About calculation the optimal labor time in the Mirrlees model

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In this paper we analyze and propose new method and algorithm of selecting the optimal labor time as a function of skills following our main references Mirrlees (1971), Saez (2001) and Stancheva (2014). The optimal labor time is a situation when the utility function of individual reaches a maximum. One of the main differences with Saez (2001) is our algorithm for creation the skill distribution.

Key words: optimal taxation, welfare analysis, utility function.

Introduction.

Saez (2001) proposed and considered the distribution of skills with using empirical distribution of income and the approximation the labor income tax by linear model. The author proved that with using the presented utility function of consumption and labor effort, it will be possible to obtain some level of skills which will be adaptable with observed (public information) taxable revenue which corresponds proposed linear tax schedule.

Let's denote that the similar algorithm and selection don't exist for every function. Individual utility is assumed to be increasing and concave in consumption (after-tax income) and decreasing, convex in labor effort. The utility function should be also twice differentiable and grows slower than the linear function, the graph is under any straight line and it means convex upward (concave). If $u(x)$ is a convex upward function, so the second derivative is ***negative***.

Two examples of utility function which satisfy the conditions and requirements proposed by our references have been considered.

Note also, that in the recent papers the distribution of skills is the Pareto distribution has been proposed.

We assume that skills are assumed to be a random value distributed over gamma law.

The task of tax optimization has been formulated in the recent publications *[1, 3]* on the basis of some optimality criterion. Let's denote the main ***definitions:***

The parameter ***n*** is the productivity parameter or skill. This productivity is a positive random value (the various level for different employees) with probability density function $f(x), x>0; f(x)=0, x\le 0$. Then, $P\{a<n<b\}=\int_a^b f(x)\,dx$.

The earnings are defined as $z=nl(n)$, $l(n)$ is the labor supply. $T(z)$ is the income tax, $c=z-T(z)$ is after-tax income.

Let's consider a model where each taxpayer maximizes utility function $u(n)=u(c(n),l(n))$, which depends positively on consumption c and negatively on labor l.

Let's rewrite utility function:

$u(c,l)=u(nl-T(nl),l)$ and depends from the variables n, l.

Remark. Let's choose the density *f(x)* as a Gamma distribution:

$$f(x)=\frac{\mu^{p+1}}{\Gamma(p+1)}x^p e^{-\mu x}, x>0; \mathrm{f(x)}=0, \mathrm{x}\le 0.$$

The parameters p, μ are estimated by sample.

The main assumption: Every employee chooses his optimal (which depends from productivity) labor time $l_n=\varphi(n)$. Thus, the composition $v(n)=u(nl_n-T(nl_n),l_n)$ depends only from random value n (productivity parameter).

The quality criteria for construction a mathematical algorithm of optimal tax model is a maximum of the functional

$$W(T)=\mathrm{E}G(u(nl_n-T(nl_n),l_n))=\int_0^\infty G(u(nl_n-T(nl_n),l_n))\,f(n)dn \qquad \textbf{\textit{(1)}}$$

subject to budget constraint

$$\mathrm{E}(T(nl_n)) = \int_0^\infty T(nl_n)\, f(n)\, dn \ge R \qquad \textbf{(2)}$$

where *G(t)* is a positive increasing concave function, *E* is a symbol of expected value corresponding to the distribution of random productivity *n.*

Let's define optimal labor time l_n of the employee:

$$u(nl_n - T(nl_n), l_n) > u(nl - T(nl), l),$$

so the choice l_n is maximizing utility function with given productivity. Denote:

$$u(nl - T(nl), l) = q(l)$$

and find a maximum of the function $q(l)$ with fixed *n.*

The general conditions for maximum:

$$q'(l) = \frac{d}{dl} u(nl - T(nl), l) = n\frac{\partial u}{\partial c}(1 - T'(nl)) + \frac{\partial u}{\partial l} = 0 \qquad \textbf{(3)}$$

$$q''(l) = \frac{d^2}{dl^2} u(nl - T(nl), l) = n^2(1 - T')^2 \frac{\partial^2 u}{\partial c^2} + 2n(1 - T')\frac{\partial^2 u}{\partial c \partial l} + \frac{\partial^2 u}{\partial l^2} - n^2 T'' \frac{\partial u}{\partial c} < 0 \qquad \textbf{(4)}$$

If inequality ***(4)*** is satisfied, so equality ***(3)*** is solvable with respect to *l* by the implicit function theorem: $l = \varphi(n) = l_n$. In particular, Saez (2001) considered example of the utility function:

$$u(c, l) = \log(c - \frac{l^k}{k}), k > 1$$

and equation $\frac{d}{dl} u(nl - T(nl), l) = 0$, which can represent as:

