \varepsilon(q)\varphi_B(b)$. Similarly, if we apply φ_C on the second leg, we will get $\varphi_C(q \triangleright c) = \varepsilon(q)\varphi_C(c)$. In other words, the distinguished linear functionals φ_B and φ_C are invariant under the actions of Q.

Define ε_P and ε'_P as in the proof of Proposition 3.7. We will use the above results to give a proof of the equality of these counits.

We find on the one hand

$$\begin{aligned}
\varepsilon_P(cqb) &= \sum_{(q)} \varepsilon_P(q_{(2)}(S^{-1}(q_{(1)}) \triangleright c)b) \\
&= \sum_{(q)} \varepsilon(q_{(2)})\varphi_C((S^{-1}(q_{(1)}) \triangleright c)S_B(b)) \\
&= \varphi_C((S^{-1}(q) \triangleright c)S_B(b))
\end{aligned}$$

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while on the other hand

$$\begin{aligned}\varepsilon'_P(cqb) &= \sum_{(q)} \varepsilon'_P(c(b \triangleleft S^{-1}(q_{(2)}))q_{(1)}) \\ &= \sum_{(q)} \varepsilon(q_{(1)})\varphi_B(S_C(c)(b \triangleleft S^{-1}(q_{(2)}))) \\ &= \varphi_B(S_C(c)(b \triangleleft S^{-1}(q))).\end{aligned}$$

So, we need to show that

$$\varphi_C((S^{-1}(q) \triangleright c)S_B(b)) = \varphi_B(S_C(c)(b \triangleleft S^{-1}(q))) \tag{3. 6}$$

for all b, c, q.

For the left hand side of (3.6) we find

$$\begin{aligned}\varphi_C((S^{-1}(q) \triangleright c)S_B(b)) &= \sum_{(q)} \varphi_C(q_{(2)} \triangleright ((S^{-1}(q_{(1)}) \triangleright c)S_B(b)) \\ &= \varphi_C(c(q \triangleright S_B(b)).\end{aligned}$$

We have used that φ_C is invariant under the action of Q. For the right hand side of (3.6) we get

$$\begin{aligned}\varphi_B(S_C(c)(b \triangleleft S^{-1}(q))) &= \sum_{(q)} \varphi_B((S_C(c)(b \triangleleft S^{-1}(q_{(2)})) \triangleleft q_{(1)} \\ &= \varphi_B(S_C(c) \triangleleft q)b).\end{aligned}$$

Here we have used that φ_B is invariant under the action of Q.

Then the equation (3.6) follows from the equation (3.5)

Again, the argument does not seem to work if E is not regular. Fortunately, we do not need it as we have obtained the equality in another way.

We have not included examples of weak multiplier Hopf *-algebras. In fact, the basic examples (Example 3.1) are weak multiplier Hopf *-algebras for the obvious involutive structures. If in the example of Theorem 3.2, the algebras B and C are *-algebras and if E is self-adjoint, then the associated pair (P, Δ_P) will be a weak multiplier Hopf *-algebra for the involutive structure on $B \otimes C$ obtained from the ones on the factors B and C. For a discrete quantum group (as in Proposition 3.5), we obtain a weak multiplier Hopf *-algebra if the original discrete quantum group is a multiplier Hopf *-algebra of discrete type. Finally, if in Proposition 3.7 we start with a self-adjoint separability idempotent and and with appropriate actions of a multiplier Hopf *-algebra, again we will end up with a weak multiplier Hopf *-algebra.

All these statements are more or less straightforward and we leave the verification as an exercise to the reader.

4. Conclusions and further research

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In this paper, we have studied the source and target maps, as well as the source and target algebras, associated with a weak multiplier Hopf algebra. We have obtained results in the general case in Section 2. And we have payed special attention to the regular case. It is still not clear if the nicer results, obtained in the regular case, can be pushed forward to the non-regular case so that also there, better results can be shown. We expect however that this will not be easy, neither to prove these results if they are true, nor to find counter examples if they are not.

In fact, non-regular examples are not so easy to construct. Of course, there are the examples of Hopf algebras with a non-invertible antipode. But at this moment, we do not know of examples of multiplier Hopf algebras with a non-regular coproduct, that is with a coproduct Δ on a non-degenerate algebra A so that elements of the form $\Delta(a)(b \otimes 1)$ and $(1 \otimes c)\Delta(a)$ do not always are in $A \otimes A$ for $a, b, c \in A$. More research here is needed.

Section 3 of the paper is devoted to examples. All of the examples we give are generalizations of known examples of finite-dimensional weak Hopf algebras. The duals of some of these examples, included in [VD-W5], are probably not yet considered, even in the case of finite-dimensional weak Hopf algebras. Nevertheless, it would still be desirable to find more examples and in particular, examples that are not simply generalizations of known examples of weak Hopf algebras. We refer also to the modification procedure as explained in [VD5] to construct new examples of regular weak multiplier Hopf algebras.

The separability elements for non-unital algebras play an important role in Section 3. It is certainly worthwhile to carry out a more thorough study of these separable non-unital algebras and the associated separability idempotents (and to relate our approach with other approaches in the literature). This is partly done already in [VD4.v1]. A new version of this paper contains more information [VD4.v2]. However, there is still the open question of the existence of non-regular separability idempotents as posed in Section 5 of [VD4.v2].

Some of the examples suggest certain generalizations of the theory. Consider e.g. a multiplier Hopf algebra (A, Δ) of discrete type. Denote by h a left cointegral. Either it can be normalized so that $\varepsilon(h) = 1$ and hence $h^2 = h$ (where ε is the counit), or we have $\varepsilon(h) = 0$ and then $h^2 = 0$. The first case is considered in Proposition 3.5 of this paper. The other case does not fit into this theory because h and hence $\Delta(h)$ is not an idempotent. Still, it has most of the other properties of a separability idempotent. The two antipodal maps exist. Indeed, on one side we simply have

$$(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h).$$

The other side is different because h is not necessarily a right cointegral. However, by the uniqueness of cointegrals, there is a homomorphism $\gamma : A \to \mathbb{C}$ defined by $ha = \gamma(a)h$ for all a. Then

$$\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S'(a))$$

where $S'(a) = \sum_{(a)} \gamma(a_{(1)})S(a_{(2)})$. This is discussed in Section 5 of [VD4.v2].

Finally, as we mentioned already in the introduction, the material studied in this paper relates intimately with other research. On the one hand there is the study of weak multiplier bialgebras as introduced in [B-G-L]). We also have [K-VD] where a Larson-Sweedler type theorem is proven. Roughly it says that a weak multiplier bialgebra with enough integrals

is a weak multiplier Hopf algebra. Here we have properties of the source and target maps and source and target algebras, proven in the context of weak multiplier bialgebras and separability idempotents.

The other obvious link with the literature is the theory of multiplier Hopf algebroids as developed in [T-VD1]. In particular, there is the paper [T-VD2] where the relation between weak multiplier Hopf algebras and multiplier Hopf algebroids is studied. It seems interesting to observe that there are various possible reasons why a multiplier Hopf algebroid does not have an underlying weak multiplier Hopf algebra.

We would like to emphasize again the importance of this paper, with the results on the source and target algebras and source and target maps for the study of integrals on weak multiplier Hopf algebras and the construction of the dual in the case enough such integrals exist. We refer to [VD-W5].

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