Two removal and cancellation laws associated with a complex matrix and its conjugate transpose

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Abstract. A complex square matrix A is said to be Hermitian if $A = A^*$, the conjugate transpose of A. The topic of the present note is concerned with the characterization of Hermitian matrix. In this note, the we show that each of the two triple matrix product equalities $AA^*A = A^*AA^*$ and $A^3 = AA^*A$ implies that A is Hermitian by means of decompositions and determinants of matrices, which are named the two-sided removal and cancellation laws associated with Hermitian matrix, respectively. We also present several general removal and cancellation laws as the extensions of the preceding two facts about Hermitian matrix.

Keywords: Hermitian matrix; matrix decomposition; cancellation property

AMS classifications: 15A24; 15A27; 15B57

Throughout this note, let $\mathbb{C}^{m\times n}$ denote the collections of all $m \times n$ matrices over the the field of complex numbers; A^* denote the conjugate transpose $A \in \mathbb{C}^{m\times n}$; $|A|$ denote the determinant of $A \in \mathbb{C}^{m\times n}$; and I_m denote the identity matrix of order m.

Recall that an $A \in \mathbb{C}^{m\times m}$ is said to be Hermitian if and only if it satisfies the following equality

$$A = A^*, \tag{1}$$

or element-wise, $a_{ij} = \overline{a}_{ji}$ for $i, j = 1, 2, \ldots, m$. Hermitian matrices are basic conceptual objects and building blocks in matrix theory and linear algebra, which have many elegant and pleasing formulas and facts, and have many significant applications in the research areas of both theoretical and applied mathematics. Although there have many studies and results concerning Hermitian matrices, there are various new and challenging problems that can be proposed or encountered in matrix analysis and its many areas of application. Given a square matrix $A \in \mathbb{C}^{m\times m}$, the assumption that A is Hermitian is a strong requirement from the matrix equation point of view, but it occurs naturally in the representations of quadratic forms, as well as in the Toeplitz decomposition of any square matrix A:

$$A = (A + A^*)/2 - i(iA + (iA)^*)/2, \tag{2}$$

where two matrices $(A+A^*)/2$ and $(iA+(iA)^*)/2$ are Hermitian (cf. [4]). Hence, it is well known that Hermitian matrices play central role in the developments of matrix theory and applications.

Apart from the definition, there are some characterizations of Hermitian matrix in the literature. Here we should mention the following two well-known cases

$$A^2 = AA^* \Leftrightarrow A = A^*, \tag{3}$$
$$A^2 = A^*A \Leftrightarrow A = A^*. \tag{4}$$

The underlying meaning of these two facts is that we can cancel A from the left- and right-hand sides of the first two equalities in (3) and (6) to yield the definition equality for Hermitian matrix without assuming that A is invertible. Obviously, these two cancellation laws can be utilized to simplify matrix equalities that involve the corresponding matrix products. The reader is to referred to [1–3, 14, 15] for their expositions and derivations. The above cancellation laws seem to be certain generalizations of the cancellation properties associated with invertible matrices although they are obtained by some other indirect matrix operations, so that they let people perceive relationships between different kinds of matrix equalities. It has been realized (cf. [8]) that this kind of cancellation problems can be regarded as special cases of the following two-sided implication facts

$$f(A, A^*) = 0 \Leftrightarrow A = \pm A^*, \tag{5}$$

where $f(\cdot)$ is certain conventional algebraic operation of A and A^* namely, Hermitian/skew-Hermitian matrices are exclusive solutions of the matrix equation on the left-hand side.

In this note, the author considers some extensions of the two cancellation laws to the cases for triple matrix products composed of A and A^*, and establishes some new equivalent facts for a matrix to be Hermitian as follows.

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Theorem 1. *Let $A \in \mathbb{C}^{m\times m}$. Then the following three statements are equivalent:*

(a) *A is Hermitian.*

(b) *AA^*A is Hermitian, i.e., $AA^*A = A^*AA^*$.*

(c) *$A^3 = AA^*A$.*

To show Theorem 1, we need to use the following known results.

Lemma 2. *Let $A \in \mathbb{C}^{m\times n}$ and $B, C \in \mathbb{C}^{n\times p}$. Then the following results hold:*

(a) [7] *$A^*AB = A^*AC \Leftrightarrow AB = AC$.*

(b) [13] *The principal nth root of positive semi-definite matrix exists and is unique.*

Lemma 3 ([5]). *Let $A \in \mathbb{C}^{m\times m}$. Then there exists a unitary matrix U such that A admits the following decomposition:*

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{6}$$

where Σ is a positive diagonal matrix, K is square matrix, and $KK^* + LL^* = I_s$ with $s = \operatorname{rank}(A)$.