$n(1 - T'(nl)) = l^{k-1}$ is solvable with respect to *l,* if $nl - T(nl) - \frac{l^k}{k} > 0.$

In this paper we analyze the relations ***(3),(4)*** and consider the examples of utility function *u(c,l).*

The statement of problem: when the system (3), (4) has a unique solution with respect to *l* as a function of *n* (if the conditions of income tax : $0 < T'(z) \le M < 1, T''(z) > 0$) and how to describe utility function *u(c,l)?*

Remark. Let's consider an example, when equation (3) is not solvable with respect to *l.* It's linear tax case: $T(z) = kz, 0 < k < 1$. So, $u(nl - T(nl), l) = u((1-k)nl, l)$ and we can rewrite (3) as:

$$\frac{d}{dl}u((1-\mathrm{k})\,\mathrm{nl},\mathrm{l}) = \mathrm{n}(1-\mathrm{k})\frac{\partial u}{\partial c} + \frac{\partial u}{\partial l} = 0$$

and solution of this equation is the utility function

$u(c,l) = g(c - n(k-1)l), g(t)$, *g(t)* is an arbitrary differentiable function of one variable.

Then for $c = (1-k)nl$, $u((1-k)\,\mathrm{nl},\mathrm{l}) = \mathrm{g}(0) = \mathrm{const}$ ■

Next, let's consider model examples of the utility function and the solvability conditions for them of the equation (3).

Example 1.

$$u(c,l) = \frac{c^{\alpha}}{(b+l)^{\beta}}, 0 < \alpha < 1, \beta > 0, b > 0 \qquad \textbf{(5)}$$

The derivatives of this function with respect to the variable *l:*

$$q'(l) = \frac{(nl + T(nl))^{\alpha-1}}{(b+l)^{\beta+1}} Q_1(n,l); q''(l) = \frac{(nl + T(nl))^{\alpha-2}}{(b+l)^{\beta+2}} Q_2(n,l), where$$

$$Q_1(n,l) = \alpha nl + \alpha nb - \alpha nlT'(nl) - \alpha nbT'(nl) - \beta nl + \beta T(nl);$$
$$Q_2(n,l) = \alpha(\alpha-1)n^2(b+l)^2(1-T'(nl))^2 -$$
$$-2\alpha\beta n(b+l)(nl - T(nl))(1-T'(nl)) - \alpha n^2(b+l)^2(nl - T(nl))T''(nl) + \beta(\beta+1)(nl - T(nl))^2.$$

Let's analyze obtained formulas, using a priory relations for *T(z):* : $0 < T'(z) \le M < 1, T''(z) > 0$.

The maximum of the function *q(l):*

$$\alpha nl+\alpha nb-\alpha nlT'(nl)-\alpha nbT'(nl)-\beta nl+\beta T(nl)=0 \qquad \textbf{(3A)}$$

$$\begin{aligned}&\alpha(\alpha-1)n^2(b+l)^2(1-T'(nl))^2-2\alpha\beta n(b+l)(nl-T(nl))(1-T'(nl))-\\&-\alpha n^2(b+l)^2(nl-T(nl))T''(nl)+\beta(\beta+1)(nl-T(nl))^2<0\end{aligned} \qquad \textbf{(4A)}$$

It follows from (3A) that

$$1-T'(nl)=\frac{\beta}{\alpha}\frac{nl-T(nl)}{nl+nb}, T'(nl)=\frac{nl(1-\frac{\beta}{\alpha})+nb+\frac{\beta}{\alpha}T(nl)}{nl+nb}$$

and the condition $\beta\le\alpha$ is sufficient to perform a priori relations. Only the last term is positive in (4A).

Let's substitute from (3A)

$$nl-T(nl)=\frac{\alpha}{\beta}n(b+l)(1-T'(nl))$$

in the second and fourth terms (4A). We obtain:

$$\begin{aligned}&Q_2(n,l)=\alpha(\frac{\alpha}{\beta}-1)n^2(b+l)^2(1-T'(nl))^2-\alpha n^2(b+l)^2(nl-\\&-T(nl))T''(nl)\end{aligned}$$

and the condition $\beta\ge\alpha$ is sufficient for inequality $Q_2(n,l)<0$. Let's get a conclusion $\alpha=\beta$ and the following formulas, taking into account the previous relation $\beta\le\alpha$: for utility function

$$u(c,l)=\frac{c^{\alpha}}{(b+l)^{\alpha}}, 0<\alpha<1, b>0 \qquad \textbf{(5A)}$$

and the derivatives with respect to a variable *l:*

$$\frac{d}{dl}u(nl-T(nl),l)=\alpha\frac{(nl-T(nl))^{\alpha-1}}{(b+l)^{\alpha+1}}(nb+T(nl)-n(b+l)T'(nl))$$

$$\frac{d^2}{dl^2}u(nl-T(nl),l)=-\frac{(nl-T(nl))^{\alpha-1}}{(b+l)^{\alpha}}\alpha n^2T''(nl)$$

and the equation for l_n :

$nb + T(nl) - b(b+l)T'(nl) = 0$

Convert this equation to the form:

$\frac{n(b+l)T'(nl) - T(nl)}{(b+l)^2} = \frac{nb}{(b+l)^2}$ and integrating to respect to variable *l*,

$\frac{T(nl)}{b+l} = C - \frac{nb}{b+l}$, C is an constant of integration.

If $T(0) = -A$, $C = n - \frac{A}{b}$ and finally get the equation for determining the optimal labor time

$$l = \varphi(n); \mathrm{T(nl)} = \mathrm{l}(\mathrm{n} - \frac{A}{b}) - A. \qquad \textbf{(6)}$$

Statement. The equation (6) has a unique solution $l = \varphi(n) \equiv \mathrm{l}_n \; for \; n \geq n_0$.

Proof. Let's denote $\Phi(n,l) = T(nl) - \mathrm{l}(\mathrm{n} - \frac{A}{b}) + A$. The equation $\Phi(n,l) = 0$ is uniquely solvable with respect to variable *l* by the implicit function theorem, if

$$\frac{\partial}{\partial l}\Phi(n,l) = n(T'(nl) - 1) + \frac{A}{b} \neq 0.$$

The first term $n(T'(nl) - 1) \leq -n_0(1 - M)$, so to choose sufficiently large parameter b in the model (5A) (A is determined by law, lumpsum grant): $\frac{\partial}{\partial l}\Phi(n,l) < 0$.

The derivative $\varphi(n)$ is calculated by the formula:

$$\varphi'(n) = -\frac{\partial\Phi}{\partial n}\bigg/\frac{\partial\Phi}{\partial l} = -\frac{l(T'(nl) - 1)}{n(T'(nl) - 1) + \frac{A}{b}} \qquad \blacksquare$$

Example 2, taking into account the limited labor time: $0 \leq l \leq T$;

$u(c,l) = g(c)h(l); g: R^+ \to R^+, g$ is increasing, concave;

$h: [0;T] \to [0; h_{\max}], h$ is decreaing,concave, $h(\mathrm{l}) = 0, if\; l \geq T$

(for example, $g(c) = \ln(a + c), a \geq 1; h(l) = T^\beta - l^\beta, \beta > 1$).

Let's calculate the derivatives:

$$\frac{d}{dl}g(nl-T(nl))h(l)=ng'(nl-T(nl))(1-T'(nl))h(l)+g(nl-T(nl))h'(l);$$
$$\frac{d^2}{dl^2}g(nl-T(nl))h(l)=n^2(g''(1-T')^2-g'T''')h+2ng'(1-T')h'+gh''.$$

If $\frac{d^2}{dl^2}g(nl-T(nl))h(l)<0$ depends from the properties of functions *g,h,* so the equation $\frac{d}{dl}g(nl-T(nl))h(l)=0$ determines the maximum. Let's rewrite this equation:

$$\frac{ng'(nl-T(nl))(1-T'(nl))}{g(nl-T(nl))}=-\frac{h'(l)}{h(l)}$$

After integration:

$$g(nl-T(nl))h(l)h(l)=const=gh\big|_{l=0}=g(0)h_{\max}$$

The variable can be expressed by this equation.

Thus, the investigation of optimal income tax *T(z)* consists for solving the extremal problem (1) subject to budget constraint (2) or to study the functional with Lagrange multiplier λ

$$J(T)=\frac{\mu^{p+1}}{\Gamma(p+1)}\int_0^\infty (G(u(nl_n-T(nl_n),l_n))+\lambda(T(n\varphi(n))-R))n^p e^{-\mu n}dn,$$

where $l_n=\varphi(n)$ satisfies the equation

$$\frac{d}{dl}u(nl-T(nl),l)\Big|_{l=l_n}=0 \ \ and\ condition\ \frac{d^2}{dl^2}g(nl-T(nl))h(l)\Big|_{l=l_n}<0$$

In this case:

$$\frac{\mu^{p+1}}{\Gamma(p+1)}\int_0^\infty T(n\varphi(n))n^p e^{-\mu n}dn=R.$$

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