Proof of Theorem 1. Result (a) obviously implies (b) and (c) by the definition of Hermitian matrix. On the contrary, if (b) holds, then we have the following implication facts:

$$\begin{aligned} AA^*A = A^*AA^* &\Rightarrow (AA^*A)(A^*AA^*) = (A^*AA^*)(AA^*A) \\ &\Rightarrow (AA^*)^3 = (A^*A)^3 \Rightarrow AA^* = A^*A \quad \text{(by Lemma 2(b))} \\ &\Rightarrow A^*A^2 = A^*AA^* \Rightarrow A^2 = AA^* \quad \text{(by Lemma 2(a))} \\ &\Rightarrow A = A^* \quad \text{(by (3))}, \end{aligned}$$

namely, (b) implies (a). By (6), the following decomposition equalities for the four matrix products A^2, AA^*, A^3A^*, and $AA^*)^2$:

$$A^2 = U\begin{bmatrix} (\Sigma K)^2 & \Sigma K\Sigma L \\ 0 & 0 \end{bmatrix}U^*, \qquad AA^* = U\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{7}$$

$$A^3A^* = U\begin{bmatrix} (\Sigma K)^2\Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}U^*, \qquad (AA^*)^2 = U\begin{bmatrix} \Sigma^4 & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

hold. Based on these decomposition equalities, we have the following derivations:

$$\begin{aligned} & A^3 = AA^*A \\ & \Rightarrow A^3A^* = (AA^*)^2 \Rightarrow U\begin{bmatrix} (\Sigma K)^2\Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} \Sigma^4 & 0 \\ 0 & 0 \end{bmatrix}U^* \text{ and } KK^* + LL^* = I_s \\ & \Rightarrow (\Sigma K)^2\Sigma^2 = \Sigma^4 \text{ and } KK^* + LL^* = I_s \\ & \Rightarrow (\Sigma K)^2 = \Sigma^2 \text{ and } KK^* + LL^* = I_s \\ & \Rightarrow |(\Sigma K)^2| = |\Sigma|^2|K|^2 = |\Sigma|^2 \text{ and } KK^* + LL^* = I_s \\ & \Rightarrow |K|^2 = 1,\ 0 \leq KK^* \leq I_s,\ 0 \leq LL^* \leq I_s, \text{ and } |I_s - LL^*| = |KK^*| = |K||K^*| = |K|\overline{|K|} \\ & \Rightarrow 0 \leq LL^* \leq I_s \text{ and } |I_s - LL^*| = 1 \\ & \Rightarrow (\Sigma K)^2 = \Sigma^2 \text{ and } L = 0 \\ & \Rightarrow A^2 = AA^* \quad \text{(by (7))} \\ & \Rightarrow A = A^* \quad \text{(by (3))}, \end{aligned}$$

namely, (c) implies (a). □

The equivalences of the three assertions in Theorem 1 shows that we can remove A and A^* from both sides of $AA^*A = A^*AA^*$ simultaneously to yield $A^* = A$, while the equivalence of Theorem 1(a) and (c) shows that we can also cancel A from both sides of $A^3 = AA^*A$ simultaneously to yield $A = A^*$. Hence, we call the two facts in Theorem 1 (a) and (b), (a) and (c) the two-sided removal and cancellation laws, respectively. It should be pointed out that the two removal/cancellation laws are not isolated facts associated with conjugate transpose operation of a square matrix, but we are able to propose and prove many types of removal/cancellation facts for the multiplications of complex square matrices and their conjugate transposes. As direct consequences of the preceding results, we present the following three groups of removal and cancellation facts:

(a) Let $A \in \mathbb{C}^{m\times m}$. Then following facts hold:

$$(AA^*A)^2 = (AA^*)^3 \Leftrightarrow (AA^*A)^2 = (A^*A)^3 \Leftrightarrow A = A^*,$$
$$(AA^*A)^3 = (AA^*)^2A(A^*A)^2 \Leftrightarrow A^2A^*A^2 = AA^*AA^*A \Leftrightarrow A = A^*,$$
$$A^3 = A^*AA^* \text{ and } A^5 = A^*AA^*AA^* \Leftrightarrow A = A^*,$$
$$A^5 = AA^*AA^*A \text{ and } A^7 = AA^*AA^*AA^*A \Leftrightarrow A = A^*,$$

which are obtained by replacing A with AA^*A in (3) and Theorem 1 and some algebraic derivations.

(b) Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then the following fact holds:

$$(BA)^2 = BB^*A^*A \Leftrightarrow AB = B^*A^*.$$

(c) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times m}$. Then the following fact holds:

$$CABCA = CC^*B^*A^*A \Leftrightarrow ABC = C^*B^*A^*.$$

As concrete cases of the equivalence problems described in (5), there are some other mixed cancellation laws associated with Hermitian matrices that were proposed and proved (cf. [1,3,15]). In comparison, the preceding removal/cancellation laws link several fundamental matrix equalities together, so that they should be recognized as some fundamental facts and common knowledge regarding Hermitian matrices and their algebraic operations in matrix theory and applications.

Recall that as an extension of the concept of the conjugate transpose of complex matrix, the $*$-involution operation of an element in an associative ring R (or semigroups and algebras) is defined to be a mapping $a \to a^*$ if it satisfies the equalities $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, and $(ab)^* = b^*a^*$ for all $a, b \in R$. In this case, it would be interest to consider the previous removal/cancellation laws in the algebraic setting with $*$-involution, where self-adjoint elements can be defined (cf. [2, 8–12]). Notice that the proof of Theorem 1 uses some well-known facts regarding the existence and uniqueness of the p-th root of positive semi-definite Hermitian matrix, as well as matrix decompositions, and determinants of matrices, which are no longer available to use in general algebraic settings. Thus, the proofs or disproofs of the preceding removal/cancellation laws for self-adjoint elements in general algebraic settings should be given by means of other kinds of analysis and calculation methods.

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